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Reducing subspaces of de Branges-Rovnyak spaces

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For $b \in H_1^\infty$, the closed unit ball of H^∞, the de Branges-Rovnyak space $\mathcal{H}(b)$ is a Hilbert space contractively contained in the Hardy space H^2 that is invariant by the backward shift operator S^*. We consider the reducing subspaces of the operator $S^{*2}|_{\mathcal{H}(b)}$. When b is an inner function, $S^{*2}|_{\mathcal{H}(b)}$ is a truncated Toeplitz operator and its reducibility was characterized by Douglas and Foias using model theory. We use another approach to extend their result to the case where b is extreme. We prove that if b is extreme but not inner, then $S^{*2}|_{\mathcal{H}(b)}$ is reducible if and only if b is even or odd, and describe the structure of reducing subspaces.

1. Introduction

Let $\mathbb{D}$ denote the unit disk. Let L^2 denote the Lebesgue space of square integrable functions on the unit circle $\mathbb{T}$. The Hardy space H^2 is the subspace of analytic functions on $\mathbb{D}$ whose Taylor coefficients are square summable. Then it can also be identified with the subspace of L^2 of functions whose negative Fourier coefficients vanish. The space of bounded analytic functions on the unit disk is denoted by H^∞. The Toeplitz operator on the Hardy space H^2 with symbol f in $L^\infty(\mathbb{T})$ is defined by

$$T_f(h) = P(fh),$$

for $h \in H^2$. Here P is the orthogonal projection from L^2 to H^2. The unilateral shift operator on H^2 is $S = T_z$.

Let A be a bounded operator on a Hilbert space H. We define the range space $\mathcal{M}(A) = AH$, and endow it with the inner product

$$\langle Af, Ag\rangle_{\mathcal{M}(A)} = \langle f, g\rangle_H, \qquad f, g \in H \ominus \mathrm{Ker} A.$$

$\mathcal{M}(A)$ has a Hilbert space structure that makes A a coisometry on H.

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Let b be a function in H_1^∞, the closed unit ball of H^∞. The de Branges-Rovnyak space $\mathcal{H}(b)$ is defined to be the space

$$\mathcal{M}((I - T_b T_{\bar{b}})^{1/2}).$$

We also define the space $\mathcal{H}(\bar{b})$ in the same way as $\mathcal{H}(b)$, but with the roles of b and $\bar{b}$ interchanged, i.e.

$$\mathcal{H}(\bar{b}) = \mathcal{M}((I - T_{\bar{b}} T_b)^{1/2}).$$

The spaces $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ are also called sub-Hardy Hilbert spaces (the terminology comes from the title of Sarason's book [9]).

The space $\mathcal{H}(b)$ was introduced by de Branges and Rovnyak [2]. Sarason and several others made essential contributions to the theory [9]. A recent two-volume monograph [4], [5] presents most of the main developments in this area.

There are two special cases for $\mathcal{H}(b)$ spaces. If $||b||_\infty < 1$, then $\mathcal{H}(b)$ is just a re-normed version of H^2. If b is an inner function, then

$$\mathcal{H}(b) = H^2 \ominus bH^2$$

is a closed subspace of H^2, the so-called model space (see [6] for a brief survey).

Let T be a bounded linear operator on a Hilbert space $\mathcal{H}$. A closed subspace M of $\mathcal{H}$ is called a reducing subspace of T if $TM \subset M$ and $T^*M \subset M$, which is equivalent to the fact that M and $M^\perp$ are both invariant by T. If T has a proper reducing subspace, T is called reducible. The reducing subspaces of shift operators or multiplication operators have been studied in various function spaces: for weighted unilateral shift operators of finite multiplicity, see [10]; for multiplication operators induced by finite Blaschke products on the Bergman space, see [14], [8] and the references therein.

Our motivation is the study of reducing subspaces of truncated Toeplitz operators on the model space. For an inner function θ and $\varphi \in L^2$, the truncated Toeplitz operator A_φ^θ with symbol φ is defined by

$$A_\varphi^\theta f = P_\theta(\varphi f),$$

for f on the dense subset $\mathcal{H}(\theta) \cap H^\infty$ of $\mathcal{H}(\theta)$. Here P_θ is the orthogonal projection from H^2 to $\mathcal{H}(\theta)$. It is known that A_z^θ is always irreducible (see e.g. [7]). A function $f \in L^2$ is called even if $f(z) = f(-z)$, for every $z \in \mathbb{D}$, and f is called odd if $f(z) = -f(-z)$, for every $z \in \mathbb{D}$. The operator A_z^θ is called the compressed shift operator. The reducibility of $A_{z^2}^\theta$ is characterized by Douglas and Foias [3] using model theory for contractions [12] as the following.

Theorem 1.1. *The operator $A_{z^2}^\theta$ is reducible if and only if either θ is even or there exists $\mu \in \mathbb{D}$ such that*

$$\theta(z) = p(z)\frac{z+\mu}{1+\bar{\mu}z},$$

where p is even.

Recently, Li, Yang and Lu found a different proof of Theorem 1.1 and extended it to the case where the symbol of a truncated Toeplitz operator is a Blaschke product of order 2 or 3 [13].

The theory of $\mathcal{H}(b)$ spaces is pervaded by a fundamental dichotomy, when b is an extreme point of H_1^∞ and when it is not. The nonextreme case includes $||b||_\infty < 1$ and the extreme case includes b is an inner function. Roughly speaking, when b is nonextreme, $\mathcal{H}(b)$ behaves similar to H^2, while in the extreme case,

$\mathcal{H}(b)$ is more closely related to the model space. For example, the polynomials belong to $\mathcal{H}(b)$ if and only if b is non-extreme (see [9, Chapter IV, V]).

Notice that $(A^{\theta}_{z^2})^* = S^{*2}|_{\mathcal{H}(\theta)}$. Thus, in view of Theorem 1.1, it is natural to consider reducing subspaces of $S^{*2}|_{\mathcal{H}(b)}$ when b is extreme. The main purpose of this paper is to characterize the reducibility of $S^{*2}|_{\mathcal{H}(b)}$ on $\mathcal{H}(b)$ in the extreme case and describe the reducing subspaces (Theorem 4.1). We also show that X_b is irreducible for every b.

2. Background on de Branges-Rovnyak spaces

In this section, we present some basic theory of de Branges-Rovnyak spaces and the results we shall use later.

The relation between $\mathcal{H}(b)$ and $\mathcal{H}(\bar{b})$ can be found in [9, II-4]. Here we use $\langle\ ,\ \rangle_b$ to denote the inner product of $\mathcal{H}(b)$.

Theorem 2.1. *A function f belongs to $\mathcal{H}(b)$ if and only if $T_{\bar{b}}f$ belongs to $\mathcal{H}(\bar{b})$. If $f_1, f_2 \in \mathcal{H}(b)$, then*

$$\langle f_1, f_2\rangle_b = \langle f_1, f_2\rangle_2 + \langle T_{\bar{b}}f_1, T_{\bar{b}}f_2\rangle_{\bar{b}}.$$

Let $b \in H_1^\infty$. Let $\rho = 1-|b|^2$ on $\mathbb{T}$ and let $H^2(\rho)$ be the closure of polynomials in $L^2(\rho) = L^2(\mathbb{T}, \rho\frac{d\theta}{2\pi})$ (we will keep using these notations in the remaining of this paper). The Cauchy transform K_ρ is the mapping from $L^2(\rho)$ to H^2 defined by

$$K_\rho f = P(\rho f).$$

In the theory of $\mathcal{H}(b)$ spaces, $\mathcal{H}(\bar{b})$ is often more amenable than $\mathcal{H}(b)$ because of a representation theorem for $\mathcal{H}(\bar{b})$ [9, III-2].

Theorem 2.2. *The operator K_ρ is an isometry from $H^2(\rho)$ to $\mathcal{H}(\bar{b})$.*

The operator on $H^2(\rho)$ of multiplication by the independent variable will be denoted by Z_ρ. We have the intertwining relation [9, III-3]

$$K_\rho Z_\rho^* = S^* K_\rho. \tag{2.1}$$

The space $\mathcal{H}(b)$ is invariant under $S^* = T_{\bar{z}}$ [9, II-7], and the restriction of S^* is a contraction. We use X_b to denote $S^*|_{\mathcal{H}(b)}$. This operator can serve as a model for a large class of Hilbert space contractions [2], [1].

The following identity shows the difference between X_b and S^* [9, II-9].

Theorem 2.3. *Let $b \in H_1^\infty$. For every $f \in \mathcal{H}(b)$,*

$$X_b^* f = Sf - \langle f, S^*b\rangle_b b.$$

If x and y are in a Hilbert space $\mathcal{H}$, we shall use $x \otimes y$ to be the following rank one operator on $\mathcal{H}$

$$(x \otimes y)(f) = \langle f, y\rangle_{\mathcal{H}} \cdot x, \quad f \in \mathcal{H}.$$

It is obvious that

$$(x \otimes y)^* = y \otimes x,$$

and if A, B are bounded linear operators on $\mathcal{H}$, then

$$A(x \otimes y)B = (Ax) \otimes (B^*y).$$

It could be misleading to write the identity in Theorem 2.3 as $X_b^* = S - b \otimes S^*b$ because b may not be in $\mathcal{H}(b)$. But it is known that $S^*b \in \mathcal{H}(b)$ [9, II-8], and we have

$$I - X_b X_b^* = (S^*b) \otimes (S^*b). \tag{2.2}$$

The space $\mathcal{H}(b)$ is a reproducing kernel Hilbert space with kernel function:

$$k_w^b(z) = \frac{1 - \overline{b(w)}b(z)}{1 - \bar{w}z}.$$

When b is extreme, we have the following identity (see e.g. [5, Theorem 25.11]).

Lemma 2.1. *Let b be an extreme point in H_1^∞. Then*

$$I - X_b^* X_b = k_0^b \otimes k_0^b.$$

For an inner function θ, $S^*\theta$ is a cyclic vector of $(A_z^\theta)^*$. A similar result holds for extreme functions (see e.g. [5, Section 26.6]).

Theorem 2.4. *If b is extreme, then*

$$\mathcal{H}(b) = Span\{S^{*n}b : n \geqslant 1\}.$$

Here Span denotes the closed linear span.

3. An equivalent condition for the reducibility

In this section we first prove that X_b is irreducible for every b. The idea in the proof will be used to study X_b^2.

Theorem 3.1. *Let $b \in H_1^\infty$. Then X_b is not reducible.*

Proof. Suppose X_b is reducible. Then

$$\mathcal{H}(b) = M_1 \oplus_b M_2,$$

where M_1, M_2 are nontrivial reducing subspaces of X_b.

Note that for every $f \in M_1, g \in M_2$,

$$(I - X_b X_b^*)f \perp (I - X_b X_b^*)g$$

in $\mathcal{H}(b)$. By equation (2.2),

$$\dim((I - X_b X_b^*)\mathcal{H}(b)) \leqslant 1.$$

Then one of the two range spaces

$$(I - X_b X_b^*)M_1, (I - X_b X_b^*)M_2$$

must be 0. WLOG, we may assume

$$(I - X_b X_b^*)M_1 = 0,$$

i.e. for every $f \in M_1$,

$$0 = (I - X_b X_b^*)f = \langle f, S^* b\rangle_b S^* b.$$

Thus f is orthogonal to S^*b in $\mathcal{H}(b)$ and then $S^*b \in M_2$. Since M_2 is invariant under S^*, we have

$$\mathrm{Span}\{S^{*n}b : n \geqslant 1\} \subset M_2.$$

If b is extreme, it follows from Theorem 2.4 that $M_2 = \mathcal{H}(b)$, which is a contradiction.

If b is nonextreme, then polynomials are dense in $\mathcal{H}(b)$. For every $f \in M_1$, since f is orthogonal to S^*b, we see from Theorem 2.3 that $Sf = X_b f$. Thus M_1 is invariant under both S and S^*. Pick a nonzero function $h \in M_1$, then

$$h(z) = \sum_{j=k}^{\infty} h_j z^j,$$

for some $k \geqslant 0$ with $h_k \neq 0$. Thus

$$\frac{1}{h_k}(I - S^{k+1}S^{*k+1})h = z^k \in M_1,$$

which implies that M_1 contain all the polynomials. So $M_1 = \mathcal{H}(b)$, which is a contradiction. □

Lemma 3.1. *Let b be an extreme point in H_1^∞. Then $S^*b, S^{*2}b$ are linearly dependent if and only if b is a single Blaschke product.*

Proof. By Theorem 2.4, $S^*b, S^{*2}b$ are linearly dependent if and only if $\mathcal{H}(b)$ is a one-dimensional space, which is equivalent to b being a single Blaschke product. □

For the extreme case, we have the following equivalent condition for the reducibility of X_b^2.

Theorem 3.2. *Let b be an extreme point in H_1^∞. Suppose b is not a single Blaschke product. Then X_b^2 is reducible if and only if there exist complex numbers α, β, $\alpha\beta \neq 1$, such that for every $n, m \geqslant 0$,*

$$S^{*2m}(S^*b + \alpha S^{*2}b) \perp S^{*2n}(\beta S^*b + S^{*2}b) \tag{3.1}$$

in $\mathcal{H}(b)$. In this case the reducing subspaces of X_b^2 are given by

$$\mathcal{H}(b) = M_1 \oplus_b M_2,$$

where

$$M_1 = \mathit{Span}\{S^{*2n}(S^*b + \alpha S^{*2}b) : n \geqslant 0\} \tag{3.2}$$

and

$$M_2 = Span\{S^{*2n}(\beta S^* b + S^{*2} b) : n \geqslant 0\}. \tag{3.3}$$

Proof. Suppose (3.1) holds, then take M_1, M_2 as in (3.2), (3.3). It is clear that M_1, M_2 are invariant under X_b^2 (or S^{*2}) and are orthogonal in $\mathcal{H}(b)$. By Theorem 2.4, we have

$$\mathcal{H}(b) = \mathrm{Span}\{M_1, M_2\}.$$

Thus

$$\mathcal{H}(b) = M_1 \oplus_b M_2,$$

and X_b^2 is reducible.

Next we assume X_b^2 is reducible. Then

$$\mathcal{H}(b) = M_1 \oplus_b M_2,$$

where M_1, M_2 are nontrivial reducing subspaces of X_b^2. Note that for every $f \in M_1, g \in M_2$,

$$(I - X_b^2 X_b^{*2}) f \in M_1, (I - X_b^2 X_b^{*2}) g \in M_2.$$

Then

$$(I - X_b^2 X_b^{*2}) f \perp (I - X_b^2 X_b^{*2}) g$$

in $\mathcal{H}(b)$. Using (2.2), we have

$$\begin{aligned} I - X_b^2 X_b^{*2} &= I - X_b X_b^* + X_b (I - X_b X_b^*) X_b^* \\ &= S^* b \otimes S^* b + X_b (S^* b \otimes S^* b) X_b^* \\ &= S^* b \otimes S^* b + S^{*2} b \otimes S^{*2} b. \end{aligned} \tag{3.4}$$

By Lemma 3.1, $S^* b$ and $S^{*2} b$ are linearly independent. Thus

$$\dim (I - X_b^2 X_b^{*2}) \mathcal{H}(b) = 2.$$

Suppose one of the two range spaces

$$(I - X_b^2 X_b^{*2}) M_1, (I - X_b^2 X_b^{*2}) M_2$$

is zero, say

$$(I - X_b^2 X_b^{*2}) M_1 = 0.$$

By (3.4), we see that every function in M_1 is orthogonal to $S^* b$ and $S^{*2} b$ in $\mathcal{H}(b)$, which implies $S^* b, S^{*2} b$ are in M_2. Since M_2 is invariant for X_b^2, using Theorem 2.4 we see that

$$\mathcal{H}(b) = \mathrm{Span}\{S^{*n} b : n \geqslant 1\} \subset M_2.$$

This is a contradiction. Therefore, we must have

$$\dim (I - X_b^2 X_b^{*2}) M_1 = \dim (I - X_b^2 X_b^{*2}) M_2 = 1.$$

This means, WLOG, there exist complex numbers α, β such that

$$(I - X_b^2 X_b^{*2})M_1 = \text{Span}\{S^* b + \alpha S^{*2} b\} \subset M_1,$$
$$(I - X_b^2 X_b^{*2})M_1 = \text{Span}\{\beta S^* b + S^{*2} b\} \subset M_2.$$

Since M_1, M_2 are invariant under X_b^2, we have

$$\text{Span}\{S^{*2n}(S^* b + \alpha S^{*2} b) : n \geqslant 0\} \subset M_1,$$
$$\text{Span}\{S^{*2n}(\beta S^* b + S^{*2} b) : n \geqslant 0\} \subset M_2.$$

Using Theorem 2.4, we obtain

$$\mathcal{H}(b) = M_1 \cup M_2,$$

and thus (3.2), (3.3) hold. The relation (3.1) follows from $M_1 \perp_b M_2$. Note that $\alpha\beta \neq 1$; otherwise $\beta S^* b + S^{*2} b \in M_1 \cap M_2 = \{0\}$, which contradicts Lemma 3.1. $\square$

4. Main results

In this section, we analyze the condition (3.1) and characterize the reducibility of X_b^2 when b is extreme but not inner.

Lemma 4.1. *Let b be an extreme point in H_1^∞. Then for every $n \geqslant 1$,*

$$I - X_b^{*n} X_b^n = \sum_{j=0}^{n-1} (X_b^{*j} k_0^b) \otimes (X_b^{*j} k_0^b).$$

Proof. This proof is by induction on n. For $n = 1$, the equality is exactly the one in Lemma 2.1. Assume that the equality holds for some $n \geqslant 2$. Then, using once again Lemma 2.1 and the induction hypothesis, we have

$$\begin{aligned}
X_b^{*n} X_b^n &= X_b^* (X_b^{*n-1} X_b^{n-1}) X_b \\
&= X_b^* (I - \sum_{j=0}^{n-2} (X_b^{*j} k_0^b) \otimes (X_b^{*j} k_0^b)) X_b \\
&= X_b^* X_b - \sum_{j=0}^{n-2} X_b^* (X_b^{*j} k_0^b) \otimes (X_b^{*j} k_0^b) X_b \\
&= I - k_0^b \otimes k_0^b - \sum_{j=0}^{n-2} (X_b^{*(j+1)} k_0^b) \otimes (X_b^{*(j+1)} k_0^b) \\
&= I - \sum_{j=0}^{n-1} (X_b^{*j} k_0^b) \otimes (X_b^{*j} k_0^b). \quad \square
\end{aligned}$$

Lemma 4.2. *Let b be an extreme point in H_1^∞ and let $f, g \in \mathcal{H}(b)$. Suppose b is not a single Blaschke product. Then*

$$\langle X_b^{2m} f, X_b^{2n} g \rangle_b = 0, \tag{4.1}$$

for every $m, n \geqslant 0$ if and only if the following hold

(1) *for every $k \geqslant 0$,*

$$\langle T_{\bar{b}} f, T_{\bar{b}} X_b^{2k} g\rangle_{\bar{b}} = \langle T_{\bar{b}} g, T_{\bar{b}} X_b^{2k} f\rangle_{\bar{b}} = 0. \tag{4.2}$$

(2) *for every $m, n \geqslant 0$,*

$$\langle S^{*2m} f, S^{*2n} g\rangle_2 = 0,$$

i.e. there exist functions $F, G \in H^2$ and complex numbers a_0, b_0, a_1, b_1 such that

$$f(z) = F(z^2)(a_0 + a_1 z), \quad g(z) = G(z^2)(b_0 + b_1 z) \tag{4.3}$$

and

$$a_0 \bar{b_0} + a_1 \bar{b_1} = 0.$$

Proof. Let

$$f(z) = \sum_{k=0}^{\infty} f_k z^k \quad \text{and} \quad g(z) = \sum_{k=0}^{\infty} g_k z^k.$$

Suppose (4.1) holds. Then for $m \leqslant n$, we have

$$0 = \langle X_b^{2m} f, X_b^{2n} g\rangle_b = \langle X_b^{*2m} X_b^{2m} f, X_b^{2n-2m} g\rangle_b. \tag{4.4}$$

By Lemma 4.1, we have

$$\begin{aligned}
(I - X_b^{*2m} X_b^{2m}) f &= f - X_b^{*2m} X_b^{2m} f \\
&= \sum_{j=0}^{2m-1} \langle f, X_b^{*j} k_0^b\rangle_b \cdot (X_b^{*j} k_0^b) \\
&= \sum_{j=0}^{2m-1} \langle X_b^j f, k_0^b\rangle_b \cdot (X_b^{*j} k_0^b) \\
&= \sum_{j=0}^{2m-1} (S^{*j} f)(0) \cdot (X_b^{*j} k_0^b) \\
&= \sum_{j=0}^{2m-1} f_j \cdot X_b^{*j} k_0^b.
\end{aligned}$$

Then

$$X_b^{*2m} X_b^{2m} f = f - \sum_{j=0}^{2m-1} f_j \cdot X_b^{*j} k_0^b.$$

This together with (4.4) implies

$$
\begin{aligned}
0 &= \langle f - \sum_{j=0}^{2m-1} f_j \cdot X_b^{*j} k_0^b, X_b^{2n-2m} g\rangle_b \\
&= \langle f, X_b^{2n-2m} g\rangle_b - \langle \sum_{j=0}^{2m-1} f_j \cdot X_b^{*j} k_0^b, X_b^{2n-2m} g\rangle_b \\
&= \langle f, X_b^{2n-2m} g\rangle_b - \sum_{j=0}^{2m-1} f_j \cdot \langle X_b^{*j} k_0^b, X_b^{2n-2m} g\rangle_b \\
&= \langle f, X_b^{2n-2m} g\rangle_b - \sum_{j=0}^{2m-1} f_j \cdot \langle k_0^b, X_b^{2n-2m+j} g\rangle_b \\
&= \langle f, X_b^{2n-2m} g\rangle_b - \sum_{j=0}^{2m-1} f_j \cdot \overline{\langle X_b^{2n-2m+j} g, k_0^b\rangle_b} \\
&= \langle f, X_b^{2n-2m} g\rangle_b - \sum_{j=0}^{2m-1} f_j \cdot \overline{g}_{2n-2m+j}.
\end{aligned} \tag{4.5}
$$

Replacing n, m in (4.5) by $n+1, m+1$ respectively, we have

$$
0 = \langle f, X_b^{2n-2m} g\rangle_b - \sum_{j=0}^{2m+1} f_j \cdot \overline{g}_{2n-2m+j}. \tag{4.6}
$$

Subtracting (4.6) by (4.5) implies

$$
f_{2m}\overline{g}_{2n} + f_{2m+1}\overline{g}_{2n+1} = 0, \tag{4.7}
$$

for $m \leqslant n$. A similar argument shows that (4.7) also holds for $n \leqslant m$. Thus we have for every $n, m \geqslant 0$, the two vectors

$$
(f_{2m}, f_{2m+1}), (g_{2n}, g_{2n+1})
$$

are orthogonal in $\mathbb{C}^2$. It is easy to check f, g must have the form (4.3). In particular, we have

$$
\langle f, X_b^{2k} g\rangle_2 = \langle g, X_b^{2k} f\rangle_2 = 0, \quad \text{for every } k \geqslant 0. \tag{4.8}
$$

It follows from (4.5) and (4.7) that

$$
\langle f, X_b^{2k} g\rangle_b = \langle g, X_b^{2k} f\rangle_b = 0, \quad \text{for every } k \geqslant 0.
$$

This together with (4.8) and Theorem 2.1 give (4.2).

The sufficiency follows easily from the calculation in (4.5). $\square$

Remark 4.1. When b is an inner function, $\mathcal{H}(\bar{b})$ is trivial and then (4.2) is automatically satisfied. One may expect the reducibility of X_b^2 is more restrictive if b is not inner. We shall see it is true in the remaining of this section.

When b is extreme, the following Lemma will be used to calculate the inner products in (4.2).

Lemma 4.3. *Let b be an extreme point in H_1^∞. Let $\rho = 1 - |b|^2$ on $\mathbb{T}$. Then for every $m, n \geqslant 1$,*

$$\langle T_{\bar{b}} S^{*m} b, T_{\bar{b}} S^{*n} b \rangle_{\bar{b}} = \begin{cases} -\langle z^{n-m}, |b|^2 \rangle_2, & m < n, \\ -\langle |b|^2, z^{m-n} \rangle_2, & m > n, \\ 1 - ||b||_2^2, & m = n. \end{cases}$$

Proof. Suppose $m \leqslant n$. Using the intertwining relation (2.1), we can easily get

$$K_\rho Z_\rho^{*n} = S^{*n} K_\rho.$$

Thus

$$\begin{aligned} K_\rho Z_\rho^{*n} 1 =& S^{*n} K_\rho 1 = S^{*n} P(\rho) = S^{*n} P(1 - |b|^2) \\ =& - S^{*n} P(|b|^2) = -S^{*n} T_{\bar{b}} b = -T_{\bar{b}} S^{*n} b. \end{aligned}$$

By Theorem 2.2, we have

$$\langle T_{\bar{b}} S^{*m} b, T_{\bar{b}} S^{*n} b \rangle_{\bar{b}} = \langle K_\rho Z_\rho^{*m} 1, K_\rho Z_\rho^{*n} 1 \rangle_{\bar{b}} = \langle Z_\rho^{*m} 1, Z_\rho^{*n} 1 \rangle_{L^2(\rho)}.$$

If b is extreme, then $H^2(\rho) = L^2(\rho)$ [11], which implies Z_ρ is a unitary operator. Then

$$\begin{aligned} \langle T_{\bar{b}} S^{*m} b, T_{\bar{b}} S^{*n} b \rangle_{\bar{b}} =& \langle Z_\rho^{*m} 1, Z_\rho^{*n} 1 \rangle_{L^2(\rho)} = \langle Z_\rho^{n-m} 1, 1 \rangle_{L^2(\rho)} = \langle z^{n-m}, 1 \rangle_{L^2(\rho)} \\ =& \langle z^{n-m}, 1 \rangle_2 - \langle z^{n-m}, |b|^2 \rangle_2 \\ =& \begin{cases} -\langle z^{n-m}, |b|^2 \rangle_2, & m < n, \\ 1 - ||b||_2^2, & m = n. \end{cases} \quad \square \end{aligned}$$

We also need the following three elementary results.

Lemma 4.4. *Let $b \in H_1^\infty$. Then*

$$\lim_{n \to \infty} \langle z^n, |b|^2 \rangle_2 = 0.$$

Proof. This is just Riemann Lebesgue lemma on Fourier coefficients. $\square$

Lemma 4.5. *Let $b \in H^\infty$. Then $|b|^2$ is even if and only if b is even or odd.*

Proof. Let $b(z) = b_0(z) + z b_1(z)$, where b_0, b_1 are even functions. Then $|b|^2$ is even if and only if

$$|b_0(z) + z b_1(z)|^2 = |b_0(z) - z b_1(z)|^2,$$

which is equivalent to $b_0 \overline{z b_1} \equiv 0$. Then the conclusion follows easily. $\square$

Lemma 4.6. *Let $\alpha, \beta \in \mathbb{C}$ with $\alpha\beta \neq 0$ and $\alpha\beta \neq 1$. Let $\{a_n\}_{n=0}^\infty$ be a sequence of complex numbers but not the zero sequence. Suppose*

$$\lim_{n \to \infty} a_n = 0$$

and for every $n \geqslant 1$, the following conditions hold.

$$a_{2n+1} + (\alpha + \bar{\beta})a_{2n} + \alpha\bar{\beta}a_{2n-1} = 0, \tag{4.9}$$
$$a_{2n+1} + (\frac{1}{\bar{\alpha}} + \frac{1}{\beta})a_{2n} + \frac{1}{\bar{\alpha}\beta}a_{2n-1} = 0. \tag{4.10}$$

Then we have either

$$\beta = -\bar{\alpha}, \quad \textit{and} \quad a_{2n-1} = 0, \quad \textit{for every } n \geqslant 1$$

or

$$|\alpha| = |\beta| = 1.$$

Proof. Subtracting (4.10) from (4.9), we have

$$(\alpha + \bar{\beta} - \frac{1}{\bar{\alpha}} - \frac{1}{\beta})a_{2n} + (\alpha\bar{\beta} - \frac{1}{\bar{\alpha}\beta})a_{2n-1} = 0. \tag{4.11}$$

Since $\{a_n\}_{n=0}^{\infty}$ is nonzero, we have the following four cases.

Case I: $\alpha + \bar{\beta} - \frac{1}{\bar{\alpha}} - \frac{1}{\beta} = \alpha\bar{\beta} - \frac{1}{\bar{\alpha}\beta} = 0$. Then we have $|\alpha\beta| = 1$ and

$$\begin{aligned} 0 =& \alpha + \bar{\beta} - \frac{1}{\bar{\alpha}} - \frac{1}{\beta} = \frac{1}{\bar{\alpha}}(|\alpha|^2 - 1) + \frac{1}{\beta}(|\beta|^2 - 1) \\ =& \frac{1}{\bar{\alpha}}(|\alpha|^2 - 1) + \frac{1}{\beta}(\frac{1}{|\alpha|^2} - 1) = \frac{|\alpha|^2 - 1}{\bar{\alpha}}(1 - \frac{1}{\alpha\beta}). \end{aligned}$$

Thus $|\alpha| = |\beta| = 1$.

Case II: $\alpha + \bar{\beta} = \frac{1}{\bar{\alpha}} + \frac{1}{\beta}$ and $a_{2n-1} = 0$, for every $n \geqslant 1$. Then (4.9) implies $\beta = -\bar{\alpha}$.

Case III: $\alpha\bar{\beta} = \frac{1}{\bar{\alpha}\beta}$ and $a_{2n} = 0$, for every $n \geqslant 1$. Then $|\alpha\beta| = 1$ and by (4.9), we have

$$|a_{2n+1}| = |\alpha\beta| \cdot |a_{2n-1}| = |a_{2n-1}|.$$

Since a_n tends to 0, we have $a_{2n-1} = 0$ and thus $\{a_n\}_{n=0}^{\infty}$ is the zero sequence, which contradicts the assumption.

Case IV: $\alpha + \bar{\beta} - \frac{1}{\bar{\alpha}} - \frac{1}{\beta} \neq 0$ and $\alpha\bar{\beta} - \frac{1}{\bar{\alpha}\beta} \neq 0$. Then by (4.11),

$$a_{2n} = \frac{\frac{1}{\bar{\alpha}\beta} - \alpha\bar{\beta}}{\alpha + \bar{\beta} - \frac{1}{\bar{\alpha}} - \frac{1}{\beta}} a_{2n-1} = \frac{1 - |\alpha\beta|^2}{\beta|\alpha|^2 + \bar{\alpha}|\beta|^2 - \beta - \bar{\alpha}} a_{2n-1}.$$

Put this in (4.9), we have

$$\begin{aligned} a_{2n+1} &= -(\alpha + \bar{\beta})a_{2n} - \alpha\bar{\beta}a_{2n-1} \\ &= -\Big((\alpha + \bar{\beta})\frac{1 - |\alpha\beta|^2}{\beta|\alpha|^2 + \bar{\alpha}|\beta|^2 - \beta - \bar{\alpha}} + \alpha\bar{\beta}\Big)a_{2n-1} \\ &= \frac{\alpha|\beta|^2 + \bar{\beta}|\alpha|^2 - \alpha - \bar{\beta}}{\beta|\alpha|^2 + \bar{\alpha}|\beta|^2 - \beta - \bar{\alpha}} a_{2n-1}. \end{aligned}$$

Thus $|a_{2n+1}| = |a_{2n-1}|$ and, similar to Case III, $a_{2n-1} = 0$, for every $n \geqslant 1$. From (4.9), (4.10), we see that either $a_{2n} = 0$, for every $n \geqslant 1$ or $\alpha + \bar{\beta} = \frac{1}{\bar{\alpha}} + \frac{1}{\beta} = 0$. They are both excluded by the assumptions. $\square$

Now we are ready to prove the main Theorem.

Theorem 4.1. *Let b be an extreme point in H_1^∞. If b is not an inner function, then X_b^2 is reducible if and only if b is even or odd. If b is even, the reducing subspaces of X_b^2 are*

$$M = Span\{(S^{*2n}b)(z+\alpha) : n \geqslant 1\}$$

with

$$M^\perp = Span\{(S^{*2n}b)(-\bar{\alpha}z+1) : n \geqslant 1\},$$

for some $\alpha \in \mathbb{C}$.

If b is odd, the reducing subspaces of X_b^2 are

$$M = Span\{S^{*2n-1}b : n \geqslant 1\}$$

with

$$M^\perp = Span\{S^{*2n}b : n \geqslant 1\}.$$

Proof. Necessity. We assume X_b^2 is reducible and b is not inner. Let

$$b(z) = \sum_{k=0}^{\infty} b_k z^k.$$

By Theorem 3.2, there exists $\alpha, \beta \in \mathbb{C}$ such that $\alpha\beta \neq 1$ and (3.1) holds. Let

$$f = S^*b + \alpha S^{*2}b, \quad g = \beta S^*b + S^{*2}b.$$

Then f, g are in $\mathcal{H}(b)$, and using Lemma 4.2, we have

$$\langle T_{\bar{b}}f, T_{\bar{b}}X_b^{2n}g\rangle_{\bar{b}} = \langle T_{\bar{b}}g, T_{\bar{b}}X_b^{2n}f\rangle_{\bar{b}} = 0,$$

for every $n \geqslant 0$. If $n \geqslant 1$, using Lemma 4.3, we have

$$\begin{aligned}
0 =& \langle T_{\bar{b}}f,\ T_{\bar{b}}X_b^{2n}g\rangle_{\bar{b}} \\
=& \langle T_{\bar{b}}S^*b + \alpha T_{\bar{b}}S^{*2}b,\ \beta T_{\bar{b}}(S^*)^{2n+1}b + T_{\bar{b}}(S^*)^{2n+2}b\rangle_{\bar{b}} \\
=& \bar{\beta}\langle T_{\bar{b}}S^*b, T_{\bar{b}}(S^*)^{2n+1}b\rangle_{\bar{b}} + \langle T_{\bar{b}}S^*b, T_{\bar{b}}(S^*)^{2n+2}b\rangle_{\bar{b}} \\
&+ \alpha\bar{\beta}\langle T_{\bar{b}}S^{*2}b, T_{\bar{b}}(S^*)^{2n+1}\rangle_{\bar{b}} + \alpha\langle T_{\bar{b}}S^{*2}b, T_{\bar{b}}(S^*)^{2n+2}b\rangle_{\bar{b}} \\
=& -\bar{\beta}\langle z^{2n}, |b|^2\rangle_2 - \langle z^{2n+1}, |b|^2\rangle_2 - \alpha\bar{\beta}\langle z^{2n-1}, |b|^2\rangle_2 - \alpha\langle z^{2n}, |b|^2\rangle_2,
\end{aligned}$$

which can be simplified to

$$\langle z^{2n+1}, |b|^2\rangle_2 + (\alpha + \bar{\beta})\langle z^{2n}, |b|^2\rangle_2 + \alpha\bar{\beta}\langle z^{2n-1}, |b|^2\rangle_2 = 0. \tag{4.12}$$

Similarly,

$$\langle T_{\bar{b}}g, T_{\bar{b}}X_b^{2n}f\rangle_{\bar{b}} = 0$$

implies

$$\bar{\alpha}\beta\langle z^{2n+1}, |b|^2\rangle_2 + (\bar{\alpha}+\beta)\langle z^{2n}, |b|^2\rangle_2 + \langle z^{2n-1}, |b|^2\rangle_2 = 0. \tag{4.13}$$

If $\alpha = \beta = 0$, then (4.13) implies for every $n \geqslant 1$,

$$0 = \langle z^{2n-1}, |b|^2\rangle_2 = \langle |b|^2, \bar{z}^{2n-1}\rangle_2.$$

This means b is even or odd by Lemma 4.5.

If $\alpha = 0$ and $\beta \neq 0$, using (4.12), (4.13), we have

$$\langle z^{2n+1}, |b|^2\rangle_2 + \bar{\beta}\langle z^{2n}, |b|^2\rangle_2 = 0,$$

and

$$\beta\langle z^{2n}, |b|^2\rangle_2 + \langle z^{2n-1}, |b|^2\rangle_2 = 0.$$

Thus

$$|\langle z^{2n+1}, |b|^2\rangle_2| = |\frac{\bar{\beta}}{\beta}\langle z^{2n-1}, |b|^2\rangle_2| = |\langle z^{2n-1}, |b|^2\rangle_2|.$$

By Lemma 4.4, we see that for every $n \geqslant 1$, $\langle z^{2n-1}, |b|^2\rangle_2 = 0$. Thus (4.12) shows that $\langle z^n, |b|^2\rangle_2 = 0$, which implies b is inner. A similar argument works for the case when $\beta = 0$ and $\alpha \neq 0$.

Next, suppose $\alpha\beta \neq 0$. Rewrite (4.13) as

$$\langle z^{2n+1}, |b|^2\rangle_2 + (\frac{1}{\bar{\alpha}} + \frac{1}{\beta})\langle z^{2n}, |b|^2\rangle_2 + \frac{1}{\bar{\alpha}\beta}\langle z^{2n-1}, |b|^2\rangle_2 = 0. \tag{4.14}$$

Consider the sequence $\{\langle z^n, |b|^2\rangle_2\}_{n=1}^{\infty}$. If it is the zero sequence, then b is an inner function. Otherwise by Lemma 4.4 and (4.12), (4.14), it satisfies the assumptions in Lemma 4.6. Then we have the following two cases:

Case I: $\beta = -\bar{\alpha}$, $\langle z^{2n-1}, |b|^2\rangle_2 = 0$, for every $n \geqslant 1$.

Case II: $|\alpha| = |\beta| = 1$.

By condition (2) in Lemma 4.2, we have for every $n \geqslant 0$,

$$\langle S^{*2n} f, S^{*2n} g\rangle_2 = 0.$$

Then

$$\begin{aligned} 0 =& \langle S^{*2n} f, S^{*2n} g\rangle_2 = \langle S^{*2n+1} b + \alpha S^{*2n+2} b, \beta S^{*2n+1} b + S^{*2n+2} b\rangle_2 \\ =& \bar{\beta}||S^{*2n+1} b||_2^2 + \alpha||S^{*2n+2} b||_2^2 + \langle S^{*2n+1} b, S^{*2n+2} b\rangle_2 + \alpha\bar{\beta}\langle S^{*2n+2} b, S^{*2n+1} b\rangle_2. \end{aligned}$$

For simplicity, let

$$c_n = \langle S^{*2n+1} b, S^{*2n+2} b\rangle_2.$$

Since

$$||S^{*2n+2} b||_2^2 = ||S^{*2n+1} b||_2^2 - |b_{2n+1}|^2,$$

we obtain

$$(\bar{\beta}+\alpha)||S^{*2n+1}b||_2^2 - \alpha|b_{2n+1}|^2 + c_n + \alpha\bar{\beta}\bar{c_n} = 0. \tag{4.15}$$

In Case I, $|b|^2$ is even, and Lemma 4.5 implies that b is even or odd. Thus $c_n = 0$ and (4.15) becomes $\alpha|b_{2n+1}|^2 = 0$, which implies $b_{2n+1} = 0$ and b is even.

In Case II, taking conjugate on (4.15), we get

$$(\beta+\bar{\alpha})||S^{*2n+1}b||_2^2 - \bar{\alpha}|b_{2n+1}|^2 + \bar{\alpha}\beta c_n + \bar{c_n} = 0. \tag{4.16}$$

Multiplying (4.15) by $\bar{\alpha}\beta$ and using $|\alpha| = |\beta| = 1$, we have

$$(\bar{\alpha}+\beta)||S^{*2n+1}b||_2^2 - \beta|b_{2n+1}|^2 + \bar{\alpha}\beta c_n + \bar{c_n} = 0. \tag{4.17}$$

By (4.16), (4.17), we have

$$(\bar{\alpha}-\beta)|b_{2n+1}|^2 = 0.$$

Note that $\bar{\alpha} \neq \beta$ because $\alpha\beta \neq 1$. We see that $b_{2n+1} = 0$, which means b is even. Using (4.12), we see that if b is not inner, then $\beta = -\bar{\alpha}$.

Sufficiency. Let

$$M_1 = \operatorname{Span}\{S^{*2n}(S^*b + \alpha S^{*2}b) : n \geqslant 0\}$$

and

$$M_2 = \operatorname{Span}\{S^{*2n}(-\bar{\alpha}S^*b + S^{*2}b) : n \geqslant 0\}.$$

We show that M_1, M_2 are reducing subspaces of X_b^2 for appropriate choices of α. By Theorem 3.2 and Lemma 4.2, we need to verify (4.2) and (4.3) when $\beta = -\bar{\alpha}$.

Note that

$$\langle z^{2n-1}, |b|^2\rangle_2 = 0, \quad \text{for every } n \geqslant 1,$$

whenever b is even or odd.

For (4.2), if $n \geqslant 1$, (4.2) follows from (4.12), (4.13) and the above relation. When $n = 0$, using Lemma 4.3, we have

$$\begin{aligned}
&\langle T_{\bar{b}}(S^*b + \alpha S^{*2}b),\ T_{\bar{b}}(-\bar{\alpha}S^*b + S^{*2}b)\rangle_{\bar{b}} \\
&= -\alpha||T_{\bar{b}}S^*b||_{\bar{b}}^2 + \alpha||T_{\bar{b}}S^{*2}b||_{\bar{b}}^2 + \langle T_{\bar{b}}S^*b, T_{\bar{b}}S^{*2}b\rangle_{\bar{b}} - \alpha^2\langle T_{\bar{b}}S^{*2}b, T_{\bar{b}}S^*b\rangle_{\bar{b}} \\
&= -\alpha(1-||b||_2^2) + \alpha(1-||b||_2^2) - \langle z, |b|^2\rangle_2 + \alpha^2\langle \bar{z}, |b|^2\rangle_2 = 0.
\end{aligned}$$

If b is odd and $\alpha = 0$, it is obvious that (4.3) holds.

If b is even, then $S^{*2}b$ is also even and

$$S^*b = zS^{*2}b.$$

We can write

$$S^*b + \alpha S^{*2}b = (S^{*2}b)(z+\alpha),$$

and

$$-\bar{\alpha}S^*b + S^{*2}b = (S^{*2}b)(-\bar{\alpha}z + 1).$$

Thus (4.3) is satisfied. □

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