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The Riemann problem for the two-dimensional zero-pressure Euler equations ☆

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A B S T R A C T

The Riemann problem for the two-dimensional zero-pressure Euler equations is considered. The initial data are constant values in each quadrant, which satisfy an assumption that each initial discontinuity projects only one two-dimensional wave. The phenomenon of two-dimensional delta shock wave with a Dirac delta function in both density and internal energy is identified. Both generalized Rankine–Hugoniot relation and entropy condition for this type of two-dimensional delta shock wave are proposed. The qualitative behavior of entropy solutions to this relation with certain special initial data is established. Based on these preparations, we obtain twenty-three explicit solutions and their corresponding criteria. In particular, the Mach-reflection-like patterns arise in the exact solutions.

1. Introduction

It is well known that, there are some nonclassical situations, where in contrast to Lax's and Glimm's classical results [12,17], the initial value problem for some physical models does not possess a weak L^∞-solution. While a type of generalized solution called delta shock wave solution is introduced as its solution. Roughly speaking, a delta shock wave solution is a solution such that at least one of the state variables contains a Dirac delta function [35]. The investigations of one-dimensional delta shock waves have been intensively developed in the past two decades [4–6,21,30,32,36] and the references therein. Interestingly, Danilov and Shelkovich [10], Panov and Shelkovich [26] introduced the one-dimensional $\delta^{(n)}$-shock wave solution, where $\delta^{(n)}$, $n = 1, 2, \cdots$, is nth derivative of the Dirac delta function, and shown that the solution to a one-dimensional system of conservation laws involves not only Dirac delta functions but also their derivatives. Moreover, the one-dimensional delta shock wave with a Dirac delta function in multiple state variables has

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been studied recently [8,24,25,28,37,38]. However, only little work has contributed to the investigations of two-dimensional and high dimensional delta shock waves.

One typical example of the systems admitting a two-dimensional delta shock wave solutions is the two-dimensional zero-pressure gas dynamics

$$\begin{cases} \rho_t + \nabla \cdot (\rho U) = 0, \\ (\rho U)_t + \nabla \cdot (\rho U \otimes U) = 0, \end{cases} \tag{1.1}$$

where the variables $\rho \geq 0, U = (u, v)$ denote density and velocity, the ∇ is the gradient operator over the space variable $X = (x, y)$, and the $\otimes$ is the usual tensor product of two vectors. The system (1.1) is obtained by flux-splitting numerical schemes for the two-dimensional compressible Euler equations [4], or by letting pressure $p = 0$ on the two-dimensional isentropic Euler equations [18]. This system is used to describe the motion of free particles sticking together under collision and the formation of large-scale structures in the universe [1,27]. It has also been used for modeling dusty media which can be considered as having no pressure [16]. There have been numerous studies on this system. In one-dimensional case, with random initial data, Bouchut [4] given the Riemann solution and checked the solution satisfying (1.1) in the sense of measures. The behavior of global weak solutions with random initial data was analyzed by Weinan, Sinai and Rykov [11]. By viscous vanishing method, Sheng and Zhang [32] studied the Riemann problem for system (1.1) in one-dimensional case, and also constructed the solutions for system (1.1) with four constant values only involving contact discontinuities. Li and Zhang [20] established a complete form of generalized Rankine–Hugoniot relation of the two-dimensional delta shock wave, and obtained all the Riemann solutions to system (1.1) by solving this relation. Moreover, Li and Yang [19] considered the Riemann problem with two constant values separated by a hyperplane, and obtained the n-dimensional ($n \geq 3$) plane delta shock wave dependent on a family of one parameter. For more investigations of high dimensional delta shock waves, the readers are also referred to articles [2,14,29,33,34]. It is noticed that the common feature of above-mentioned multi-dimensional delta shock waves is that only one state variable contains a Dirac delta function.

Mach-reflection-like pattern is a kind of important nonlinear phenomenon in multi-dimensional systems of conservation laws. Li, Zhang and Yang [21] studied the two-dimensional Riemann problem for system (1.1) with four constant values, where, the two-dimensional delta shock wave with a Dirac delta function only in density emerges in the solution. Meanwhile, in some solutions, one two-dimensional delta shock wave bifurcates somewhere into two new two-dimensional delta shock waves, and there exists a triple-wave point where three delta shock waves match together. This structure is named as the Mach-reflection-like pattern, which is similar to the Mach reflection configuration in gas dynamics [9,40]. It also resembles the Guckenheimer structure which issues in two-dimensional single conservation law [13,31,39]. All of these result from the global (local and global) interactions of two-dimensional waves.

As was shown in [16], for modeling media which can be considered as having no pressure, it is necessary to consider energy transport. This motivates us to study the following two-dimensional zero-pressure Euler equations

$$\begin{cases} \rho_t + \nabla \cdot (\rho U) = 0, \\ (\rho U)_t + \nabla \cdot (\rho U \otimes U) = 0, \\ (\frac{1}{2}\rho||U||^2 + \rho e)_t + \nabla \cdot ((\frac{1}{2}\rho||U||^2 + \rho e)U) = 0, \end{cases} \tag{1.2}$$

where the variable $e \geq 0$ represents specific energy, and $||U||^2 = u^2 + v^2$. The third equation of system (1.2) expresses the conservation of energy. The system (1.2) can be derived by letting the pressure $p = 0$ on the two-dimensional compressible Euler equations. Kraiko [16] studied the system (1.2) in one-dimensional case, and the discontinuities which would be different from classical ones and carry mass, impulse and energy, were used to construct the solution for arbitrary initial data. Nilsson, Rozanova and Shelkovich [22, 23] considered (1.2) in one-dimensional case by denoting the internal energy ρe by a new variable H, and shown the concentration processes of both mass and energy on the one-dimensional delta shock wave front. Subsequently, the corresponding one-dimensional Riemann problem was completely solved by Cheng [7]. Motivated by the works [22,23], we consider (1.2) in the following form

$$\begin{cases} \rho_t + \nabla \cdot (\rho U) = 0, \\ (\rho U)_t + \nabla \cdot (\rho U \otimes U) = 0, \\ (\frac{1}{2}\rho||U||^2 + H)_t + \nabla \cdot ((\frac{1}{2}\rho||U||^2 + H)U) = 0, \end{cases} \tag{1.3}$$

where the state variable $H \geq 0$ denotes internal energy.

Albeverio, Rozanova and Shelkovich [3] introduced the integral identities to define a multi-dimensional delta shock wave solution for system (1.3), and then derived the corresponding Rankine–Hugoniot condition. They also shown the concentration processes of both mass and internal energy on the multi-dimensional delta shock wave front. In the present paper, we study the two-dimensional Riemann problem for (1.3) with initial data

$$(\rho, U, H)(0, X) = (\rho_i, U_i, H_i), \text{ in the } i\text{th quadrant}, \ i = 1, 2, 3, 4, \tag{1.4}$$

where $\rho_i > 0, U_i, H_i > 0$ are constant values, satisfying assumption (H): *Each initial discontinuity projects only one two-dimensional wave.*

The self-similar bounded solutions are firstly analyzed. It is also proven that the $\rho, u_x, v_x, u_y, v_y, H$ blow up simultaneously in a finite time under certain assumptions on the initial data. These facts show that two-dimensional delta shock wave with a Dirac delta function in both density and internal energy takes place in this situation. While, both generalized Rankine–Hugoniot relation and entropy condition for this type of two-dimensional delta shock wave are proposed. To construct the Riemann solutions in self-similar plane, the generalized Rankine–Hugoniot relation is reformulated to a system of ordinary differential equations, and the qualitative behavior of entropy solutions to this reformulated relation with four kinds of special initial data is analyzed. Finally, under the assumption (H), twenty-three explicit solutions and their corresponding criteria are obtained.

In the next section, we first present the self-similar bounded solutions. We analyze rigorously the phenomenon of two-dimensional delta shock wave with a Dirac delta function in both density and internal energy, and propose both generalized Rankine–Hugoniot relation and entropy condition for this type of two-dimensional delta shock wave. We also reformulate the generalized Rankine–Hugoniot relation to a system of ordinary differential equations. In Section 3, we discuss the qualitative behavior of entropy solutions to the reformulated generalized Rankine–Hugoniot relation with four kinds of special initial data. In Section 4, under the assumption (H), we divide the two-dimensional Riemann problem (1.3) and (1.4) into five cases, and construct all of exact solutions in self-similar plan. A brief conclusion is given in Section 5.

2. Preliminaries

Under the conditions $\rho > 0$ and $H > 0$, the system (1.3) in direction $(\mu, \nu)(\mu^2 + \nu^2 = 1)$ has a quadruple eigenvalue $\lambda^{\mu,\nu} = \mu u + \nu v$, with the associated right eigenvectors $r_1^{\mu,\nu} = (1, 0, 0, 0)^T, r_2^{\mu,\nu} = (0, -\nu, \mu, 0)^T, r_3^{\mu,\nu} = (0, 0, 0, 1)^T$, satisfying $\nabla\lambda^{\mu,\nu} \cdot r_i^{\mu,\nu} = 0, i = 1, 2, 3$, where ∇ is the gradient operator with respect to the variables ρ, u, v, H. Thus (1.3) is a linearly degenerate and non-strictly hyperbolic system. Along the characteristic curve of system (1.3) defined as

$$\frac{\mathrm{d}X}{\mathrm{d}t} = U, \tag{2.1}$$

it holds that

$$\frac{\mathrm{d}U}{\mathrm{d}t} = 0,\ \frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho\nabla\cdot U,\ \frac{\mathrm{d}H}{\mathrm{d}t} = -H\nabla\cdot U. \tag{2.2}$$

These imply that all the characteristic curves are straight, and U keeps constant values along the characteristic curve.

2.1. Self-similar bounded solutions

Since both (1.3) and (1.4) are self-similar, we seek self-similar solutions $(\rho, U, H)(t, x, y) = (\rho, U, H)(\Xi)$ $(\Xi = (\xi, \eta), \xi = x/t, \eta = y/t)$. Considering smooth solutions, the two-dimensional Riemann problem (1.3) and (1.4) becomes the following boundary value problem

$$AW_\xi + BW_\eta = 0, \tag{2.3}$$

with the boundary value at infinity

$$\lim_{\xi^2+\eta^2\to\infty,\ \xi/\eta=\text{const.}} W = W_i,\ \text{if } (\xi, \eta) \text{ is in the } i\text{th quadrant},\ i = 1, ..., 4,$$

where $W = (\rho, u, v, H)^T$,

$$A = \begin{pmatrix} -\xi + u & \rho & 0 & 0 \\ 0 & \rho(-\xi + u) & 0 & 0 \\ 0 & 0 & \rho(-\xi + u) & 0 \\ 0 & H & 0 & -\xi + u \end{pmatrix},$$

$$B = \begin{pmatrix} -\eta + v & 0 & \rho & 0 \\ 0 & \rho(-\eta + v) & 0 & 0 \\ 0 & 0 & \rho(-\eta + v) & 0 \\ 0 & 0 & H & -\eta + v \end{pmatrix}.$$

The system (2.3) has a quadruple eigenvalue $(\rho > 0, H > 0)$,

$$\lambda = (v - \eta)(u - \xi)^{-1}, \tag{2.4}$$

with the associated right eigenvectors, $r_1 = (1, 0, 0, 0)^T, r_2 = (0, u - \xi, v - \eta, 0)^T, r_3 = (0, 0, 0, 1)^T$, which satisfy $\nabla\lambda \cdot r_i = 0, i = 1, 2, 3$. Thus (2.3) is a linearly degenerate and non-strictly hyperbolic system.

Define the pseudo-characteristic curve of system (1.3) in (ξ, η)-plane by

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \lambda, \tag{2.5}$$

which has a singular point (u, v). Along each pseudo-characteristic curve, it holds that $\mathrm{d}U/\mathrm{d}\xi = 0$. Therefore, all pseudo-characteristic curves are straight. We postulate that the pseudo-characteristic line comes from the infinity and ends at its singular point.

We can see from (2.3) that, besides a constant state $(\rho, U, H)(\xi, \eta) = Const.$ $(\rho > 0, H > 0)$, the smooth solution of (1.3) contains a vacuum solution $\rho, U, H)(\xi, \eta) = (0, U(\xi, \eta), 0)$, where $U(\xi, \eta)$ is an arbitrary smooth vector-valued function.

Let us consider the bounded discontinuous solution. Let $\eta = \eta(\xi)$, with the states (ρ_-, u_-, v_-, H_-) and (ρ_+, u_+, v_+, H_+) on its two sides, be a discontinuity of the bounded discontinuous solution. It is a surface

$y = t\eta(x/t)$ with the normal direction $(\eta - \xi\sigma, \sigma, -1)(\sigma = \eta'(\xi))$ in (t, x, y)-space. We solve the following Rankine–Hugoniot condition

$$\begin{cases} (\eta - \xi\sigma, \sigma, -1) \cdot ([\rho], [\rho u], [\rho v]) = 0, \\ (\eta - \xi\sigma, \sigma, -1) \cdot ([\rho u], [\rho u^2], [\rho uv]) = 0, \\ (\eta - \xi\sigma, \sigma, -1) \cdot ([\rho v], [\rho uv], [\rho v^2]) = 0, \\ (\eta - \xi\sigma, \sigma, -1) \cdot ([\frac{1}{2}\rho||U||^2 + H], [(\frac{1}{2}\rho||U||^2 + H)u], [(\frac{1}{2}\rho||U||^2 + H)v]) = 0, \end{cases} \tag{2.6}$$

to obtain

$$\begin{cases} \sigma = \frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \frac{\eta - v_-}{\xi - u_-} = \frac{\eta - v_+}{\xi - u_+}, \\ [v] = \sigma[u], \end{cases} \tag{2.7}$$

hereafter, $[q] = q_- - q_+$ be the jump of q across the discontinuity. This is a two-dimensional contact discontinuity, denoted by J, which is the pseudo-characteristic lines for both sides. It passes through the singular points $\Xi_- = (u_-, v_-)$ and $\Xi_+ = (u_+, v_+)$. We orient that a two-dimensional contact discontinuity comes from the infinity and ends at one of these two singular points.

2.2. Two-dimensional delta shock wave solution

As pointed out in the last subsection, the bounded discontinuous solution only involves the two-dimensional contact discontinuity, where the pseudo-characteristic lines for both sides are coincident. We want to know that what type of singularity of solution to system (1.3) will develop when the pseudo-characteristic lines from the different states overlap each other in (ξ, η)-plan. To this end, we first give some useful lemmas.

Lemma 2.1. *[15] A C^1 map $X = f(\alpha)$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ is a C^1 diffeomorphism if and only if f is a proper map, namely, $||f(\alpha)|| \to \infty$ as $||\alpha|| \to \infty$, and the* $\det(\partial f/\partial\alpha)$ *never vanishes.*

Lemma 2.2. *Assume that initial data $(\rho, U, H)(0, X) = (\rho_0, U_0, H_0)(X) \in C^1(\mathbb{R}^2)$ with bounded C^0 norm, where $\rho_0(X) > 0, H_0(X) > 0$, then the characteristic lines passing through all the points on the initial plane $t = 0$ never intersect for all time $t > 0$ if and only if none of the eigenvalues of the Jacobi matrix*

$$\frac{\partial U_0}{\partial X} = \begin{pmatrix} \frac{\partial u_0}{\partial x} & \frac{\partial u_0}{\partial y} \\ \frac{\partial v_0}{\partial x} & \frac{\partial v_0}{\partial y} \end{pmatrix}, \forall X \in \mathbb{R}^2, \tag{2.8}$$

is negative.

Proof. For any fixed $t \geq 0$, the (2.1) with initial value $X_0 = (x_0, y_0)$ can define a map

$$f(X) = X_0 + tU_0(X_0). \tag{2.9}$$

Obviously, the (2.9) is a proper map for any fixed $t \geq 0$ due to the bounded C^0 norm of the initial data. Besides, we have by (2.9) that

$$\frac{\partial f(X)}{\partial X_0} = \begin{pmatrix} 1 + t\frac{\partial u_0}{\partial x_0} & t\frac{\partial u_0}{\partial y_0} \\ t\frac{\partial v_0}{\partial x_0} & 1 + t\frac{\partial v_0}{\partial y_0} \end{pmatrix}. \tag{2.10}$$

Let a_1, a_2 be eigenvalues of $\frac{\partial U_0}{\partial X}$, and b_1, b_2 be eigenvalues of $\frac{\partial f(X)}{\partial X_0}$. We have

$$\det(\tfrac{\partial f(X)}{\partial X_0}) = b_1 b_2 = (1+ta_1)(1+ta_2) \geq 1. \tag{2.11}$$

Thus, the map (2.9) is a C^1 diffeomorphism by Lemma 2.1, that is, the inverse of f exists, denoted by f^{-1}.

For any given bounded domain Ω_0 on the initial plane $t = 0$, let $A_0 > 0$ be the area of the domain Ω_0, Ω_{t_0} be a domain generated by the projections of the characteristic lines passing through the domain Ω_0 onto the plane $t = t_0$, and $A(t_0)$ be the area of the domain Ω_{t_0}. We calculate that

$$\begin{aligned} A(t_0) &= \iint\limits_{\Omega_{t_0}} \mathrm{d}X = \iint\limits_{\Omega_0} \Big| \det\big(\tfrac{\partial f(X)}{\partial X_0}\big)\big|_{t=t_0} \Big| \mathrm{d}X_0 \\ &= \iint\limits_{\Omega_0} 1 + t_0(a_1+a_2) + a_1 a_2 t_0^2 \ \mathrm{d}X_0 \\ &= A_0 + \iint\limits_{\Omega_0} t_0(a_1+a_2) + t_0^2 a_1 a_2 \ \mathrm{d}X_0. \end{aligned}$$

This shows that the area of the domain Ω_{t_0} is a nondecreasing function of t_0, that is, the characteristic lines passing through all the points on the initial plane $t = 0$ never intersect each other for all time $t > 0$.

On the other hand, assume that (2.8) fails. Then, there exists a bounded domain Ω_0^* on the initial plane $t = 0$, such that at least one of the a_1 and a_2 is negative in the domain Ω_0^*. This fact implies that there exists a $T_0 > 0$, such that

$$\det\big(\tfrac{\partial f(X)}{\partial X_0}\big) > 0, X_0 \in \Omega_0^* \qquad \text{for } 0 \leq t < T_0,$$

and

$$\det\big(\tfrac{\partial f(X)}{\partial X_0}\big) = 0, X_0 \in \Omega_0^* \quad \text{for } t = T_0.$$

Thus, as $0 \leq t_0 < T_0, X_0 \in \Omega_0^*$, we deduce that

$$\begin{aligned} A^*(t_0) &= \iint\limits_{\Omega_{t_0}^*} \mathrm{d}X = \iint\limits_{\Omega_0^*} \Big| \det\big(\tfrac{\partial f(X)}{\partial X_0}\big)\big|_{t=t_0} \Big| \mathrm{d}X_0 \\ &= \iint\limits_{\Omega_0^*} 1 + t_0(a_1+a_2) + a_1 a_2 t_0^2 \ \mathrm{d}X_0 \\ &\to 0, \quad \text{as } t_0 \to T_0, \end{aligned}$$

where $\Omega_{t_0}^*$ is a domain generated by the projections of the characteristic lines passing through the domain Ω_0^* onto the plane $t = t_0$, and $A^*(t_0)$ is the area of the domain $\Omega_{t_0}^*$. Hence, the area of the domain $\Omega_{t_0}^*$ is not a nondecreasing function of t_0, which contradicts to the fact that the characteristic lines passing through all the points on the initial plane $t = 0$ never intersect each other for all time $t > 0$. The proof is complete. □

Theorem 2.3. *Assume that initial data $(\rho, U, H)(0, X) = (\rho_0, U_0, H_0)(X) \in C^1(\mathbb{R}^2)$ with bounded C^0 norm, where $\rho_0(X) > 0, H_0(X) > 0$, then (1.3) admits uniquely a global smooth solution on $t > 0$ if and only if the (2.8) holds.*

Proof. If the (2.8) holds, we can see from Lemma 2.2 that the map (2.9) is a C^1 diffeomorphism. We then calculate that

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \frac{1}{\det\left(\frac{\partial f(X)}{\partial X_0}\right)} \begin{pmatrix} \frac{\partial u_0}{\partial x_0}\left(1+t\frac{\partial v_0}{\partial y_0}\right) - t\frac{\partial u_0}{\partial y_0}\frac{\partial v_0}{\partial x_0} & \frac{\partial u_0}{\partial y_0} \\ \frac{\partial v_0}{\partial x_0} & \frac{\partial v_0}{\partial y_0}\left(1+t\frac{\partial u_0}{\partial x_0}\right) - t\frac{\partial v_0}{\partial x_0}\frac{\partial u_0}{\partial y_0} \end{pmatrix}. \tag{2.12}$$

By solving (2.2) with the given initial data, we obtain uniquely a global smooth solution,

$$\rho = \frac{\rho_0(f^{-1}(X))}{\det\left(\frac{\partial f(X)}{\partial X_0}(f^{-1}(X))\right)}, \quad U = U_0(f^{-1}(X)), \quad H = \frac{H_0(f^{-1}(X))}{\det\left(\frac{\partial f(X)}{\partial X_0}(f^{-1}(X))\right)}. \tag{2.13}$$

On the other hand, suppose that (2.8) fails. There exists a point $A = (\alpha_1, \alpha_2)$ on the initial plane $t = 0$, such that at least one of the a_1 and a_2 is negative at the point A. This implies that there exists a $T_0 > 0$, such that it holds that

$$\det\left(\frac{\partial f(X)}{\partial X_0}\right)\Big|_{X_0=A} > 0 \qquad \text{for } 0 \le t < T_0,$$

and

$$\det\left(\frac{\partial f(X)}{\partial X_0}\right)\Big|_{X_0=A} = 0 \qquad \text{for } t = T_0.$$

Thus, along the characteristic line passing through point A, we get $\det(\frac{\partial f(X)}{\partial X_0})|_{X_0=A} \to 0$ as $t \to T_0$, then $\rho \to \infty, \frac{\partial u}{\partial x} \to \infty, \frac{\partial u}{\partial y} \to \infty, \frac{\partial v}{\partial x} \to \infty, \frac{\partial v}{\partial y} \to \infty, H \to \infty$ as $t \to T_0$ by (2.12) and (2.13). Thus the density ρ, the internal energy H and the first order partial derivatives of velocity U blow up simultaneously at a finite time, which contradicts to the fact that the (1.3) admits uniquely a global smooth solution on $t > 0$. The proof is complete. $\square$

By a method as used in [21], we can prove that the overlapping of pseudo-characteristic lines in (ξ, η)-plan is equivalent to that of characteristic lines in (t, x, y)-space. Thus, when the pseudo-characteristic lines from the different states overlap each other, we can know from Lemma 2.2 and Theorem 2.3 that, the density ρ, the internal energy H and the first order partial derivatives of velocity U blow up simultaneously. This motivates us to seek two-dimensional delta shock wave solution which contains a Dirac delta function in both ρ and H.

Definition 2.4. The three-dimensional weighted Dirac delta function $w(t, s)\delta$ supported on a smooth surface S parameterized as $x = x(t, s), y = y(t, s)(t \in [0, +\infty), s \in [0, +\infty))$ is defined as

$$< w(t,s)\delta, \phi >= \int_0^{+\infty}\int_0^{+\infty} w(t,s)\phi(t, x(t,s), y(t,s))\,\mathrm{d}\,t\mathrm{d}s,$$

for all the test functions $\phi \in C_0^\infty((0,\infty) \times \mathbb{R}^2)$.

Definition 2.5. The distribution (ρ, U, H) is a two-dimensional delta shock wave solution of (1.3) and (1.4) in the sense of distribution if there exist a surface S and two functions $w(t, s), h(t, s) \in C^1(S)$, such that $\rho = \bar{\rho}(t, X) + w(t, s)\delta, U = \bar{U}(t, X), H = \bar{H}(t, X) + h(t, s)\delta$, and

$$\begin{cases} <\rho, \phi_t> + < \rho u, \phi_x> + < \rho v, \phi_y> = 0, \\ <\rho u, \phi_t> + < \rho u^2, \phi_x> + < \rho uv, \phi_y> = 0, \\ <\rho v, \phi_t> + < \rho uv, \phi_x> + < \rho v^2, \phi_y> = 0, \\ <\frac{1}{2}\rho||U||^2 + H, \phi_t> + < (\frac{1}{2}\rho||U||^2 + H)u, \phi_x> \\ \quad + < (\frac{1}{2}\rho||U||^2 + H)v, \phi_y> = 0, \end{cases} \tag{2.14}$$

for all the test functions $\phi \in C_0^\infty((0, +\infty) \times \mathbb{R}^2)$, where $\bar{\rho}, \bar{U}, \bar{H} \in L^\infty([0, +\infty) \times \mathbb{R}^2; R), U|_S = U_\delta(t, s)$,

$$< \rho, \phi > = \int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \bar{\rho}\phi \mathrm{d}X\mathrm{d}t + < w(t, s)\delta, \phi >,$$

$$< \rho u, \phi > = \int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \bar{\rho}\bar{u}\phi \mathrm{d}X\mathrm{d}t + < w(t, s)u_\delta(t, s)\delta, \phi >,$$

and H has the similar integral identities as above.

With Definitions 2.4–2.5, in the region where the characteristic lines intersect each other, we introduce a surface $S : X = X(t, s)$ which divides this region into two subregion Ω_- and Ω_+, and seek the two-dimensional delta shock wave solution,

$$(\rho, U, H) = \begin{cases} (\rho_-, U_-, H_-), (t, X) \in \Omega_-, \\ (w(t, s)\delta(X - X(t, s)), U_\delta(t, s), h(t, s)\delta(X - X(t, s))), (t, X) \in S, \\ (\rho_+, u_+, H_+), (t, X) \in \Omega_+, \end{cases} \tag{2.15}$$

where $(\rho_i, U_i, H_i), i = -, +$ are smooth bounded solutions, δ is the standard Dirac measure supported on the surface S, and $w(t, s), h(t, s)$ are the weights of the two-dimensional delta shock wave on the state variables ρ and H.

Applying Gauss's theorem, we can obtain easily the following lemma.

Lemma 2.6. *If the (2.15) satisfies the system of partial differential equations with unknowns X, w, U_δ, h,*

$$\begin{cases} \frac{\partial X}{\partial t} = U_\delta(t, s), \\ \frac{\partial w}{\partial t} = ([\rho], [\rho U]) \cdot (N_t, N_X), \\ \frac{\partial (wU_\delta)}{\partial t} = (\rho U \otimes U) \cdot (N_t, N_X), \\ \frac{\partial (\frac{1}{2}w||U_\delta||^2 + h)}{\partial t} = ([\frac{1}{2}\rho||U||^2 + H], [(\frac{1}{2}\rho||U||^2 + H)U]) \cdot (N_t, N_X), \end{cases} \tag{2.16}$$

where $(N_t, N_x, N_y) = (u_\delta \frac{\partial y}{\partial s} - v_\delta \frac{\partial x}{\partial s}, -\frac{\partial y}{\partial s}, \frac{\partial x}{\partial s})$ is the normal direction of S, then, it is a two-dimensional delta shock wave solution of (1.3) and (1.4) in the sense of distribution.

In addition, we propose the following condition

$$U_+ \cdot N_X < U_\delta \cdot N_X < U_- \cdot N_X, \tag{2.17}$$

to guarantee the uniqueness of two-dimensional delta shock wave solution, which means that all the characteristic lines from the different states are incoming on both sides of the surface S.

Definition 2.7. A discontinuity surface S satisfying (2.16) and (2.17) is called a two-dimensional delta shock wave, denoted by δ.

The (2.16) is called the generalized Rankine–Hugoniot relation of two-dimensional delta shock wave. It describes the location, propagation speed, and the weights of the two-dimensional delta shock wave. Meanwhile, the (2.17) is called the entropy condition of two-dimensional delta shock wave.

2.3. The Riemann problem with two pieces of initial data

Denote

$$[U_-, U_+, \frac{\partial X}{\partial s}] = \begin{vmatrix} u_- & v_- & 1 \\ u_+ & v_+ & 1 \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & 0 \end{vmatrix}, [U_-, U_+, \Xi] = \begin{vmatrix} u_- & v_- & 1 \\ u_+ & v_+ & 1 \\ \xi & \eta & 1 \end{vmatrix}.$$

The generalized Rankine–Hugoniot relation (2.16) is converted into

$$\begin{cases} \frac{\partial X}{\partial t} = U_\delta(t,s), \\ \frac{\partial w}{\partial t} = \rho_-[U_-, U_\delta, \frac{\partial X}{\partial s}] - \rho_+[U_+, U_\delta, \frac{\partial X}{\partial s}], \\ \frac{\partial (wU_\delta)}{\partial t} = \rho_- U_-[U_-, U_\delta, \frac{\partial X}{\partial s}] - \rho_+ U_+[U_+, U_\delta, \frac{\partial X}{\partial s}], \\ \frac{\partial (\frac{w}{2}||U_\delta||^2 + h)}{\partial t} = (\frac{\rho_-}{2}||U_-||^2 + H_-)[U_-, U_\delta, \frac{\partial X}{\partial s}] - (\frac{\rho_+}{2}||U_+||^2 + H_+)[U_+, U_\delta, \frac{\partial X}{\partial s}], \end{cases} \tag{2.18}$$

and the entropy condition (2.17) becomes

$$[U_-, U_\delta, \frac{\partial X}{\partial s}] > 0, \quad [U_+, U_\delta, \frac{\partial X}{\partial s}] < 0. \tag{2.19}$$

Under the entropy condition (2.19), we can deduce from (2.18) that

$$\begin{cases} \frac{\partial w}{\partial t} = \rho_-[U_-, U_\delta, \frac{\partial X}{\partial s}] - \rho_+[U_+, U_\delta, \frac{\partial X}{\partial s}] \geq 0, \\ \frac{\partial h}{\partial t} = (\frac{\rho_-}{2}||U_- - U_\delta||^2 + H_-)[U_-, U_\delta, \frac{\partial X}{\partial s}] - (\frac{\rho_+}{2}||U_\delta - U_+||^2 + H_+)[U_+, U_\delta, \frac{\partial X}{\partial s}] \geq 0. \end{cases} \tag{2.20}$$

These show that both w and h are non-decreasing functions of t.

Let the line $L : \mu x + \nu y = 0$ divide (x, y)-plane into two infinite regions Ω_- and Ω_+, and the normal direction $N = (\mu, \nu)(\mu^2 + \nu^2 = 1)$ of L points from Ω_- to Ω_+. We consider the Riemann problem for system (1.3) with initial data

$$(\rho, U, H)(0, X) = \begin{cases} (\rho_-, U_-, H_-), X \in \Omega_-, \\ (\rho_+, U_+, H_+), X \in \Omega_+, \end{cases} \tag{2.21}$$

where $\rho_i > 0, H_i > 0, i = -, +$, are constant values. We can prove easily that the necessary and sufficient condition of the overlapping of characteristic lines from Ω_- and Ω_+ is

$$[U] \cdot N > 0. \tag{2.22}$$

Therefore, this Riemann problem can be divided into the following two cases.

(i) $[U] \cdot N \leq 0$. In virtue of $[U] \cdot N \leq 0$, the characteristic lines from the domains Ω_- and Ω_+ do not overlap. The vacuum will develop. The solution is expressed as

$$(\rho, U, H)(t, X) = \begin{cases} (\rho_-, U_-, H_-), & X \cdot N \le U_1 \cdot Nt, \\ Vac., & U_1 \cdot Nt < X \cdot N < U_2 \cdot Nt, \\ (\rho_+, U_+, H_+), & X \cdot N \ge U_2 \cdot Nt. \end{cases} \tag{2.23}$$

(ii) $[U] \cdot N > 0$. In this case, the characteristic lines from Ω_- and Ω_+ will overlap each other. Thus we seek a two-dimensional delta shock wave solution. We describe the initial data on L as

$$t = 0 : X(0, s) = (\nu s, -\mu s),\ U(0, s) = U_0(s),\ w(0, s) = 0,\ h(0, s) = 0, \tag{2.24}$$

with parameter $s > 0$, where $U_0(s)$ is undetermined. Under entropy condition (2.19), we solve initial value problem (2.18) and (2.24) to obtain

$$\begin{cases} X(t, s) = U_\delta(t, s)t + X(0, s), \\ U_\delta(t, s) = \frac{\sqrt{\rho_-}U_- + \sqrt{\rho_+}U_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}, \\ w(t, s) = ([U] \cdot N)\sqrt{\rho_-\rho_+}t, \\ h(t, s) = \Big(\big(\frac{\rho_-\rho_+||U_- - U_+||^2}{2(\sqrt{\rho_-} + \sqrt{\rho_+})} + H_-\sqrt{\rho_+} + H_+\sqrt{\rho_-}\big)\frac{[U]\cdot N}{\sqrt{\rho_-} + \sqrt{\rho_+}}t. \end{cases} \tag{2.25}$$

Thus the solution can be expressed as follows,

$$(\rho, U, H)(t, X) = \begin{cases} (\rho_-, U_-, H_-), X \cdot N < U_\delta \cdot Nt, \\ (w(t, s)\delta(X - X(t, s)), U_\delta, h(t, s)\delta(X - X(t, s))), X \cdot N = U_\delta \cdot Nt, \\ (\rho_+, U_+, H_+),\ X \cdot N > U_\delta \cdot Nt. \end{cases} \tag{2.26}$$

It is shown that the two-dimensional delta shock wave solution (2.26) is self-similar except the weights w and h those are linear functions of t. To construct the solutions in (ξ, η)-plane, motivated by [21], introducing a pseudo-self-similar transformation $X(t, s) = t\Xi(\bar{s}), U_\delta(t, s) = U_\delta(\bar{s}), w(t, s) = tm(\bar{s}), h(t, s) = tn(\bar{s})$, where $\bar{s} = \ln(s/t)$, we reformulate (2.18) and (2.19) as

$$\begin{cases} \Xi' = \Xi - U_\delta, \\ m' = m - (\rho_-[U_-, U_\delta, \Xi] - \rho_+[U_+, U_\delta, \Xi])e^{-\bar{s}}, \\ (mU_\delta)' = mU_\delta - (\rho_-U_-[U_-, U_\delta, \Xi] - \rho_+U_+[U_+, U_\delta, \Xi])e^{-\bar{s}}, \\ (\frac{m}{2}||U_\delta||^2 + n)' = \frac{m}{2}||U_\delta||^2 + n - \big((\frac{\rho_-}{2}||U_-||^2 + H_-)[U_-, U_\delta, \Xi] \\ \qquad - (\frac{\rho_+}{2}||U_+||^2 + H_+)[U_+, U_\delta, \Xi]\big)e^{-\bar{s}}, \end{cases} \tag{2.27}$$

and

$$[U_-, U_\delta, \Xi] > 0,\quad [U_+, U_\delta, \Xi] < 0. \tag{2.28}$$

With the second and third equalities in (2.27), it holds that

$$m[U_-, U_+, U_\delta] = Ce^{\bar{s}}, \tag{2.29}$$

where $C = m(\bar{s}_0)[U_-, U_+, U_\delta(\bar{s}_0)]$. Using the properties of the mixed product of the vectors, one can check easily that the solution of (1.3) is entropic if and only if the ordering of U_-, U_δ, Ξ is counterclockwise and that of U_+, U_δ, Ξ is clockwise.

3. The qualitative behavior of solutions to relation (2.27)

We discuss the qualitative behavior of solutions to relation (2.27) with initial data

$$\bar{s}=0:\Xi(0)=\Xi_0, U_\delta(0)=U_\delta^0, m(0)=m_0>0, n(0)=n_0>0. \tag{3.1}$$

The following four special cases will be discussed in this section.

3.1. $\rho_-H_->0$, $\rho_+=0$, $H_+=0$, $[U_-,U_\delta^0,\Xi_0]>0$

Since the algebraic system

$$\begin{cases} (m-m')\cdot 1 =1\cdot\rho_-[U_-,U_\delta,\Xi]e^{-\bar{s}}-1\cdot\rho_+[U_+,U_\delta,\Xi]e^{-\bar{s}},\\ (mu_\delta-(mu_\delta)')\cdot 1 =u_-\cdot\rho_-[U_-,U_\delta,\Xi]e^{-\bar{s}}-u_\delta^0\cdot\rho_+[U_+,U_\delta,\Xi]e^{-\bar{s}},\\ (mv_\delta-(mv_\delta)')\cdot 1 =v_-\cdot\rho_-[U_-,U_\delta,\Xi]e^{-\bar{s}}-v_\delta^0\cdot\rho_+[U_+,U_\delta,\Xi]e^{-\bar{s}}, \end{cases}$$

which consists of the second and third equalities of system (2.27), with unknowns $1,\rho_-[U_-,U_\delta,\Xi]e^{-\bar{s}}$ and $\rho_+[U_+,U_\delta,\Xi]e^{-\bar{s}}$, has non-zero solution $1,\rho_-[U_-,U_\delta,\Xi]e^{-\bar{s}}$ and 0. This yields that

$$\begin{vmatrix} m-m' & 1 & 1\\ mu_\delta-(mu_\delta)' & u_- & u_\delta^0\\ mv_\delta-(mv_\delta)' & v_- & v_\delta^0 \end{vmatrix}=0,$$

which implies

$$m[U_-,U_\delta^0,U_\delta]=m_0[U_-,U_\delta^0,U_\delta^0]\equiv 0. \tag{3.2}$$

Inserting the second equation of (2.27) into the third one, and using $\rho_+=0$, we get

$$m(U_\delta-U_-)=m_0(U_\delta^0-U_-)e^{\bar{s}}\not\equiv 0. \tag{3.3}$$

A combination of (3.2) and (3.3) shows $[U_-,U_\delta^0,U_\delta]=0,[U_-,U_\delta^0,\Xi]=[U_-,U_\delta^0,\Xi_0]e^{\bar{s}}$, then

$$\begin{aligned} m^2-mm' =&\rho_-m[U_-,U_\delta,\Xi]e^{-\bar{s}}=\rho_-[U_-,m(U_\delta-U_-),\Xi]e^{-\bar{s}}\\ =&\rho_-[U_-,m_0(U_\delta^0-U_-),\Xi]=\rho_-m_0[U_-,U_\delta^0,\Xi]\\ =&\rho_-m_0[U_-,U_\delta^0,\Xi_0]e^{\bar{s}}. \end{aligned}$$

Solving this differential equation yields that

$$m=\pm\big(m_0^2+2\rho_-m_0[U_-,U_\delta^0,\Xi_0](e^{-\bar{s}}-1)\big)^{1/2}e^{\bar{s}}. \tag{3.4}$$

Inserting this fact into (3.3) and (2.27), we derive that

$$U_\delta=U_-+m_0(U_\delta^0-U_-)\tfrac{e^{\bar{s}}}{m},\ \Xi=U_-+e^{\bar{s}}(\Xi_0-U_-)+\tfrac{U_\delta^0-U_-}{\rho_-[U_-,U_\delta^0,\Xi_0]}(m-m_0e^{\bar{s}}).$$

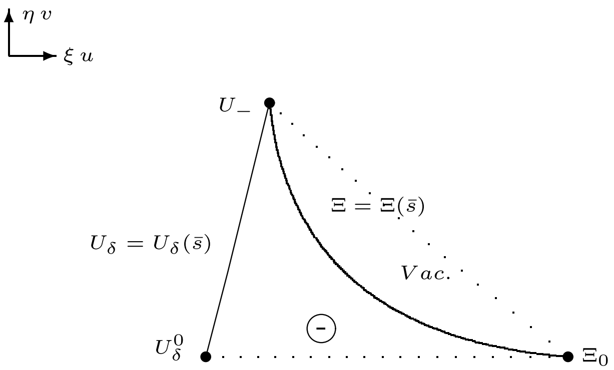

Fig. 3.1. The qualitative behavior of solution to Case 3.1.

Notice that,

$$\begin{aligned}[U_-, U_\delta, \Xi] =&[U_-, U_- + m_0(U_\delta^0 - U_-)\tfrac{e^{\bar{s}}}{m}, U_- + e^{\bar{s}}(\Xi_0 - U_-) + \tfrac{U_\delta^0 - U_-}{\rho_-[U_-,U_\delta^0,\Xi_0]}(m - m_0e^{\bar{s}})]\\ =&\tfrac{m_0e^{\bar{s}}}{m}([U_-, U_\delta^0, (\Xi_0 - U_-)e^{\bar{s}}] + [U_-, U_\delta^0, \tfrac{U_\delta^0 - U_-}{\rho_-[U_-,U_\delta^0,\Xi_0]}(m - m_0e^{\bar{s}})])\\ =&\tfrac{m_0e^{2\bar{s}}}{m}[U_-, U_\delta^0, \Xi_0],\end{aligned}$$

we can see $m > 0$ using entropy condition (2.28). Hence, the entropy solution can be shown as

$$\begin{cases} \Xi =U_- + e^{\bar{s}}(\Xi_0 - U_-) + \frac{U_\delta^0 - U_-}{\rho_-[U_-,U_\delta^0,\Xi_0]}\Big(m - m_0e^{\bar{s}}\Big),\\ m =\big(m_0^2 + 2\rho_-m_0[U_-, U_\delta^0, \Xi_0](e^{-\bar{s}} - 1)\big)^{1/2}e^{\bar{s}},\\ U_\delta =U_- + m_0(U_\delta^0 - U_-)\frac{e^{\bar{s}}}{m},\\ n =\Big(\frac{\|U_\delta^0\|^2}{2}m_0 + n_0 - \big(\frac{\|U_-\|^2}{2} + \frac{H_-}{\rho_-}\big)m_0\Big)e^{\bar{s}} + \big(\frac{\|U_-\|^2}{2} + \frac{H_-}{\rho_-} - \frac{\|U_\delta\|^2}{2}\big)m. \end{cases} \tag{3.5}$$

With a simple calculation, we can obtain the following lemma.

Lemma 3.1. *Under the conditions $\rho_-H_- > 0, \rho_+ = 0, H_+ = 0, [U_-, U_\delta^0, \Xi_0] > 0$, the solution to (2.27) and (3.1) has the following properties, shown in Fig. 3.1.*

(1) $\lim_{\bar{s}\to-\infty}(\Xi, m, U_\delta, n) = (U_-, 0, U_-, 0)$*;*
(2) The $U_\delta = U_\delta(\bar{s})$ lies on the line $U_\delta^0U_-$ and approaches U_- asymptotically;
(3) The $\Xi = \Xi(\bar{s})$ protrudes to the line $\Xi_0U_\delta^0$ and approaches U_- asymptotically.

Remark 3.2. Under the conditions $\rho_+H_+ > 0, \rho_- = 0, H_- = 0, [U_+, U_\delta^0, \Xi_0] < 0$, we can solve (2.27) and (3.1) in a similar manner as before.

3.2. $\rho_-H_- > 0,\ \rho_+H_+ > 0,\ [U_-, U_\delta^0, \Xi_0] > 0,\ [U_+, U_\delta^0, \Xi_0] < 0,\ [U_-, U_+, U_\delta^0] = 0$

The combination of $[U_-, U_\delta^0, \Xi_0] > 0, [U_+, U_\delta^0, \Xi_0] < 0$ and $[U_-, U_+, U_\delta^0] = 0$ implies that there exist $\mu_-^0 > 0, \mu_+^0 > 0$ with $\mu_-^0 + \mu_+^0 = 1$ such that

$$[U_-, U_+, \Xi_0] > 0, \quad U_\delta^0 = \mu_-^0U_- + \mu_+^0U_+. \tag{3.6}$$

While, it follows from (2.29) that

$$m[U_-, U_+, U_\delta] = m_0[U_-, U_+, U_\delta^0]e^{\bar{s}} \equiv 0. \tag{3.7}$$

Substituting the second equality of (2.27) into the third one, and integrating it from 0 to $\bar{s}$, we obtain

$$m(U_\delta - U_-) = e^{\bar{s}}\Big(m_0(U_\delta^0 - U_-) - \int_0^{\bar{s}} \rho_+(U_- - U_+)[U_+, U_\delta, \Xi]e^{-2\tau}\mathrm{d}\tau\Big) \not\equiv 0. \tag{3.8}$$

This fact together with (3.7) yields that

$$[U_-, U_+, U_\delta] = 0, \tag{3.9}$$

namely,

$$U_\delta = \mu_- U_- + \mu_+ U_+, \mu_- + \mu_+ = 1. \tag{3.10}$$

Then, using (3.9) and (3.6), we deduce from $\Xi - \Xi' = U_\delta$ that

$$[U_-, U_+, \Xi] = [U_-, U_+, \Xi_0]e^{\bar{s}} > 0, \tag{3.11}$$

then

$$[U_-, U_\delta, \Xi] = [U_+, U_\delta, \Xi] + [U_-, U_+, \Xi_0]e^{\bar{s}}. \tag{3.12}$$

Hence, we can further derive from (2.27) that

$$m - m' = \rho_-[U_-, U_+, \Xi_0] + (\rho_- - \rho_+)[U_+, U_\delta, \Xi]e^{-\bar{s}}, \tag{3.13}$$

$$mU_\delta - (mU_\delta)' = \rho_- U_-[U_-, U_+, \Xi_0] + (\rho_- U_- - \rho_+ U_+)[U_+, U_\delta, \Xi]e^{-\bar{s}}, \tag{3.14}$$

$$\begin{aligned}(\tfrac{m}{2}||U_\delta||^2 + n) - (\tfrac{m}{2}||U_\delta||^2 + n)' &= (\tfrac{\rho_-}{2}||U_-||^2 + H_-)[U_-, U_+, \Xi_0] \\ &+ (\tfrac{\rho_-}{2}||U_-||^2 + H_- - \tfrac{\rho_+}{2}||U_+||^2 - H_+)[U_+, U_\delta, \Xi]e^{-\bar{s}}.\end{aligned} \tag{3.15}$$

When $\rho_- = \rho_+$. Integrating (3.13) from 0 to $\bar{s}$ gives

$$m = e^{\bar{s}}(m_0 + \rho_-[U_-, U_+, \Xi_0](e^{-\bar{s}} - 1)). \tag{3.16}$$

Using (3.10) and (3.11), one can calculate from (3.14) that

$$m(\mu_+ U_- + \mu_- U_+) = e^{\bar{s}}\Big(m_0 U_\delta^0 - \rho_-[U_-, U_+, \Xi_0]\int_0^{\bar{s}}(\mu_+ U_- + \mu_- U_+)e^{-\tau}\mathrm{d}\tau\Big).$$

Differentiating this equation with respect to $\bar{s}$, we derive

$$\frac{\mu_-'}{2\mu_- - 1} = \frac{\rho_-[U_-, U_+, \Xi_0]e^{-\bar{s}}}{m_0 + \rho_-[U_-, U_+, \Xi_0](e^{-\bar{s}} - 1)}.$$

Solving this equation yields that

$$\mu_- = \frac{1}{2}\Big(1 + \frac{m_0^2(2\mu_-^0 - 1)}{(m_0 + \rho_-[U_-, U_+, \Xi_0](e^{-\bar{s}} - 1))^2}\Big),$$

hence,

$$U_\delta = \tfrac{1}{2}(U_- + U_+) + \tfrac{1}{2}\tfrac{m_0^2(2\mu_-^0-1)}{(m_0+\rho_-[U_-,U_+,\Xi_0](e^{-\bar{s}}-1))^2}(U_- - U_+). \tag{3.17}$$

By a simple calculation, we have that

$$\begin{aligned}\Xi =&e^{\bar{s}}\Big(\Xi_0 + \tfrac{m_0^2(2\mu_-^0-1)}{2\rho_-[U_-,U_+,\Xi_0]}\big(\tfrac{1}{m_0} - \tfrac{1}{m_0+\rho_-[U_-,U_+,\Xi_0](e^{-\bar{s}}-1)}\big)(U_- - U_+)\\ &+ \tfrac{1}{2}(e^{-\bar{s}}-1)(U_- + U_+)\Big),\end{aligned} \tag{3.18}$$

$$\begin{aligned}n =&e^{\bar{s}}\Big(\tfrac{(2\mu_-^0-1)m_0^2}{2\rho_-}\big(\tfrac{1}{m_0} - \tfrac{1}{m_0+\rho_-[U_-,U_+,\Xi_0](e^{-\bar{s}}-1)}\big)\big(\tfrac{\rho_+}{2}||U_+||^2 + H_+ - \tfrac{\rho_-}{2}||U_-||^2 - H_-\big) + n_0\\ &+ \tfrac{m_0}{2}||U_\delta^0||^2 + \tfrac{e^{-\bar{s}}-1}{2}[U_-,U_+,\Xi_0]\big(\tfrac{\rho_-}{2}||U_-||^2 + H_- + \tfrac{\rho_+}{2}||U_+||^2 + H_+\big)\Big) - \tfrac{m}{2}||U_\delta||^2.\end{aligned} \tag{3.19}$$

Besides, we can calculate that

$$\begin{aligned}[U_-,U_\delta,\Xi]=&\tfrac{e^{\bar{s}}}{2}\big(1 - \tfrac{m_0^2(2\mu_-^0-1)}{(m_0+\rho_-[U_-,U_+,\Xi_0](e^{-\bar{s}}-1))^2}\big)[U_-,U_+,\Xi_0]\\ =&\tfrac{2m_0^2(1-\mu_-^0)+\rho_-[U_-,U_+,\Xi_0](e^{-\bar{s}}-1)(2m_0+\rho_-[U_-,U_+,\Xi_0](e^{-\bar{s}}-1))}{2(m_0+\rho_-[U_-,U_+,\Xi_0](e^{-\bar{s}}-1))^2}[U_-,U_+,\Xi_0]e^{\bar{s}}\\ >&0,\\ [U_+,U_\delta,\Xi]=&-\tfrac{e^{\bar{s}}}{2}\big(1 + \tfrac{m_0^2(2\mu_-^0-1)}{(m_0+\rho_-[U_-,U_+,\Xi_0](e^{-\bar{s}}-1))^2}\big)[U_-,U_+,\Xi_0]\\ =&-\tfrac{2\mu_-^0 m_0^2+\rho_-[U_-,U_+,\Xi_0](e^{-\bar{s}}-1)(2m_0+\rho_-[U_-,U_+,\Xi_0](e^{-\bar{s}}-1))}{2(m_0+\rho_-[U_-,U_+,\Xi_0](e^{-\bar{s}}-1))^2}[U_-,U_+,\Xi_0]e^{\bar{s}}\\ <&0.\end{aligned}$$

Therefore, the solution expressed in (3.16), (3.17), (3.18) and (3.19) is an entropy solution.

When $\rho_- \neq \rho_+$. Comparing (3.13) with (3.14), we obtain

$$\begin{aligned}(m((\rho_- - \rho_+)U_\delta - (\rho_-U_- - \rho_+U_+)))' =&m((\rho_- - \rho_+)U_\delta - (\rho_-U_- - \rho_+U_+))\\ &+ \rho_-\rho_+(U_- - U_+)[U_-,U_+,\Xi_0].\end{aligned} \tag{3.20}$$

Integrating it from 0 to $\bar{s}$ leads to

$$\begin{aligned}m(\rho_- - \rho_+)U_\delta =&m_0(\rho_- - \rho_+)U_\delta^0 e^{\bar{s}} + (m - m_0e^{\bar{s}})(\rho_-U_- - \rho_+U_+)\\ &+ \rho_-\rho_+(U_- - U_+)[U_-,U_+,\Xi_0](e^{\bar{s}}-1),\end{aligned}$$

then,

$$\begin{aligned}m^2 - mm' =&\rho_-[U_-,U_+,\Xi_0]m + [U_+,(\rho_- - \rho_+)mU_\delta,\Xi]e^{-\bar{s}}\\ =&\rho_-[U_-,U_+,\Xi_0]m + [U_+,m_0(\rho_- - \rho_+)U_\delta^0 e^{\bar{s}} + (m - m_0e^{\bar{s}})(\rho_-U_- - \rho_+U_+)\\ &+ \rho_-\rho_+(U_- - U_+)[U_-,U_+,\Xi_0](e^{\bar{s}}-1),\Xi]e^{-\bar{s}}\\ =&m_0(\rho_-\mu_+^0 + \rho_+\mu_-^0)[U_-,U_+,\Xi_0]e^{\bar{s}} + \rho_-\rho_+[U_-,U_+,\Xi_0]^2(1-e^{\bar{s}}).\end{aligned} \tag{3.21}$$

One solves (3.21) to derive

$$m = \pm e^{\bar{s}}\sqrt{G_0(\bar{s})}, \tag{3.22}$$

where $G_0(\bar{s}) = m_0^2 + 2m_0(\rho_-\mu_+^0 + \rho_+\mu_-^0)[U_-, U_+, \Xi_0](e^{-\bar{s}} - 1) + \rho_-\rho_+[U_-, U_+, \Xi_0]^2(e^{-\bar{s}} - 1)^2$. We then get

$$U_\delta = \tfrac{1}{\rho_- - \rho_+}\Big(\rho_- U_- - \rho_+ U_+ + \tfrac{G_1(\bar{s})}{m}\Big), \tag{3.23}$$

here $G_1(\bar{s}) = m_0 e^{\bar{s}}((\rho_- - \rho_+)U_\delta^0 - \rho_- U_- + \rho_+ U_+) + \rho_-\rho_+(U_- - U_+)[U_-, U_+, \Xi_0](e^{\bar{s}} - 1)$. Again note $\Xi - \Xi' = U_\delta$, we obtain that

$$\Xi = \Xi_0 e^{\bar{s}} + \tfrac{e^{\bar{s}}}{\rho_- - \rho_+}\Big((\rho_- U_- - \rho_+ U_+)(e^{-\bar{s}} - 1) - \int_0^{\bar{s}} \tfrac{G_1}{m} e^{-\tau}\mathrm{d}\tau\Big). \tag{3.24}$$

Also, inserting (3.13) into (3.15), we arrive at

$$\begin{aligned}
&((\rho_- - \rho_+)(\tfrac{m}{2}||U_\delta||^2 + n) - m(\tfrac{\rho_-}{2}||U_-||^2 + H_- - \tfrac{\rho_+}{2}||U_+||^2 - H_+))' \\
=&(\rho_- - \rho_+)(\tfrac{m}{2}||U_\delta||^2 + n) - m(\tfrac{\rho_-}{2}||U_-||^2 + H_- - \tfrac{\rho_+}{2}||U_+||^2 - H_+) \\
&+ (\rho_+(\tfrac{\rho_-}{2}||U_-||^2 + H_-) - \rho_-(\tfrac{\rho_+}{2}||U_+||^2 + H_+))[U_-, U_+, \Xi_0],
\end{aligned}$$

which is solved to be

$$\begin{aligned}
n =& -\tfrac{m}{2}||U_\delta||^2 + (\tfrac{m_0}{2}||U_\delta^0||^2 + n_0)e^{\bar{s}} \\
&+ \tfrac{1}{\rho_- - \rho_+}\Big((m - m_0 e^{\bar{s}})(\tfrac{\rho_-}{2}||U_-||^2 + H_- - \tfrac{\rho_+}{2}||U_+||^2 - H_+) \\
&+ (\rho_-(\tfrac{\rho_+}{2}||U_+||^2 + H_+) - \rho_+(\tfrac{\rho_-}{2}||U_-||^2 + H_-))[U_-, U_+, \Xi_0](1 - e^{\bar{s}})\Big).
\end{aligned} \tag{3.25}$$

Let us single out an entropy solution. As $m = e^{\bar{s}}\sqrt{G_0(\bar{s})}$, from

$$\begin{aligned}
&G_0(\bar{s}) - (m_0 + \rho_-[U_-, U_+, \Xi_0](e^{-\bar{s}} - 1))^2 \\
=&\big(2m_0\mu_-^0[U_-, U_+, \Xi_0](e^{-\bar{s}} - 1) + \rho_-[U_-, U_+, \Xi_0]^2(e^{-\bar{s}} - 1)^2\big)(\rho_+ - \rho_-), \\
&G_0(\bar{s}) - (m_0 + \rho_+[U_-, U_+, \Xi_0](e^{-\bar{s}} - 1))^2 \\
=&\big(2m_0\mu_+^0[U_-, U_+, \Xi_0](e^{-\bar{s}} - 1) + \rho_+[U_-, U_+, \Xi_0]^2(e^{-\bar{s}} - 1)^2\big)(\rho_- - \rho_+),
\end{aligned}$$

we can deduce that

$$\begin{aligned}
[U_-, U_\delta, \Xi] =& \tfrac{e^{2\bar{s}}}{m}(m_0\mu_+^0 - \rho_+\tfrac{me^{-\bar{s}} - m_0 - \rho_-[U_-, U_+, \Xi_0](e^{-\bar{s}} - 1)}{\rho_- - \rho_+})[U_-, U_+, \Xi_0] \\
=& \tfrac{e^{2\bar{s}}}{m}(m_0\mu_+^0 - \rho_+\tfrac{\sqrt{G_0(\bar{s})} - (m_0 + \rho_-[U_-, U_+, \Xi_0](e^{-\bar{s}} - 1))}{\rho_- - \rho_+})[U_-, U_+, \Xi_0] \\
>& 0, \\
[U_+, U_\delta, \Xi] =& -\tfrac{e^{2\bar{s}}}{m}(m_0\mu_-^0 + \rho_-\tfrac{me^{-\bar{s}} - m_0 - \rho_+[U_-, U_+, \Xi_0](e^{-\bar{s}} - 1)}{\rho_- - \rho_+})[U_-, U_+, \Xi_0] \\
=& -\tfrac{e^{2\bar{s}}}{m}(m_0\mu_-^0 + \rho_-\tfrac{\sqrt{G_0(\bar{s})} - (m_0 + \rho_+[U_-, U_+, \Xi_0](e^{-\bar{s}} - 1))}{\rho_- - \rho_+})[U_-, U_+, \Xi_0] \\
<& 0.
\end{aligned}$$

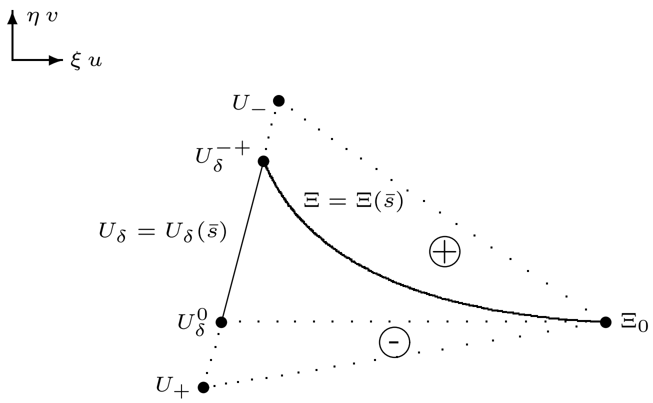

Fig. 3.2. The qualitative behavior of solution to Case 3.2.

However, as $m = -e^{\bar{s}}\sqrt{G_0(\bar{s})}$, if $\rho_- - \rho_+ > 0$, it yields that

$$[U_-, U_\delta, \Xi] = \frac{e^{2\bar{s}}}{m}(m_0\mu_+^0 - \rho_+ \frac{me^{-\bar{s}} - m_0 - \rho_-[U_-, U_+, \Xi_0](e^{-\bar{s}} - 1)}{\rho_- - \rho_+})[U_-, U_+, \Xi_0] < 0. \tag{3.26}$$

Therefore, we choose $m = e^{\bar{s}}\sqrt{G_0(\bar{s})}$ by entropy condition (2.28). The entropy solution is given as follows,

$$\begin{cases} \Xi = \Xi_0 e^{\bar{s}} + \frac{e^{\bar{s}}}{\rho_- - \rho_+}\Big((\rho_- U_- - \rho_+ U_+)(e^{-\bar{s}} - 1) - \int\limits_0^{\bar{s}} \frac{G_1}{m} e^{-\tau} \mathrm{d}\tau\Big), \\ m = e^{\bar{s}}\sqrt{G_0(\bar{s})}, \\ U_\delta = \frac{1}{\rho_- - \rho_+}\Big(\rho_- U_- - \rho_+ U_+ + \frac{G_1(\bar{s})}{m}\Big), \\ n = -\frac{m}{2}||U_\delta||^2 + (\frac{m_0}{2}||U_\delta^0||^2 + n_0)e^{\bar{s}} \\ \qquad + \frac{1}{\rho_- - \rho_+}\Big((m - m_0 e^{\bar{s}})(\frac{\rho_-}{2}||U_-||^2 + H_- - \frac{\rho_+}{2}||U_+||^2 - H_+) \\ \qquad + (\rho_-(\frac{\rho_+}{2}||U_+||^2 + H_+) - \rho_+(\frac{\rho_-}{2}||U_-||^2 + H_-))[U_-, U_+, \Xi_0](1 - e^{\bar{s}})\Big). \end{cases} \tag{3.27}$$

With a simple calculation, we can obtain the following lemma.

Lemma 3.3. *Under the conditions $\rho_- H_- > 0, \rho_+ H_+ > 0, [U_-, U_\delta^0, \Xi_0] > 0, [U_+, U_\delta^0, \Xi_0] < 0, [U_-, U_+, U_\delta^0] = 0$, the solution to (2.27) and (3.1) given in (3.16), (3.17), (3.18) and (3.19) as $\rho_- = \rho_+$, or (3.27) as $\rho_- \neq \rho_+$, has the following properties, as illustrated in Fig. 3.2.*

(1) $\lim_{\bar{s}\to-\infty}(\Xi, m, U_\delta, n) = (U_\delta^{-+}, \sqrt{\rho_-\rho_+}[U_-, U_+, \Xi_0], U_\delta^{-+}, n_{-+})$, *where* $U_\delta^{-+} = \frac{\sqrt{\rho_-}U_- + \sqrt{\rho_+}U_+}{\sqrt{\rho_-} + \sqrt{\rho_+}}, n_{-+} = \frac{[U_-, U_+, \Xi_0]}{\sqrt{\rho_-} + \sqrt{\rho_+}}\big(\sqrt{\rho_-}(\frac{\rho_+}{2}||U_+||^2 + H_+ - \frac{\rho_+}{2}||U_\delta^{-+}||^2) + \sqrt{\rho_+}(\frac{\rho_-}{2}||U_-||^2 + H_- - \frac{\rho_-}{2}||U_\delta^{-+}||^2)\big)$;
(2) The $U_\delta = U_\delta(\bar{s})$ lies on the line U_-U_+ and approaches the point U_δ^{-+} asymptotically;
(3) The $\Xi = \Xi(\bar{s})$ protrudes to the line $\Xi_0 U_+$ and approaches U_δ^{-+} asymptotically.

Remark 3.4. Under the conditions $\rho_- H_- > 0, \rho_+ H_+ > 0, [U_-, U_+, \Xi_0] > 0, \Xi(0) = \Xi_0, U_\delta(0) = U_\delta^0$, $m(0) = 0, n(0) = 0$, one can solve (2.27) and (3.1) in a similar way. The entropy solution is shown as follows,

$$
\begin{cases}
\Xi = e^{\bar{s}}\big(\Xi_0 + U_\delta(e^{-\bar{s}}-1)\big),\\
m = \sqrt{\rho_-\rho_+}[U_-,U_+,\Xi_0](1-e^{\bar{s}}),\\
U_\delta = \frac{\sqrt{\rho_-}U_- + \sqrt{\rho_+}U_+}{\sqrt{\rho_-}+\sqrt{\rho_+}},\\
n = \begin{cases}
\frac{1}{2}\big(\frac{\rho_-}{2}||U_-||^2 + H_- + \frac{\rho_+}{2}||U_+||^2 + H_+\big)[U_-,U_+,\Xi_0](1-e^{\bar{s}})\\
-\frac{m}{2}||U_\delta||^2, \text{ if } \rho_- = \rho_+,\\
\frac{1}{\rho_- - \rho_+}\Big(m(\frac{\rho_-}{2}||U_-||^2 + H_- - \frac{\rho_+}{2}||U_+||^2 - H_+)\\
+\big(\rho_-(\frac{\rho_+}{2}||U_+||^2 + H_+) - \rho_+(\frac{\rho_-}{2}||U_-||^2 + H_-)\big)[U_-,U_+,\Xi_0](1-e^{\bar{s}})\Big)\\
-\frac{m}{2}||U_\delta||^2, \text{ if } \rho_- \neq \rho_+.
\end{cases}
\end{cases}
\tag{3.28}
$$

3.3. $\rho_-H_- > 0, \rho_+H_+ > 0,\ [U_-,U_\delta^0,\Xi_0] > 0,\ [U_+,U_\delta^0,\Xi_0] < 0,\ [U_-,U_+,U_\delta^0] > 0$

The conditions $[U_-,U_\delta^0,\Xi_0] > 0, [U_+,U_\delta^0,\Xi_0] < 0$ imply that the straight line $\Xi_0U_\delta^0$ intersects with the segment U_-U_+ in the interior, labeled $\tilde{U_\delta^0}$ by the intersection point, that is, $\tilde{U_\delta^0} = \mu_-^0U_- + \mu_+^0U_+, \mu_-^0 > 0, \mu_+^0 > 0, \mu_-^0 + \mu_+^0 = 1$. Let μ_0 be the ratio of the length of segment $U_-\tilde{U_\delta^0}$ and that of $\tilde{U_\delta^0}U_+$, namely, $\mu_0 = \mu_+^0/\mu_-^0$. If the solution of (2.27) and (3.1) satisfies entropy condition (2.28), we can define a similar $\mu = \mu_+/\mu_-$, the ratio of the length of the segment $U_-\tilde{U_\delta}$ and that of $\tilde{U_\delta}U_+$, where $\tilde{U_\delta}$ is the intersection point of the straight segments ΞU_δ and U_-U_+, namely, $\tilde{U_\delta} = \mu_-U_- + \mu_+U_+, \mu_- > 0, \mu_+ > 0, \mu_- + \mu_+ = 1$.

Denote $b_-^0 = [U_-,U_\delta^0,\Xi_0], b_+^0 = -[U_+,U_\delta^0,\Xi_0], b_- = [U_-,U_\delta,\Xi]$ and $b_+ = -[U_+,U_\delta,\Xi]$. Then it holds that, $\mu_0 = \mu_+^0/\mu_-^0 = b_-^0/b_+^0, \mu = \mu_+/\mu_- = b_-/b_+$. We then deduce from (2.27) that

$$\mu' = (\rho_-\mu^2 - \rho_+)\frac{[U_-,U_+,\Xi]}{me^{\bar{s}}}. \tag{3.29}$$

Solving this differential equation with initial value $\mu|_{\bar{s}=0} = \mu_0 > 0$, we get

$$
\begin{cases}
\mu = \sqrt{\rho_+/\rho_-}, & \text{if } \mu_0 = \sqrt{\rho_+/\rho_-},\\
\frac{\sqrt{\rho_+/\rho_-}+\mu}{\sqrt{\rho_+/\rho_-}-\mu} = \frac{\sqrt{\rho_+/\rho_-}+\mu_0}{\sqrt{\rho_+/\rho_-}-\mu_0}\exp\Big(\int_0^{\bar{s}} -2\sqrt{\rho_-\rho_+}\frac{[U_-,U_+,\Xi]}{e^{\bar{s}}m}\mathrm{d}s\Big), & \text{if } \mu_0 \neq \sqrt{\rho_+/\rho_-}.
\end{cases}
\tag{3.30}
$$

We now present some lemmas.

Lemma 3.5. *Under the conditions $\rho_- > 0, \rho_+ > 0, [U_-,U_\delta^0,\Xi_0] > 0, [U_+,U_\delta^0,\Xi_0] < 0, [U_-,U_+,U_\delta^0] > 0$, let $[U_-,U_\delta,\Xi] > 0, [U_+,U_\delta,\Xi] < 0$, then the inequality holds*

$$([U_-,U_+,\Xi] - [U_-,U_+,U_\delta])e^{-\bar{s}} > [U_-,U_+,\Xi_0] - [U_-,U_+,U_\delta^0] > 0, \forall \bar{s} < 0. \tag{3.31}$$

Proof. From the conditions $[U_-,U_\delta^0,\Xi_0] > 0, [U_+,U_\delta^0,\Xi_0] < 0$, we can see that $[U_-,U_+,\Xi_0] - [U_-,U_+,U_\delta^0] = [U_-,U_\delta^0,\Xi_0] - [U_+,U_\delta^0,\Xi_0] > 0$. Similarly, the conditions $[U_-,U_\delta,\Xi] > 0$ and $[U_+,U_\delta,\Xi] < 0$ yield

$$[U_-,U_+,\Xi] - [U_-,U_+,U_\delta] = [U_-,U_\delta,\Xi] - [U_+,U_\delta,\Xi] > 0, \tag{3.32}$$

$$m = e^{\bar{s}}\Big(m_0 + \int_{\bar{s}}^0 (\rho_-[U_-,U_\delta,\Xi] - \rho_+[U_+,U_\delta,\Xi])e^{-2\tau}\mathrm{d}\tau\Big) > 0, \tag{3.33}$$

that is,

$$\tfrac{[U_-,U_+,\Xi]}{m} > \tfrac{[U_-,U_+,U_\delta]}{m} > 0. \tag{3.34}$$

Besides, by $[U_-,U_+,U_\delta^0] > 0$, it follows from (2.29) that

$$[U_-,U_+,U_\delta]/m > 0, [U_-,U_+,U_\delta]' = (m-m')[U_-,U_+,U_\delta]/m. \tag{3.35}$$

In virtue of $m - m' = \rho_-[U_-,U_\delta,\Xi] - \rho_+[U_+,U_\delta,\Xi] > 0$, we have $[U_-,U_+,U_\delta]' > 0$, then, $(([U_-,U_+,\Xi] - [U_-,U_+,U_\delta])e^{-\bar{s}})' = -[U_-,U_+,U_\delta]'e^{-\bar{s}} < 0, \forall \bar{s} < 0$. The proof is complete. $\square$

Using (3.29), (3.30) and (3.34), we arrive at the following result.

Lemma 3.6. *Under the conditions $[U_-,U_\delta^0,\Xi_0] > 0$, $[U_+,U_\delta^0,\Xi_0] < 0$, $[U_-,U_+,U_\delta^0] > 0$, $\rho_- > 0$, if $[U_-,U_\delta,\Xi] > 0, [U_+,U_\delta,\Xi] < 0$, then for all $\bar{s} < 0$, the solution to (3.29) with initial value $\mu|_{\bar{s}=0} = \mu_0 > 0$ possesses the following properties.*

(1) If $\mu_0 < \sqrt{\rho_+/\rho_-}$, then $\mu' < 0, 0 < \mu_0 < \mu < \sqrt{\rho_+/\rho_-}$;
(2) If $\mu_0 = \sqrt{\rho_+/\rho_-}$, then $\mu' = 0, \mu = \sqrt{\rho_+/\rho_-}$;
(3) If $\mu_0 > \sqrt{\rho_+/\rho_-}$, then $\mu' > 0, \sqrt{\rho_+/\rho_-} < \mu < \mu_0$.

Lemma 3.7. *Assume that $[U_-,U_\delta^0,\Xi_0] > 0, [U_+,U_\delta^0,\Xi_0] < 0, [U_-,U_+,U_\delta^0] > 0, \rho_- > 0, \rho_+ > 0, \mu_0 > 0$. If there exists a unique solution to (2.27) and (3.1) for all $\bar{s} \in (-\infty, 0]$, then this solution satisfies entropy condition (2.28).*

Proof. The contradiction method is used to prove this lemma. We only discuss the case $\mu_0 < \sqrt{\rho_+/\rho_-}$. The other cases can be done similarly. Let $G = \{\bar{s} \le 0 | b_-(\tau) > 0, b_+(\tau) > 0, \forall \tau \in [\bar{s}, 0]\}$. With the conditions $[U_-,U_\delta^0,\Xi_0] > 0, [U_+,U_\delta^0,\Xi_0] < 0$, the continuity of b_-, b_+ shows that there exists a $\bar{s}_1 < 0$, such that $b_-(\tau) > 0, b_+(\tau) > 0$ for all $\tau \in [\bar{s}_1, 0]$. Thus G is a non-empty set.

We proceed to prove $\inf G = -\infty$. Let $\bar{s}_* = \inf G > -\infty$. According to conditions $[U_-,U_+,U_\delta^0] > 0$, $\rho_- > 0, \rho_+ > 0, \mu_0 > 0$, when $\bar{s} \in (\bar{s}_*, 0]$, it holds that

$$0 < \mu_0 < \mu = b_-/b_+ < \sqrt{\rho_+/\rho_-}, \tag{3.36}$$

and $([U_-,U_+,\Xi] - [U_-,U_+,U_\delta])e^{-\bar{s}} > [U_-,U_+,\Xi_0] - [U_-,U_+,U_\delta^0] > 0$ by Lemma 3.5, namely,

$$(b_- + b_+)e^{-\bar{s}} > b_-^0 + b_+^0 > 0. \tag{3.37}$$

In virtue of continuity of b_- and b_+, we take respectively the limits in (3.36) and (3.37) to obtain $b_-(\bar{s}_*)/b_+(\bar{s}_*) > 0, b_-(\bar{s}_*) + b_+(\bar{s}_*) > 0$. These facts lead to $b_-(\bar{s}_*) > 0, b_+(\bar{s}_*) > 0$, which mean that G is a closed set. Thus, there exists a constant $h > 0$, such that $b_-(\tau) > 0, b_+(\tau) > 0, \forall \tau \in (\bar{s}_* - h, 0]$, which contradicts to $\bar{s}_* = \inf G$. The proof is complete. $\square$

Theorem 3.8. *Assume that $[U_-,U_\delta^0,\Xi_0] > 0, [U_+,U_\delta^0,\Xi_0] < 0, [U_-,U_+,U_\delta^0] > 0, \rho_-H_- > 0, \rho_+H_+ > 0$, $\mu_0 > 0$. For all $\bar{s} < 0$, there exists uniquely an entropy solution to (2.27) and (3.1).*

Proof. It is trivially checked that the right side of (2.27) is a continuous vector-valued function and satisfies a local Lipschtiz condition. By the theorem on the existence and unique of solution to the system of ordinary differential equations, there exists uniquely a solution to (2.27) and (3.1) on an interval I, where the solution can be extended to the boundary of the region $\Omega = \{(\Xi, U_\delta, m, n, \bar{s}) | m > 0, n > 0, \bar{s} < 0\}$.

If I is a finite interval, namely $I = (A_0, 0]$, where $A_0 < 0$ is a constant, then it holds $\lim_{\bar{s}\to A_0}(||\Xi||+||U_\delta||+|m|+|n|) = \infty$. With a similar proof as in Lemma 3.7, we can see that $b_-(\bar{s}) > 0, b_+(\bar{s}) > 0, \forall \bar{s} \in (A_0, 0]$. Hence as $\bar{s} \in (A_0, 0]$, it shows from Lemma 3.5 that

$$m - m' \geq \min\{\rho_-, \rho_+\}(b_- + b_+)e^{-\bar{s}} \geq \min\{\rho_-, \rho_+\}(b_-^0 + b_+^0) = a_0 > 0, \tag{3.38}$$

hence,

$$m = m_0 e^{A_0} + e^{A_0}\int_0^{A_0} -(m - m')e^{-\tau}\mathrm{d}\tau \geq m_0 e^{A_0} + a_0(1 - e^{A_0}) > 0. \tag{3.39}$$

Besides, notice that $[U_-, U_+, U_\delta^0] > 0, m_0 > 0$, it gives from (2.29) that

$$0 < [U_-, U_+, U_\delta]e^{-\bar{s}} = \tfrac{m_0[U_-,U_+,U_\delta^0]}{m} \leq \tfrac{m_0[U_-,U_+,U_\delta^0]}{m_0 e^{A_0}+a_0(1-e^{A_0})} = l_0, \tag{3.40}$$

then, by Lemma 3.5,

$$\begin{aligned} 0 <&[U_-, U_+, U_\delta] < [U_-, U_+, \Xi] = e^{A_0}\Big([U_-, U_+, \Xi_0] - \int_0^{A_0}[U_-, U_+, U_\delta]e^{-\tau}\mathrm{d}\tau\Big) \\ \leq& e^{A_0}([U_-, U_+, \Xi_0] - A_0 l_0). \end{aligned} \tag{3.41}$$

This gives that

$$\begin{aligned} (b_- + b_+)e^{-\bar{s}} =&([U_-, U_+, \Xi] - [U_-, U_+, U_\delta])e^{-\bar{s}} \leq ([U_-, U_+, \Xi])e^{-\bar{s}} \\ \leq&[U_-, U_+, \Xi_0] - A_0 l_0. \end{aligned} \tag{3.42}$$

Therefore, by these inequalities, it can be derived easily that

$$\begin{aligned} |m| =&e^{A_0}\Big(m_0 - \int_0^{A_0}(\rho_- b_- + \rho_+ b_+)e^{-2\tau}\mathrm{d}\tau\Big) \\ \leq&e^{A_0}\Big(m_0 - \int_0^{A_0}\max\{\rho_-, \rho_+\}(b_- + b_+)e^{-2\tau}\mathrm{d}\tau\Big) \\ \leq&m_0 e^{A_0} + \max\{\rho_-, \rho_+\}([U_-, U_+, \Xi_0] - A_0 l_0)(1 - e^{A_0}), \end{aligned}$$

$$\begin{aligned} ||U_\delta|| =&\Big|\Big|\tfrac{e^{A_0}}{m}\Big(m_0 U_\delta^0 - \int_0^{A_0}(\rho_- U_- b_- + \rho_+ U_+ b_+)e^{-2\tau}\mathrm{d}\tau\Big)\Big|\Big| \\ \leq&\tfrac{e^{A_0}}{m_0 e^{A_0}+a_0(1-e^{A_0})}\Big(m_0||U_\delta^0|| + \int_{A_0}^{0}\max\{\rho_-||U_-||, \rho_+||U_+||\}(b_- + b_+)e^{-2\tau}\mathrm{d}\tau\Big) \\ \leq&\tfrac{1}{m_0 e^{A_0}+a_0(1-e^{A_0})}\Big(([U_-, U_+, \Xi_0] - A_0 l_0)(1 - e^{A_0})\max\{\rho_-||U_-||, \rho_+||U_+||\} \\ &+ e^{A_0} m_0||U_\delta^0||\Big), \end{aligned}$$

$$\begin{aligned}
||\Xi|| =& \Big|\Big| e^{A_0}\Big(\Xi_0 - \int_0^{A_0} U_\delta e^{-\tau}\mathrm{d}\tau\Big)\Big|\Big| \leq e^{A_0}\Big(||\Xi_0|| + \int_{A_0}^{0} ||U_\delta|| e^{-\tau}\mathrm{d}\tau\Big) \\
=& \tfrac{1-e^{A_0}}{m_0 e^{A_0} + a_0(1-e^{A_0})}\Big(([U_-, U_+, \Xi_0] - A_0 l_0)(1-e^{A_0}) \max\{\rho_-||U_-||, \rho_+||U_+||\} \\
&+ e^{A_0} m_0 ||U_\delta^0||\Big) + e^{A_0}||\Xi_0||, \\
|n| =& \Big| e^{A_0}\Big(- \int_0^{A_0} ((\tfrac{\rho_-}{2}||U_-||^2 + H_-)b_- + (\tfrac{\rho_+}{2}||U_+||^2 + H_+)b_+)e^{-2\tau}\mathrm{d}\tau\Big) \\
&+ e^{A_0}(\tfrac{m_0}{2}||U_\delta^0||^2 + n_0) - \tfrac{m}{2}||U_\delta||^2 \Big| \\
\leq& e^{A_0}\Big(- \int_0^{A_0} \max\{(\tfrac{\rho_-}{2}||U_-||^2 + H_-), (\tfrac{\rho_+}{2}||U_+||^2 + H_+)\}(b_- + b_+)e^{-2\tau}\mathrm{d}\tau\Big) \\
&+ e^{A_0}(\tfrac{m_0}{2}||U_\delta^0||^2 + n_0) + \tfrac{|m|}{2}||U_\delta||^2 \\
\leq& \max\{(\tfrac{\rho_-}{2}||U_-||^2 + H_-), (\tfrac{\rho_+}{2}||U_+||^2 + H_+)\}([U_-, U_+, \Xi_0] - A_0 l_0)(1 - e^{A_0}) \\
&+ \tfrac{|m|}{2}||U_\delta||^2 + e^{A_0}(\tfrac{m_0}{2}||U_\delta^0||^2 + n_0).
\end{aligned}$$

These show that $\lim_{\bar{s}\to A_0}(||\Xi|| + ||U_\delta|| + |m| + |n|) < \infty$, which contradicts to $\lim_{\bar{s}\to A_0}(||\Xi|| + ||U_\delta|| + |m| + |n|) = \infty$. Hence, there exists uniquely a solution to (2.27) and (3.1) for all $\bar{s} < 0$.

Furthermore, we can verify by Lemma 3.7 that the solution satisfies entropy condition (2.28). The proof is complete. □

We discuss subsequently the trajectory of $\Xi = \Xi(\bar{s}), \bar{s} < 0$, in (ξ, η)-plane. To make the following presentation more convenient, we choose a coordinate system such that U_δ^{-+} is on the origin of (ξ, η)-plane, the line through U_- and U_+ coincides with η-axis, and the direction of $\overrightarrow{U_+U_-}$ is the direction of η-axis.

Lemma 3.9. *Suppose that $[U_-, U_\delta^0, \Xi_0] > 0, [U_+, U_\delta^0, \Xi_0] < 0, [U_-, U_+, U_\delta^0] > 0, \rho_- > 0, \rho_+ > 0, \mu_0 > 0$. Denote $\Xi = \Xi(\bar{s})$ by $\eta = \eta(\xi), \bar{s} < 0$, then $\eta = \eta(\xi)$ possesses the following properties.*

(1) If $\mu_0 < \sqrt{\rho_+/\rho_-}$, then $\frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2} < 0$;
(2) If $\mu_0 = \sqrt{\rho_+/\rho_-}$, then $\frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2} = 0$;
(3) If $\mu_0 > \sqrt{\rho_+/\rho_-}$, then $\frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2} > 0$.

Proof. We only prove (1). The other cases can be treated similarly. In our coordinate system, $\tilde{u_\delta}(\bar{s}) = 0$, $\tilde{v_\delta}(\bar{s}) = \mu_- v_- + \mu_+ v_+$. Since $\mu_0 < \sqrt{\rho_+/\rho_-}$, it gives by Lemma 3.6 that

$$\tilde{v_\delta}' > 0, \forall \bar{s} < 0. \tag{3.43}$$

Besides, by Theorem 3.8, it holds from (3.34) that

$$[U_-, U_+, U_\delta] > 0, \forall \bar{s} < 0. \tag{3.44}$$

This fact with (3.32) leads to $[U_-, U_+, \Xi] = [U_-, U_+, U_\delta] + [U_-, U_\delta, \Xi] - [U_+, U_\delta, \Xi] > 0, [U_-, U_+, \Xi'] = [U_-, U_+, \Xi] - [U_-, U_+, U_\delta] = [U_-, U_\delta, \Xi] - [U_+, U_\delta, \Xi] > 0$, hence, $\xi > 0, \xi' > 0, \forall \bar{s} < 0$. Thus, it yields that

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \frac{\eta'}{\xi'} = \frac{\eta - v_\delta}{\xi - u_\delta} = \frac{\eta - \tilde{v_\delta}}{\xi - \tilde{u_\delta}} = \frac{\eta - \tilde{v_\delta}}{\xi}, \ \frac{\mathrm{d}^2\eta}{\mathrm{d}\xi^2} = -\frac{\tilde{v_\delta}'}{\xi\xi'} < 0. \tag{3.45}$$

The proof is complete. □

We next study the trajectory of $U_\delta = U_\delta(\bar{s}), \bar{s} < 0$, in (u, v)-plane. The (u, v)-plane is chosen in the same way as the previous (ξ, η)-plane. Hence, these two coordinate systems can be coincided. Let $U_\delta = U_\delta(\bar{s}) = (u_\delta(\bar{s}, v_\delta(\bar{s})$ by $v_\delta = v_\delta(u_\delta)$. Its tangent line intersects with the line U_-U_+ at some point $\bar{U_\delta}$, that is, $\bar{U_\delta} = (\bar{u_\delta}, \bar{v_\delta}) = k_-U_- + k_+U_+, k_- + k_+ = 1$. We have that

$$\begin{aligned}\frac{k_+}{k_-} &= \lim_{h\to 0} -\frac{[U_-, U_\delta(\bar{s}+h), U_\delta(\bar{s})]}{[U_+, U_\delta(\bar{s}+h), U_\delta(\bar{s})]} = \lim_{h\to 0} -\frac{([U_-, U_\delta(\bar{s}+h), U_\delta(\bar{s})] - [U_-, U_\delta(\bar{s}), U_\delta(\bar{s})])/h}{([U_+, U_\delta(\bar{s}+h), U_\delta(\bar{s})] - [U_+, U_\delta(\bar{s}), U_\delta(\bar{s})])/h} \\ &= -\frac{[U_-, U_\delta', U_\delta]}{[U_+, U_\delta', U_\delta]} = -\frac{[U_-, mU_\delta', U_\delta]}{[U_+, mU_\delta', U_\delta]}. \end{aligned} \tag{3.46}$$

Using (2.27), it follows that $k_+/k_- = \rho_+b_+(\rho_-b_-)^{-1} = \rho_+(\rho_-\mu)^{-1} > 0$, which means that $\bar{U_\delta}$ lies in the interior of the segment $\overline{U_-U_+}$.

Lemma 3.10. *Assume that $[U_-, U_\delta^0, \Xi_0] > 0, [U_+, U_\delta^0, \Xi_0] < 0, [U_-, U_+, U_\delta^0] > 0, \rho_-\rho_+ > 0, \mu|_{\bar{s}=0} = \mu_0 > 0$. In (u, v)-plane, the trajectory of curve $v_\delta = v_\delta(u_\delta)$ is shown as follows.*

(1) If $\mu_0 < \sqrt{\rho_+/\rho_-}$, then $\frac{\mathrm{d}^2 v_\delta}{\mathrm{d}u_\delta^2} > 0$;
(2) If $\mu_0 = \sqrt{\rho_+/\rho_-}$, then $\frac{\mathrm{d}^2 v_\delta}{\mathrm{d}u_\delta^2} = 0$;
(3) If $\mu_0 > \sqrt{\rho_+/\rho_-}$, then $\frac{\mathrm{d}^2 v_\delta}{\mathrm{d}u_\delta^2} < 0$.

Proof. In the chosen coordinate system, we have $\bar{u}_\delta(\bar{s}) = 0, \bar{v}_\delta(\bar{s}) = k_-v_- + k_+v_+$. As $\mu_0 < \sqrt{\rho_+/\rho_-}$, it holds that $(k_+/k_-)' > 0$ by Lemma 3.6, then

$$\bar{v}_\delta' < 0, \forall \bar{s} < 0. \tag{3.47}$$

Notice that,

$$\frac{\mathrm{d}v_\delta}{\mathrm{d}u_\delta} = \frac{v_\delta'}{u_\delta'} = \frac{v_\delta - \bar{v}_\delta}{u_\delta - \bar{u}_\delta} = \frac{v_\delta - \bar{v}_\delta}{u_\delta}, \tag{3.48}$$

it is easy to see that $\frac{\mathrm{d}^2 v_\delta}{\mathrm{d}u_\delta^2} = -\bar{v}_\delta'(u_\delta u_\delta')^{-1}$. In view of Theorem 3.8, we have $[U_-, U_\delta, \Xi] > 0, [U_+, U_\delta, \Xi] < 0$. These facts with (3.35) and (3.33) yield $[U_-, U_+, U_\delta] > 0, [U_-, U_+, U_\delta'] > 0$, which show $u_\delta > 0, u_\delta' > 0$. Thus, we have $\frac{\mathrm{d}^2 v_\delta}{\mathrm{d}u_\delta^2} > 0$. The other cases can be done similarly. The proof is complete. □

Finally, we analyze the limit behaviors of $\Xi(\bar{s})$ and $U_\delta(\bar{s})$ as $\bar{s} \to -\infty$.

Lemma 3.11. *Under the conditions $[U_-, U_\delta^0, \Xi_0] > 0, [U_+, U_\delta^0, \Xi_0] < 0, [U_-, U_+, U_\delta^0] > 0, \rho_-H_- > 0, \rho_+H_+ > 0, \mu_0 > 0$, the $\Xi(\bar{s})$ and $U_\delta(\bar{s})$ satisfy*

$$\lim_{\bar{s}\to -\infty} \Xi(\bar{s}) = U_\delta^{-+}, \lim_{\bar{s}\to -\infty} U_\delta(\bar{s}) = U_\delta^{-+}. \tag{3.49}$$

Proof. In the chosen coordinate system, according to Theorem 3.8, it shows $[U_-, U_\delta, \Xi] > 0$, $[U_+, U_\delta, \Xi] < 0$. Using Lemma 3.5, the combination of (2.29) and (3.33) shows that $0 \leq \lim_{\bar{s}\to -\infty}[U_-, U_+, U_\delta] \leq \lim_{\bar{s}\to -\infty} Ce^{\bar{s}}(m_0e^{\bar{s}} + a_0(1 - e^{\bar{s}}))^{-1} = 0$, so $\lim_{\bar{s}\to -\infty} u_\delta = 0$. We further calculate that

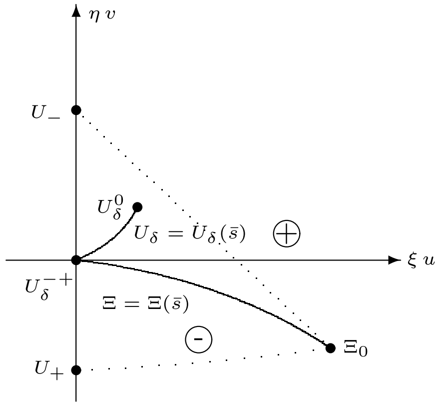

Fig. 3.3. The qualitative behavior of solution to Case 3.3 with $\mu_0 < \sqrt{\rho_+/\rho_-}$.

$$[U_-, U_+, \Xi] = e^{\bar{s}}[U_-, U_+, \Xi_0] - e^{\bar{s}} \int_0^{\bar{s}} [U_-, U_+, U_\delta] e^{-\tau} \mathrm{d}\tau, \tag{3.50}$$

which implies $\lim_{\bar{s}\to-\infty}[U_-, U_+, \Xi] = 0$, hence $\lim_{\bar{s}\to-\infty} \xi = 0, \lim_{\bar{s}\to-\infty}(b_- + b_+) = \lim_{\bar{s}\to-\infty}([U_-, U_+, \Xi] - [U_-, U_+, U_\delta]) = 0$.

Next, we prove $\lim_{\bar{s}\to-\infty} v_\delta = 0, \lim_{\bar{s}\to-\infty} \eta = 0$. Here we only deal with the case $\mu_0 < \sqrt{\rho_+/\rho_-}$. The other cases can be done similarly. Both (3.45) and (3.48) imply that $\eta = \tilde{v}_\delta + \eta'\xi(\xi')^{-1}, v_\delta = \bar{v}_\delta + v_\delta' u_\delta (u_\delta')^{-1}$. In virtue of $\mu_0 < \sqrt{\rho_+/\rho_-}, \Xi - \Xi' = U_\delta$, we deduce from Lemmas 3.9 and 3.10 that

$$-\infty < \eta'(0)/\xi'(0) < \eta'/\xi' < v_\delta'/u_\delta' < v_\delta'(0)/u_\delta'(0) < +\infty, \tag{3.51}$$

which shows both v_δ'/u_δ' and η'/ξ' are bounded. Besides, it follows from (3.43) and (3.47) that

$$0 = \mu_- \tfrac{\sqrt{\rho_-}+\sqrt{\rho_+}}{\sqrt{\rho_-}} \tfrac{\sqrt{\rho_-}v_- + \sqrt{\rho_+}v_+}{\sqrt{\rho_-}+\sqrt{\rho_+}} < \mu_-(v_- + \mu v_+) = \tilde{v}_\delta < \tilde{v}_\delta(0),$$
$$\bar{v}_\delta(0) < \bar{v}_\delta = k_-(v_- + \tfrac{\rho_+}{\rho_-\mu} v_+) < k_- \tfrac{\sqrt{\rho_-}+\sqrt{\rho_+}}{\sqrt{\rho_-}} \tfrac{\sqrt{\rho_-}v_- + \sqrt{\rho_+}v_+}{\sqrt{\rho_-}+\sqrt{\rho_+}} = 0,$$

hence $\lim_{\bar{s}\to-\infty} \eta = \lim_{\bar{s}\to-\infty} \tilde{v}_\delta \geq 0, \lim_{\bar{s}\to-\infty} v_\delta = \lim_{\bar{s}\to-\infty} \bar{v}_\delta \leq 0$. Since $\lim_{\bar{s}\to-\infty} \xi' = \lim_{\bar{s}\to-\infty}(\xi - u_\delta) = 0$, we can obtain from (3.51) that $\lim_{\bar{s}\to-\infty} \eta - v_\delta = \lim_{\bar{s}\to-\infty} \eta' = 0$. Hence, we arrive at $\lim_{\bar{s}\to-\infty} \eta = 0$ and $\lim_{\bar{s}\to-\infty} v_\delta = 0$. We finish the proof of this lemma. □

In all, the qualitative behavior of solution is obtained. Here we only present the qualitative behavior of solution for Case $\mu_0 < \sqrt{\rho_+/\rho_-}$ in Fig. 3.3.

3.4. $\rho_- H_- > 0,\ \rho_+ H_+ > 0,\ [U_-, U_\delta^0, \Xi_0] > 0,\ [U_+, U_\delta^0, \Xi_0] < 0,\ [U_-, U_+, U_\delta^0] < 0$

We first prove some lemmas.

Lemma 3.12. *Under the conditions $\rho_- H_- > 0, \rho_+ H_+ > 0, \mu_0 > 0, [U_-, U_\delta^0, \Xi_0] > 0, [U_+, U_\delta^0, \Xi_0] < 0$, $[U_-, U_+, U_\delta^0] < 0$. If $[U_-, U_+, \Xi_0] > 0$, then there exists a constant $\bar{s}_1 > -\infty$, such that $[U_-, U_+, \Xi(\bar{s}_1)] = 0$.*

Proof. The contradiction method is used to prove this lemma. Here we only consider the case $\mu_0 < \sqrt{\rho_+/\rho_-}$. The other cases can be proved similarly. In virtue of $[U_-, U_+, \Xi_0] > 0$, assume that

$$[U_-, U_+, \Xi] > 0, \forall \bar{s} < 0. \tag{3.52}$$

Define $G = \{\bar{s} < 0 | b_-(\tau) > 0, b_+(\tau) > 0, \mu_0 \leq \mu < \sqrt{\rho_+/\rho_-}, \forall \tau \in [\bar{s}, 0]\}$. Obviously, the set G is a non-empty set.

We show $\inf G = -\infty$. Suppose $\bar{s}_* = \inf G > -\infty$. Then as $\tau \in (\bar{s}_*, 0]$, it holds that $\mu_0 \leq \mu = b_-(\tau)/b_+(\tau) < \sqrt{\rho_+/\rho_-}, b_-(\tau) + b_+(\tau) = [U_-, U_+, \Xi] - [U_-, U_+, U_\delta] > 0, b_-(\tau) > 0, b_+(\tau) > 0$. Notice that (3.29), we obtain $b_-(\bar{s}_*) > 0, b_+(\bar{s}_*) > 0, \mu_0 \leq \mu(\bar{s}_*) < \sqrt{\rho_+/\rho_-}$ by the continuity of b_- and b_+, which imply that G is a closed set. Thus there exists a constant $h > 0$, such that $b_-(\tau) > 0, b_+(\tau) > 0, \mu_0 \leq \mu < \sqrt{\rho_+/\rho_-}, \forall \tau \in (\bar{s}_* - h, 0]$, which contradicts to $\bar{s}_* = \inf G$.

We substitute (2.29) into (3.50), and conduce that

$$[U_-, U_+, \Xi] = e^{\bar{s}}\Big([U_-, U_+, \Xi_0] + C\int_{\bar{s}}^{0} m^{-1} \mathrm{d}\tau\Big),$$

where, $C = m_0[U_-, U_+, U_\delta^0] < 0$. Noting $[U_-, U_+, U_\delta^0] < 0, b_- > 0, b_+ > 0, \forall \bar{s} < 0$, it shows that

$$\begin{aligned} 0 < m &= e^{\bar{s}}\Big(m_0 + \int_{\bar{s}}^{0} (m - m')e^{-\tau} \mathrm{d}\tau\Big) \\ &< e^{\bar{s}} m_0 + \max\{\rho_-, \rho_+\}([U_-, U_\delta^0, \Xi_0] - [U_+, U_\delta^0, \Xi_0])(1 - e^{\bar{s}}) \\ &< m_0 + \max\{\rho_-, \rho_+\}([U_-, U_\delta^0, \Xi_0] - [U_+, U_\delta^0, \Xi_0]), \end{aligned}$$

which implies that $\int_{\bar{s}}^{0} m^{-1} \mathrm{d}\tau$ is divergent. Thus, we have $\lim_{\bar{s}\to\infty}[U_-, U_+, \Xi] < 0$, which contradicts to (3.52). We complete the proof of this lemma. □

Lemma 3.13. *Under the conditions $\rho_- H_- > 0, \rho_+ H_+ > 0, \mu_0 > 0, [U_-, U_\delta^0, \Xi_0] > 0, [U_+, U_\delta^0, \Xi_0] < 0$, $[U_-, U_+, U_\delta^0] < 0$. There exists a constant $\bar{s}_2 < \bar{s}_1$, such that the entropy condition is violated, that is,*

(1) If $\mu_0 < \sqrt{\rho_+/\rho_-}$, there exists a constant $\bar{s}_2 < \bar{s}_1$, such that $b_-(\bar{s}_2) = 0$;
(2) If $\mu_0 = \sqrt{\rho_+/\rho_-}$, there exists a constant $\bar{s}_2 < \bar{s}_1$, such that $b_-(\bar{s}_2) = 0$ and $b_+(\bar{s}_2) = 0$;
(3) If $\mu_0 > \sqrt{\rho_+/\rho_-}$, there exists a constant $\bar{s}_2 < \bar{s}_1$, such that $b_+(\bar{s}_2) = 0$.

Proof. We consider $\mu_0 < \sqrt{\rho_+/\rho_-}$. Let $G = \{\bar{s} < \bar{s}_1 | b_-(\tau) > 0, b_+(\tau) > 0, \tau \in (\bar{s}, \bar{s}_1]\}$. Obviously, G is a non-empty set. Denote $\bar{s}_2 = \inf G$. We claim that $\bar{s}_2 > -\infty$. In fact, if $\bar{s}_2 = -\infty$, then it holds that $b_-(\bar{s}) > 0, b_+(\bar{s}) > 0, \forall \bar{s} < 0$. We then get $[U_-, U_+, \Xi'] = [U_-, U_+, \Xi] - [U_-, U_+, U_\delta] = b_- + b_+ > 0$. Hence, letting $\bar{s}_3 < \bar{s}_1$, it holds that $[U_-, U_+, \Xi(\bar{s}_3)] < [U_-, U_+, \Xi(\bar{s}_1)] = 0, \mu(\bar{s}_3) < \sqrt{\rho_+/\rho_-}, [U_-, U_+, U_\delta(\bar{s}_3)] < [U_-, U_+, \Xi(\bar{s}_3)]$. Integrating (3.29) from $\bar{s}_3$ to $\bar{s}$, we obtain

$$\frac{\sqrt{\rho_+/\rho_-} + \mu}{\sqrt{\rho_+/\rho_-} - \mu} = \frac{\sqrt{\rho_+/\rho_-} + \mu(\bar{s}_3)}{\sqrt{\rho_+/\rho_-} - \mu(\bar{s}_3)} \exp\Big(-\int_{\bar{s}_3}^{\bar{s}} 2\sqrt{\rho_-\rho_+}\frac{[U_-, U_+, \Xi]}{e^{\tau} m} \mathrm{d}\tau\Big). \tag{3.53}$$

Considering

$$\begin{aligned} -\int_{\bar{s}_3}^{\bar{s}} 2\sqrt{\rho_-\rho_+}\frac{[U_-, U_+, \Xi]}{e^{\tau} m} \mathrm{d}\tau &= \int_{\bar{s}}^{\bar{s}_3} 2\sqrt{\rho_-\rho_+}\frac{[U_-, U_+, \Xi][U_-, U_+, U_\delta]}{e^{\tau} m[U_-, U_+, U_\delta]} \mathrm{d}\tau \\ &\leq \int_{\bar{s}}^{\bar{s}_3} 2\sqrt{\rho_-\rho_+}\frac{[U_-, U_+, \Xi(\bar{s}_3)]^2}{e^{2\tau} C} \mathrm{d}\tau, \end{aligned}$$

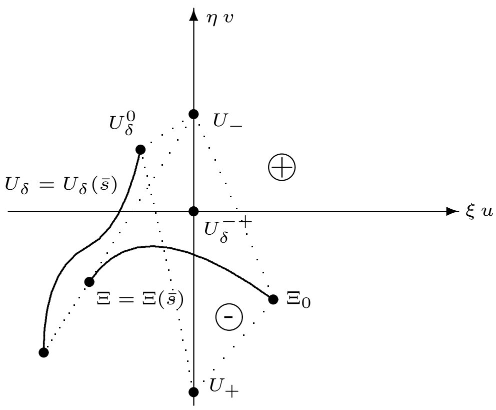

Fig. 3.4. The qualitative behavior of solution to Case 3.4 with $\mu_0 < \sqrt{\rho_+/\rho_-}$.

where $C = m_0[U_-, U_+, U_\delta^0] < 0$, it gives that the right-hand side of (3.53) tends to 0 as $\bar{s} \to -\infty$. This fact implies that there exists a constant $\bar{s}_4$ such that $(\sqrt{\rho_+/\rho_-} + \mu)(\sqrt{\rho_+/\rho_-} - \mu)^{-1} = 1$, that is $b_-(\bar{s}_4) = 0$, which contradicts to $\bar{s}_4 \in G$. Thus, there exists a constant $\bar{s}_2 < \bar{s}_1$, such that $b_-(\bar{s}_2) = 0$. The other cases can be treated similarly. We finish the proof of this lemma. □

With the similar manner as in Lemmas 3.9 and 3.10, we obtain the following lemmas.

Lemma 3.14. *Under the conditions $\rho_-H_- > 0, \rho_+H_+ > 0, \mu_0 > 0, [U_-, U_\delta^0, \Xi_0] > 0, [U_+, U_\delta^0, \Xi_0] < 0$, $[U_-, U_+, U_\delta^0] < 0$, as $[U_-, U_+, \Xi_0] > 0$, then for all $\bar{s} \in (\bar{s}_2, 0]$, the curve $\eta = \eta(\xi)$ preserves its convexity, namely, all the conclusions of Lemma 3.9 are true.*

Lemma 3.15. *Under the conditions $\rho_-H_- > 0, \rho_+H_+ > 0, \mu_0 > 0, [U_-, U_\delta^0, \Xi_0] > 0, [U_+, U_\delta^0, \Xi_0] < 0$, $[U_-, U_+, U_\delta^0] < 0$, as $[U_-, U_+, \Xi_0] > 0$, the trajectory of curve $v_\delta = v_\delta(u_\delta)$ is shown as follows.*

(1) When $\mu_0 < \sqrt{\rho_+/\rho_-}$, as $\bar{s} \in (\bar{s}_1, 0]$, then $\frac{\mathrm{d}^2 v_\delta}{\mathrm{d}u_\delta^2} > 0$, while as $\bar{s} \in (\bar{s}_2, \bar{s}_1)$, then $\frac{\mathrm{d}^2 v_\delta}{\mathrm{d}u_\delta^2} < 0$;
(2) When $\mu_0 = \sqrt{\rho_+/\rho_-}$, then $\frac{\mathrm{d}^2 v_\delta}{\mathrm{d}u_\delta^2} = 0, \bar{s} \in (\bar{s}_2, 0]$;
(3) When $\mu_0 > \sqrt{\rho_+/\rho_-}$, as $\bar{s} \in (\bar{s}_1, 0]$, then $\frac{\mathrm{d}^2 v_\delta}{\mathrm{d}u_\delta^2} < 0$, while as $\bar{s} \in (\bar{s}_2, \bar{s}_1)$, then $\frac{\mathrm{d}^2 v_\delta}{\mathrm{d}u_\delta^2} > 0$.

Remark 3.16. Under the conditions $\rho_-H_- > 0, \rho_+H_+ > 0, \mu_0 > 0, [U_-, U_\delta^0, \Xi_0] > 0, [U_+, U_\delta^0, \Xi_0] < 0$, $[U_-, U_+, U_\delta^0] < 0$, as $[U_-, U_+, \Xi_0] < 0$, we can prove that for all $\bar{s} \in (\bar{s}_2, 0)$, the convexity of curve $v_\delta = v_\delta(u_\delta)$ is the same as that of curve $\eta = \eta(\xi)$.

Therefore, we obtain clearly the qualitative behavior of solution. Here we only illustrate the qualitative behavior of solution for Case $\mu_0 < \sqrt{\rho_+/\rho_-}$ in Fig. 3.4.

4. The structures of solutions

In this section, we construct the solutions to (1.3) and (1.4) in (ξ, η)-plane. The construction of the solutions involves frequently the interactions of two two-dimensional waves, where their interaction point Ξ_0 becomes a straight line $X = \Xi_0 t$ in (t, X)-space. We will continue the solution of (1.3) at the Cauchy support $t = s : X = \Xi_0 s$ up to $t > s$ which corresponds to $\bar{s} < 0$. This is why we solve the reformulated Rankine–Hugoniot relation (2.27) $\bar{s} < 0$.

We present the following Schauder's fixed point theorem.

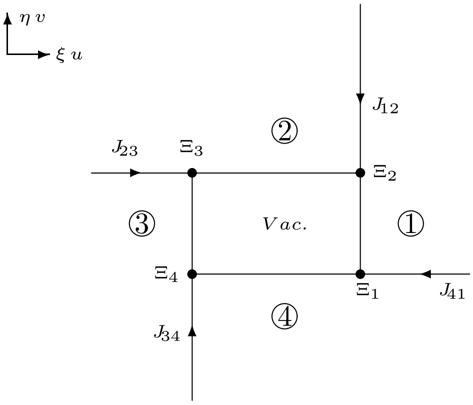

Fig. 4.1. The solution for Case 4.1(i).

Theorem 4.1. *Suppose that K is a non-empty convex closed set in Banach space B, T is a continuous operator, and $T(K)$ is a precompact set in B. Then, there exists $x \in K$ such that $Tx = x$.*

Besides, for the presentation more convenient, we orient the normal direction of discontinuity $x = 0, y \geq 0$ as $N = (1, 0)$, the normal direction of ray $y = 0, x \geq 0$ as $N = (0, -1)$, the normal direction of ray $x = 0, y \leq 0$ as $N = (-1, 0)$, and the normal direction of ray $y = 0, x \leq 0$ as $N = (0, 1)$. We also list the following notations,

ⓘ: the state (ρ_i, U_i, H_i), Ξ_i: the point (u_i, v_i), $\overline{\Xi_i\Xi_j}$: the segment connecting points Ξ_i and Ξ_j,
Ω_i: the determination region of the state ⓘ, where $\Omega_1 = \{(\xi, \eta)|\xi > u_1, \eta > v_1\}$, $\Omega_2 = \{(\xi, \eta)|\xi < u_2, \eta > v_2\}$, $\Omega_3 = \{(\xi, \eta)|\xi < u_3, \eta < v_3\}$, $\Omega_4 = \{(\xi, \eta)|\xi > u_4, \eta < v_4\}$,
J_{ij}: the two-dimensional contact discontinuity connecting the states ⓘ and ⓙ,
δ_{ij}: the two-dimensional delta shock wave connecting the states ⓘ and ⓙ,
δ_i: the two-dimensional delta shock wave connecting the state ⓘ and the vacuum,
δ_{ij}^A: the two-dimensional delta shock wave emitting from point A and connecting the states ⓘ and ⓙ,
δ_i^A: the two-dimensional delta shock wave emitting from point A and connecting the state ⓘ and the vacuum,
m_{ij}: the weight of δ_{ij} on the variable ρ,
n_{ij}: the weight of δ_{ij} on the variable H,
the fine curve standing for the two-dimensional contact discontinuity,
the black curve representing the two-dimensional delta shock wave.

Next, we divide the two-dimensional Riemann problem (1.3) and (1.4) into five cases by the different combinations of two-dimensional waves, and construct the solutions to each case.

4.1. Four two-dimensional contact discontinuities

By (2.7), the initial data satisfy $u_1 = u_2, v_1 = v_4, u_3 = u_4, v_2 = v_3$. It can be divided into the following two cases.

(i) $u_1 = u_2 > u_3 = u_4, v_1 = v_4 < v_2 = v_3$. In this case, we can see that four determination regions Ω_i, $i = 1, 2, 3, 4$ do not overlap, and that the pseudo-characteristic lines from the states ⓘ $(i = 1, 2, 3, 4)$ do not come into the rectangle $\Xi_1\Xi_2\Xi_3\Xi_4$. Thus, the contact discontinuities J_{12}, J_{23}, J_{34} and J_{41} stop at their singular points Ξ_1, Ξ_2, Ξ_3 and Ξ_4, while, the vacuum develops in the rectangle $\Xi_1\Xi_2\Xi_3\Xi_4$. The solution is shown in Fig. 4.1.

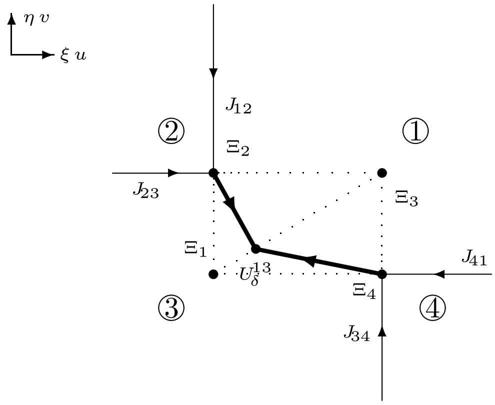

Fig. 4.2. The solution for Case 4.1(ii).

(ii) $u_1 = u_2 < u_3 = u_4, v_1 = v_4 < v_2 = v_3$. J_{12} meets with J_{23} at the point Ξ_2. One can see that the pseudo-characteristic lines from the states ① and ③ intersect each other in the rectangle $\Xi_1\Xi_2\Xi_3\Xi_4$. Hence, a delta shock wave issues in this rectangle. The initial data for this delta shock wave are defined as

$$\bar{s} = 0, \begin{cases} \Xi(0) = \Xi_2,\ U_\delta(0) = 0, m(0) = 0,\ n(0) = 0, \\ (\rho_-, U_-, H_-) = (\rho_1, U_1, H_1), (\rho_+, U_+, H_+) = (\rho_3, U_3, H_3). \end{cases} \tag{4.1}$$

Besides, the $u_1 = u_2 < u_3, v_1 < v_2 = v_3$ lead to $[U_1, U_3, \Xi_2] > 0$. Using Remark 3.4 and solving initial value problem (2.27) and (4.1), we obtain the exact solution expressed in (3.28), labeled by delta shock wave $\delta_{13}^{\Xi_2}$.

Similarly, J_{34} and J_{41} interact at the point Ξ_4. The inequalities $u_1 < u_3 = u_4, v_1 = v_4 < v_3$ imply $[U_3, U_1, \Xi_4] > 0$. By Remark 3.4, we solve initial value problem (2.27) with initial data

$$\bar{s} = 0, \begin{cases} \Xi(0) = \Xi_4,\ U_\delta(0) = 0, m(0) = 0,\ n(0) = 0, \\ (\rho_-, U_-, H_-) = (\rho_3, U_3, H_3), (\rho_+, U_+, H_+) = (\rho_1, U_1, H_1), \end{cases} \tag{4.2}$$

and obtain a delta shock wave $\delta_{13}^{\Xi_4}$. This $\delta_{13}^{\Xi_4}$ finally matches with $\delta_{13}^{\Xi_2}$ at the singular point U_δ^{13}. The solution is shown in Fig. 4.2.

4.2. Four two-dimensional delta shock waves

From (2.22), the initial data satisfy $u_1 < u_2, v_1 < v_4, u_4 < u_3, v_2 < v_3$. It involves the following three cases.

(i) $u_4 < u_3 < u_1 < u_2, v_1 < v_4 < v_2 < v_3$. Let A be the intersection point of the delta shock wave δ_{12}:

$$\begin{cases} \xi = \frac{\sqrt{\rho_1}u_1 + \sqrt{\rho_2}u_2}{\rho_1+\rho_2}, \quad U_\delta^{12} = \frac{\sqrt{\rho_1}U_1 + \sqrt{\rho_2}U_2}{\sqrt{\rho_1}+\sqrt{\rho_2}}, \\ m_{12} = \sqrt{\rho_1\rho_2}(u_2 - u_1), \quad n_{12} = \left(\frac{\rho_1\rho_2||U_1-U_2||^2}{2(\sqrt{\rho_1}+\sqrt{\rho_2})} + \sqrt{\rho_2}H_1 + \sqrt{\rho_1}H_2\right)\frac{u_2-u_1}{\sqrt{\rho_1}+\sqrt{\rho_2}}, \end{cases} \tag{4.3}$$

and the line $\Xi_2\Xi_3$, B the intersection point of the delta shock wave δ_{41}:

$$\begin{cases} \eta = \frac{\sqrt{\rho_4}v_4 + \sqrt{\rho_1}v_1}{\rho_4+\rho_1}, \quad U_\delta^{41} = \frac{\sqrt{\rho_4}U_4 + \sqrt{\rho_1}U_1}{\sqrt{\rho_4}+\sqrt{\rho_1}}, \\ m_{41} = \sqrt{\rho_4\rho_1}(v_4 - v_1), \quad n_{41} = \left(\frac{\rho_4\rho_1||U_4-U_1||^2}{2(\sqrt{\rho_4}+\sqrt{\rho_1})} + \sqrt{\rho_1}H_4 + \sqrt{\rho_4}H_1\right)\frac{v_4-v_1}{\sqrt{\rho_4}+\sqrt{\rho_1}}, \end{cases} \tag{4.4}$$

and the line $\Xi_1\Xi_2$, C the intersection point of the delta shock wave δ_{34}:

$$\begin{cases} \xi = \frac{\sqrt{\rho_3}u_3+\sqrt{\rho_4}u_4}{\rho_3+\rho_4}, \quad U_\delta^{34} = \frac{\sqrt{\rho_3}U_3+\sqrt{\rho_4}U_4}{\sqrt{\rho_3}+\sqrt{\rho_4}}, \\ m_{34} = \sqrt{\rho_3\rho_4}(u_3-u_4), \quad n_{34} = \left(\frac{\rho_3\rho_4||U_3-U_4||^2}{2(\sqrt{\rho_3}+\sqrt{\rho_4})} + \sqrt{\rho_4}H_3 + \sqrt{\rho_3}H_4\right)\frac{u_3-u_4}{\sqrt{\rho_3}+\sqrt{\rho_4}}, \end{cases} \tag{4.5}$$

and the line $\Xi_1\Xi_4$, D the intersection point of the delta shock wave δ_{23}:

$$\begin{cases} \eta = \frac{\sqrt{\rho_2}v_2+\sqrt{\rho_3}v_3}{\rho_2+\rho_3}, \quad U_\delta^{23} = \frac{\sqrt{\rho_2}U_2+\sqrt{\rho_3}U_3}{\sqrt{\rho_2}+\sqrt{\rho_3}}, \\ m_{23} = \sqrt{\rho_2\rho_3}(v_3-v_2), \quad n_{23} = \left(\frac{\rho_2\rho_3||U_2-U_3||^2}{2(\sqrt{\rho_2}+\sqrt{\rho_3})} + \sqrt{\rho_2}H_3 + \sqrt{\rho_3}H_2\right)\frac{v_3-v_2}{\sqrt{\rho_2}+\sqrt{\rho_3}}, \end{cases} \tag{4.6}$$

and the line $\Xi_3\Xi_4$, $E = (u_\delta^{12}, v_2), F = (u_1, v_\delta^{41}), G = (u_\delta^{34}, v_4)$ and $H = (u_3, v_\delta^{23})$.

We analyze the structure of solution. Since $u_4 < u_3 < u_1 < u_2, v_1 < v_4 < v_2 < v_3$, the delta shock waves $\delta_{12}, \delta_{23}, \delta_{34}, \delta_{41}$ do not meet. While, there exists a domain, inside which no pseudo-characteristic lines from the states (i) $(i = 1, 2, 3, 4)$ comes. This fact implies that the vacuum emerges in this domain. Thus, the structure of solution is that these four delta shock waves separate the vacuum from the states (i), $i = 1, 2, 3, 4$, and finally match together.

We now construct rigorously the solution. Sine the weights and velocities of delta shock waves are actually changed at their intersection points, the delta shock waves can not propagate along their previous routes. Without loss of generality, let point $\Xi_0^1 \in \overline{AE}$. By the laws of conservation of mass, momentum and energy, the initial data at this point are given as follows,

$$\bar{s} = 0, \begin{cases} \Xi(0) = \Xi_0^1, \ U_\delta(0) = U_{\delta_1}^0 = \frac{m_{12}U_\delta^{12}+m_1^*U_{\delta_1}^*}{m_{12}+m_1^*}, \ m(0) = m_0^1 = m_{12} + m_1^*, \\ n(0) = n_0^1 = n_{12} + n_1^*, (\rho_-, U_-, H_-) = (\rho_1, U_1, H_1), \\ (\rho_+, U_+, H_+) = (0, U(\xi, \eta), 0), [U_-, U_\delta^0, \Xi_0] = [U_1, U_{\delta_1}^0, \Xi_0^1] > 0, \end{cases} \tag{4.7}$$

where $U_{\delta_1}^* \in \triangle AE\Xi_2, m_1^* \in [0, M^*], n_1^* \in [0, N^*]$, and M^* and N^* are determined. By the result in Subsection 3.1, we solve initial value problem (2.27) and (4.7) to obtain the solution which is given in (3.5), denoted by delta shock wave $\delta_1^{\Xi_0^1}$.

Let Ξ_0^2 be the intersection point of $\delta_1^{\Xi_0^1}$ and δ_{41}, where the weights and velocity of $\delta_1^{\Xi_0^1}$ are m_2^*, n_2^* and $U_{\delta_2}^*$. It shows by Lemma 3.1 that $\Xi_0^2 \in \overline{BF}, U_{\delta_2}^* \in \triangle BF\Xi_1$. We claim that $m_2^* \in [0, M^*], n_2^* \in [0, N^*]$.

In fact, denote $m_* = \max\{m_{12}, m_{23}, m_{34}, m_{41}\}, n_* = \max\{n_{12}, n_{23}, n_{34}, n_{41}\}$, S_0 the area of quadrilateral $\Xi_1\Xi_2\Xi_3\Xi_4$,

$$\begin{aligned} L_0 &= \max\left\{\max_{U_{\delta_1}^* \in \triangle AE\Xi_2} ||U_{\delta_1}^*||, ||U_1||, ||U_2||, ||U_3||, ||U_4||\right\}, \\ p_* &= \max\left\{\frac{\sqrt{\rho_4}(v_4-v_1)}{(\sqrt{\rho_4}+\sqrt{\rho_1})(v_2-v_1)}, \frac{\sqrt{\rho_3}(u_3-u_4)}{(\sqrt{\rho_3}+\sqrt{\rho_4})(u_1-u_4)}, \frac{\sqrt{\rho_2}(v_3-v_2)}{(\sqrt{\rho_2}+\sqrt{\rho_3})(v_3-v_4)}, \frac{\sqrt{\rho_1}(u_2-u_1)}{(\sqrt{\rho_1}+\sqrt{\rho_2})(u_2-u_3)}\right\}, \\ q_* &= \max\left\{\frac{||U_1||}{2} + \frac{H_1}{\rho_1}, \frac{||U_2||}{2} + \frac{H_2}{\rho_2}, \frac{||U_3||}{2} + \frac{H_3}{\rho_3}, \frac{||U_4||}{2} + \frac{H_4}{\rho_4}\right\}. \end{aligned}$$

In virtue of $u_4 < u_3 < u_1 < u_2, v_1 < v_4 < v_2 < v_3$, we have

$$\lim_{M\to\infty}\left\{\frac{(M+m_*)^2p_*^2 2\rho_1(M+m_*)S_0}{M^2}\right\} = p_*^2 < 1.$$

This fact implies that there exists a $M^* > 0$ such that

$$(M^* + m_*)^2p_*^2 + 2\rho_1(M^* + m_*)S_0 < (M^*)^2. \tag{4.8}$$

Similarly, since

$$\lim_{N\to\infty} \frac{(N+n_*)p_*+L_0p_*(M^*+m_*)/2+q_*M^*}{N} = p_* < 1,$$

there exists a $N^* > 0$ such that

$$(N^* + n_*)p_* + L_0p_*(M^* + m_*)/2 + q_*M^* < N^*. \tag{4.9}$$

Denote by $\bar{s}_2$ the value of $\bar{s}$ at the point Ξ_0^2. It follows from (3.5) that

$$v_1 + e^{\bar{s}_2}(\eta_0^1 - v_1) + \frac{v_{\delta_1}^0 - v_1}{\rho_1[U_1, U_{\delta_1}^0, \Xi_0^1]}(m(\bar{s}_2) - m_0^1 e^{\bar{s}_2}) = \frac{\sqrt{\rho_1}v_1 + \sqrt{\rho_4}v_4}{\rho_1+\rho_4}.$$

Since $\eta_0^1 - v_1 > 0, v_{\delta_1}^0 - v_1 > 0, m(\bar{s}_2) - m_0^1 e^{\bar{s}_2} > 0, [U_1, U_{\delta_1}^0, \Xi_0^1] > 0$, we have

$$0 < e^{\bar{s}_2} < \frac{\sqrt{\rho_4}(v_4-v_1)}{(\rho_1+\rho_4)(\eta_0^1-v_1)} < \frac{\sqrt{\rho_4}(v_4-v_1)}{(\rho_1+\rho_4)(v_2-v_1)} \le p_* < 1. \tag{4.10}$$

Combining this fact with (4.8) and (4.9), we deduce

$$\begin{aligned}
m_2^* =& \Big((m_0^1)^2 e^{2\bar{s}} + 2\rho_1 m_0^1 [U_1, U_{\delta_1}^0, \Xi_0^1](e^{\bar{s}} - e^{2\bar{s}})\Big)^{1/2} \\
\le& \Big((M^* + m_*)^2 p_*^2 + 2\rho_1(M^* + m_*)S_0\Big)^{1/2} < M^*, \\
n_2^* =& n_0^1 e^{\bar{s}_2} + \frac{||U_{\delta_1}^0||^2}{2} m_0^1 e^{\bar{s}_2} + \Big(\frac{||U_1||^2}{2} + \frac{H_1}{\rho_1}\Big)(m_2^* - m_0^1 e^{\bar{s}_2}) - \frac{||U_{\delta_2}^*||^2}{2} m_2^* \\
\le& (N^* + n_*)p_* + L_0p_*(M^* + m_*)/2 + q_*M^* < N^*,
\end{aligned}$$

which arrive at the desired claim.

We proceed to solve initial value problem (2.27) with initial data

$$\bar{s} = 0, \begin{cases} \Xi(0) = \Xi_0^2,\ U_\delta(0) = U_{\delta_2}^0 = \frac{m_{41}U_\delta^{41} + m_2^* U_{\delta_2}^*}{m_{41}+m_2^*}, \quad m(0) = m_0^2 = m_{41} + m_2^*, \\ n(0) = n_0^2 = n_{41} + n_2^*, (\rho_-, U_-, H_-) = (\rho_4, U_4, H_4), \\ (\rho_+, U_+, H_+) = (0, U(\xi, \eta), 0),\ [U_-, U_\delta^0, \Xi(0)] = [U_4, U_{\delta_2}^0, \Xi_0^2] > 0, \end{cases}$$

and obtain a delta shock wave $\delta_2^{\Xi_0^2}$ which intersects with δ_{34} at some point Ξ_0^3. We don't stop this process until the delta shock wave $\delta_4^{\Xi_0^4}$ intersects with δ_{12} at a point $\Xi_0^5 \in \overline{AE}$, where $U_{\delta_5}^* \in \triangle AE\Xi_2, m_5^* \in [0, M^*], n_5^* \in [0, N^*]$.

Let $K = K_1 \times K_2 \times K_3 \times K_4$, where $K_1 = \overline{AE}, K_2 = \triangle AE\Xi_2, K_3 = [0, M^*], K_4 = [0, N^*]$. An operator $T : K \to K$ is defined as

$$T(\Xi_0^1, U_{\delta_1}^*, m_1^*, n_1^*) = (\Xi_0^5, U_{\delta_5}^*, m_5^*, n_5^*). \tag{4.11}$$

We claim that there exists a fixed point $(\Xi_*, U_\delta^*, m^*, n^*) \in K$ such that $T(\Xi_*, U_\delta^*, m^*, n^*) = (\Xi_*, U_\delta^*, m^*, n^*)$. In fact, choose $\mathbb{R}^6$ as the Banach space B with the usual metric topology. K is a non-empty convex closed set in B. According to the property of the dependence of solutions of ordinary differential equations on its initial data continuously, we can deduce that the operator T is continuous. Since TK is bounded, TK is a precompact set in $\mathbb{R}^6$. Thus, by Theorem 4.1, there exists a fixed point $(\Xi_*, U_\delta^*, m^*, n^*) \in K$ such that $T(\Xi_*, U_\delta^*, m^*, n^*) = (\Xi_*, U_\delta^*, m^*, n^*)$.

Therefore, the global solution is constructed, as illustrated in Fig. 4.3.

We observe from Fig. 4.3 that, each of the $\delta_{12}, \delta_{23}, \delta_{34}$ and δ_{41} splits somewhere into two new delta shock waves before they reach their own singular points or end-points. While, there exists a triple-wave point

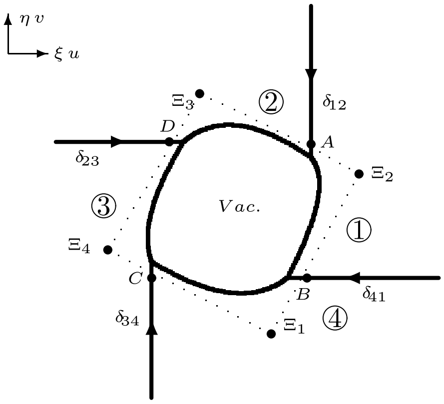

Fig. 4.3. The solution for Case 4.2(i).

where three delta shock waves match together. This structure is called the Mach-reflection-like pattern [21], which is similar to Mach reflection configuration in gas dynamics. Meanwhile, the mechanism for the formation of this configuration results from the global interactions of two-dimensional delta shock waves.

(ii) $u_1 < u_2, u_4 < u_3, v_1 < v_4, v_2 < v_3, [U_1, U_3, U_2] \geq 0$ and $[U_1, U_3, U_4] \leq 0$. The δ_{12} and δ_{23} collide at some point A. The initial data at this point can be shown as follows,

$$\bar{s} = 0, \begin{cases} \Xi(0) = A = (\frac{\sqrt{\rho_1}u_1 + \sqrt{\rho_2}u_2}{\rho_1+\rho_2}, \frac{\sqrt{\rho_2}v_2 + \sqrt{\rho_3}v_3}{\rho_2+\rho_3}),\ U_\delta(0) = \frac{m_{12}U_\delta^{12} + m_{23}U_\delta^{23}}{m_{12}+m_{23}}, \\ m(0) = m_{12} + m_{23},\ n(0) = n_{12} + n_{23}, \\ (\rho_-, U_-, H_-) = (\rho_1, U_1, H_1), (\rho_+, U_+, H_+) = (\rho_3, U_3, H_3). \end{cases} \tag{4.12}$$

The $[U_1, U_3, U_2] \geq 0$ implies that $[U_1, U_\delta(0), A] > 0, [U_3, U_\delta(0), A] < 0, [U_1, U_3, U_\delta(0)] > 0$. Using the result given in Subsection 3.3, we solve initial value problem (2.27) and (4.12), and obtain a delta shock wave δ_{13}^A. Without loss of generality, we only consider the case that the trajectory of δ_{13}^A protrudes to $\overline{\Xi_1\Xi_3}$. This delta shock wave stops at the singular point U_δ^{13}.

Meanwhile, the δ_{34} intersects with the δ_{41} at some point B. The initial data at this point are given as follows,

$$\bar{s} = 0, \begin{cases} \Xi(0) = B = (\frac{\sqrt{\rho_3}u_3 + \sqrt{\rho_4}u_4}{\rho_3+\rho_4}, \frac{\sqrt{\rho_4}v_4 + \sqrt{\rho_1}v_1}{\rho_4+\rho_1}),\ U_\delta(0) = \frac{m_{34}U_\delta^{34} + m_{41}U_\delta^{41}}{m_{34}+m_{41}}, \\ m(0) = m_{34} + m_{41},\ n(0) = n_{34} + n_{41}, \\ (\rho_-, U_-, H_-) = (\rho_3, U_3, H_3), (\rho_+, U_+, H_+) = (\rho_1, U_1, H_1). \end{cases} \tag{4.13}$$

It shows by $[U_1, U_3, U_4] \leq 0$ that, $[U_3, U_\delta(0), B] > 0, [U_1, U_\delta(0), B] < 0, [U_3, U_1, U_\delta(0)] > 0$. We solve initial value problem (2.27) and (4.13), and obtain a delta shock wave δ_{13}^B. This delta shock wave also stops at the singular point U_δ^{13}. See Fig. 4.4.

(iii) $u_4 < u_1 < u_2 < u_3, v_1 < v_4, v_2 < v_3, [U_1, U_3, U_2] \geq 0$ and $[U_1, U_3, U_4] \geq 0$. The δ_{12} overtakes δ_{23} at the point A. The initial data at this point are (4.12). The $[U_1, U_3, U_2] \geq 0$ yields that $[U_1, U_\delta(0), A] > 0$, $[U_3, U_\delta(0), A] < 0, [U_1, U_3, U_\delta(0)] > 0$. Solving initial value problem (2.27) and (4.12), we obtain a delta shock wave δ_{13}^A.

When $v_\delta^{41} < v_\delta^{13}$ and $u_\delta^{34} < u_1$. The $\delta_{13}^A, \delta_{34}$ and δ_{41} do not meet, and stop at their own end-points. Based on the result in Case (i) of this subsection, we can see that the Mach-reflection-like pattern appears in the solution. Specially, we take a point C on δ_{13}^A satisfying $[\Xi_3, \Xi_4, C] > 0$, such that a new delta shock wave δ_1^C emits from this point, which separates the state ① from the vacuum. This δ_1^C overtakes δ_{41} at some point

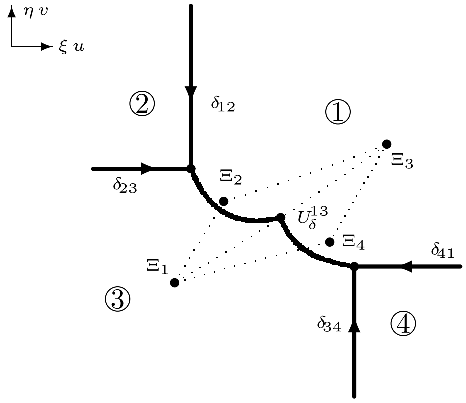

Fig. 4.4. The solution for Case 4.2(ii).

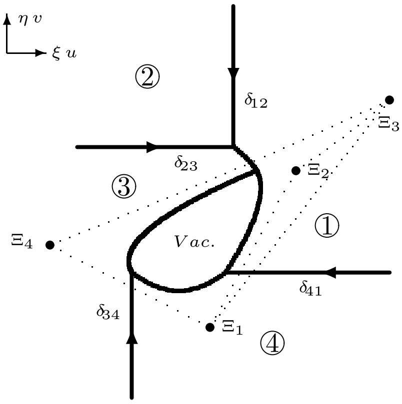

Fig. 4.5. The solution for Case 4.2(iii)a.

D and produces a new delta shock wave δ_4^D connecting the state ④ and the vacuum. Subsequently, the δ_4^D intersects with δ_{34} at some point E, and generates a new delta shock wave δ_3^E which separates the state ④ from the vacuum. This δ_3^E finally matches with δ_{13}^A at the point C. See Fig. 4.5.

When $v_\delta^{41} < v_\delta^{13}$ and $u_\delta^{34} > u_1$. The δ_{41} and δ_{34} interact at some point B, and the initial data at this point are shown in (4.13). Since $[U_1, U_3, U_2] \geq 0, [U_1, U_3, U_4] \geq 0$, it shows that $[U_3, U_\delta(0), B] > 0$, $[U_1, U_\delta(0), B] < 0, [U_3, U_1, U_\delta(0)] < 0$. By means of the result shown in Subsection 3.4, we solve initial value problem (2.27) and (4.13), and obtain a delta shock wave δ_{13}^B. Without loss of generality, we only consider the case that the trajectory of δ_{13}^B protrudes to $\overline{\Xi_1\Xi_4}$. According to Lemma 3.13, we know that the entropy condition of δ_{13}^B is violated before it reaches the singular point U_δ^{13}. As δ_{13}^B meets with δ_{13}^A before its entropy condition fails, if we solve initial value problem at the intersection point locally, then no solution exists for this initial value problem. Based on the result in Case (i) of this subsection, it shows that the Mach-reflection-like pattern takes place in the solution. More precisely, let C be a point on δ_{13}^B satisfying $[\Xi_1, \Xi_3, C] < 0$, such that a new delta shock wave δ_3^C emits from this point connecting the state ③ with the vacuum. The δ_3^C overtakes δ_{13}^A at some point D, and generates another new delta shock wave δ_1^D connecting the state ① and the vacuum. This δ_1^D finally matches with δ_{13}^B at the point C. See Fig. 4.6.

As the entropy condition of δ_{13}^B fails before it intersects with δ_{13}^A, this situation can be treated in a similar way as the last case. The solution is shown in Fig. 4.6.

Therefore, this case has two types of solutions.

It is shown from Fig. 4.5 that the local interaction of δ_{12} and δ_{23} generates a new δ_{13}^A and then the global interactions among $\delta_{13}^A, \delta_{34}, \delta_{41}$ result in the Mach-reflection-like pattern. Besides, it is also observed

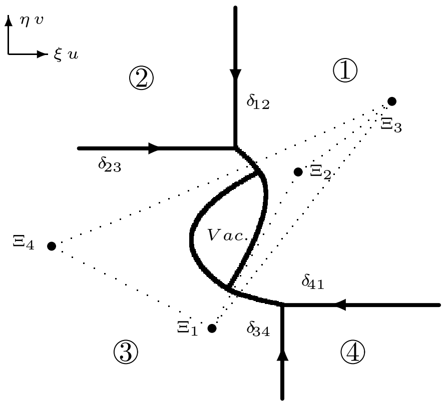

Fig. 4.6. The solution for Case 4.2(iii)b.

from Fig. 4.6 that the local interactions of δ_{12} and δ_{23}, as well as the δ_{34} and δ_{41} generate the $\delta_{13}^A, \delta_{13}^B$ and then the global interactions of $\delta_{13}^A, \delta_{13}^B$ bring about the Mach-reflection-like pattern. The mechanism for the formation of this configuration results from the local and global interactions of two-dimensional delta shock waves.

4.3. Three two-dimensional delta shock waves and one two-dimensional contact discontinuity

Without loss of generality, suppose that the initial continuity connecting the states ① and ② projects a contact discontinuity J_{12}. By (2.7) and (2.22), the initial data satisfy $u_1 = u_2, v_1 < v_4, u_4 < u_3, v_2 < v_3$. We consider the following three cases.

(i) $[U_1, U_2, U_3] > 0, [U_1, U_2, U_4] > 0$ and $v_1 < v_\delta^{41} < v_2$. Denote by $A = (u_2, v_\delta^{41})$ the intersection point of δ_{41} and J_{12}. Since all the pseudo-characteristic lines from the state ② stop at the point Ξ_2, the $v_1 < v_\delta^{41} < v_2$ shows that the state on the right of the point A is the vacuum. By solving initial value problem (2.27) with initial data

$$\bar{s} = 0, \begin{cases} \Xi(0) = A = (u_2, v_\delta^{41}),\ U_\delta(0) = U_\delta^{41}, m(0) = m_{41}, n(0) = n_{41}, \\ (\rho_-, U_-, H_-) = (\rho_4, U_4, H_4),\ (\rho_+, U_+, H_+) = (0, U(\xi, \eta), 0), \end{cases} \tag{4.14}$$

we obtain a delta shock wave δ_4^A.

When δ_{34} meets with δ_4^A earlier than it meets with δ_{23}. The δ_{34} collides with δ_4^A at some point B, and forms a new delta shock wave δ_3^B which connects the state ③ with the vacuum. Then this δ_3^B overtakes δ_{23} at some point C, and generates another new delta shock wave δ_2^C which connects the state ② with the vacuum. The δ_2^C finally stops at the point Ξ_2, as illustrated in Fig. 4.7.

When δ_{34} meets with δ_{23} earlier than it meets with δ_4^A. The δ_{34} overtakes δ_{23} at some point B. Solving the initial value problem (2.27) with initial data,

$$\bar{s} = 0, \begin{cases} \Xi(0) = B = (u_\delta^{34}, v_\delta^{23}),\ U_\delta(0) = \frac{m_{23}U_\delta^{23} + m_{34}U_\delta^{34}}{m_{23} + m_{34}}, \\ m(0) = m_{23} + m_{34},\ n(0) = n_{23} + n_{34}, \\ (\rho_-, U_-, H_-) = (\rho_2, U_2, H_2),\ (\rho_+, U_+, H_+) = (\rho_4, U_4, H_4), \end{cases} \tag{4.15}$$

we obtain a delta shock wave δ_{24}^B using the results in Subsection 3.3 if $[U_2, U_4, U_\delta(0)] > 0$, or the results in Subsection 3.4 if $[U_2, U_4, U_\delta(0)] < 0$. The interactions of δ_{24}^B and δ_4^A at some point C form a new delta shock wave δ_2^C which connects the state ② with the vacuum. This δ_2^C finally vanishes at the point Ξ_2. See Fig. 4.8.

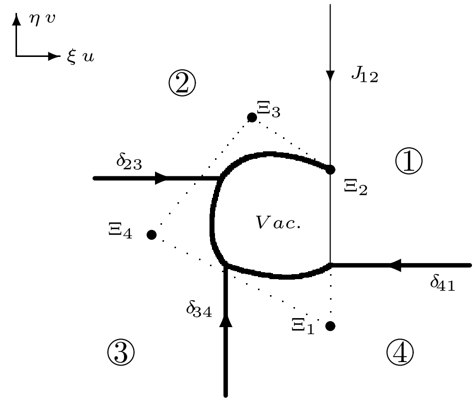

Fig. 4.7. The solution for Case 4.3(i)a.

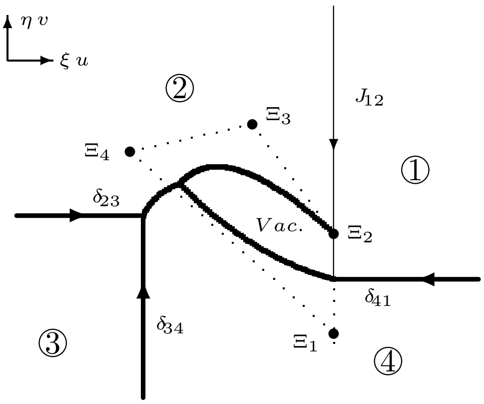

Fig. 4.8. The solution for Case 4.3(i)b.

Therefore, the solutions show two exact structures.

(ii) $[U_1, U_2, U_3] > 0, [U_1, U_2, U_4] > 0$ and $v_1 < v_2 < v_\delta^{41}$. The J_{12} and δ_{41} interact at some point A, where the initial data are given as follows,

$$\bar{s} = 0, \begin{cases} \Xi(0) = A = (u_1, v_\delta^{41}), \quad U_\delta(0) = U_\delta^{41}, m(0) = m_{41}, n(0) = n_{41}, \\ (\rho_-, U_-, H_-) = (\rho_4, U_4, H_4), \ (\rho_+, U_+, H_+) = (\rho_2, U_2, H_2). \end{cases} \tag{4.16}$$

The $u_4 < u_1 = u_2, v_1 < u_2 < v_\delta^{41}$ lead to $[U_4, U_2, U_\delta(0)] < 0$. We then solve initial value problem (2.27) and (4.16) to obtain a delta shock wave δ_{24}^A. By Lemma 3.13, we can see that the entropy condition of δ_{24}^A is violated before it reaches the singular point U_δ^{23}.

The interactions among δ_{24}^A, δ_{34} and δ_{23} are the same as those of $\delta_{13}^A, \delta_{41}$ and δ_{34} in Case (iii) of Subsection 4.2. We only depict the solutions in Figs. 4.9 and 4.10.

(iii) $[U_1, U_2, U_3] < 0, [U_1, U_2, U_4] < 0$ and $v_1 < v_2$. The J_{12} meets with δ_{23} at some point A, where the initial data are given below,

$$\bar{s} = 0, \begin{cases} \Xi(0) = A = (u_1, v_\delta^{23}), \quad U_\delta(0) = U_\delta^{23}, m(0) = m_{23}, n(0) = n_{23}, \\ (\rho_-, U_-, H_-) = (\rho_1, U_1, H_1), \ (\rho_+, U_+, H_+) = (\rho_3, U_3, H_3). \end{cases} \tag{4.17}$$

The $[U_1, U_2, U_3] < 0, u_1 < u_2$ and $v_1 < v_2$ yield $[U_1, U_3, U_\delta(0)] > 0$. We solve initial value problem (2.27) and (4.17) to obtain a delta shock wave δ_{13}^A.

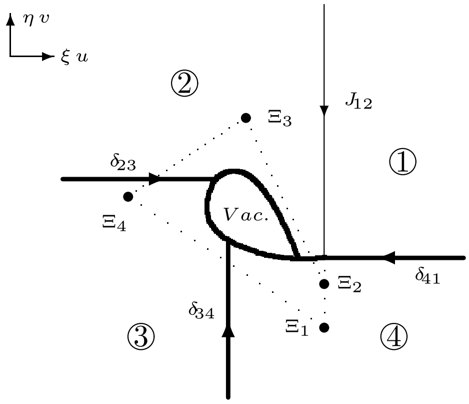

Fig. 4.9. The solution for Case 4.3(ii)a.

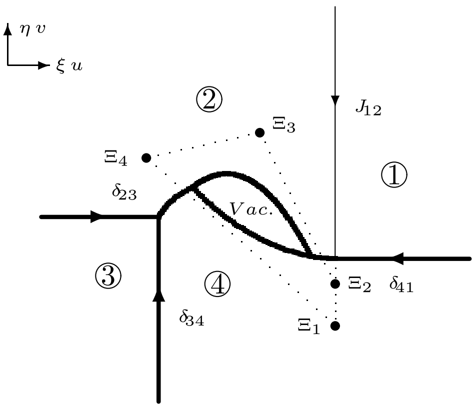

Fig. 4.10. The solution for Case 4.3(ii)b.

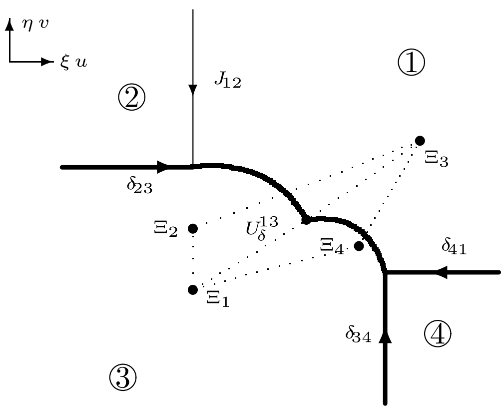

Fig. 4.11. The solution for Case 4.3(iii).

Similarly, the δ_{34} overtakes δ_{41} at some point B, where the initial data are given in (4.13). There are two subcases: $[U_1, U_3, U_4] > 0, [U_1, U_3, U_4] < 0$. For the former, the structure of the solution is similar to Fig. 4.10. For the latter, it shows that $[U_3, U_1, U_\delta^0] > 0$. We solve initial value problem (2.27) and (4.13) to obtain a new delta shock wave δ_{13}^B, which matches with δ_{13}^A at the singular point U_δ^{13}. See Fig. 4.11.

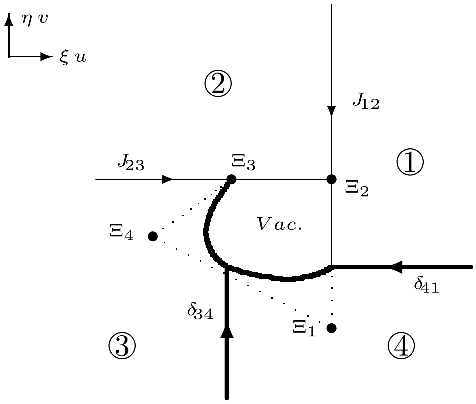

Fig. 4.12. The solution for Case 4.4(1)(i)a.

4.4. Two two-dimensional delta shock waves and two two-dimensional contact discontinuities

We classify this situation into the following two cases, then construct the solution for each case.

(1) Two delta shock waves are neighboring. Without loss of generality, two delta shock waves δ_{34}, δ_{41} connect the states ③ and ④, as well as the states ④ and ①, respectively. According to (2.7) and (2.22), we have $u_1 = u_2, u_4 < u_3, v_2 = v_3, v_1 < v_4$. This case is divided into the following three subcases.

(i) $u_4 < u_3 < u_1 = u_2, v_1 < v_4, v_1 < v_\delta^{41} < v_2 = v_3$. Let $A = (u_2, v_\delta^{41})$ be the intersection point of δ_{41} and J_{12}. In virtue of $v_1 < v_\delta^{41} < v_2$, the state on the right of the point A is the vacuum. We solve initial value problem (2.27) and (4.14), and obtain a delta shock wave δ_4^A.

As δ_{34} meets with δ_4^A earlier than it meets with J_{23}. The δ_{34} collides with δ_4^A at some point C, and forms a new delta shock wave δ_3^C. It connects the state ③ with the vacuum, which finally stops at the point Ξ_3. Meanwhile, the J_{23} stops at the singular point Ξ_2. See Fig. 4.12.

As δ_{34} interacts with J_{23} earlier than it interacts with δ_4^A, which implies $v_2 = v_3 < v_4$. Denote by B the intersection point of δ_{34} and J_{23}, where the initial data are presented as follows,

$$\bar{s} = 0, \begin{cases} \Xi(0) = B = (u_\delta^{34}, v_2,), \quad U_\delta(0) = U_\delta^{34}, m(0) = m_{34}, n(0) = n_{34}, \\ (\rho_-, U_-, H_-) = (\rho_2, U_2, H_2), \ (\rho_+, U_+, H_+) = (\rho_4, U_4, H_4). \end{cases} \tag{4.18}$$

Besides, it has $[U_2, U_4, U_\delta(0)] > 0$ by $u_4 < u_3 < u_2, v_2 = v_3 < v_4$. Solving initial value problem (2.27) and (4.18), we obtain a delta shock wave δ_{24}^B.

The δ_{24}^B meets with δ_4^A at some point C, and forms a new delta shock wave δ_2^C. This δ_2^C connects the state ② with the vacuum, which finally stops at the point Ξ_2. See Fig. 4.13.

Therefore, there exist two exact solutions.

(ii) $u_4 < u_3 < u_1 = u_2, v_2 = v_3, v_1 < v_4, v_1 < v_2 < v_\delta^{41}$. The J_{12} intersects with δ_{41} at some point A, where the initial data are given in (4.16). Besides, it holds $[U_4, U_2, U_\delta^0] < 0$ due to $u_4 < u_1 = u_2, v_1 < v_2 < v_4$. So, we solve initial value problem (2.27) and (4.16), and then obtain a delta shock wave δ_{24}^A. It is noticed that the entropy condition of δ_{24}^A is violated before it reaches the singular point U_δ^{24}.

Denote by B the intersection point of J_{23} and δ_{34}, where the initial data are given by (4.18). From $v_2 = v_3 < v_4, u_4 < u_3 < u_2$, one has $[U_2, U_4, U_\delta^0] > 0$. Solving initial value problem (2.27) and (4.18), we obtain a delta shock wave δ_{24}^B.

The interactions of δ_{24}^A and δ_{24}^B are the same as those of δ_{13}^A and δ_{13}^B in Case (iii) of Subsection 4.2. We only describe the solution in Fig. 4.14.

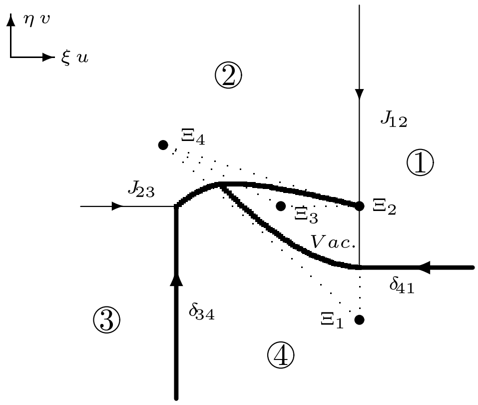

Fig. 4.13. The solution for Case 4.4(1)(i)b.

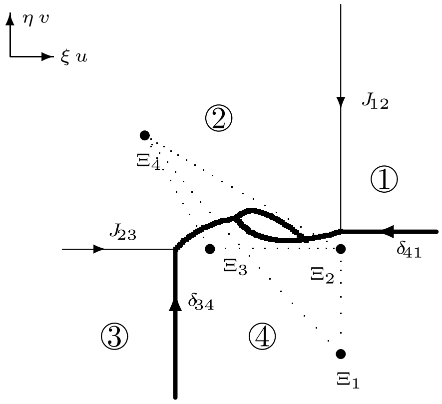

Fig. 4.14. The solution for Case 4.4(1)(ii).

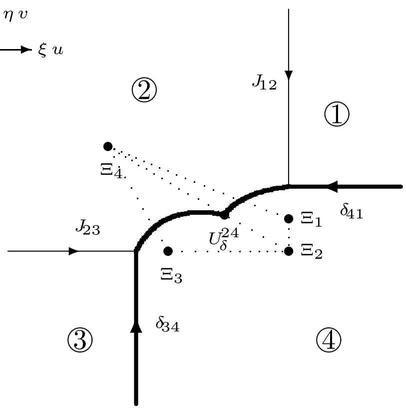

Fig. 4.15. The solution for Case 4.4(1)(iii).

(iii) $u_4 < u_3 < u_1 = u_2, v_3 = v_2 < v_1 < v_4$. In this case, the δ_{41} overtakes J_{12} at some point A. The initial data at this point are (4.16). By $u_4 < u_1 = u_2, v_2 < v_1 < v_4$, it shows $[U_4, U_2, U_\delta^0] > 0$. We solve initial value problem (2.27) and (4.16) to obtain a delta shock wave δ_{24}^A. Symmetrically, the δ_{34} and J_{23} interact at some point B. The $u_4 < u_3 < u_2, v_2 = v_3 < v_4$ lead to $[U_2, U_4, U_\delta^0] > 0$. Solving initial value problem (2.27) and (4.18), we get a delta shock wave δ_{24}^B. This δ_{24}^B finally matches with δ_{24}^A at the singular point U_δ^{24}. The solution is shown in Fig. 4.15.

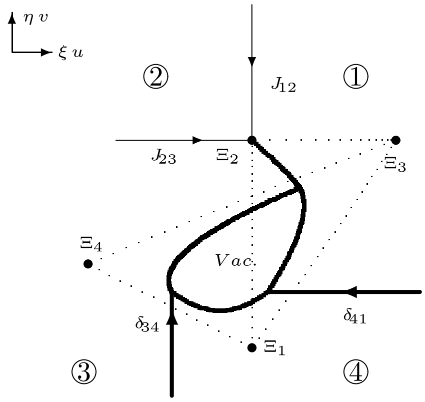

Fig. 4.16. The solution for Case 4.4(1)(iv)a.

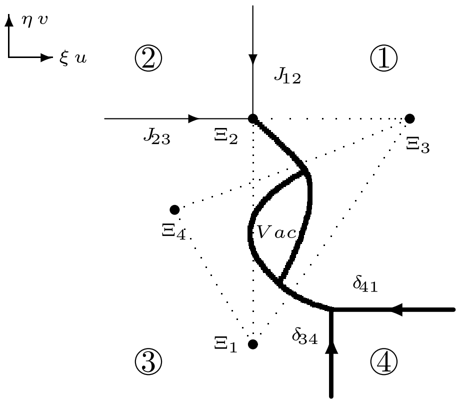

Fig. 4.17. The solution for Case 4.4(1)(iv)b.

(iv) $u_4 < u_1 = u_2 < u_3, v_1 < v_2 = v_3, v_1 < v_4$. The J_{12} meets with J_{23} at the point Ξ_2. The $u_1 = u_2 < u_3$, $v_1 < v_2 = v_3$ yield that $[U_1, U_3, \Xi_2] > 0$. We solve initial value problem (2.27) and (4.1) to obtain a delta shock wave $\delta_{13}^{\Xi_2}$.

The sequent interactions among $\delta_{13}^{\Xi_2}$, δ_{34} and δ_{41} are the same as those of $\delta_{13}^{A}, \delta_{34}$ and δ_{41} in Case (iii) of Subsection 4.2. We only depict the solutions in Figs. 4.16 and 4.17.

(v) $u_1 = u_2 < u_4 < u_3, v_1 < v_2 = v_3, v_1 < v_4$. The J_{12} and J_{23} interact at the point Ξ_2. The $u_1 = u_2 < u_3$, $v_1 < v_2 = v_3$ lead to $[U_1, U_3, \Xi_2] > 0$. Hence, we solve initial value problem (2.27) and (4.1) at this point to obtain a delta shock wave $\delta_{13}^{\Xi_2}$.

Meanwhile, the δ_{34} overtakes δ_{41} at some point B, where the initial data are shown in (4.13). There are two subcases: $[U_1, U_3, U_4] > 0, [U_1, U_3, U_4] < 0$. For the former, the structure of the solution is similar to Fig. 4.17. For the latter, one has $[U_3, U_1, U_\delta^0] > 0$. We solve initial value problem (2.27) and (4.13), and obtain a delta shock wave δ_{13}^B. This δ_{13}^B finally matches with δ_{13}^A at the singular point U_δ^{13}. The solution is illustrated in Fig. 4.18.

(2) Two delta shock waves are not neighboring. Without loss of generality, assume that two delta shock waves δ_{23}, δ_{41} connect the states ② and ③, as well as the states ④ and ① respectively. From (2.7) and (2.22), the initial data satisfy $u_3 = u_4, u_1 = u_2, v_2 < v_3, v_1 < u_4$. We only consider the subcase $u_3 = u_4 < u_1 = u_2, v_1 < v_\delta^{41} < v_2 < v_3, v_1 < v_4 < v_\delta^{23} < v_3$, because the solutions to the other subcases are similar to those in Cases (1)(i)–(iii) of this subsection.

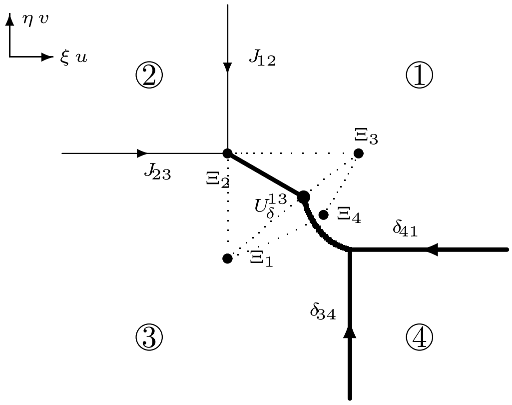

Fig. 4.18. The solution for Case 4.4(1)(v).

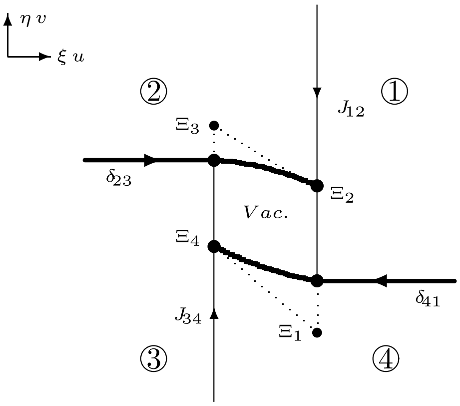

Fig. 4.19. The solution for Case 4.4(2).

In this subcase, the δ_{41} and J_{12} interact at some point A. Since $v_1 < v_\delta^{41} < v_2$, we solve initial value problem (2.27) and (4.14), and obtain a delta shock wave δ_4^A. This delta shock wave vanishes at the point Ξ_4.

Meanwhile, the J_{34} meets with δ_{23} at some point B. In virtue of $v_4 < v_\delta^{23} < v_3$, the initial data at this point are as follows,

$$\bar{s} = 0, \begin{cases} \Xi(0) = B = (u_3, v_\delta^{23}),\ U_\delta(0) = U_\delta^{23}, m(0) = m_{23}, n(0) = n_{23}, \\ (\rho_-, U_-, H_-) = (\rho_2, U_2, H_2),\ (\rho_+, U_+, H_+) = (0, U(\xi, \eta), 0). \end{cases} \tag{4.19}$$

We solve initial value problem (2.27) and (4.19) to get a delta shock wave δ_2^B. This delta shock wave stops finally at the point Ξ_2. The solution is depicted in Fig. 4.19.

4.5. One two-dimensional delta shock wave and three two-dimensional contact discontinuities

Without loss of generality, assume that a delta shock waves δ_{12} connects the states ① and ②. We have can see from (2.7) and (2.22) that, $u_1 < u_2, v_1 = v_4, u_3 = u_4, v_2 = v_3$. The following two cases are discussed.

(i) $u_3 = u_4 < u_1 < u_2, v_4 = v_1 < v_2 = v_3$. The δ_{12} overtakes J_{23} at some point $A = (u_\delta^{12}, v_2)$. The initial data at this point are

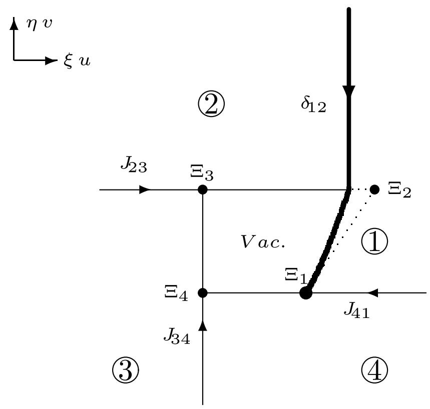

Fig. 4.20. The solution for Case 4.5(i).

$$\bar{s} = 0, \begin{cases} \Xi(0) = A = (u_\delta^{12}, v_2),\ U_\delta(0) = U_\delta^{12}, m(0) = m_{12}, n(0) = n_{12}, \\ (\rho_-, U_-, H_-) = (\rho_1, U_1, H_1),\ (\rho_+, U_+, H_+) = (0, U(\xi,\eta), 0), \end{cases} \tag{4.20}$$

due to $u_3 < u_1 < u_\delta^{12} < u_2$. Solving initial value problem (2.27) and (4.20), we get a new delta shock wave δ_1^A. This delta shock wave vanishes at the point Ξ_1.

Meanwhile, the J_{34} and J_{41} also stop finally at their respective singular points Ξ_3 and Ξ_4. The solution is given in Fig. 4.20.

(ii) $u_1 < u_2, u_1 < u_3 = u_4, v_4 = v_1 < v_2 = v_3$. The J_{34} and J_{41} meet at the point Ξ_4. The $u_1 < u_3 = u_4$, $v_4 = v_1 < v_2 = v_3$ yield $[U_3, U_1, \Xi_4] > 0$. We solve initial value problem (2.27) and (4.2) to obtain a delta shock wave $\delta_{13}^{\Xi_4}$.

Meanwhile, δ_{12} interacts with J_{23} at some point B. When $u_1 < u_\delta^{12} < u_3 < u_2$, the initial data at this point are

$$\bar{s} = 0, \begin{cases} \Xi(0) = B = (u_\delta^{12}, v_2),\ U_\delta(0) = U_\delta^{12}, m(0) = m_{12},\ n(0) = n_{12}, \\ (\rho_-, U_-, H_-) = (\rho_1, U_1, H_1), (\rho_+, U_+, H_+) = (\rho_3, U_3, H_3). \end{cases} \tag{4.21}$$

Besides, the $u_1 < u_\delta^{12} < u_3 < u_2, v_1 < v_2 = v_3$ lead to $[U_1, U_3, U_\delta(0)] < 0$. Solving initial value problem (2.27) and (4.21), we obtain a delta shock wave δ_{13}^B. It is noticed that the entropy condition of δ_{13}^B is violated before it reaches the singular point U_δ^{13}. The interactions of δ_{13}^B and $\delta_{13}^{\Xi_4}$ are the same as those of δ_{13}^B and δ_{13}^A in Case (iii) of Subsection 4.2. We only picture the solution in Fig. 4.21.

When $u_1 < u_3 < u_\delta^{12} < u_2$, the initial data at the point B are

$$\bar{s} = 0, \begin{cases} \Xi(0) = B = (u_\delta^{12}, v_2),\ U_\delta(0) = U_\delta^{12}, m(0) = m_{12}, n(0) = n_{12}, \\ (\rho_-, U_-, H_-) = (\rho_1, U_1, H_1),\ (\rho_+, U_+, H_+) = (0, U(\xi,\eta), 0). \end{cases} \tag{4.22}$$

We solve initial value problem (2.27) and (4.22) to obtain a new delta shock wave δ_1^B. The δ_1^B and $\delta_{13}^{\Xi_4}$ collide at some point C, and form a new delta shock wave δ_3^C. This δ_3^C connects the state ③ and the vacuum, and finally vanishes at the point Ξ_3. See Fig. 4.22.

When $u_1 < u_\delta^{12} < u_2 < u_3$, the initial data are shown in (4.21). Besides, the $u_1 < u_\delta^{12} < u_2 < u_3, v_1 < v_2 = v_3$ yield $[U_1, U_3, U_\delta(0)] > 0$. Hence, by solving initial value problem (2.27) and (4.21), we obtain a delta shock wave δ_{13}^B. The δ_{13}^B matches finally with $\delta_{13}^{\Xi_4}$ at the singular point U_δ^{13}, as shown in Fig. 4.23.

Therefore, this case contains three exact solutions.

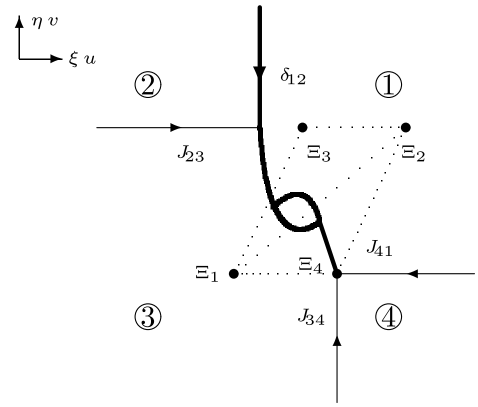

Fig. 4.21. The solution for Case 4.5(ii)a.

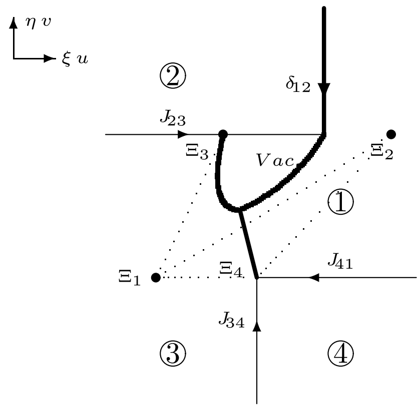

Fig. 4.22. The solution for Case 4.5(ii)b.

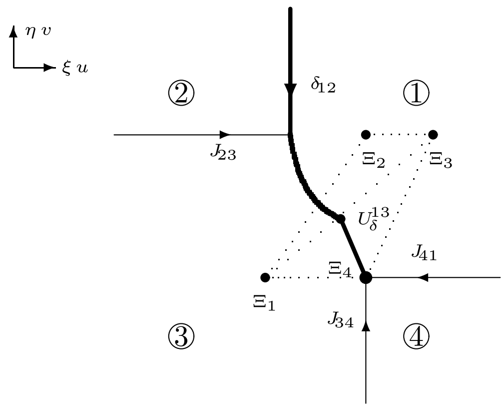

Fig. 4.23. The solution for Case 4.5(ii)c.

5. Conclusion

We analyzed completely the exact solutions to the two-dimensional Riemann problem (1.3) and (1.4). Under the assumption (H), twenty-three explicit solutions and their corresponding criteria are obtained. The occurrence of two-dimensional delta shock wave with a Dirac delta function in both density and internal energy is confirmed rigorously. The mechanism for the formation of this kind of two-dimensional delta shock wave results from the overlapping of the linearly degenerate characteristic lines. It is also shown that the Mach-reflection-like patterns appear in solutions. Precisely, in some solutions, one two-dimensional delta shock wave splits somewhere into two new two-dimensional delta shock waves, that is, there exists a triple-wave point where three two-dimensional delta shock waves match together. The mechanism for the formation of this pattern results from the global (or local and global) interactions of this type of two-dimensional delta shock waves, or the interactions of this type of two-dimensional delta shock waves with the two-dimensional contact discontinuities. To the best of our knowledge, this type of mechanism for the formation of Mach-reflection-like pattern has not been found in the previous studies.

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