



Asymptotic behavior of global solutions of aerotaxis equations

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ABSTRACT

We study asymptotic behavior of global solutions of one-dimensional aerotaxis model proposed in Knosalla and Nadzieja (2015) [9].

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1. Introduction and main results

Aerotaxis is the movement of bacteria toward the optimal concentration of oxygen for their growth [4], [8], [9], [11], [13]. In [9] we introduced a mathematical model of the aerotaxis, which is a modification of the models proposed in [14], [16].

We denote by $u(x, t)$ the density of the colony of bacteria living in a capillary of the unit length and by $p(x, t)$ the density of oxygen, $x \in [0, 1]$, $t > 0$.

The evolution of $u(x, t)$ is given by the following one-dimensional drift-diffusion equation

$$u_t = u_{xx} - (u(E(p))_x)_x. \quad (1)$$

Here $E(p)$ is the *energy function*, which has a single maximum at a point p^* of the optimal concentration of oxygen.

We make the following assumptions on the function E :

1. E is a non-negative C^3 function on \mathbb{R} with the bounded first derivative,
2. $E(p) \equiv 0$ for $p \leq 0$ and $E(p) > 0$ for $p > 0$,
3. E has only one local maximum at a point p^* ,
4. $\lim_{p \rightarrow \infty} E(p) = 0$.

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The diffusion term u_{xx} is responsible for random walk of bacteria, and the drift term $(u(E(p))_x)_x$ – for their tendency to achieve the optimal concentration of oxygen. Assumptions on the function E and the form of drift term describe the impact of oxygen concentration on bacteria movement. Namely, bacteria escape from regions with too high (too low, respectively) concentrations of oxygen, and move to the optimal one.

Oxygen diffuses in water filling the capillary, and is consumed by bacteria at the rate proportional to the density of bacteria and the value of energy function at a given density of oxygen. Thus, the evolution of oxygen density $p(x, t)$ is described by the following equation

$$p_t = p_{xx} - E(p)u. \quad (2)$$

The density $u(x, t)$ is assumed to satisfy the no-flux boundary condition, i.e.

$$u_x(0, t) - u(0, t)(E(p))_x(0, t) = u_x(1, t) - u(1, t)(E(p))_x(1, t) = 0, \quad (3)$$

which guarantees that total mass of bacteria is conserved. The left end of the capillary is closed. Hence at $x = 0$ we impose the no-flux boundary value condition

$$p_x(0, t) = 0. \quad (4)$$

At the point $x = 1$ we may consider two distinct boundary value conditions, either a constant level of the oxygen at the right hand side of the capillary:

$$\text{the Dirichlet condition } p(1, t) = \bar{p}, \quad (5)$$

or a constant flow of the oxygen across the right hand externity of the capillary:

$$\text{the Neumann condition } p_x(1, t) = \bar{p}. \quad (6)$$

Here \bar{p} is a given positive constant.

The equations (1), (2) are supplemented with the initial density of bacteria

$$u(x, 0) = u_0(x), \quad (7)$$

and the initial density of oxygen

$$p(x, 0) = p_0(x). \quad (8)$$

Here $u_0(x)$, $p_0(x)$ are given continuous non-negative functions on the interval $[0, 1]$.

The equations (1), (2) together with boundary data (3)–(5), (6) and initial data (7), (8) describe the evolution of the densities of bacteria and of oxygen.

A proof of global well-posedness can be found in [10]. In this paper we prove some results concerning asymptotic behavior of solutions. Energy methods are the main tool of our analysis. Existence and uniqueness of the steady states of the model were studied in [9].

We denote by (U, P) the stationary solutions of the Dirichlet problem (1)–(5) or the Neumann problem (1)–(4), (6) and by M the total mass of bacteria i.e. $M = \int_0^1 U(x)dx$. In [9] we proved that if M is sufficiently small then the Dirichlet problem has a unique steady state (U, P) . Our main result concerning the Dirichlet problem is the local asymptotic stability of small mass stationary solutions.

Theorem 1.1. *Let (u, p) be a classical solution of (1)–(5) with initial condition (u_0, p_0) . There exist $M_s > 0$, $\varepsilon_s > 0$ such that if $M \in (0, M_s)$ and the initial condition (u_0, p_0) is sufficiently close in $L^2 \times H^1$ norm to the steady state (U, P) , i.e.*

$$|u_0 - U|_2 + |(p_0 - P)_x|_2 \leq \varepsilon_s,$$

then

$$|u(t) - U|_2 + |(p(t) - P)_x|_2 \leq Ce^{-\gamma t}$$

for some positive constants C, γ .

In [9] we showed that the stationary solutions of the Neumann problem (1)–(4), (6) do not exist when $\bar{p} > ME(p^*)$. Second main result of this paper correspond to this situation. Under some additional assumption on *energy function* we prove that when time tends to infinity, p grows to infinity and u behaves like the Neumann heat semigroup, so, in particular, $|u(\cdot, t) - M|_2 \rightarrow 0$.

Theorem 1.2. *Let (u, p) be a classical solution of (1)–(4), (6) with initial condition (u_0, p_0) . Assume that $\lim_{p \rightarrow \infty} E'(p) = 0$ and there exists $p_{\min} > p^*$ such that $E''(p) > 0$ for $p > p_{\min}$. If $\bar{p} > E(p^*)M$ then $\lim_{t \rightarrow \infty} |p(\cdot, t)|_\infty = \infty$ and $\lim_{t \rightarrow \infty} |u(\cdot, t) - M|_2 = 0$.*

Neumann problem (1)–(4), (6) but with $E(p) = p$ and $u_x(x, t)|_{x=0,1} = p_x(x, t)|_{x=0,1} = 0$ was analyzed in [12]. They showed that (u, p) converges to the unique steady state $(M, 0)$ in one, two and under some assumptions in three-dimensional domains.

2. Preliminaries

In this section we collect some facts which will be used in our analysis. In [10] using abstract Amann theory [3] and Moser–Alikakos iteration procedure [1,2] we proved the following theorem.

Theorem 2.1. *Let $(u_0, p_0) \in (W^{1,r}(0, 1))^2$ for some $r > 1$. Then, the Dirichlet problem (1)–(5) (Neumann problem (1)–(4), (6)) have a unique global in time classical solution*

$$(u, p) \in (C(W^{1,r}(0, 1) \times [0, \infty)) \cap C^{2,1}([0, 1] \times (0, \infty)))^2.$$

Moreover $u_0 \geq 0$, $p_0 \geq 0$ implies $u \geq 0$, $p \geq 0$. Total mass of bacteria is conserved in time, i.e.

$$M = \int_0^1 u_0(x) dx = \int_0^1 u(x, t) dx$$

for each $t > 0$.

Remark 1. In this paper we extend the result of the above theorem. In Lemma 4.2 we prove that $|u(t)|_2$ is uniformly bounded in time ([10] implies uniform boundedness of $|u(t)|_\infty$). By the maximum principle we have $p \leq \max\{\bar{p}, |p_0|_\infty\}$ for the Dirichlet problem ((2), (4), (5)). When $\bar{p} > ME(p^*)$ the L^∞ norm of solution of the Neumann problem ((2), (4), (6)) may grow to infinity as time tends to infinity (see Lemma 1.2).

Now we recall Gronwall and Young inequality in the form which we will use in the proofs.

Lemma 2.2 ([6]). Let $y(t)$ be a non-negative, absolutely continuous function on $[0, T]$ which satisfies for a.e. t the differential inequality

$$y'(t) \leq h(t)y(t) + k(t),$$

where $h(t)$ and $k(t)$ are nonnegative, integrable functions on $[0, T]$. Then

$$y(t) \leq \exp \left(\int_0^t h(s) ds \right) \left(y(0) + \int_0^t k(s) ds \right).$$

Lemma 2.3 ([6]). Let $1 < q_1, q_2 < \infty$, $\frac{1}{q_1} + \frac{1}{q_2} = 1$. Then

$$ab \leq \varepsilon a^{q_1} + \frac{b^{q_2}}{(\varepsilon q_1)^{q_2/q_1} q_2} \quad (a, b > 0, \varepsilon > 0).$$

We will also need one-dimensional Gagliardo–Nirenberg interpolation inequality [7, Theorem I.10.1].

Lemma 2.4. Let $\Omega = (0, 1)$, and let $l, q, r \geq 1$. Then for any $f \in W^{1,q}(\Omega) \cap L^r(\Omega)$, there exist positive constants c_1, c_2 , such that

$$|f|_l \leq c_1 |f'|_q^a |f|_r^{1-a} + c_2 |f|_r \quad (9)$$

with $a \in [0, 1]$ satisfying

$$\frac{1}{l} = a \left(\frac{1}{q} - 1 \right) + \frac{1}{r} (1 - a). \quad (10)$$

Lemma 2.5. Let $\Omega = (0, 1)$ and $f \in W^{1,r}(\Omega)$ for some $r \in [1, \infty)$. For any $x, x_0 \in [0, 1]$ we have

$$|f(x) - f(x_0)| \leq |f'|_r. \quad (11)$$

In particular, if $f(x_0) = 0$ for some $x_0 \in [0, 1]$, we have

$$|f(x)| \leq |f'|_r, \quad (12)$$

and inequality (9) reads as follows

$$|f|_l \leq c_3 |f'|_q^a |f|_r^{1-a}, \quad (13)$$

where $c_3 = 2 \max(c_1, c_2)$.

Proof. The Hölder inequality and [5, Theorem 8.2] gives us

$$|f(x) - f(x_0)| \leq \int_{x_0}^x |f'| \leq \int_0^1 |f'| \leq |f'|_r$$

for $r \in [1, \infty)$, thus we have (11). By the assumption we get (12).

From (9) and (12) we obtain

$$\begin{aligned} |f|_l &\leq c_1 |f'|_q^a |f|_r^{1-a} + c_2 |f|_r \leq \max(c_1, c_2) (|f'|_q^a + |f|_r^a) |f|_r^{1-a} \\ &\leq \max(c_1, c_2) (|f'|_q^a + |f|_\infty^a) |f|_r^{1-a} \\ &\leq \max(c_1, c_2) (|f'|_q^a + |f'|_q^a) |f|_r^{1-a} \\ &\leq c_3 |f'|_q^a |f|_r^{1-a}. \quad \square \end{aligned}$$

3. Dirichlet problem

In this section we analyze stability of steady states of the following Dirichlet problem

$$\begin{cases} u_t = (u_x - uE(p))_x, \\ p_t = p_{xx} - E(p)u, \\ (u_x(x, t) - u(x, t)E(p(x, t)))_{|x=0,1} = 0, \\ p_x(0, t) = 0, \quad p(1, t) = \bar{p}, \quad t \geq 0, \\ u(x, 0) = u_0(x), \quad p(x, 0) = p_0(x). \end{cases} \quad (14)$$

By $U(x)$, $P(x)$ we denote the stationary densities of bacteria and oxygen. The stationary problem for (14) reads as follows

$$\begin{cases} 0 = (U_x - UE(P))_x, \\ 0 = P_{xx} - E(P)U, \\ (U_x(x) - U(x)E(P(x)))_{|x=0,1} = 0, \\ P_x(0) = 0, \quad P(1) = \bar{p}. \end{cases} \quad (15)$$

By direct calculations (see [9]) the bacteria density has the form

$$U = M \frac{e^{E(P)}}{\int_0^1 e^{E(P(s))} ds}, \quad (16)$$

where $P(x)$ solves

$$P_{xx} = UE(P), \quad P_x(0) = 0, \quad P(1) = \bar{p}, \quad (17)$$

and $M = \int_0^1 U(x) dx$ is the total mass of bacteria.

From (16), (17) and assumptions (1)–(4) on *energy function* E we obtain estimates

$$\begin{aligned} |U|_\infty &\leq M e^{E(p^*)}, \\ |P_x|_\infty &\leq M E(p^*) \leq M e^{E(p^*)}. \end{aligned}$$

Now we prove main result concerning the local asymptotic stability of small mass solution.

Proof of Theorem 1.1. We consider solution (u, p) of (14) as a perturbation of the steady state, i.e. $(u, p) = (U + u_1, P + p_1)$, where (u_1, p_1) solves the system

$$\begin{cases} u_{1t} = (u_{1x} + U(E(P) - E(p))_x - u_1 E(p)_x)_x, \\ p_{1t} = p_{1xx} + U(E(P) - E(p)) - E(p)u_1, \\ (u_1(x, t)_x + U(x)(E(P(x)) - E(p(x, t))))_{|x=0,1} = 0, \\ p_{1x}(0, t) = 0, \quad p_1(1, t) = 0, \quad t \geq 0, \\ u_1(x, 0) = u_{10}(x) = u_0(x) - U(x), \quad p_1(x, 0) = p_{10}(x) = p_0(x) - P(x). \end{cases} \quad (18)$$

Let us point out that

$$\int_0^1 u_1(x, t) = \int_0^1 (u(x, t) - U(x)) dx = 0.$$

Multiplying both sides of the first equation in (18) by u_1 and integrating over Ω we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_0^1 u_1^2 dx &= \int_0^1 u_1 u_{1t} dx \\ &= \int_0^1 (u_{1x} + U(E(P) - E(p))_x - u_1 E(p)_x) u_1 dx \\ &= (u_{1x} + U(E(P) - E(p))_x - u_1 E(p)_x) u_1 \Big|_0^1 \\ &\quad - \int_0^1 (u_{1x} + U(E(P) - E(p))_x - u_1 E(p)_x) u_{1x} dx \\ &= - \int_0^1 (u_{1x})^2 + \int_0^1 U(E(P) - E(p))_x u_{1x} - u_1 E(p)_x u_{1x} dx. \end{aligned}$$

Thus we arrive at the following inequality

$$\frac{1}{2} \frac{d}{dt} |u_1|_2^2 + |u_{1x}|_2^2 \leq \int_0^1 |U| |(E(P) - E(p))_x| |u_{1x}| + |u_1| |E(p)_x| |u_{1x}| dx. \quad (19)$$

Using assumptions (1)–(4) we obtain the following estimates

$$\begin{aligned} |(E(P) - E(p))_x| &= |(E(P) - E(P + p_1))_x| \\ &= |E'(P)P_x - E'(P + p_1)(P_x + p_{1x})| \\ &= |(E'(P) - E'(P + p_1))P_x - E'(P + p_1)p_{1x}| \\ &\leq |E''|_\infty |P_x|_\infty |p_1| + |E'|_\infty |p_{1x}|. \end{aligned}$$

Applying (12) (due to $p_{1x}(0, t) = p_{1x}(1, t) = 0$) and the Young inequality to the first term of the right hand side of (19) we have

$$\begin{aligned} \int_0^1 |U| |(E(P) - E(p))_x| |u_{1x}| dx &\leq |U|_\infty \int_0^1 (|E''|_\infty |P_x|_\infty |p_1|_\infty + |E'|_\infty |p_{1x}|) |u_{1x}| dx \\ &\leq |U|_\infty \int_0^1 (|E''|_\infty |P_x|_\infty |p_1|_\infty + |E'|_\infty |p_{1x}|)^2 + \frac{1}{4} |u_{1x}|^2 dx \\ &\leq |U|_\infty \int_0^1 2|E''|_\infty^2 |P_x|_\infty^2 |p_1|_\infty^2 + 2|E'|_\infty^2 |p_{1x}|^2 + \frac{1}{4} |u_{1x}|^2 dx \end{aligned}$$

$$\begin{aligned} &\leq M e^{E(p^*)} \left(2(|E''|_\infty^2 |P_x|_\infty^2 + |E'|_\infty^2) |p_{1x}|_2^2 + \frac{1}{4} |u_{1x}|_2^2 \right) \\ &\leq M e^{E(p^*)} (C(M, |E'|_\infty^2, |E''|_\infty^2) |p_{1xx}|_2^2 + \frac{1}{4} |u_{1x}|_2^2). \end{aligned}$$

By the relations

$$|E(p)_x| = |E'(P + p_1)(P_x + p_{1x})| \leq |E'|_\infty (|P_x|_\infty + |p_{1x}|)$$

the second term of the r.h.s of (19) may be estimated in the following way

$$\begin{aligned} \int_0^1 |u_1| |E(p)_x| |u_{1x}| dx &\leq \int_0^1 |u_1|^2 |E(p)_x|^2 dx + \frac{1}{4} |u_{1x}|_2^2 \\ &\leq \int_0^1 |u_1|^2 2 |E'|_\infty^2 (|P_x|_\infty^2 + |p_{1x}|^2) dx + \frac{1}{4} |u_{1x}|_2^2 \\ &\leq \int_0^1 |u_1|^2 2 |E'|_\infty^2 (M^2 e^{2E(p^*)} + |p_{1x}|^2) dx + \frac{1}{4} |u_{1x}|_2^2 \\ &\leq \int_0^1 |u_1|^2 2 |E'|_\infty^2 |p_{1x}|^2 dx + (2 |E'|_\infty^2 M^2 e^{2E(p^*)} + \frac{1}{4}) |u_{1x}|_2^2 \\ &\leq 2 |E'|_\infty^2 |u_1|_4^2 |p_{1x}|_4^2 + (2 |E'|_\infty^2 M^2 e^{2E(p^*)} + \frac{1}{4}) |u_{1x}|_2^2 \\ &\leq |E'|_\infty^2 |u_1|_4^4 + \frac{1}{2} |p_{1x}|_4^4 + (2 |E'|_\infty^2 M^2 e^{2E(p^*)} + \frac{1}{4}) |u_{1x}|_2^2 \\ &\leq \varepsilon_1 |u_{1x}|_2^2 + C(\varepsilon_1, |E'|_\infty) |u_1|_2^6 + \varepsilon_2 |p_{1xx}|_2^2 + C(\varepsilon_2) |p_{1x}|_2^6 \\ &\quad + (2 |E'|_\infty^2 M^2 e^{2E(p^*)} + \frac{1}{4}) |u_{1x}|_2^2. \end{aligned}$$

Here, the last estimate is a consequence of Young inequality and following version of (13)

$$|f|_4^4 \leq c |f'|_2 |f|_2^3.$$

Parameters $\varepsilon_1, \varepsilon_2$ will be chosen appropriately later. Summing up the above estimates, we get from (19)

$$\frac{1}{2} \frac{d}{dt} |u_1|_2^2 + \left(\frac{3}{4} - \max(M, M^2) C_1 - \varepsilon_1 \right) |u_{1x}|_2^2 \leq (M C_2 + \varepsilon_2) |p_{1xx}|_2^2 + C_3 (|u_1|_2^2 + |p_{1x}|_2^2)^3. \quad (20)$$

Now we multiply the second equation of (18) by $-p_{1xx}$ and integrate over Ω to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |p_{1x}|_2^2 + |p_{1xx}|_2^2 &= - \int_0^1 U(E(P) - E(p)) p_{1xx} dx + \int_0^1 E(p) u_1 p_{1xx} dx \\ &\leq |U|_\infty |E'|_\infty \int_0^1 |p_1| |p_{1xx}| dx + E(p^*) \int_0^1 |u_1| |p_{1xx}| dx \\ &\leq |U|_\infty |E'|_\infty |p_1|_\infty |p_{1xx}|_1 + E(p^*) |u_1|_\infty |p_{1xx}|_1 \end{aligned}$$

$$\begin{aligned}
&\leq MC|p_{1xx}|_2^2 + E(p^*)|u_{1x}|_1|p_{1xx}|_1 \\
&\leq MC|p_{1xx}|_2^2 + E(p^*)^2|u_{1x}|_2^2 + \frac{1}{4}|p_{1xx}|_2^2,
\end{aligned}$$

thus the inequality

$$\frac{1}{2} \frac{d}{dt} |p_{1x}|_2^2 + \left(\frac{3}{4} - MC_4 \right) |p_{1xx}|_2^2 \leq E(p^*)^2 |u_{1x}|_2^2. \quad (21)$$

Adding (21) multiplied by $\frac{1}{8E(p^*)^2}$ to (20) we get

$$\frac{1}{2} \frac{d}{dt} \left(|u_1|_2^2 + \frac{1}{8E(p^*)^2} |p_{1x}|_2^2 \right) + \delta_1 |u_{1x}|_2^2 + \delta_2 |p_{1xx}|_2^2 \leq C_3 (|u_1|_2^2 + |p_{1x}|_2^2)^3, \quad (22)$$

with

$$\begin{aligned}
\delta_1 &= \frac{1}{2} - \max(M, M^2)C_1, \\
\delta_2 &= \frac{1}{8E(p^*)^2} \left(\frac{1}{2} - MC_4 \right) - MC_2,
\end{aligned}$$

where we choose $\varepsilon_1 = 1/8$ and $\varepsilon_2 = \frac{1}{32E(p^*)^2}$. The numbers δ_1, δ_2 are positive provided M is sufficiently small. Let $\delta_0 = \min(\delta_1, 8E(p^*)^2\delta_2)$, $C_5 = C_3 \max(1, 8E(p^*)^2)^3$, then from (22)

$$\frac{1}{2} \frac{d}{dt} \left(|u_1|_2^2 + \frac{1}{8E(p^*)^2} |p_{1x}|_2^2 \right) + \delta_0 \left(|u_1|_2^2 + \frac{1}{8E(p^*)^2} |p_{1x}|_2^2 \right) \leq C_5 \left(|u_1|_2^2 + \frac{1}{8E(p^*)^2} |p_{1x}|_2^2 \right)^3. \quad (23)$$

Denoting $y(t) = |u_1|_2^2 + \frac{1}{8E(p^*)^2} |p_{1x}|_2^2$, we see that inequality (23) has the following form

$$y'(t) \leq -2\delta_0 y(t) + 2C_5 y(t)^3.$$

The right hand side of the above inequality is negative for $y \in (0, \sqrt{\frac{\delta_0}{C_5}})$, i.e. $y'(t) \leq 0$, thus taking $y(0) \in (0, \sqrt{\frac{\delta_0}{C_5}})$ we can estimate

$$\begin{aligned}
y'(t) &\leq -2\delta_0 y(t) + 2C_5 y(t)^3 \leq -2\delta_0 y(t) + 2C_5 y(t)y(0)^2 \\
y'(t) &\leq (-2\delta_0 + 2C_5 y(0)^2)y(t) := -\gamma y(t)
\end{aligned}$$

for a positive γ . Now we apply the Gronwall lemma and obtain the desired convergence. \square

4. Neumann problem

In this section we analyze the asymptotic behavior of the following system

$$\begin{cases} u_t = (u_x - uE(p)_x)_x, \\ p_t = p_{xx} - E(p)u, \\ (u_x(x, t) - u(x, t)E(p(x, t))_x)|_{x=0,1} = 0, \\ p_x(0, t) = 0, \quad p_x(1, t) = \bar{p}, \quad t \geq 0, \\ u(x, 0) = u_0(x), \quad p(x, 0) = p_0(x). \end{cases} \quad (24)$$

We begin with proving estimates on $p_x(x, t)$, where $p(x, t)$ is a solution of the initial value problem

$$\begin{cases} p_t = p_{xx} - E(p)u \\ p_x(0, t) = 0, \quad p_x(1, t) = \bar{p}, \quad t \geq 0, \\ p(x, 0) = p_0(x). \end{cases} \quad (25)$$

Lemma 4.1. *Let p be a classical solution of (25) on $[0, T]$. For $t \in [\tau, T]$, where $0 < \tau < \min\{1, T\}$, we have*

$$|p_x(\cdot, t)|_{q_2} \leq C_1(\tau, p_0, M), \text{ for } q_2 \in [1, \infty) \quad (26)$$

and

$$|p_x(\cdot, t)|_\infty \leq C_2(\tau, p_0, \sup_{t \in [0, T]} |u(t)|_2). \quad (27)$$

Proof. In [15] Winkler proved following estimates for the Neumann heat semigroup $\{S_N(t)\}_{t \geq 0}$ in a bounded domain $\Omega \subset \mathbb{R}^n$

$$|\nabla S_N(t)f|_{q_2} \leq C(1 + t^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{q_1} - \frac{1}{q_2})})e^{-\lambda_1 t}|f|_{q_1} \text{ for } t > 0, \quad (28)$$

where $1 \leq q_1 \leq q_2 \leq \infty$, $f \in L^{q_1}(\Omega)$ and λ_1 is the first nonzero eigenvalue of the Laplacian with the homogeneous Neumann boundary condition.

By substitution $Q(x, t) := p(x, t) - \frac{\bar{p}x^2}{2}$ problem (25) takes the following form

$$\begin{cases} Q_t = Q_{xx} + \bar{p} - E(Q + \frac{\bar{p}x^2}{2})u \\ Q_x(0, t) = Q_x(1, t) = 0, \quad t \geq 0, \\ Q(x, 0) = p_0(x) - \frac{\bar{p}x^2}{2}. \end{cases} \quad (29)$$

Applying the Duhamel formula we get

$$Q(t) = S_N(t)Q_0 + \int_0^t S_N(t-s)[\bar{p} - E\left(Q(s) + \frac{\bar{p}x^2}{2}\right)u(s)]ds \text{ for } t \in (0, T). \quad (30)$$

Differentiating (30) with respect to x and applying (28) (for $n = 1$, $q_1 = 1$) leads to

$$\begin{aligned} |Q_x(t)|_{q_2} &\leq C(1 + t^{-\frac{1}{2} - \frac{1}{2}(1 - \frac{1}{q_2})})e^{-\lambda_1 t}|Q_0|_1 \\ &\quad + \int_0^t C(1 + (t-s)^{-\frac{1}{2} - \frac{1}{2}(1 - \frac{1}{q_2})})e^{-\lambda_1(t-s)} \left| E\left(Q(s) + \frac{\bar{p}x^2}{2}\right)u(s) \right|_1 ds \\ &\leq C(1 + t^{-\frac{1}{2} - \frac{1}{2}(1 - \frac{1}{q_2})})e^{-\lambda_1 t}|Q_0|_1 \\ &\quad + E(p^*)M \int_0^t C\left(1 + (t-s)^{-\frac{1}{2} - \frac{1}{2}(1 - \frac{1}{q_2})}\right)e^{-\lambda_1(t-s)} ds \\ &\leq C_1(\tau, Q_0, M) \text{ for } q_2 \in [1, \infty). \end{aligned}$$

Repeating the above reasoning with $q_1 = 2$ $q_2 = \infty$, we obtain

$$|Q_x(\cdot, t)|_\infty \leq C_2 \left(\tau, Q_0, \sup_{t \in [0, T]} |u(t)|_2 \right). \quad \square$$

With the help of above lemma we can prove the uniform in time estimates on u .

Lemma 4.2. *The norm $|u(\cdot, t)|_2$ is uniformly bounded in time for any mass $M > 0$.*

Proof. We multiply (1) by u , and integrate over $(0, 1)$ to obtain

$$\frac{1}{2} \frac{d}{dt} |u|_2^2 + |u_x|_2^2 = \int_0^1 u_x u E(p)_x dx.$$

By the Schwarz and Young inequality ($ab \leq a^2/4 + b^2$) we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |u|_2^2 + |u_x|_2^2 &\leq \frac{1}{4} |u_x|_2^2 + \int_0^1 u^2 |E(p)_x|^2 dx \\ &\leq \frac{1}{4} |u_x|_2^2 + |u|_4^2 |E(p)_x|_4^2. \end{aligned}$$

Adding $|u|_2^2$ to both sides we get

$$\frac{1}{2} \frac{d}{dt} |u|_2^2 + \frac{3}{4} |u_x|_2^2 + |u|_2^2 \leq |u|_2^2 + |u|_4^2 |E(p)_x|_4^2. \quad (31)$$

To estimate the right hand side of (31) we need following versions of the Gagliardo–Nirenberg inequality (9)

$$|u|_4^2 \leq c(|u_x|_2 |u|_1 + |u|_1^2) = c(|u_x|_2 M + M^2), \quad (32)$$

$$|u|_2^2 \leq c(|u_x|_2^{\frac{2}{3}} |u|_1^{\frac{4}{3}} + |u|_1^2) = c(|u_x|_2^{\frac{2}{3}} M^{\frac{4}{3}} + M^2). \quad (33)$$

Applying Lemma 4.1, (32) and (33) to (31) we arrive at

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |u|_2^2 + \frac{3}{4} |u_x|_2^2 + |u|_2^2 &\leq c(|u_x|_2^{\frac{2}{3}} M^{\frac{4}{3}} + M^2) + c(|u_x|_2 M + M^2) \\ &= |u_x|_2^{\frac{2}{3}} c M^{\frac{4}{3}} + |u_x|_2 c M C_1(\tau, p_0, M)^2 + c M^2 (C_1(\tau, p_0, M)^2 + 1) \end{aligned} \quad (34)$$

for $t \geq \tau$.

Now to the right hand side of (34) we apply twice the Young inequality ($ab \leq a^3/2 + Cb^{3/2}$ and $ab \leq a^2/4 + b^2$)

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |u|_2^2 + \frac{3}{4} |u_x|_2^2 + |u|_2^2 &\leq \frac{3}{4} |u_x|_2^2 + C c^{\frac{3}{2}} M^2 \\ &\quad + (cM)^2 C_1(\tau, p_0, M)^4 + c M^2 (C_1(\tau, p_0, M)^2 + 1) \\ &= \frac{3}{4} |u_x|_2^2 + \tilde{C}(\tau, p_0, M). \end{aligned}$$

We have

$$\frac{d}{dt}|u|_2^2 + 2|u|_2^2 \leq 2\tilde{C}(\tau, p_0, M), \quad (35)$$

and the constant $\tilde{C}(\tau, p_0, M)$ does not depend on t . Inequality (35) is equivalent to

$$e^{-2t} \frac{d}{dt} (e^{2t}|u|_2^2) \leq 2\tilde{C}(\tau, p_0, M).$$

Denoting $y(t) := e^{2t}|u|_2^2$ and using the Gronwall lemma

$$\begin{aligned} y(t) &\leq y(\tau) + \int_{\tau}^t e^{2s} 2\tilde{C}(\tau, p_0, M) ds = y(\tau) + \tilde{C}(\tau, p_0, M)(e^{2t} - e^{2\tau}) \\ &\leq y(\tau) + e^{2t}\tilde{C}(\tau, p_0, M), \end{aligned}$$

thus

$$|u(t)|_2^2 \leq e^{-2t}|u(\tau)|_2^2 + \tilde{C}(\tau, p_0, M) \text{ for } t \geq \tau.$$

Since u is classical solution of (1) we have $\sup_{t \in [0, \tau]} |u(t)|_2 < \infty$, thus we have uniform boundedness on $[0, \infty]$. \square

Remark 2. From Lemma 4.2 and 4.1 the quantity $|p_x(\cdot, t)|_{\infty}$ is uniformly bounded on $[\tau, \infty)$. Moreover, Lemma 4.2 and 4.1 are true for (1.1), since estimates similar to (28) are true for the Dirichlet problem.

Now we prove the main theorem concerning Neumann problem.

Proof of Theorem 1.2. Integration of (25) with respect to x leads to

$$\frac{d}{dt}|p|_1 = \bar{p} - |E(p)u|_1 \geq \bar{p} - E(p^*)M$$

thus $|p(\cdot, t)|_1 \geq |p_0|_1 + c_M t$ if $c_M := \bar{p} - E(p^*)M > 0$. Since $p(x, t)$ is a classical solution thus for each $t \geq 0$ there exist $x_t \in [0, 1]$ such that $p(x_t, t) = |p(\cdot, t)|_1$. On the other hand $|p_x(x, t)|_{\infty} < C_2$, and C_2 is time independent. By the mean value theorem

$$\begin{aligned} ||p(\cdot, t)|_1 - p(x, t)| &= |p(x_t, t) - p(x, t)| \leq |p_x(x, t)|_{\infty} \leq C_2, \\ C_2 + |p(\cdot, t)|_1 &\geq p(x, t) \geq |p(\cdot, t)|_1 - C_2 \geq |p_0|_1 + c_M t - C_2 \end{aligned} \quad (36)$$

and $\lim_{t \rightarrow \infty} |p(\cdot, t)|_{\infty} = \infty$. Let $u_1 = u - M$, then $u = u_1 + M$, and equation (1) takes the form

$$u_{1t} = u_{1xx} - ((u_1 + M)E(p)_x)_x \quad (37)$$

with the boundary condition

$$u_{1x} - (u_1 + M)E(p)_x|_{x=0,1} = 0.$$

Now, we multiply (37) by u_1 and integrate over $(0, 1)$

$$\frac{1}{2} \frac{d}{dt} |u_1|_2^2 + |u_{1x}|_2^2 = \int_0^1 u_{1x}(u_1 + M)E(p)_x dx,$$

$$\frac{1}{2} \frac{d}{dt} |u_1|_2^2 + \frac{3}{4} |u_{1x}|_2^2 \leq \int_0^1 (u_1 + M)^2 |E'(p)| |p_x|_\infty dx,$$

so from (12)

$$\frac{1}{2} \frac{d}{dt} |u_1|_2^2 + \frac{3}{4} |u_{1x}|_2^2 \leq \int_0^1 (2u_1^2 + 2M^2) |E'(p)| C_2 dx \text{ for } t \geq \tau \quad (38)$$

follows. By (36) and $\lim_{p \rightarrow \infty} E'(p) = 0$ for an arbitrary $\varepsilon > 0$ there exists $s > p_{\min}$ such that for $t > s$ we have $|E'(p)| < |E'(|p_0|_1 + c_M t - C_2)| < \varepsilon$. We may pick s suitably large such that from (38) we get

$$\frac{d}{dt} |u_1|_2^2 + 2\tilde{C}_1 |u_1|_2^2 \leq \int_0^1 4M^2 |E'(p)| C_2 dx$$

for $t > s$ and some positive constant \tilde{C}_1 . The above inequality is equivalent to the following

$$\frac{d}{dt} (e^{2\tilde{C}_1 t} |u_1|_2^2) \leq e^{2\tilde{C}_1 t} \int_0^1 4M^2 |E'(p)| C_2 dx.$$

The Gronwall lemma for $[s, t]$ implies

$$e^{2\tilde{C}_1 t} |u_1(t)|_2^2 \leq e^{-2\tilde{C}_1 s} |u_1(s)|_2^2 + \int_s^t e^{2\tilde{C}_1 \varsigma} \int_0^1 4M^2 |E'(p(x, \varsigma))| C_2 dx d\varsigma,$$

$$e^{2\tilde{C}_1 t} |u_1(t)|_2^2 \leq e^{2\tilde{C}_1 s} |u_1(s)|_2^2 + e^{2\tilde{C}_1 t} \int_s^t \int_0^1 4M^2 |E'(p(x, \varsigma))| C_2 dx d\varsigma,$$

$$|u_1(t)|_2^2 \leq e^{2\tilde{C}_1(s-t)} |u_1(s)|_2^2 + \int_s^t \int_0^1 4M^2 |E'(p(x, \varsigma))| C_2 dx d\varsigma.$$

Due to $E''(p) > 0$ and $E'(p) < 0$, for sufficiently large p we have $|E'(p)| = -E'(p)$, and

$$|u_1(t)|_2^2 \leq e^{2\tilde{C}_1(s-t)} |u_1(s)|_2^2 - \int_s^t \int_0^1 4M^2 E'(p(x, \varsigma)) C_2 dx d\varsigma.$$

By inequality (36)

$$|u_1(t)|_2^2 \leq e^{2\tilde{C}_1(s-t)} |u_1(s)|_2^2 - \int_s^t 4M^2 E'(|p_0|_1 + c_M \varsigma - C_2) C_2 d\varsigma$$

$$= e^{2\tilde{C}_1(s-t)} |u_1(s)|_2^2 - \frac{4M^2 C_2}{c_M} (E(|p_0|_1 + c_M t - C_2) - E(|p_0|_1 + c_M s - C_2)),$$

and finally we arrive at

$$|u_1(t)|_2^2 \leq e^{2\tilde{C}_1(s-t)} |u_1(s)|_2^2 + \frac{4M^2C_2}{c_M} E(|p_0|_1 + c_M s - C_2). \quad (39)$$

Since $|u(t)|_2^2$ is uniformly bounded and $\lim_{p \rightarrow \infty} E(p) = 0$ we see that the right hand side of (39) tends to 0 as $s \rightarrow \infty$. \square

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