

Characterization of Banach spaces Y satisfying that the pair (ℓ_∞^4, Y) has the Bishop-Phelps-Bollobás property for operators

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ABSTRACT. We study the Bishop-Phelps-Bollobás property for operators from ℓ_∞^4 to a Banach space. For this reason we introduce an appropriate geometric property, namely the $\text{AHSp-}\ell_\infty^4$. We prove that spaces Y satisfying $\text{AHSp-}\ell_\infty^4$ are precisely those spaces Y such that (ℓ_∞^4, Y) has the Bishop-Phelps-Bollobás property. We also provide classes of Banach spaces satisfying this condition. For instance, finite-dimensional spaces, uniformly convex spaces, $C_0(L)$ and $L_1(\mu)$ satisfy $\text{AHSp-}\ell_\infty^4$.

1. Introduction

Bishop-Phelps theorem [10] states that every continuous linear functional on a Banach space can be approximated (in norm) by norm attaining functionals. Bollobás proved a “quantitative version” of that result [11]. In order to state such result, we denote by B_X , S_X and X^* the closed unit ball, the unit sphere and the topological dual of a Banach space X , respectively. If X and Y are both real or both complex Banach spaces, $L(X, Y)$ denotes the space of (bounded linear) operators from X to Y , endowed with its usual operator norm.

Bishop-Phelps-Bollobás Theorem (see [12, Theorem 16.1], or [15, Corollary 2.4]). Let X be a Banach space and $0 < \varepsilon < 1$. Given $x \in B_X$ and $x^* \in S_{X^*}$ with $|1 - x^*(x)| < \frac{\varepsilon^2}{4}$, there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$.

A lot of attention has been devoted to extending Bishop-Phelps theorem to operators and interesting results have been obtained about that topic. For instance, we mention the remarkable results by Lindenstrauss [23], Bourgain [13] and Gowers [19]. In 2008 the study of extensions of Bishop-Phelps-Bollobás theorem to operators was initiated by Acosta, Aron, García and Maestre [2]. In order to state some of these extensions it will be convenient to recall the following notion.

DEFINITION 1.1. ([2, Definition 1.1]). Let X and Y be either real or complex Banach spaces. The pair (X, Y) is said to have the Bishop-Phelps-Bollobás property for operators (BPBp) if for every $0 < \varepsilon < 1$

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there exists $0 < \eta(\varepsilon) < \varepsilon$ such that for every $T \in S_{L(X,Y)}$, if $x_0 \in S_X$ satisfies $\|T(x_0)\| > 1 - \eta(\varepsilon)$, then there exist an element $u_0 \in S_X$ and an operator $S \in S_{L(X,Y)}$ satisfying the following conditions

$$\|S(u_0)\| = 1, \quad \|u_0 - x_0\| < \varepsilon \quad \text{and} \quad \|S - T\| < \varepsilon.$$

Acosta, Aron, García and Maestre showed that the pair (X, Y) has the BPBp whenever X and Y are finite-dimensional spaces [2, Proposition 2.4]. They also proved that the pair (X, Y) has the BPBp in case that Y has a certain isometric property (called property β of Lindenstrauss), for every Banach space X [2, Theorem 2.2]. For instance, the spaces c_0 and ℓ_∞ have such property. In case that the domain is ℓ_1 they obtained a characterization of the Banach spaces Y such that (ℓ_1, Y) has the BPBp [2, Theorem 4.1]. There was also proved that for several classical spaces Y , and for any positive measure μ , the pair $(L_1(\mu), Y)$ have the BPBp, for instance when $Y = L_\infty(\nu)$ or $Y = L_1(\nu)$ (see [16], [8] and [17]). For those results the proofs are involved and interesting. Even so, as far as we know, there are no characterization of the spaces Y such that the pair $(L_1([0, 1]), Y)$ has the BPBp.

However, the case of $X = c_0$ is quite different from the case $X = \ell_1$ and seems to be much more difficult. Now we will list results about this topic where the domain is a space $C_0(L)$ (L is a locally compact Hausdorff space). It was shown that (ℓ_∞^n, Y) has the BPBp for every positive integer n whenever Y is uniformly convex [2, Theorem 5.2]. In fact Kim proved that (c_0, Y) also has this property under the same assumption [20, Corollary 2.6]. Aron, Cascales and Kozhushkina showed that $(X, C_0(L))$ has the BPBp if X is Asplund [7, Corollary 2.6] (see also [14] for an extension of this result). As a consequence, the pair $(c_0, C_0(L))$ has the Bishop-Phelps-Bollobás property for operators. In the real case, it is also known that the pair $(C(K), C(S))$ also satisfies the BPBp, for any compact Hausdorff spaces K and S [3, Theorem 2.5]. Kim, Lee and Lin proved that the pair $(L_\infty(\mu), Y)$ has BPBp for every positive measure μ , whenever Y is uniformly convex [22, Theorem 5]. Kim and Lee extended such result for the pair $(C(K), Y)$ (K is compact Hausdorff spaces) [21, Theorem 2.2]. In the complex case, Acosta proved that the pair $(C_0(L), Y)$ has the BPBp for every locally compact Hausdorff space L and any \mathbb{C} -uniformly convex space Y [1, Theorem 2.4]. In the complex case $L_1(\mu)$ is \mathbb{C} -uniformly convex space, so the previous result can be applied. There are some other sufficient conditions on a Banach space Y in order that the pair (c_0, Y) satisfies the BPBp (see for instance [6, Theorem 2.4]).

However until now there is no characterization of the spaces Y such that (c_0, Y) has the BPBp. Indeed in the real case it is not known whether or not the pair (c_0, ℓ_1) has the BPBp. As a consequence of [9, Theorem 2.1], in case that the pair (c_0, ℓ_1) has the BPBp, then the pairs (ℓ_∞^n, ℓ_1) satisfy the BPBp “uniformly” for every n . For this reason in this paper we approach this problem by using in the domain appropriate finite-dimensional spaces.

Notice that for dimension 2, since ℓ_∞^2 is isometrically isomorphic to ℓ_1^2 , as a consequence of [2, Theorem 4.1], it is known that (ℓ_∞^2, Y) has the BPBp if and only if Y has the approximate hyperplane series property for convex combinations of two elements. A characterization of the spaces Y such that (ℓ_∞^3, Y) has the BPBp was shown in [5, Theorem 2.9]. As a consequence of that result, classical Banach spaces satisfying the previous property were provided. The goal of this paper is to extend the above mentioned results for the pair (ℓ_∞^4, Y) .

Now we briefly describe the content of the paper. In section 2 we introduce a geometric property on a Banach space Y , namely the $\text{AHSp-}\ell_\infty^4$ (see Definition 2.3). We also provide several reformulations of such property in Proposition 2.12. That result is essential to prove in section 3 that the pair (ℓ_∞^4, Y) has the BPBp for operators if and only if Y has the $\text{AHSp-}\ell_\infty^4$ (see Theorem 3.3). In section 4 we provide examples of classical spaces satisfying the $\text{AHSp-}\ell_\infty^4$. For instance, we check that finite-dimensional spaces and uniformly convex spaces have this property. It is also satisfied that $C_0(L, Y)$ has the $\text{AHSp-}\ell_\infty^4$ whenever Y has the same property, for any locally compact Hausdorff space L (Proposition 4.4). Lastly we prove that ℓ_1 has the $\text{AHSp-}\ell_\infty^4$ (Proposition 4.6). The proof of that result requires some effort. As a consequence of the result for ℓ_1 , we obtain that $(\ell_\infty^4, L_1(\mu))$ has the Bishop-Phelps-Bollobás property for any positive measure μ .

Throughout this paper we follow the spirit of the results obtained in [5] for ℓ_∞^3 , but the proofs are much more complicated. We provide a few arguments for that. One reason is that for ℓ_∞^3 any subset of three extreme points contained in the same face of the unit ball can be applied to any other subset of the same kind by using an appropriate linear isometry. For dimension 4 the previous statement is not satisfied. Because of that the image under an operator of any three extreme points in the same face of $B_{\ell_\infty^3}$ is a good choice in order to identify an operator whose domain is ℓ_∞^3 . In case that the domain of the operator is ℓ_∞^4 we had to find an appropriate choice of the basis in order to identify such operator and to compute its norm (Proposition 2.11). Another reason is that due to the bigger amount of extreme points in the unit ball of ℓ_∞^4 the description used to identify operators whose domain is ℓ_∞^4 is more involved. As a consequence, the property of Y equivalent to the fact that (ℓ_∞^4, Y) has the BPBp is more complicated. That reason also makes to provide examples more difficult.

Throughout the paper we consider *real* normed spaces. By ℓ_∞^4 we denote the space \mathbb{R}^4 , endowed with the norm given by $\|x\| = \max\{|x_i| : i \leq 4\}$.

2. The approximate hyperplane sum property for ℓ_∞^4

In this section we identify the unit ball of $L(\ell_\infty^4, Y)$ with a certain family of elements in Y^4 called M_Y^4 (Proposition 2.11). We also introduce an intrinsic property on a Banach space Y , namely the $\text{AHSp-}\ell_\infty^4$ and show a characterization of that property (see Proposition 2.12).

NOTATION 2.1. If Y is a Banach space, we will denote by

$$M_Y^4 = \{(y_i)_{i \leq 4} \in (B_Y)^4 : y_{i_1} - y_{i_2} + y_{i_3} \in B_Y, \forall 1 \leq i_1 < i_2 < i_3 \leq 4\}.$$

REMARK 2.2. It is clear that $(y_i)_{i \leq 4} \in M_Y^4$ if $(-y_i)_{i \leq 4} \in M_Y^4$.

The following notion is analogous to the $\text{AHSp-}\ell_\infty^3$ that was used to characterize those spaces Y such that the pair (ℓ_∞^3, Y) has the BPBp for operators (see [5, Definition 2.1]).

DEFINITION 2.3. A Banach space Y has the *approximate hyperplane sum property for ℓ_∞^4* ($\text{AHSp-}\ell_\infty^4$) if for every $0 < \varepsilon < 1$ there is $0 < \gamma(\varepsilon) < \varepsilon$ satisfying the following condition

For every $(y_i)_{i \leq 4} \in M_Y^4$, if there exist a nonempty subset A of $\{1, 2, 3, 4\}$ and $y^* \in S_{Y^*}$ such that $y^*(y_i) > 1 - \gamma(\varepsilon)$ for each $i \in A$, then there exists an element $(z_i)_{i \leq 4} \in M_Y^4$ satisfying $\|z_i - y_i\| < \varepsilon$ for every $i \leq 4$ and $\|\sum_{i \in A} z_i\| = |A|$.

REMARK 2.4. We recall that Y has the $\text{AHSp-}\ell_\infty^3$ if for every $\varepsilon > 0$ there is $\gamma > 0$ satisfying the following condition:

For a subset $\{y_i : i \leq 3\} \subset B_Y$ with $\|y_1 + y_2 - y_3\| \leq 1$, if there exist a nonempty subset A of $\{1, 2, 3\}$ and $y^* \in S_{Y^*}$ such that $y^*(y_i) > 1 - \gamma$ for every $i \in A$, then there exists $\{z_i : i \leq 3\} \subset B_Y$ with $\|z_1 + z_2 - z_3\| \leq 1$ satisfying $\|z_i - y_i\| < \varepsilon$ for every $i \leq 3$ and $\|\sum_{i \in A} z_i\| = |A|$.

If $(y_i)_{i \leq 3}$ satisfies the assumption in the definition of the $\text{AHSp-}\ell_\infty^3$, it is immediate that $(y_1, y_3, y_2, y_2) \in M_Y^4$. This is the key idea to check that $\text{AHSp-}\ell_\infty^4$ implies $\text{AHSp-}\ell_\infty^3$.

Now we state some basic but useful results.

LEMMA 2.5. If $\{y_i : 1 \leq i \leq 4\} \subset B_Y$, the following conditions are equivalent

- (1) $(y_1, y_2, y_3, y_4) \in M_Y^4$
- (2) $(y_2, y_3, y_4, -y_1) \in M_Y^4$
- (3) $(y_3, y_4, -y_1, -y_2) \in M_Y^4$
- (4) $(y_4, -y_1, -y_2, -y_3) \in M_Y^4$

PROOF. Statement (1) is satisfied exactly when the following elements belong to B_Y

$$(2.1) \quad y_1 - y_2 + y_3, \quad y_1 - y_2 + y_4, \quad y_1 - y_3 + y_4, \quad y_2 - y_3 + y_4.$$

On the other hand, condition (2) means that each of the following elements belongs to B_Y

$$(2.2) \quad y_2 - y_3 + y_4, \quad y_2 - y_3 - y_1, \quad y_2 - y_4 - y_1, \quad y_3 - y_4 - y_1.$$

As a consequence conditions (1) and (2) are equivalent. By applying this fact we obtain that (2) and (3) are equivalent. Again by the same argument (3) and (4) are equivalent. \square

Next result shows that the condition stated in Definition 2.3 is trivially satisfied in case that the set A contains a unique element.

PROPOSITION 2.6. *Let Y be a Banach space. Let $0 < \varepsilon < 1$ and $(y_i)_{i \leq 4} \in M_Y^4$. Assume that $A \subset \{1, 2, 3, 4\}$ contains only one element and it is satisfied that*

$$\|y_j\| > 1 - \frac{\varepsilon}{6}, \quad j \in A.$$

Then there is an element $(z_i)_{i \leq 4} \in M_Y^4$ such that

$$\|z_j\| = 1 \text{ for } j \in A \quad \text{and} \quad \|z_i - y_i\| < \varepsilon, \quad \forall 1 \leq i \leq 4.$$

PROOF. Let $0 < \varepsilon < 1$ and $(y_i)_{i \leq 4} \in M_Y^4$. In view of Lemma 2.5 it suffices to show the statement in case that $A = \{1\}$.

Assume that the element y_1 satisfies that

$$\|y_1\| > 1 - \frac{\varepsilon}{6} > 0.$$

We define the following real numbers

$$a = \frac{\varepsilon}{3}, \quad a_2 = \frac{\varepsilon}{3}, \quad a_3 = \frac{\varepsilon}{6}, \quad a_4 = -\frac{\varepsilon}{6},$$

and the elements in Y

$$z_1 = \frac{y_1}{\|y_1\|} \quad \text{and} \quad z_i = (1 - a)y_i + a_i z_1, \quad i \in \{2, 3, 4\}.$$

It is trivially satisfied that $z_1 \in S_Y$. We also have that

$$(2.3) \quad \|z_1 - y_1\| = 1 - \|y_1\| < \frac{\varepsilon}{6} < \varepsilon.$$

In case that $2 \leq i \leq 4$ we obtain that

$$\|z_i\| \leq (1 - a) + |a_i| \leq 1 - \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = 1$$

and

$$\|z_i - y_i\| = \|-ay_i + a_i z_1\| \leq a + |a_i| \leq \frac{2}{3}\varepsilon < \varepsilon.$$

It remains to show that for every $1 \leq i_1 < i_2 < i_3 \leq 4$ it is satisfied that

$$\|z_{i_1} - z_{i_2} + z_{i_3}\| \leq 1.$$

In case that $\{i_1, i_2, i_3\} = \{2, 3, 4\}$ we obtain

$$\|z_2 - z_3 + z_4\| = \|(1 - a)(y_2 - y_3 + y_4) + (a_2 - a_3 + a_4)z_1\| = \|(1 - a)(y_2 - y_3 + y_4)\| \leq 1 - a < 1.$$

Otherwise $i_1 = 1$. Notice that for every $1 < i_2 < i_3 \leq 4$ we have that

$$|a - a_{i_2} + a_{i_3}| \leq \frac{\varepsilon}{6}.$$

Hence

$$\begin{aligned}
 \|z_1 - z_{i_2} + z_{i_3}\| &= \|(1-a)(z_1 - y_{i_2} + y_{i_3}) + (a - a_{i_2} + a_{i_3})z_1\| \\
 &\leq (1-a)(1 + \|z_1 - y_1\|) + |a - a_{i_2} + a_{i_3}| \\
 &< \left(1 - \frac{\varepsilon}{3}\right)\left(1 + \frac{\varepsilon}{6}\right) + \frac{\varepsilon}{6} \quad (\text{by (2.3)}) \\
 &= 1 - \frac{\varepsilon}{3} + \frac{\varepsilon}{6} - \frac{\varepsilon^2}{18} + \frac{\varepsilon}{6} \\
 &= 1 - \frac{\varepsilon^2}{18} \\
 &< 1.
 \end{aligned}$$

We checked that $(z_i)_{i \leq 4} \in M_Y^4$ and it satisfies all the required conditions. \square

NOTATION 2.7. In what follows we will denote by E_1 the subset of $B_{\ell_\infty^4}$ given by

$$E_1 = \{x \in \ell_\infty^4 : 1 = x(1) = \|x\|\}.$$

In the sequel it will be convenient to use the following notation for the following elements in E_1

$$\begin{aligned}
 v_1 &= (1, 1, 1, 1), \quad v_2 = (1, -1, 1, 1), \quad v_3 = (1, -1, -1, 1), \quad v_4 = (1, -1, -1, -1), \\
 v_5 &= (1, 1, -1, 1), \quad v_6 = (1, 1, -1, -1), \quad v_7 = (1, 1, 1, -1), \quad v_8 = (1, -1, 1, -1),
 \end{aligned}$$

By \mathcal{B} we will denote the set given by $\mathcal{B} = \{v_i : 1 \leq i \leq 4\}$.

The proofs of next assertions are straightforward. For the first one it suffices to check that every coordinate of an extreme point of E_1 belongs to $\{1, -1\}$.

LEMMA 2.8. It is satisfied that

$$\text{Ext}(E_1) = \{v_i : i \leq 8\}.$$

LEMMA 2.9. The set \mathcal{B} is a basis of \mathbb{R}^4 contained in $S_{\ell_\infty^4}$. Moreover the functionals given by

$$v_1^*(x) = \frac{x(1) + x(2)}{2}, \quad v_i^*(x) = \frac{x(i+1) - x(i)}{2}, \quad i = 2, 3 \quad \text{and} \quad v_4^*(x) = \frac{x(1) - x(4)}{2},$$

are elements in $S_{(\ell_\infty^4)^*}$ that are the biorthogonal functionals of the basis \mathcal{B} . Hence each x in \mathbb{R}^4 can be expressed as

$$x = \frac{x(1) + x(2)}{2}v_1 + \sum_{i=2}^3 \frac{x(i+1) - x(i)}{2}v_i + \frac{x(1) - x(4)}{2}v_4.$$

As a consequence of the last assertion in the previous result we obtain the following.

REMARK 2.10. The following equalities are satisfied

$$v_5 = v_1 - v_2 + v_3, \quad v_6 = v_1 - v_2 + v_4,$$

and

$$v_7 = v_1 - v_3 + v_4, \quad v_8 = v_2 - v_3 + v_4.$$

Hence $(v_i)_{i \leq 4} \in M_Y^4$ for $Y = \ell_\infty^4$.

The next result shows the connection between operators from ℓ_∞^4 to Y and the set M_Y^4 .

PROPOSITION 2.11. *Every element $T \in L(\ell_\infty^4, Y)$ satisfies*

$$\|T\| = \max\{\|T(v_{i_1} - v_{i_2} + v_{i_3})\| : 1 \leq i_1 \leq i_2 \leq i_3 \leq 4\}.$$

So the mapping given by $\Phi(T) = (T(v_i))_{i \leq 4}$ identifies $B_{L(\ell_\infty^4, Y)}$ with M_Y^4 .

PROOF. In view of Lemma 2.9 the set $\mathcal{B} = \{v_i : i \leq 4\}$ is a basis of \mathbb{R}^4 , so every operator T from ℓ_∞^4 to Y is determined by the element $(T(v_i))_{i \leq 4}$.

It is clear $\text{Ext}(B_{\ell_\infty^4}) = \text{Ext}(E_1) \cup \text{Ext}(-E_1)$. Hence

$$\begin{aligned} \|T\| &= \max\{\|T(e)\| : e \in \text{Ext}(B_{\ell_\infty^4})\} \\ (2.4) \quad &= \max\{\|T(e)\| : e \in \text{Ext}(E_1)\} \\ &= \max\{\|T(v_i)\| : i \leq 8\}, \end{aligned}$$

where we used Lemma 2.8.

In view of Remark 2.10 we have that

$$\{v_i : 5 \leq i \leq 8\} = \{v_{i_1} - v_{i_2} + v_{i_3} : 1 \leq i_1 < i_2 < i_3 \leq 4\}.$$

From (2.4) and the previous equality we obtain that

$$\|T\| = \max\{\|T(v_{i_1} - v_{i_2} + v_{i_3})\| : 1 \leq i_1 \leq i_2 \leq i_3 \leq 4\}.$$

It follows that $T \in B_{L(\ell_\infty^4, Y)}$ if and only if $\Phi(T)$ is an element in M_Y^4 . \square

We recall that a subset $B \subset B_{Y^*}$ is 1-norming if $\|y\| = \sup\{|y^*(y)| : y^* \in B\}$ for each $y \in Y$. Now we provide a characterization of $\text{AHSp-}\ell_\infty^4$ which will be used in the following sections.

PROPOSITION 2.12. *Let Y be a Banach space. The following conditions are equivalent:*

- 1) Y has the approximate hyperplane sum property for ℓ_∞^4 .
- 2) There is a 1-norming subset $B \subset S_{Y^*}$ such that the condition stated in Definition 2.3 is satisfied for every $y^* \in B$.
- 3) For every $0 < \varepsilon < 1$ there exists $0 < \nu(\varepsilon) < \varepsilon$ such that for every element $(y_i)_{i \leq 4} \in M_Y^4$ and each convex combination $\sum_{i=1}^4 \alpha_i y_i$ satisfying

$$\left\| \sum_{i=1}^4 \alpha_i y_i \right\| > 1 - \nu(\varepsilon),$$

there exist a set $A \subset \{1, 2, 3, 4\}$ and an element $(z_i)_{i \leq 4} \in M_Y^4$ such that

- i) $\sum_{i \in A} \alpha_i > 1 - \varepsilon$,
- ii) $\|z_i - y_i\| < \varepsilon$ for each $i \leq 4$,
- iii) $\|\sum_{i \in A} z_i\| = |A|$.

Moreover, if ρ is the function satisfying condition 2), then condition 3) is also satisfied with the function $\nu = \rho^2$. In case that 3) is satisfied with a function ν , Y has the $AHSp\text{-}\ell_\infty^4$ with the function $\gamma(\varepsilon) = \nu(\frac{\varepsilon}{4})$.

PROOF. Clearly 1) implies 2).

2) \Rightarrow 3)

Assume that Y satisfies condition 2). For each $0 < \varepsilon < 1$ let be $\rho(\varepsilon) < \varepsilon$ the positive real number satisfying Definition 2.3 for every element $y^* \in B$. We take $\nu(\varepsilon) = (\rho(\varepsilon))^2$.

Let $(y_i)_{i \leq 4} \in M_Y^4$ and assume that the convex combination $\sum_{i=1}^4 \alpha_i y_i$ satisfies $\|\sum_{i=1}^4 \alpha_i y_i\| > 1 - \nu(\varepsilon)$. Since $(-y_i)_{i \leq 4} \in M_Y^4$, by using $(-y_i)$ instead of (y_i) , if needed, since B is a 1-norming set, there is $y^* \in B$ such that

$$y^* \left(\sum_{i=1}^4 \alpha_i y_i \right) = \sum_{i=1}^4 \alpha_i y^*(y_i) > 1 - \nu(\varepsilon) = 1 - \rho(\varepsilon)^2.$$

By [2, Lemma 3.3] the set A given by $A := \{i \leq 4 : y^*(y_i) > 1 - \rho(\varepsilon)\}$ satisfies

$$\sum_{i \in A} \alpha_i \geq 1 - \frac{\nu(\varepsilon)}{\rho(\varepsilon)} > 1 - \varepsilon.$$

By assumption there is an element $(z_i)_{i \leq 4} \in M_Y^4$ such that $\|z_i - y_i\| < \varepsilon$ for each $i \leq 4$ and $\|\sum_{i \in A} z_i\| = |A|$.

3) \Rightarrow 1)

Now we assume that Y satisfies condition 3). Let be $0 < \varepsilon < 1$ and $\nu(\varepsilon)$ the positive real number satisfying the assumption. We will show that $\gamma(\varepsilon) = \nu(\frac{\varepsilon}{4})$ satisfies Definition 2.3.

Let $(y_i)_{i \leq 4} \in M_Y^4$ and assume that for some nonempty set $A \subset \{1, 2, 3, 4\}$ and $y^* \in S_{Y^*}$ it is satisfied that $y^*(y_i) > 1 - \gamma(\varepsilon)$ for each $i \in A$. We define the following nonnegative real numbers

$$\alpha_i = \begin{cases} \frac{1}{|A|} & \text{if } i \in A \\ 0 & \text{if } i \in \{1, 2, 3, 4\} \setminus A. \end{cases}$$

Clearly $\sum_{i=1}^4 \alpha_i = 1$ and we also have that

$$\left\| \sum_{i=1}^4 \alpha_i y_i \right\| = \frac{\left\| \sum_{i \in A} y_i \right\|}{|A|} \geq \frac{y^* \left(\sum_{i \in A} y_i \right)}{|A|} > 1 - \nu\left(\frac{\varepsilon}{4}\right).$$

By assumption there is a set $C \subset \{1, 2, 3, 4\}$ and $(z_i)_{i \leq 4} \in M_Y^4$ such that

- i) $\sum_{i \in C} \alpha_i > 1 - \frac{\varepsilon}{4} > \frac{3}{4}$,
- ii) $\|z_i - y_i\| < \frac{\varepsilon}{4}$ for each $i \leq 4$,
- iii) $\|\sum_{i \in C} z_i\| = |C|$.

If $A \subset C$ then the proof will be finished since condition iii) implies that $\|\sum_{i \in A} z_i\| = |A|$. In case that there is some $i_0 \in A \setminus C$, we put $B = \{i \leq 4 : i \neq i_0\}$ and so

$$1 - \frac{1}{|A|} = \frac{|A| - 1}{|A|} = \sum_{i \in B} \alpha_i \geq \sum_{i \in C} \alpha_i > \frac{3}{4}.$$

Then $|A| > 4$, which is a contradiction. Hence $A \subset C$ and we proved that Y has the $\text{AHSp-}\ell_\infty^4$.

Of course, if γ satisfies Definition 2.3, then $\rho = \gamma$ also satisfies condition 2). In case that 2) is satisfied with the function ρ , we showed that condition 3) is also satisfied with the function $\nu = \rho^2$. Lastly if we assume that 3) is true with a function ν we know that Y has the $\text{AHSp-}\ell_\infty^4$ with the function $\varepsilon \mapsto \nu(\frac{\varepsilon}{4})$ because of the proof of $3) \Rightarrow 1)$. \square

3. A characterization of the spaces Y such that the pair (ℓ_∞^4, Y) has the Bishop-Phelps-Bollobás property

In this section we show that the pair (ℓ_∞^4, Y) has the Bishop-Phelps-Bollobás property for operators if and only if Y has the $\text{AHSp-}\ell_\infty^4$. Some technical results will make the proof easier. As usual, we denote by $\text{co}(A)$ the convex hull of a subset A of a linear space.

LEMMA 3.1. *If $x \in E_1$ and $x(i) \leq x(i+1)$ for $2 \leq i \leq 3$, then $x \in \text{co}(\mathcal{B})$. It is satisfied that $E_1 = \cup_{k=1}^6 \text{co}(\{v_i : i \in A_k\})$, where*

$$A_1 = \{1, 2, 3, 4\}, \quad A_2 = \{1, 3, 4, 5\}, \quad A_3 = \{1, 4, 6, 7\},$$

$$A_4 = \{1, 2, 4, 8\}, \quad A_5 = \{1, 4, 5, 6\} \quad \text{and} \quad A_6 = \{1, 4, 7, 8\}.$$

Indeed for every $1 \leq k \leq 6$, $\{v_i : i \in A_k\}$ is the image under an appropriate linear isometry on ℓ_∞^4 of \mathcal{B} .

PROOF. If $x \in E_1$, by Lemma 2.9 we know that

$$(3.1) \quad x = \frac{1+x(2)}{2}v_1 + \sum_{i=2}^3 \frac{x(i+1)-x(i)}{2}v_i + \frac{1-x(4)}{2}v_4.$$

In case that $x(2) \leq x(3) \leq x(4)$ notice that x is expressed in (3.1) as a convex combination of $\{v_i : 1 \leq i \leq 4\}$.

For each permutation σ of $\{2, 3, 4\}$ we define the linear isometry T_σ on ℓ_∞^4 given by

$$T_\sigma(x) = (x(1), x(\sigma(2)), x(\sigma(3)), x(\sigma(4))) \quad (x \in \mathbb{R}^4).$$

Notice that T_σ preserves E_1 .

If $x \in E_1$ and σ is a permutation of $\{2, 3, 4\}$ is such that $x(\sigma(2)) \leq x(\sigma(3)) \leq x(\sigma(4))$ we know that the element $T_\sigma(x)$ can be expressed as a convex combination of $\{v_i : 1 \leq i \leq 4\}$. Hence $x = T_{\sigma^{-1}}(T_\sigma(x))$ can be expressed as a convex combination of $\{T_{\sigma^{-1}}(v_i) : 1 \leq i \leq 4\}$. So it suffices to compute the images by T_σ of $\{v_i : 1 \leq i \leq 4\}$, where σ is any permutation of $\{2, 3, 4\}$. Notice that the elements v_1 and v_4 are invariant by all these isometries. So it suffices to evaluate the image of the elements $\{v_2, v_3\}$.

We include the results in the following table, where we denote by I the identity and $\tau_{i,j}$ the transposition of the elements i and j on $\{2, 3, 4\}$

σ	$T_\sigma(\{v_2, v_3\})$
I	$\{v_2, v_3\}$
$\tau_{2,3}$	$\{v_3, v_5\}$
$\tau_{2,4}$	$\{v_6, v_7\}$
$\tau_{3,4}$	$\{v_2, v_8\}$
$\tau_{2,3} \circ \tau_{3,4}$	$\{v_5, v_6\}$
$\tau_{2,3} \circ \tau_{2,4}$	$\{v_7, v_8\}$

Then we obtained that $E_1 = \cup_{i=1}^6 \text{co}\{v_j : j \in A_i\}$, where the sets A_i are given by

$$A_1 = \{1, 2, 3, 4\}, \quad A_2 = \{1, 3, 4, 5\}, \quad A_3 = \{1, 4, 6, 7\},$$

$$A_4 = \{1, 2, 4, 8\}, \quad A_5 = \{1, 4, 5, 6\} \quad \text{and} \quad A_6 = \{1, 4, 7, 8\}.$$

□

The next result gives a procedure to change an element u_0 close to $\text{co}\{v_i : 1 \leq i \leq 4\}$ by a new one satisfying more requirements.

LEMMA 3.2. Assume that $0 < \varepsilon < \frac{1}{2}$, $x_0 \in \text{co}\{v_i : 1 \leq i \leq 4\}$ and $u_0 \in E_1$ satisfies that $\|u_0 - x_0\| < \varepsilon$. Then there is $v_0 \in E_1$ such that $\|v_0 - u_0\| < 3\varepsilon$ and such that there is a set $A \subset \{i \in \mathbb{N} : i \leq 8\}$ satisfying that u_0 and v_0 can be written as convex combinations as follows

$$u_0 = \sum_{i \in A} \beta_i v_i, \quad v_0 = \sum_{i \in A, i \leq 4} \gamma_i v_i$$

and also

$$\gamma_i > 0 \quad \text{for some } i \Rightarrow \beta_i > 0.$$

PROOF. By assumption we know that x_0 can be written as $x_0 = \sum_{i=1}^4 \alpha_i v_i$, where $\alpha_i \geq 0$ for $i \leq 4$ and $\sum_{i=1}^4 \alpha_i = 1$.

By Lemma 3.1 $u_0 \in \cup_{i=1}^6 \text{co}(\{v_i : i \in A_k\})$. If $u_0 \in \text{co}\{v_i : i \in A_1\} = \{v_i : i \leq 4\}$ then the element $v_0 = u_0$ satisfies the statement. Otherwise we can express $u_0 = \sum_{i \in A} \beta_i v_i$, where $\beta_i \geq 0$ for each $i \in A$ and $\sum_{i \in A} \beta_i = 1$ and it suffices to prove the claim in the following cases:

Case a) $A = A_2 \cup A_5 = \{1, 3, 4, 5, 6\}$. Since the functional v_2^* given by $v_2^*(x) = \frac{x(3)-x(2)}{2}$ belongs to the unit ball of $(\ell_\infty^4)^*$, in view of Lemma 2.9 and Remark 2.10 we obtain that

$$(3.2) \quad \alpha_2 + \beta_5 + \beta_6 = v_2^*(x_0 - u_0) \leq \|x_0 - u_0\| < \varepsilon.$$

Case b) Assume that $A = A_3 = \{1, 4, 6, 7\}$. The functional v^* given by $v^*(x) = \frac{x(4)-x(2)}{2}$ belongs to the unit ball of $(\ell_\infty^4)^*$ and satisfies

$$v^*(v_i) = 0, \quad i = 1, 4 \quad v^*(v_i) = 1, \quad i = 2, 3, \quad \text{and} \quad v^*(v_i) = -1, \quad i = 6, 7.$$

As a consequence we have that

$$(3.3) \quad \alpha_2 + \alpha_3 + \beta_6 + \beta_7 = v^*(x_0 - u_0) \leq \|x_0 - u_0\| < \varepsilon.$$

Case c) Assume that $A = A_4 \cup A_6 = \{1, 2, 4, 7, 8\}$. By Lemma 2.9 and Remark 2.10 we obtain that

$$(3.4) \quad \alpha_3 + \beta_7 + \beta_8 = v_3^*(x_0 - u_0) \leq \|x_0 - u_0\| < \varepsilon.$$

In each of the above cases, we take $B = A \cap \{1, 2, 3, 4\}$. Notice that in view of (3.2), (3.3) and (3.4), it is satisfied that

$$(3.5) \quad \sum_{i \in A \setminus B} \beta_i < \varepsilon \Rightarrow \sum_{i \in B} \beta_i > 1 - \varepsilon > \frac{1}{2}.$$

We will check now that the element $v_0 = \frac{1}{\sum_{i \in B} \beta_i} \sum_{i \in B} \beta_i v_i$ satisfies the requirements of the claim. Since $B \subset \{1, 2, 3, 4\}$, clearly $v_0 \in \text{co}\{v_1, v_2, v_3, v_4\} \subset E_1$.

Since $B \subset A$ the element v_0 also satisfies

$$\begin{aligned} \|v_0 - u_0\| &\leq \left\| \frac{1}{\sum_{i \in B} \beta_i} \sum_{i \in B} \beta_i v_i - \sum_{i \in B} \beta_i v_i \right\| + \left\| \sum_{i \in A \setminus B} \beta_i v_i \right\| \\ &\leq \frac{1}{\sum_{i \in B} \beta_i} - 1 + \sum_{i \in A \setminus B} \beta_i \\ &= \frac{1}{\sum_{i \in B} \beta_i} \sum_{i \in A \setminus B} \beta_i + \sum_{i \in A \setminus B} \beta_i \\ &= \sum_{i \in A \setminus B} \beta_i \left(1 + \frac{1}{\sum_{i \in B} \beta_i} \right) \\ &< \varepsilon \left(1 + \frac{1}{1 - \varepsilon} \right) < 3\varepsilon \quad (\text{by (3.5)}). \end{aligned}$$

Let us notice that $B \subset \{1, 2, 3, 4\} \cap A$. If we take $\gamma_i = \frac{\beta_i}{\sum_{i \in B} \beta_i}$ for every $i \in B$ then $v_0 = \sum_{i \in B} \gamma_i v_i$. As a consequence, in case that $\gamma_i > 0$ for some $i \in B$ we obtain that $\beta_i > 0$. So the element v_0 satisfies all the required conditions. \square

Next result is the version of [5, Theorem 2.9] for ℓ_∞^4 , where the analogous result was obtained for ℓ_∞^3 . In our case, the fact that the domain has dimension 4, and so the norm of an element $T \in L(\ell_\infty^4, Y)$ is the maximum of the norm of the evaluation of T at eight extreme points of $B_{\ell_\infty^4}$ makes the proof more complicated comparing to the case that the domain has dimension 3.

THEOREM 3.3. *For every Banach space Y , the pair (ℓ_∞^4, Y) has the Bishop-Phelps-Bollobás property if and only if Y has the approximate hyperplane sum property for ℓ_∞^4 .*

Moreover, if (ℓ_∞^4, Y) satisfies Definition 1.1 with the function η , then Y has the $AHSp\text{-}\ell_\infty^4$ with $\gamma(\varepsilon) = \eta(\frac{\varepsilon}{48})$. In case that Y has the $AHSp\text{-}\ell_\infty^4$ for the function γ (see Definition 2.3), the pair (ℓ_∞^4, Y) satisfies BPBp with the function $\eta(\varepsilon) = \gamma^2(\frac{\varepsilon}{3})$.

PROOF. Assume that the pair (ℓ_∞^4, Y) satisfies the BPBp with the function η . We will prove that Y satisfies condition 3) in Proposition 2.12 with the function $\nu(\varepsilon) = \eta(\frac{\varepsilon}{12})$. As a consequence Y has the $AHSp\text{-}\ell_\infty^4$ with the function $\gamma(\varepsilon) = \eta(\frac{\varepsilon}{48})$.

Let us fix $0 < \varepsilon < 1$. Assume that $(y_i)_{i \leq 4} \in M_Y^4$ and that $\sum_{i=1}^4 \alpha_i y_i$ is a convex combination satisfying that

$$\left\| \sum_{i=1}^4 \alpha_i y_i \right\| > 1 - \nu(\varepsilon).$$

Let T be the element in $B_{L(\ell_\infty^4, Y)}$ that $(y_i)_{i \leq 4}$ represents in view of Proposition 2.11. The element $x_0 = \sum_{i=1}^4 \alpha_i v_i$ satisfies that $x_0 \in S_{\ell_\infty^4}$ and by assumption we know that

$$\|T(x_0)\| = \left\| \sum_{i=1}^4 \alpha_i y_i \right\| > 1 - \nu(\varepsilon) = 1 - \eta\left(\frac{\varepsilon}{12}\right) > 1 - \frac{\varepsilon}{12} > 0.$$

Since the pair (ℓ_∞^4, Y) has the BPBp, there are $u_0 \in S_{\ell_\infty^4}$ and $S \in S_{L(\ell_\infty^4, Y)}$ satisfying the following conditions

$$(3.6) \quad \|Su_0\| = 1, \quad \|u_0 - x_0\| < \frac{\varepsilon}{12} \quad \text{and} \quad \left\| S - \frac{T}{\|T\|} \right\| < \frac{\varepsilon}{12}.$$

Notice that $|u_0(1) - 1| = |u_0(1) - x_0(1)| \leq \|u_0 - x_0\| < 1$ and $\|u_0\| = 1$, so $0 < u_0(1) \leq 1$. Since we clearly have that the element u_0 can be written as a convex combination as follows

$$u_0 = \frac{1 + u_0(1)}{2} (1, u_0(2), u_0(3), u_0(4)) + \frac{1 - u_0(1)}{2} (-1, u_0(2), u_0(3), u_0(4)),$$

$1 + u_0(1) > 0$ and S attains its norm at u_0 , then S also attains its norm at $(1, u_0(2), u_0(3), u_0(4))$. The previous element belongs to $S_{\ell_\infty^4}$ and satisfies that

$$\|(1, u_0(2), u_0(3), u_0(4)) - x_0\| \leq \|u_0 - x_0\| < \frac{\varepsilon}{12}.$$

As a consequence, by changing u_0 by $(1, u_0(2), u_0(3), u_0(4))$, if needed, we can assume that $u_0 \in E_1$.

By Lemma 3.2 there is $v_0 \in E_1$ such that $\|v_0 - u_0\| < \frac{\varepsilon}{4}$ and such that there is a set $A \subset \{i \in \mathbb{N} : i \leq 8\}$ satisfying that u_0 and v_0 can be written as convex combinations as follows

$$u_0 = \sum_{i \in A} \beta_i v_i, \quad v_0 = \sum_{i \in A, i \leq 4} \gamma_i v_i$$

and also

$$(3.7) \quad \gamma_i > 0 \text{ for some } i \Rightarrow \beta_i > 0.$$

By (3.6) S attains its norm at u_0 , hence by Hahn-Banach Theorem there is an element $y^* \in S_{Y^*}$ such that $y^*(S(u_0)) = 1$. Since

$$1 = y^*(S(u_0)) = \sum_{i \in A} \beta_i y^*(S(v_i)),$$

$\beta_i \geq 0$ for each $i \in A$ and $\sum_{i \in A} \beta_i = 1$, we have that

$$\beta_i \in A, \beta_i > 0 \Rightarrow y^*(S(v_i)) = 1.$$

As a consequence, by (3.7) we obtain that

$$(3.8) \quad i \in A, i \leq 4, \gamma_i \neq 0 \Rightarrow y^*(S(v_i)) = 1.$$

The element v_0 satisfies that

$$(3.9) \quad \begin{aligned} \|v_0 - x_0\| &\leq \|v_0 - u_0\| + \|u_0 - x_0\| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{12} \quad (\text{by (3.6)}) \\ &= \frac{\varepsilon}{3}. \end{aligned}$$

By Lemma 2.9 the functionals $\{v_i^* : 1 \leq i \leq 4\}$ belong to $B_{(\ell_\infty^4)^*}$ and are the biorthogonal functionals of the basis \mathcal{B} . In view of (3.9) we get that

$$(3.10) \quad |\alpha_i - \gamma_i| = |v_i^*(x_0 - v_0)| \leq \|x_0 - v_0\| < \frac{\varepsilon}{3}, \quad \forall i \leq 4.$$

We write $C := \{i \leq 4 : i \in A, \gamma_i \neq 0\}$. If $C = \{1, 2, 3, 4\}$ then $\sum_{i \in C} \alpha_i = 1$. Otherwise, from (3.10) we obtain that

$$\sum_{i \in C} \alpha_i = 1 - \sum_{i \leq 4, i \notin C} \alpha_i > 1 - \frac{\varepsilon}{3}(4 - |C|) \geq 1 - \varepsilon.$$

Finally we check that $(z_i)_{i \leq 4} = (S(v_i))_{i \leq 4}$ is the desired element in M_Y^4 . Clearly $(z_i)_{i \leq 4} \in M_Y^4$ since $S \in S_{L(\ell_\infty^4, Y)}$ (see Proposition 2.11). By (3.8) we have that

$$\left\| \sum_{i \in C} z_i \right\| \geq y^* \left(\sum_{i \in C} S(v_i) \right) = |C|.$$

It remains to show only that $\|z_i - y_i\| < \varepsilon$ for each $i \leq 4$. Indeed for each $1 \leq i \leq 4$ we have that

$$\begin{aligned} \|z_i - y_i\| &= \|S(v_i) - T(v_i)\| \\ &\leq \left\| S(v_i) - \frac{T(v_i)}{\|T\|} \right\| + \left\| \frac{T(v_i)}{\|T\|} - T(v_i) \right\| \\ &\leq \left\| S - \frac{T}{\|T\|} \right\| + 1 - \|T\| \\ &< \frac{\varepsilon}{6} < \varepsilon \quad (\text{by (3.6)}). \end{aligned}$$

We proved that Y satisfies condition 3) of Proposition 2.12 with $\nu(\varepsilon) = \eta(\frac{\varepsilon}{12})$.

Assume that Y satisfies the $\text{AHSp-}\ell_\infty^4$ with the function γ . By Proposition 2.12, Y also satisfies condition 3) in that result with the function $\nu(\varepsilon) = \gamma^2(\varepsilon)$. We will show that the pair (ℓ_∞^4, Y) has the BPBp with the function $\eta(\varepsilon) = \nu(\frac{\varepsilon}{3}) = \gamma^2(\frac{\varepsilon}{3})$.

Let be $0 < \varepsilon < 1$ and assume that $T \in S_{L(\ell_\infty^4, Y)}$ and $x_0 \in S_{\ell_\infty^4}$ are such that

$$\|Tx_0\| > 1 - \eta(\varepsilon) = 1 - \nu\left(\frac{\varepsilon}{3}\right).$$

By using an appropriate isometry, if needed, in view of Lemma 3.1, we can assume that $x_0 \in \text{co}\{v_i : 1 \leq i \leq 4\}$. Let us write x_0 as a convex combination $x_0 = \sum_{i=1}^4 \alpha_i v_i$. In view of Proposition 2.11 the set $(y_i)_{i \leq 4} = (T(v_i))_{i \leq 4} \in M_Y^4$. So we have that

$$\left\| \sum_{i=1}^4 \alpha_i y_i \right\| = \|T(x_0)\| > 1 - \nu\left(\frac{\varepsilon}{3}\right).$$

By assumption Y satisfies condition 3) in Proposition 2.12, so there is a (nonempty) set $A \subset \{1, 2, 3, 4\}$, and $(z_i)_{i \leq 4} \in M_Y^4$ such that

$$(3.11) \quad \sum_{i \in A} \alpha_i > 1 - \frac{\varepsilon}{3} > 0, \quad \|z_i - y_i\| < \frac{\varepsilon}{3} \text{ for all } i \leq 4$$

and

$$(3.12) \quad \left\| \sum_{i \in A} z_i \right\| = |A|.$$

Let S be the unique linear operator from ℓ_∞^4 to Y such that $S(v_i) = z_i$ for every $1 \leq i \leq 4$. Since $(z_i)_{i \leq 4} \in M_Y^4$, $S \in B_{L(\ell_\infty^4, Y)}$ by Proposition 2.11. The element u_0 given by $u_0 = \sum_{i \in A} \frac{\alpha_i}{\sum_{i \in A} \alpha_i} v_i$ belongs to $S_{\ell_\infty^4}$. By equation (3.12) the operator S attains its norm at u_0 since

$$1 = \frac{\left\| \sum_{i \in A} \alpha_i z_i \right\|}{\sum_{i \in A} \alpha_i} = \|S(u_0)\| \leq \|S\| \leq 1.$$

So $S \in S_{L(\ell_\infty^4, Y)}$. From Proposition 2.11 and (3.11) we obtain that $\|S - T\| < \varepsilon$. We write $C = \{i \in \mathbb{N} : i \leq 4, i \notin A\}$. Finally we obtain that

$$\begin{aligned} \|x_0 - u_0\| &= \left\| \sum_{i=1}^4 \alpha_i v_i - \sum_{i \in A} \frac{\alpha_i}{\sum_{i \in A} \alpha_i} v_i \right\| \\ &= \left\| \left(1 - \frac{1}{\sum_{i \in A} \alpha_i}\right) \sum_{i \in A} \alpha_i v_i + \sum_{i \in C} \alpha_i v_i \right\| \\ &\leq \left|1 - \frac{1}{\sum_{i \in A} \alpha_i}\right| \sum_{i \in A} \alpha_i \|v_i\| + \sum_{i \in C} \alpha_i \|v_i\| \\ &\leq \frac{\sum_{i \in C} \alpha_i}{\sum_{i \in A} \alpha_i} \sum_{i \in A} \alpha_i + \sum_{i \in C} \alpha_i \\ &= 2 \sum_{i \in C} \alpha_i < \frac{2\varepsilon}{3} < \varepsilon \quad (\text{by (3.11)}). \end{aligned}$$

We proved that the pair (ℓ_∞^4, Y) has the BPBp with $\eta(\varepsilon) = \nu(\frac{\varepsilon}{3}) = \gamma^2(\frac{\varepsilon}{3})$. \square

4. Examples of spaces with the approximate hyperplane sum property for ℓ_∞^4

The goal of this section is to provide classes of Banach spaces with the approximate hyperplane sum property for ℓ_∞^4 .

As we already mentioned in the introduction the pair (X, Y) has BPBp whenever X and Y are finite-dimensional normed spaces [2, Proposition 2.4]. By applying this result to $X = \ell_\infty^4$, and in view of Theorem 3.3 we obtain that finite-dimensional spaces have the $\text{AHSp-}\ell_\infty^4$. We will also provide a simple direct proof of this fact.

PROPOSITION 4.1. *Every finite-dimensional normed space has the $\text{AHSp-}\ell_\infty^4$.*

PROOF. We will argue by contradiction. Let Y be a finite-dimensional space and assume that Y does not have the $\text{AHSp-}\ell_\infty^4$. So there is $\varepsilon_0 > 0$ for which Definition 2.3 is not satisfied. Hence there is a sequence $\{\gamma_n\}$ of positive real numbers satisfying $\{\gamma_n\} \rightarrow 0$ and also for each natural number n , there are an element $(y_i^n)_{i \leq 4} \in M_Y^4$ and a nonempty set $A_n \subset \{1, 2, 3, 4\}$ satisfying

$$(4.1) \quad \left\| \sum_{i \in A_n} y_i^n \right\| > |A_n|(1 - \gamma_n)$$

and also

$$(4.2) \quad (z_i)_{i \leq 4} \in M_Y^4, \quad \left\| \sum_{i \in A_n} z_i \right\| = |A_n| \Rightarrow \max\{\|y_i^n - z_i\| : i \leq 4\} \geq \varepsilon_0,$$

Since Y is finite dimensional, M_Y^4 is a compact set of Y^4 . By taking into account that $\{i \in \mathbb{N} : i \leq 4\}$ is finite, and passing to a subsequence, we can assume that there is a set $A \subset \{i \in \mathbb{N} : i \leq 4\}$ such that $A_n = A$, for every n , and also that for each $i \leq 4$ the sequence $\{y_i^n\}_n$ converges to y_i , so $(y_i)_{i \leq 4} \in M_Y^4$. From condition (4.1) it follows that $\|\sum_{i \in A} y_i\| = |A|$. In view of condition (4.2) this is a contradiction. \square

Recall that a Banach space Y is *uniformly convex* if for every $\varepsilon > 0$ there is $0 < \delta < 1$ such that

$$u, v \in B_Y, \quad \frac{\|u + v\|}{2} > 1 - \delta \Rightarrow \|u - v\| < \varepsilon.$$

In such a case, the *modulus of convexity* of Y is the function defined by

$$\delta_Y(\varepsilon) := \inf \left\{ 1 - \frac{\|u + v\|}{2} : u, v \in B_Y, \|u - v\| \geq \varepsilon \right\}.$$

As a consequence of [20, Theorem 2.5] the pair (ℓ_∞^4, Y) has the Bishop-Phelps-Bollobás property for operators in case that Y is uniformly convex. By Theorem 3.3 uniformly convex spaces have $\text{AHSp-}\ell_\infty^4$. However, we provide a direct proof of that fact.

LEMMA 4.2. Assume that $(y_i)_{i \leq 4} \in M_Y^4$, $y^* \in S_{Y^*}$ and $\delta > 0$ satisfies that

$$i < k \leq 4, \quad y^*(y_i), y^*(y_k) > 1 - \delta \Rightarrow y^*(y_j) > 1 - 2\delta \quad \text{for each } i < j < k.$$

PROOF. Since $(y_i)_{i \leq 4} \in M_Y^4$, then for every $i < j < k$ we have that $\|y_i - y_j + y_k\| \leq 1$. If we assume that $y^*(y_i), y^*(y_k) > 1 - \delta$ then we have that

$$2 - 2\delta - y^*(y_j) < y^*(y_i - y_j + y_k) \leq \|y_i - y_j + y_k\| \leq 1.$$

As a consequence,

$$y^*(y_j) > 1 - 2\delta.$$

□

PROPOSITION 4.3. Every uniformly convex Banach space has the approximate hyperplane sum property for ℓ_∞^4 . Moreover, if δ_Y is the modulus of convexity of Y , then Y has the $AHSp\text{-}\ell_\infty^4$ with the function $\gamma(\varepsilon) = \min\{\frac{\delta_Y(\varepsilon)}{2}, \frac{\varepsilon}{6}\}$.

PROOF. Assume that Y is a uniformly convex Banach space with modulus of convexity δ_Y . Given $0 < \varepsilon < 1$, we define $\gamma(\varepsilon) = \min\{\frac{\delta_Y(\varepsilon)}{2}, \frac{\varepsilon}{6}\}$. Assume that $y^* \in S_{Y^*}$, $(y_i)_{i \leq 4} \in M_Y^4$ and $\emptyset \neq A \subset \{1, 2, 3, 4\}$ is a set such that

$$y^*(y_i) > 1 - \gamma(\varepsilon) > 0, \quad \forall i \in A.$$

By Lemma 2.5 we can assume that $\min A = 1$. In view of Lemma 4.2, the set $C = \{i \in \mathbb{N} : 1 \leq i \leq \max A\}$ satisfies that

$$y^*(y_i) > 1 - 2\gamma(\varepsilon) > 0, \quad \forall i \in C.$$

Hence for every $i, j \in C$ we have that

$$1 - \delta_Y(\varepsilon) \leq 1 - 2\gamma(\varepsilon) < y^*\left(\frac{y_i + y_j}{2}\right) \leq y^*\left(\frac{\frac{y_i}{\|y_i\|} + y_j}{2}\right) \leq \left\|\frac{\frac{y_i}{\|y_i\|} + y_j}{2}\right\|.$$

By the definition of the modulus of convexity it follows that

$$(4.3) \quad \left\|\frac{y_i}{\|y_i\|} - y_j\right\| < \varepsilon, \quad \forall i, j \in C.$$

We will show that there exists $(z_i)_{i \leq 4} \in M_Y^4$ satisfying that

$$(4.4) \quad \|z_i - y_i\| < \varepsilon, \forall i \leq 4 \quad \text{and} \quad \left\|\sum_{i \in C} z_i\right\| = |C|.$$

By Proposition 2.6 it suffices to show (4.4) in case that there is $2 \leq k \leq 4$ such that C coincides with the set $C_k = \{i \in \mathbb{N} : i \leq k\}$.

So let us fix $2 \leq k \leq 4$ and define $z_i = \frac{y_i}{\|y_i\|}$ for every $i \in C_k$. From (4.3) it follows that

$$(4.5) \quad \|z_i - y_i\| < \varepsilon, \quad \forall i \in C_k$$

and it is trivially satisfied that

$$(4.6) \quad \left\| \sum_{i \in C_k} z_i \right\| = |C_k|.$$

In case that $k = 2$, we put $a = \frac{\gamma(\varepsilon)}{1+\gamma(\varepsilon)}$, $b = \frac{\gamma(\varepsilon)}{2(1+\gamma(\varepsilon))}$ and

$$z_3 = (1-a)y_3 + bz_1, \quad z_4 = (1-a)y_4 - bz_1.$$

For each $i \in \{3, 4\}$, it follows that

$$\|z_i\| \leq 1 - a + b = 1 - \frac{\gamma(\varepsilon)}{1+\gamma(\varepsilon)} + \frac{\gamma(\varepsilon)}{2(1+\gamma(\varepsilon))} \leq 1$$

and

$$\|z_i - y_i\| \leq a + b = \frac{\gamma(\varepsilon)}{1+\gamma(\varepsilon)} + \frac{\gamma(\varepsilon)}{2(1+\gamma(\varepsilon))} = \frac{3\gamma(\varepsilon)}{2(1+\gamma(\varepsilon))} < \frac{3\gamma(\varepsilon)}{2} < \varepsilon.$$

By the last chain of inequalities and (4.5) it is satisfied that

$$\|z_i - y_i\| < \varepsilon, \quad \forall i \leq 4.$$

Since for each $i \in \{3, 4\}$ we have

$$(4.7) \quad \|z_1 - z_2 + z_i\| = \|z_i\| \leq 1$$

and for each $j \in \{1, 2\}$ it is clear that

$$\begin{aligned} \|z_j - z_3 + z_4\| &= \|z_1 - z_3 + z_4\| \\ &= \|(1-2b)z_1 + (1-a)(-y_3 + y_4)\| \\ &= (1-a)\|z_1 - y_3 + y_4\| \\ (4.8) \quad &\leq (1-a)(\|y_1 - y_3 + y_4\| + \|z_1 - y_1\|) \\ &\leq (1-a)\left(1 + \left\| \frac{y_1}{\|y_1\|} - y_1 \right\|\right) \\ &= (1-a)(2 - \|y_1\|) < (1-a)(1 + \gamma(\varepsilon)) = 1. \end{aligned}$$

In view of (4.7) and (4.8) we obtain that $(z_i)_{i \leq 4} \in M_Y^4$.

If $k = 3$, we take $z_4 = y_4$, in this case it is immediate to check that $(z_i)_{i \leq 4} \in M_Y^4$, also by the definition of z_4 and (4.5) it follows that

$$\|z_i - y_i\| < \varepsilon, \quad \forall i \leq 4.$$

Finally, if $k = 4$, we have by definition that $z_i = \frac{y_i}{\|y_i\|}$ for every $i \in \{1, 2, 3, 4\}$, so it is trivially satisfied that $(z_i)_{i \leq 4} \in M_Y^4$ and in view of (4.5) we also have that $\|z_i - y_i\| < \varepsilon$, for each $i \in \{1, 2, 3, 4\}$.

We showed that for every $2 \leq k \leq 4$ there exists $(z_i)_{i \leq 4} \in M_Y^4$ satisfying (4.4) for $C = C_k$, so the proof is complete since $A \subset C$. \square

As a consequence of [6, Theorem 2.4], [9, Theorem 2.1] and Theorem 3.3, there is also a nontrivial class of spaces containing uniformly convex spaces and satisfying $\text{AHSp-}\ell_\infty^4$.

The next statement can be shown by using the same argument of the analogous result for $\text{AHSp-}\ell_\infty^3$ (see [5, Proposition 2.4]). For this reason we do not include the proof.

PROPOSITION 4.4. *Let L be a nonempty locally compact Hausdorff topological space and Y a Banach space. Then $\mathcal{C}_0(L, Y)$ has the $\text{AHSp-}\ell_\infty^4$ if, and only if, Y has the $\text{AHSp-}\ell_\infty^4$.*

It is clear that \mathbb{R} has the $\text{AHSp-}\ell_\infty^4$, so by the previous result $\mathcal{C}_0(L)$ has the same property for any locally compact Hausdorff space L .

Our aim now is to prove that the space ℓ_1 has the $\text{AHSp-}\ell_\infty^4$. In what follows we will denote by u^* the element in ℓ_1^* given by

$$u^*(x) = \sum_{n=1}^{\infty} x(n) \quad (x \in \ell_1).$$

The next simple result will be useful for this purpose.

LEMMA 4.5. *Let be $s, t \in \mathbb{R}^+$, $x, y, z \in \ell_1$. Assume that*

$$1 - s \leq u^*(x - y + z) \quad \text{and} \quad \|x - y + z\| \leq 1 + t.$$

Then there is $w \in \ell_1$ such that

$$w \geq z, \quad \|w - z\| \leq s + t \quad \text{and} \quad x - y + w \geq 0.$$

PROOF. Define the sets given by

$$P = \{k \in \mathbb{N} : y(k) \leq (x + z)(k)\} \quad \text{and} \quad N = \mathbb{N} \setminus P.$$

Since $u^* \geq 0$, we have that

$$\begin{aligned} 1 - s &\leq u^*(x - y + z) \leq u^*((x - y + z)\chi_P) \\ &\leq \|(x - y + z)\chi_P\| \leq \|x - y + z\| \\ &\leq 1 + t. \end{aligned}$$

Hence $\|(x - y + z)\chi_P\| \geq 1 - s$, so

$$(4.9) \quad \|(x - y + z)\chi_N\| = \|x - y + z\| - \|(x - y + z)\chi_P\| \leq 1 + t - (1 - s) = s + t.$$

Let $w \in \ell_1$ be the element given by

$$w(k) = \begin{cases} z(k) & \text{if } k \in P \\ (y - x)(k) & \text{if } k \in N. \end{cases}$$

It is clear that $w\chi_P = z\chi_P$ and $(y - x)\chi_N \geq z\chi_N$, so $w \geq z$. Since

$$(x - y + w)\chi_P = (x - y + z)\chi_P \geq 0 \quad \text{and} \quad (x - y + w)\chi_N = 0,$$

it is satisfied that $x - y + w \geq 0$.

In view of (4.9) we also have that

$$\begin{aligned}\|w - z\| &= \|(w - z)\chi_N\| = \|(y - x - z)\chi_N\| \\ &= \|(x - y + z)\chi_N\| \leq s + t.\end{aligned}$$

□

Notice that in Lemma 4.5 the element x satisfies the same assumptions that z . So under the same conditions we also obtain an element $v \geq x$ such that $\|v - x\| \leq s + t$ and $v - y + z \geq 0$.

PROPOSITION 4.6. *There is a function $\rho :]0, 1[\rightarrow \mathbb{R}^+$ such that the space ℓ_1^n satisfies condition 2) of Proposition 2.12 for such function, for each natural number n . Also the space ℓ_1 satisfies the previous statement. As a consequence, there is a function $\gamma :]0, 1[\rightarrow \mathbb{R}^+$ such that the spaces ℓ_1^n and ℓ_1 have the approximate hyperplane sum property for ℓ_∞^4 with such function (see Definition 2.3).*

PROOF. We prove the statement for ℓ_1 . To this purpose we denote by $\{e_n\}$ the usual Schauder basis of ℓ_1 . It suffices to show that ℓ_1 satisfies condition 2) in Proposition 2.12 for $\rho(\varepsilon) = \frac{\varepsilon}{226}$ and $E = \text{Ext}(B_{(\ell_1)^*})$, which is clearly a 1-norming set for ℓ_1 . In such case we deduce that ℓ_1 satisfies the $\text{AHSp-}\ell_\infty^4$ with the function γ given by $\gamma(\varepsilon) = \rho^2(\frac{\varepsilon}{4})$ by Proposition 2.12.

From now on $0 < \varepsilon < 1$ will be fixed and we simply write ρ instead of $\rho(\varepsilon)$. Assume that $y^* \in E = \text{Ext}(B_{(\ell_1)^*})$, $(a_i)_{i \leq 4} \in M_{\ell_1}^4$ and $\emptyset \neq C \subset \{1, 2, 3, 4\}$ is a nonempty set such that

$$y^*(a_i) > 1 - \rho, \quad \forall i \in C.$$

Firstly by Lemma 2.5 we can clearly assume that $\min C = 1$.

In the case that C contains only one element, by Proposition 2.6 condition 2) in Proposition 2.12 is satisfied for such set C . Otherwise C contains at least two elements.

Since $\text{Ext}(B_{(\ell_1)^*}) = \{z^* \in \ell_1^* : |z^*(e_n)| = 1, \forall n \in \mathbb{N}\}$, by using an appropriate surjective linear isometry onto ℓ_1 , we can clearly assume that $y^* = u^*$. It suffices to show that there exists $(z_i)_{i \leq 4} \in M_{\ell_1}^4$ satisfying that

$$\|z_i - a_i\| < \varepsilon, \forall i \leq 4 \quad \text{and also} \quad u^*(z_i) = 1 \quad \text{and} \quad z_i \geq 0, \quad \forall i \in C.$$

We define the set C' given by

$$C' := \{i \in \mathbb{N} : 1 \leq i \leq \max C, u^*(a_i) > 1 - 2\rho\}.$$

In case that $i, k \in C$ and $j \in \mathbb{N}$ satisfies $i < j < k$, by Lemma 4.2 we know that $j \in C'$. Since $C \subset C'$, the previous remark shows that C' has consecutive elements. In fact

$$C' = \{i \in \mathbb{N} : 1 = \min C \leq i \leq \max C\}.$$

We define

$$P_i = \{k \in \mathbb{N} : a_i(k) \geq 0\}, \quad N_i = \mathbb{N} \setminus P_i, \quad b_i = a_i \chi_{P_i}, \quad (i \in C')$$

Since $u^* \geq 0$, by assumption we have that

$$(4.10) \quad 1 - 2\rho < u^*(a_i) \leq u^*(b_i) \leq \|b_i\| \leq \|a_i\| \leq 1 \quad \text{and} \quad b_i \geq 0, \quad \forall i \in C'.$$

Hence

$$(4.11) \quad \|b_i - a_i\| = \|a_i \chi_{N_i}\| \leq 1 - \|b_i\| < 2\rho, \quad \forall i \in C'.$$

Now we will distinguish several cases, depending on the number of elements of C' . Since we assume that C contains more than one element and $C \subset C'$, then C' also contains at least two elements. So $C' = \{1, 2\}$, $C' = \{1, 2, 3\}$ or $C' = \{1, 2, 3, 4\}$.

• Case 1: Assume that $C' = \{1, 2\}$. Define the elements in ℓ_1 given by

$$x_i = \begin{cases} b_i + (1 - \|b_i\|)e_1 & \text{if } i \in \{1, 2\} \\ a_i & \text{if } i \in \{3, 4\}. \end{cases}$$

So we have that

$$x_i \geq 0, \quad \text{and} \quad \|x_i\| = 1, \quad i = 1, 2.$$

From (4.10) and (4.11) it follows that

$$(4.12) \quad \|x_i\| \leq 1 \quad \text{and} \quad \|x_i - a_i\| < 4\rho, \quad \forall i \leq 4.$$

In this case we will change twice the previous vectors in order to get the desired properties. Firstly we define the set $\{y_i : 1 \leq i \leq 4\}$ by

$$y_i = \begin{cases} \frac{x_i}{1+8\rho} + (1 - \frac{1}{1+8\rho})e_1 & \text{if } i \in \{1, 2\} \\ \frac{x_i}{1+8\rho} & \text{if } i \in \{3, 4\}. \end{cases}$$

It is clearly satisfied that

$$(4.13) \quad y_i \geq 0, \quad \|y_i\| = 1, \quad \text{for } i = 1, 2 \quad \text{and} \quad y_i \in B_{\ell_1} \quad \text{for } 1 \leq i \leq 4.$$

For each $1 \leq i \leq 4$ we also have that

$$(4.14) \quad \begin{aligned} \|y_i - a_i\| &\leq \left\| y_i - \frac{x_i}{1+8\rho} \right\| + \left\| \frac{x_i}{1+8\rho} - x_i \right\| + \|x_i - a_i\| \\ &< \left(1 - \frac{1}{1+8\rho} \right) + \left(1 - \frac{1}{1+8\rho} \right) + 4\rho \quad (\text{by (4.12)}) \\ &= \frac{16\rho}{1+8\rho} + 4\rho \\ &< 20\rho. \end{aligned}$$

Now we fix $1 \leq i_1 < i_2 < i_3 \leq 4$ and estimate the norm of $\|y_{i_1} - y_{i_2} + y_{i_3}\|$ as follows. In case that $\{1, 2\} \subset \{i_1, i_2, i_3\}$ we have

$$\begin{aligned}
 (4.15) \quad \|y_{i_1} - y_{i_2} + y_{i_3}\| &= \left\| \frac{x_{i_1} - x_{i_2} + x_{i_3}}{1 + 8\rho} \right\| \\
 &\leq \frac{\|a_{i_1} - a_{i_2} + a_{i_3}\| + \|x_1 - a_1\| + \|x_2 - a_2\|}{1 + 8\rho} \\
 &< \frac{\|a_{i_1} - a_{i_2} + a_{i_3}\| + 8\rho}{1 + 8\rho} \quad (\text{by (4.12)}) \\
 &\leq 1.
 \end{aligned}$$

Otherwise $\{1, 2\} \not\subset \{i_1, i_2, i_3\}$ and we obtain that

$$\begin{aligned}
 (4.16) \quad \|y_{i_1} - y_{i_2} + y_{i_3}\| &\leq \left\| \frac{x_{i_1} - x_{i_2} + x_{i_3}}{1 + 8\rho} \right\| + 1 - \frac{1}{1 + 8\rho} \\
 &< \frac{\|a_{i_1} - a_{i_2} + a_{i_3}\| + 4\rho}{1 + 8\rho} + \frac{8\rho}{1 + 8\rho} \quad (\text{by (4.12)}) \\
 &< 1 + 4\rho.
 \end{aligned}$$

Now we define

$$z_i = \begin{cases} \frac{y_i}{1+4\rho} + (1 - \frac{1}{1+4\rho})e_1 & \text{if } i \in \{1, 2, 3\} \\ \frac{y_4}{1+4\rho} & \text{if } i = 4. \end{cases}$$

In view of (4.13) we have that

$$(4.17) \quad z_i \geq 0, \quad u^*(z_i) = 1 \quad \text{for } i = 1, 2 \quad \text{and} \quad z_i \in B_{\ell_1} \quad \text{for } 1 \leq i \leq 4.$$

For each $1 \leq i \leq 4$ we obtain that

$$\begin{aligned}
 (4.18) \quad \|z_i - a_i\| &\leq \left\| z_i - \frac{y_i}{1 + 4\rho} \right\| + \left\| \frac{y_i}{1 + 4\rho} - y_i \right\| + \|y_i - a_i\| \\
 &< \left(1 - \frac{1}{1 + 4\rho} \right) + \left(1 - \frac{1}{1 + 4\rho} \right) + 20\rho \quad (\text{by (4.13) and (4.14)}) \\
 &= \frac{8\rho}{1 + 4\rho} + 20\rho < 28\rho < \varepsilon.
 \end{aligned}$$

Now we check that $(z_i)_{i \leq 4} \in M_{\ell_1}^4$. For each $1 \leq i_1 < i_2 < i_3 \leq 4$, we consider the following two cases. If $\{1, 2\} \subset \{i_1, i_2, i_3\}$ it is clear that

$$\begin{aligned}
 \|z_{i_1} - z_{i_2} + z_{i_3}\| &\leq \frac{\|y_{i_1} - y_{i_2} + y_{i_3}\|}{1 + 4\rho} + 1 - \frac{1}{1 + 4\rho} \\
 &< \frac{1}{1 + 4\rho} + 1 - \frac{1}{1 + 4\rho} \quad (\text{by (4.15)}) \\
 &= 1.
 \end{aligned}$$

Otherwise $\{1, 2\} \not\subset \{i_1, i_2, i_3\}$ and we obtain that

$$\begin{aligned}
 \|z_{i_1} - z_{i_2} + z_{i_3}\| &= \frac{\|y_{i_1} - y_{i_2} + y_{i_3}\|}{1 + 4\rho} \\
 &< 1 \quad (\text{by (4.16)}).
 \end{aligned}$$

In view of (4.17) and (4.18), since we checked that $(z_i)_{i \leq 4} \in M_{\ell_1}^4$, the proof is finished in case 1.

• Case 2: Assume that $C' = \{1, 2, 3\}$. We know that

$$\begin{aligned} 1 - 4\rho &= 2(1 - 2\rho) - 1 < u^*(b_1 + b_3) - u^*(b_2) \quad (\text{by (4.10)}) \\ &= u^*(b_1 - b_2 + b_3) \leq \|b_1 - b_2 + b_3\| \\ &\leq \|a_1 - a_2 + a_3\| + \sum_{i=1}^3 \|b_i - a_i\| \\ &< 1 + 6\rho \quad (\text{by (4.11)}). \end{aligned}$$

We obtained that

$$(4.19) \quad 1 - 4\rho < u^*(b_1 - b_2 + b_3) \leq \|b_1 - b_2 + b_3\| < 1 + 6\rho.$$

We can apply Lemma 4.5 with b_1 playing the role of z , so there is $x_1 \in \ell_1$ such that

$$(4.20) \quad x_1 \geq b_1 \geq 0, \quad \|x_1 - b_1\| \leq 10\rho \quad \text{and} \quad x_1 - b_2 + b_3 \geq 0.$$

We define

$$x_i = b_i \quad \text{for } i \in \{2, 3\} \quad \text{and} \quad x_4 = a_4.$$

Notice that in view of (4.11) and (4.20) we have

$$(4.21) \quad \|x_i - a_i\| < 12\rho, \quad \forall i \leq 4.$$

As a consequence, for each $i \in \{1, 2, 3\}$, from (4.10) and (4.20) it follows that

$$\begin{aligned} (4.22) \quad 1 - 2\rho &< \|b_i\| \\ &\leq \|x_i\| \\ &\leq \|x_i - a_i\| + \|a_i\| < 1 + 12\rho. \end{aligned}$$

If $1 \leq i_1 < i_2 < i_3 \leq 4$ we obtain that

$$\begin{aligned} (4.23) \quad \|x_{i_1} - x_{i_2} + x_{i_3}\| &\leq \|a_{i_1} - a_{i_2} + a_{i_3}\| + \sum_{j=1}^3 \|x_{i_j} - a_{i_j}\| \\ &< 1 + 36\rho \quad (\text{by (4.21)}). \end{aligned}$$

Now we define a new element $(y_i)_{i \leq 4}$ in order to have that $\{y_i : i \leq 3\} \subset \{x \in B_{\ell_1} : u^*(x) = 1\}$. We put

$$y_i = \begin{cases} \frac{x_i}{1+36\rho} + (1 - \frac{\|x_i\|}{1+36\rho})e_1 & \text{if } i \in \{1, 2, 3\} \\ \frac{x_4}{1+36\rho} & \text{if } i = 4. \end{cases}$$

For each $i \leq 3$, since $b_i \geq 0$, from (4.20) we also have that $x_i \geq 0$. In view of (4.22) we deduce that

$$(4.24) \quad y_i \geq 0, \quad u^*(y_i) = 1 \quad \text{for } i \in \{1, 2, 3\} \quad \text{and} \quad \|y_4\| \leq 1.$$

For each $1 \leq i \leq 4$, in view of (4.22) and (4.21) we obtain that

$$\begin{aligned}
 (4.25) \quad \|y_i - a_i\| &\leq \left\| y_i - \frac{x_i}{1+36\rho} \right\| + \left\| \frac{x_i}{1+36\rho} - x_i \right\| + \|x_i - a_i\| \\
 &< \left(1 - \frac{1-2\rho}{1+36\rho}\right) + \|x_i\| \left(1 - \frac{1}{1+36\rho}\right) + 12\rho \\
 &< \frac{38\rho}{1+36\rho} + (1+12\rho) \frac{36\rho}{1+36\rho} + 12\rho \\
 &< 86\rho.
 \end{aligned}$$

On one hand, since $x_i \geq 0$ for each $i \leq 3$ we also have that

$$\begin{aligned}
 (4.26) \quad \|y_1 - y_2 + y_3\| &= \left\| \frac{x_1 - x_2 + x_3}{1+36\rho} + \left(1 - \frac{\|x_1\| - \|x_2\| + \|x_3\|}{1+36\rho}\right) e_1 \right\| \\
 &= \left\| \frac{x_1 - x_2 + x_3}{1+36\rho} + \left(1 - \frac{\|x_1 - x_2 + x_3\|}{1+36\rho}\right) e_1 \right\| \quad (\text{by (4.20)}) \\
 &= \left\| \frac{x_1 - x_2 + x_3}{1+36\rho} \right\| + \left(1 - \frac{\|x_1 - x_2 + x_3\|}{1+36\rho}\right) \quad (\text{by (4.23)}) \\
 &= 1.
 \end{aligned}$$

On the other hand, if $1 \leq i_1 < i_2 < 4$ then

$$\begin{aligned}
 (4.27) \quad \|y_{i_1} - y_{i_2} + y_4\| &\leq \left\| \frac{x_{i_1} - x_{i_2} + x_4}{1+36\rho} \right\| + \frac{1}{1+36\rho} \left| \|x_{i_1}\| - \|x_{i_2}\| \right| \\
 &< 1 + \frac{14\rho}{1+36\rho} \quad (\text{by (4.22) and (4.23)}) \\
 &< 1 + 14\rho.
 \end{aligned}$$

Now we define the element in $M_{\ell_1}^4$ satisfying the required conditions in this case. To this purpose we take

$$z_i = \begin{cases} \frac{y_i}{1+14\rho} + \left(1 - \frac{1}{1+14\rho}\right) e_1 & \text{if } i \in \{1, 2, 3\} \\ \frac{y_4}{1+14\rho} & \text{if } i = 4. \end{cases}$$

Let us fix $1 \leq i \leq 4$. In view of (4.24) it is satisfied that

$$(4.28) \quad z_i \geq 0, \quad u^*(z_i) = 1 \quad \text{for } i \in \{1, 2, 3\} \quad \text{and} \quad \|z_4\| \leq 1.$$

By using (4.24) and (4.25) we obtain the following upper estimate

$$\begin{aligned}
 (4.29) \quad \|z_i - a_i\| &\leq \left\| z_i - \frac{y_i}{1+14\rho} \right\| + \left\| \frac{y_i}{1+14\rho} - y_i \right\| + \|y_i - a_i\| \\
 &< \left(1 - \frac{1}{1+14\rho}\right) + \|y_i\| \left(1 - \frac{1}{1+14\rho}\right) + 86\rho \\
 &< 28\rho + 86\rho = 114\rho < \varepsilon.
 \end{aligned}$$

We also have that

$$\begin{aligned}
 \|z_1 - z_2 + z_3\| &= \left\| \frac{y_1 - y_2 + y_3}{1 + 14\rho} + \left(1 - \frac{1}{1 + 14\rho}\right)e_1 \right\| \\
 (4.30) \quad &\leq \frac{\|y_1 - y_2 + y_3\|}{1 + 14\rho} + \left(1 - \frac{1}{1 + 14\rho}\right) \\
 &= 1 \quad (\text{by (4.26)}).
 \end{aligned}$$

In case that $1 \leq i_1 < i_2 < 4$ we obtain that

$$\begin{aligned}
 (4.31) \quad \|z_{i_1} - z_{i_2} + z_4\| &= \left\| \frac{y_{i_1} - y_{i_2} + y_4}{1 + 14\rho} \right\| \\
 &< 1 \quad (\text{by (4.27)}).
 \end{aligned}$$

From equations (4.28), (4.29), (4.30) and (4.31) the element $(z_i)_{i \leq 4} \in M_{\ell_1}^4$ satisfies all the required conditions.

• Case 3: Assume that $C' = \{1, 2, 3, 4\}$. For each $1 \leq i_1 < i_2 < i_3 \leq 4$, by the same argument used to obtain (4.19) in case 2, with (i_1, i_2, i_3) playing the role of $(1, 2, 3)$ there we get that

$$(4.32) \quad 1 - 4\rho < u^*(b_{i_1} - b_{i_2} + b_{i_3}) \leq \|b_{i_1} - b_{i_2} + b_{i_3}\| < 1 + 6\rho.$$

In view of Lemma 4.5, there are $x_1, x_4 \in \ell_1$ such that

$$\begin{aligned}
 (4.33) \quad x_i &\geq b_i, \quad \|x_i - b_i\| \leq 10\rho \quad \text{for } i = 1, 4, \\
 x_1 - b_2 + b_3 &\geq 0 \quad \text{and} \quad b_1 - b_3 + x_4 \geq 0.
 \end{aligned}$$

As a consequence

$$\begin{aligned}
 1 - 4\rho &< u^*(b_1 - b_2 + b_4) \leq u^*(x_1 - b_2 + b_4) \quad (\text{by (4.32) and (4.33)}) \\
 &\leq \|x_1 - b_2 + b_4\| \\
 &\leq \|b_1 - b_2 + b_4\| + \|x_1 - b_1\| \\
 &< \|a_1 - a_2 + a_4\| + 6\rho + \|x_1 - b_1\| \quad (\text{by (4.11)}) \\
 &\leq 1 + 16\rho \quad (\text{by (4.33)})
 \end{aligned}$$

and

$$\begin{aligned}
 1 - 4\rho &< u^*(b_2 - b_3 + b_4) \leq u^*(b_2 - b_3 + x_4) \quad (\text{by (4.32) and (4.33)}) \\
 &\leq \|b_2 - b_3 + x_4\| \\
 &\leq \|b_2 - b_3 + b_4\| + \|x_4 - b_4\| < 1 + 16\rho \quad (\text{by (4.11) and (4.33)}).
 \end{aligned}$$

From the last two chains of inequalities we have that

$$\begin{aligned}
 1 - 4\rho &< u^*(x_1 - b_2 + b_4) \leq \|x_1 - b_2 + b_4\| < 1 + 16\rho \\
 1 - 4\rho &< u^*(b_2 - b_3 + x_4) \leq \|b_2 - b_3 + x_4\| < 1 + 16\rho.
 \end{aligned}$$

By applying again Lemma 4.5, there are $y_1, y_4 \in \ell_1$ satisfying that

$$(4.34) \quad \begin{aligned} y_i &\geq x_i, & \|y_i - x_i\| &\leq 20\rho \quad \text{for } i = 1, 4, \\ y_1 - b_2 + b_4 &\geq 0 & \text{and} & \quad b_2 - b_3 + y_4 \geq 0. \end{aligned}$$

Now we define $y_2 = b_2$ and $y_3 = b_3$.

By using (4.34), (4.33) and (4.11), for each $i \leq 4$ it is clear that

$$(4.35) \quad \begin{aligned} \|y_i - a_i\| &\leq \|y_i - b_i\| + \|b_i - a_i\| \\ &< 20\rho + 10\rho + 2\rho = 32\rho. \end{aligned}$$

As a consequence, from (4.33) and (4.34) for each $i \leq 4$ we also have that

$$(4.36) \quad 1 - 2\rho < u^*(a_i) \leq u^*(b_i) \leq u^*(y_i) \leq \|y_i\| \leq \|y_i - a_i\| + \|a_i\| < 1 + 32\rho.$$

In view of (4.33) and (4.34) it is immediate to check that

$$(4.37) \quad y_{i_1} - y_{i_3} + y_{i_3} \geq 0 \quad \text{for any } 1 \leq i_1 < i_2 < i_3 \leq 4.$$

For every $1 \leq i_1 < i_2 < i_3 \leq 4$ we also get that

$$(4.38) \quad \begin{aligned} \|y_{i_1} - y_{i_2} + y_{i_3}\| &\leq \|a_{i_1} - a_{i_2} + a_{i_3}\| + \sum_{j=1}^3 \|y_{i_j} - a_{i_j}\| \\ &< 1 + 96\rho \quad (\text{by (4.35)}). \end{aligned}$$

Now we define the elements satisfying the required conditions by

$$z_i = \frac{y_i}{1 + 96\rho} + \left(1 - \frac{\|y_i\|}{1 + 96\rho}\right)e_1 \quad (1 \leq i \leq 4).$$

By (4.34) and (4.33) it is clear that $y_i \geq 0$ for each $i \leq 4$. From (4.36) we know that $\|y_i\| \leq 1 + 32\rho$ and so we deduce that

$$(4.39) \quad z_i \geq 0 \quad \text{and} \quad u^*(z_i) = 1, \quad \forall 1 \leq i \leq 4.$$

We also obtain that

$$(4.40) \quad \begin{aligned} \|z_i - a_i\| &\leq \left\| z_i - \frac{y_i}{1 + 96\rho} \right\| + \left\| \frac{y_i}{1 + 96\rho} - y_i \right\| + \|y_i - a_i\| \\ &< \left(1 - \frac{\|y_i\|}{1 + 96\rho}\right) + \|y_i\| \left(1 - \frac{1}{1 + 96\rho}\right) + 32\rho \quad (\text{by (4.35)}) \\ &< \left(1 - \frac{1 - 2\rho}{1 + 96\rho}\right) + \frac{1 + 32\rho}{1 + 96\rho} 96\rho + 32\rho \quad (\text{by (4.36)}) \\ &< \frac{98\rho}{1 + 96\rho} + 96\rho + 32\rho \\ &< 226\rho = \varepsilon. \end{aligned}$$

Finally, notice that for every $1 \leq i_1 < i_2 < i_3 \leq 4$, since $y_i \geq 0$ for every $i \leq 4$ we have that

$$\begin{aligned}
 \|z_{i_1} - z_{i_2} + z_{i_3}\| &= \left\| \frac{y_{i_1} - y_{i_2} + y_{i_3}}{1 + 96\rho} + \left(1 - \frac{\|y_{i_1}\| - \|y_{i_2}\| + \|y_{i_3}\|}{1 + 96\rho}\right) e_1 \right\| \\
 (4.41) \quad &= \left\| \frac{y_{i_1} - y_{i_2} + y_{i_3}}{1 + 96\rho} + \left(1 - \frac{\|y_{i_1} - y_{i_2} + y_{i_3}\|}{1 + 96\rho}\right) e_1 \right\| \quad (\text{by (4.37)}) \\
 &\leq \frac{\|y_{i_1} - y_{i_2} + y_{i_3}\|}{1 + 96\rho} + 1 - \frac{\|y_{i_1} - y_{i_2} + y_{i_3}\|}{1 + 96\rho} \quad (\text{by (4.38)}) \\
 &= 1.
 \end{aligned}$$

In view of equations (4.39), (4.40) and (4.41) the proof is also finished in case 3. So we proved the statement for ℓ_1 .

The proof for ℓ_1^n follows from the same argument that we used for ℓ_1 by considering elements in $M_{\ell_1^n}^4$ (instead of elements in $M_{\ell_1}^4$) and the description $\text{Ext}(B_{(\ell_1^n)^*}) = \{z^* \in (\ell_1^n)^* : |z^*(e_k)| = 1, \forall k \leq n\}$. \square

Next result follows from the same argument of [5, Theorem 2.7]. It will be useful, for instance, to extend the previous result to $L_1(\mu)$ for any positive measure μ .

THEOREM 4.7. *Assume that Y is a Banach space and $\gamma :]0, 1[\rightarrow \mathbb{R}^+$ is a function such that $Y = \overline{\cup\{Y_\alpha : \alpha \in \Lambda\}}$, where $\{Y_\alpha : \alpha \in \Lambda\}$ is a nested family of subspaces of Y satisfying the $\text{AHSp-}\ell_\infty^4$ with the function γ . Then Y has the $\text{AHSp-}\ell_\infty^4$ with the function $\zeta(\varepsilon) = \gamma(\frac{\varepsilon}{2})$.*

PROOF. Given $0 < \varepsilon < 1$, let $\gamma(\varepsilon)$ be the positive real number satisfying Definition 2.3 for each space Y_α .

Assume that $(a_i)_{i \leq 4} \in M_Y^4$ and that for some nonempty set $A \subset \{1, 2, 3, 4\}$ and $y^* \in S_{Y^*}$, it is satisfied that $y^*(a_i) > 1 - \gamma(\frac{\varepsilon}{2})$ for each $i \in A$. Let us choose a real number t such that

$$0 < t < \frac{1}{4} \min \left\{ \frac{\varepsilon}{2}, \min \left\{ y^*(a_i) - 1 + \gamma\left(\frac{\varepsilon}{2}\right) : i \in A \right\} \right\}.$$

By assumption there exist $\alpha_0 \in \Lambda$ and $\{b_i : i \leq 4\} \subset B_{Y_{\alpha_0}}$ satisfying

$$\|b_i - a_i\| < t \quad \text{for all } i \leq 4.$$

Now we define $y_i = \frac{b_i}{1+3t}$ for any $i \leq 4$. By using that $(a_i)_{i \leq 4} \in M_Y^4$ it is immediate to check that $(y_i)_{i \leq 4} \in M_{Y_{\alpha_0}}^4$. We clearly have

$$(4.42) \quad \|y_i - a_i\| \leq \left\| \frac{b_i}{1+3t} - b_i \right\| + \|b_i - a_i\| < 3t + t = 4t < \frac{\varepsilon}{2}, \quad \text{for all } i \leq 4.$$

For each $i \in A$ we obtain that

$$(4.43) \quad y^*(y_i) > y^*(a_i) - 4t > 1 - \gamma\left(\frac{\varepsilon}{2}\right) > 0.$$

We define the element $z^* \in Y_{\alpha_0}^*$ by

$$z^*(z) = y^*(z) \quad (z \in Y_{\alpha_0}).$$

Since $\|z^*\| \leq \|y^*\|$, we get that $z^* \in B_{Y_{\alpha_0}^*}$. In view of (4.43) we know that $z^* \neq 0$ and we also have that

$$\frac{z^*}{\|z^*\|}(y_i) = \frac{y^*}{\|y^*\|}(y_i) \geq y^*(y_i) > 1 - \gamma\left(\frac{\varepsilon}{2}\right) \quad \text{for all } i \in A.$$

By assumption there is $(z_i)_{i \leq 4} \in M_{Y_{\alpha_0}}^4$ such that

- i) $\|z_i - y_i\| < \frac{\varepsilon}{2}$ for each $i \leq 4$,
- ii) $\|\sum_{i \in A} z_i\| = |A|$.

In view of (4.42) we obtain that

$$\|z_i - a_i\| \leq \|z_i - y_i\| + \|y_i - a_i\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \quad \forall i \leq 4.$$

We proved that Y satisfies Definition 2.3 with the function $\zeta(\varepsilon) = \gamma\left(\frac{\varepsilon}{2}\right)$. □

Let us remark that the subspace of $L_1(\mu)$ generated by a finite number of characteristic functions is linearly isometric to some space ℓ_1^n . Since the space of simple functions is dense in $L_1(\mu)$, in view of Proposition 4.6, we conclude that $L_1(\mu)$ satisfies the assumption of the previous result and the function γ does not depend on μ . In view of Theorem 3.3 we obtain that the pairs $(\ell_\infty^4, L_1(\mu))$ have uniformly BPBp for operators for every measure μ . More concretely we deduce the following assertions.

COROLLARY 4.8. *There is a function $\gamma :]0, 1[\rightarrow \mathbb{R}^+$ such that $L_1(\mu)$ satisfies Definition 2.3 with such function, for any positive measure μ . Hence the pair $(\ell_\infty^4, L_1(\mu))$ has the BPBp for operators. Moreover there is a function η such that the pair $(\ell_\infty^4, L_1(\mu))$ satisfies Definition 1.1 for such function, for any positive measure μ .*

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