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A Shimorin-type analytic model on an annulus for left-invertible operators and applications

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ABSTRACT

A new analytic model for left-invertible operators, which extends both Shimorin's analytic model for left-invertible and analytic operators and Gellar's model for bilateral weighted shift is introduced and investigated. We show that a left-invertible operator T, which satisfies certain conditions can be modeled as a multiplication operator $\mathscr{M}_z$ on a reproducing kernel Hilbert space of vector-valued analytic functions on an annulus or a disc. A similar result for composition operators in ℓ^2-spaces is established.

1. Introduction

The classical Wold decomposition theorem (see [60]) states that if U is isometry on Hilbert space $\mathcal{H}$, then $\mathcal{H}$ is the direct sum of two subspaces reducing U, $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_p$ such that $U|_{\mathcal{H}_u} \in \boldsymbol{B}(\mathcal{H}_u)$ is unitary and $U|_{\mathcal{H}_p} \in \boldsymbol{B}(\mathcal{H}_p)$ is unitarily equivalent to a unilateral shift. This decomposition is unique and the canonical subspaces are defined by

$$\mathcal{H}_u := \bigcap_{n=1}^{\infty} U^n\mathcal{H} \quad \text{and} \quad \mathcal{H}_p := \bigoplus_{n=1}^{\infty} U^n E,$$

where $E := \mathcal{N}(U^*) = \mathcal{H} \ominus U\mathcal{H}$. The Wold decomposition theorem and results analogous to this theorem plays an important role in many areas of operator theory, including the invariant subspace problem for Hilbert spaces of holomorphic functions. The interested reader is referred to [11,12,33,34,38,42,50,55,52].

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One of the key ideas in operator theory is that of viewing an operator as multiplication by z on a Hilbert space consisting of (vector-valued) holomorphic functions. The point is that this multiplication operator is much easier to analyze than is the case in the original setting because of the richer structure of a space of holomorphic functions. This is its great advantage and one of the reasons why it attracts attention of researchers. An excellent example of an interplay between weighted shift operators and analytic functions is the problem of describing all invariant subspaces of weighted shifts and the celebrated Beurling-Lax theorem. There are numerous results in the literature relating analytic models for the operators in certain classes. We mention some selected models:

- the Sz.-Nagy-Foiaş model for contraction (see [3]),
- the analytic model for weighted shifts (see [24]),
- the analytic model of a pure hyponormal operator T with rank one self-commutators $[T^*, T]$ (see [46]),
- the model for the class $\mathscr{F}$ of pure operators T on a Hilbert space $\mathcal{H}$ satisfying

$$\langle T^m g, T^n h\rangle = 0, \qquad g, h \in [T^*, T]\mathcal{H},\ m \neq n,\ m, n \in \mathbb{N},$$

  where $[T^*, T] := T^*T - TT^*$ (see [64]),
- Shimorin's analytic model for left-invertible analytic operators (see [52]),
- the analytic model of doubly commuting contractions (see [4]).

The interested reader is referred to [15,32,61–64] for further information.

In [52] S. Shimorin obtain a weak analog of the Wold decomposition theorem, representing operator close to isometry in some sense as a direct sum of a unitary operator and a shift operator acting in some reproducing kernel Hilbert space of vector-valued holomorphic functions defined on a disc. The construction of the Shimorin's model for a left-invertible analytic operator $T \in \boldsymbol{B}(\mathcal{H})$ is as follows. Let $E := \mathcal{N}(T^*)$ and define a vector-valued holomorphic functions U_x as

$$U_x(z) = \sum_{n=0}^{\infty} (P_E T'^{*n} x) z^n, \quad z \in \mathbb{D}(r(T')^{-1}),$$

where T' is the Cauchy dual of T. Then we equip the obtained space of analytic functions $\mathscr{H} := \{U_x : x \in \mathcal{H}\}$ with the inner product induced by $\mathcal{H}$. The operator $U : \mathcal{H} \ni x \to U_x \in \mathscr{H}$ becomes a unitary isomorphism. It turns out that the operator T is unitarily equivalent to the operator $\mathscr{M}_z$ of multiplication by z on $\mathscr{H}$ and T'^* is unitarily equivalent to the operator $\mathscr{L}$ given by

$$(\mathscr{L} f)(z) = \frac{f(z) - f(0)}{z}, \quad f \in \mathscr{H}.$$

Moreover, Shimorin proved that $\mathscr{H}$ is a reproducing kernel Hilbert space in the following sense: *the reproducing kernel* for $\mathscr{H}$ (see [52]) is an $\boldsymbol{B}(E)$-valued function of two variables $\kappa_{\mathscr{H}} : \Omega \times \Omega \to \boldsymbol{B}(E)$ such that

(i) for any $e \in E$ and $\lambda \in \Omega$

$$\kappa_{\mathscr{H}}(\cdot, \lambda) e \in \mathscr{H},$$

(ii) for any $e \in E$, $f \in \mathscr{H}$ and $\lambda \in \Omega$

$$\langle f(\lambda), e\rangle_E = \langle f, \kappa_{\mathscr{H}}(\cdot, \lambda) e\rangle_{\mathscr{H}},$$

where $\Omega \subset \mathbb{C}$. The interested reader is referred to [56] and [57] for further facts concerning the reproducing kernel Hilbert space and its multiplication operators.

The substitution operation is basic to mathematics therefore composition operators naturally appear in many areas of mathematics. They play an important role in ergodic theory and functional analysis. The class of composition operators is related to other areas of operator theory in somewhat surprising ways. S. Banach and M. Stone proved that a surjective linear isometry $T : C(X) \to C(Y)$ is a weighted composition operator. The analogical result for the Hardy spaces $H^p(\mathbb{D})$ (with $p \geqslant 1$ and $p \neq 2$) was shown by Forelli in [22]. Furthermore, commutants of many analytic Toeplitz operators are generated by composition and multiplication operators. The literature on this subject and related topics is vast and still growing (see e.g., [2,6,8,13,14,19,21,26,36,37,35,40,41,43,53,54,59]).

The class of weighted shifts on a directed tree was introduced in [27] and intensively studied since then (see e.g., [5–7,9,15,16,23,28,39]). Z.J. Jabłoński, I.B. Jung and J. Stochel realized the importance of this class as a vehicle to collect a number of interesting examples and counterexamples (see e.g., [10,20,28–31, 43,44,58]).

The analytic aspects of the theory of composition operators, weighted shifts and weighted shifts on a directed tree were studied by many authors. As was mentioned by A.L. Shields in the paper [51] the fact that weighted shift can be viewed as multiplication by z on a Hilbert space of formal power series has been long folklore and this point of view was taken by R. Gellar (see [24,25]). He showed that the commutant of any weighted shift operator consists of certain formal power series in the operator, and hence that the commutant is abelian. According to [51] the spectrum of weighted shift operator is either an annulus or a disk. Some results on the spectrum and commutants of composition operators were obtained in [47] and [14]. In [15] S. Chavan and S. Trivedi showed that a weighted shift S_λ on a rooted directed tree with finite branching index is analytic therefore can be modeled as a multiplication operator $\mathscr{M}_z$ on a reproducing kernel Hilbert space $\mathscr{H}$ of E-valued holomorphic functions on a disc centered at the origin, where $E := \mathcal{N}(S_\lambda^*)$. Moreover, they proved that the reproducing kernel associated with $\mathscr{H}$ is multi-diagonal. It is worth pointing out that the commutant and reflexivity for n-tuples of multiplication operators by independent variables $z_1, \ldots, z_n$ on a reproducing Hilbert space of vector-valued holomorphic functions were studied in the paper [17].

Recently, the analytic structure of weighted shifts on directed trees was also studied by P. Budzyński, P. Dymek, A. Płaneta and M. Ptak. In [7] they showed that a weighted shift on a rooted directed tree is related to a multiplier algebra of coefficients of analytic functions. They used this relation to provide a kind of functional calculus for functions from multiplier algebras and to study spectral properties of weighted shift on a rooted directed tree. Moreover in [20] they extended the notion of multipliers to left-invertible and analytic operators and characterize the commutant of such operators in terms of generalized multipliers. This line of investigation was continued in [45].

In this paper, we provide a new analytic model for left-invertible operators, which extends both Shimorin's analytic model for left-invertible and analytic operators (see Theorem 3.3) and Gellar's model for a bilateral weighted shift (see Example 5.2). We show that a left-invertible operator T, which satisfies certain conditions can be modeled as a multiplication operator $\mathscr{M}_z$ on a reproducing kernel Hilbert space of vector-valued analytic functions on an annulus or a disc (see Theorem 3.2). As an application of this model, we obtain significantly improved model for weighted composition operator upon provided the symbol of this operator has finite branching index (see Theorem 4.3). In particular, we describe the inner and outer radius of convergence for weighted composition operators only in terms of its weight and symbol.

2. Preliminaries

In this paper, we use the following notation. The fields of rational, real and complex numbers are denoted by $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. The symbols $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ stand for the sets of integers, positive integers and nonnegative integers, respectively. Set $\mathbb{D}(r) := \{z \in \mathbb{C} : |z| < r\}$ and $\mathbb{A}(r^-, r^+) := \{z \in \mathbb{C} : r^- < |z| < r^+\}$ for $r, r^-, r^+ \in [0, \infty)$. The expression "a countable set" means a finite set or a countably infinite set.

All Hilbert spaces considered in this paper are assumed to be complex. Let W be a subset of $\mathcal{H}$. Then $\operatorname{lin} W$, $\bigvee W$ stands for the smallest linear subspace, closed subspace generated by W, respectively. We use the notation $\langle x \rangle$ in place of $\operatorname{lin}\{x\}$, for $x \in \mathcal{H}$. Let T be a linear operator in a complex Hilbert space $\mathcal{H}$. Denote by T^* the adjoint of T. We write $\boldsymbol{B}(\mathcal{H})$ for the C^*-algebra of all bounded operators. The spectrum, point spectrum and spectral radius of $T \in \boldsymbol{B}(\mathcal{H})$ is denoted by $\sigma(T)$, $\sigma_p(T)$ and $r(T)$ respectively. We say that $T \in \boldsymbol{B}(\mathcal{H})$ is left-invertible if there exists $S \in \boldsymbol{B}(\mathcal{H})$ such that $ST = I$. The *Cauchy dual operator* T' of a left-invertible operator $T \in \boldsymbol{B}(\mathcal{H})$ is defined by

$$T' := T(T^*T)^{-1}.$$

Note that T is left-invertible if and only if there exists a constant $c > 0$ such that $T^*T \geqslant cI$. The notion of the Cauchy dual operator has been introduced and studied by Shimorin in the context of the wandering subspace problem for Bergman-type operators [52]. We call T *analytic* if $\mathcal{H}_\infty := \bigcap_{i=1}^\infty T^i\mathcal{H} = \{0\}$. Let $\Omega \subset \mathbb{C}$ be such that $\operatorname{int}\Omega = \Omega \neq \emptyset$. A function $f : \Omega \to \mathcal{H}$ is said to be *holomorphic* on Ω if f is differentiable.

Let X be a set and $\varphi : X \to X$. If $n \in \mathbb{Z}_+$, then the n-th iterate of φ is given by $\varphi^{(n)} := \underbrace{\varphi \circ \varphi \circ \cdots \circ \varphi}_{n}$, φ composed with itself n-times and $\varphi^{(0)}$ is identity function. For $x \in X$ the set

$$[x]_\varphi := \{y \in X : \text{there exist } i, j \in \mathbb{N} \text{ such that } \varphi^{(i)}(x) = \varphi^{(j)}(y)\}$$

is called the *orbit of* φ containing x. If $x \in X$ and $\varphi^{(i)}(x) = x$ for some $i \in \mathbb{Z}_+$, then the *cycle of* φ containing x is the set

$$\mathscr{C}_\varphi := \{\varphi^{(i)}(x) : i \in \mathbb{N}\}.$$

Define the function $[\varphi] : X \to \mathbb{Z}$ by

(i) $[\varphi](x) = 0$ if x is in the cycle of φ,
(ii) $[\varphi](x^*) = 0$, where x^* is a fixed element of orbit F of φ not containing a cycle,
(iii) $[\varphi](\varphi(x)) = [\varphi](x) - 1$ if x is not in a cycle of φ.

We set

$$\operatorname{Gen}_\varphi(m, n) := \{x \in X : m \leqslant [\varphi](x) \leqslant n\}$$

for $m, n \in \mathbb{Z}$.

Let $(X, \mathscr{A}, \mu)$ be a μ-finite measure space, $\varphi : X \to X$ and $w : X \to \mathbb{C}$ be measurable transformations. By a *weighted composition operator* $C_{\varphi,w}$ in $L^2(\mu)$ we mean a mapping

$$\mathcal{D}(C_{\varphi,w}) := \{f \in L^2(\mu) : w(f \circ \varphi) \in L^2(\mu)\},$$
$$C_{\varphi,w} f := w(f \circ \varphi), \quad f \in \mathcal{D}(C_{\varphi,w}).$$

We call φ and w the *symbol* and the *weight* of $C_{\varphi,w}$ respectively.

Let us recall some useful properties of composition operator we need in this paper:

Lemma 2.1. *Let X be a countable set, $\varphi : X \to X$ and $w : X \to \mathbb{C}$ be measurable transformations. If $C_{\varphi,w} \in \boldsymbol{B}(\ell^2(X))$, then for any $x \in X$ and $n \in \mathbb{Z}_+$*

(i) $C^*_{\varphi,w}e_x = \overline{w(x)}e_{\varphi(x)}$,
(ii) $C_{\varphi,w}e_x = \sum_{y\in\varphi^{-1}(x)} w(y)e_y$,
(iii) $C^{*n}_{\varphi,w}e_x = \overline{w(x)w(\varphi(x))\cdots w(\varphi^{(n-1)}(x))}e_{\varphi^{(n)}(x)}$,
(iv) $C^n_{\varphi,w}e_x = \sum_{y\in\varphi^{-n}(x)} w(y)w(\varphi(y))\cdots w(\varphi^{(n-1)}(y))e_y$,
(v) $C^*_{\varphi,w}C_{\varphi,w}e_x = \Big(\sum_{y\in\varphi^{-1}(x)} |w(y)|^2\Big)e_x$.

Proof. (i) and (ii) See [14, page 633].
(iii) and (iv) Apply (i), (ii) and induction on n.
(v) This follows from (i) and (ii). □

We now describe Cauchy dual of weighted composition operator.

Lemma 2.2. *Let X be a countable set, $\varphi : X \to X$ and $w : X \to \mathbb{C}$ be measurable transformations. If $C_{\varphi,w} \in \boldsymbol{B}(\ell^2(X))$ is left-invertible operator, then the Cauchy dual $C'_{\varphi,w}$ of $C_{\varphi,w}$ is also a weighted composition operator $C_{\varphi',w}$ with the same symbol $\varphi : X \to X$ and weight $w' : X \to \mathbb{C}$ defined by*

$$w'(x) := \frac{w(x)}{\Big(\sum_{y\in\varphi^{-1}(\varphi(x))} |w(y)|^2\Big)}.$$

Proof. This is a direct consequence of assertions (i) and (ii) of Lemma 2.1. □

Let $\mathscr{T} = (V;E)$ be a directed tree (V and E are the sets of vertices and edges of $\mathscr{T}$, respectively). For any vertex $u \in V$ we put $\mathrm{Chi}(u) = \{v \in V : (u,v) \in E\}$. Denote by par the partial function from V to V which assigns to a vertex u a unique $v \in V$ such that $(v,u) \in E$. A vertex $u \in V$ is called a root of $\mathscr{T}$ if u has no parent. If $\mathscr{T}$ has a root, we denote it by root. Put $V^\circ = V \setminus \{\mathrm{root}\}$ if $\mathscr{T}$ has a root and $V^\circ = V$ otherwise. The Hilbert space of square summable complex functions on V equipped with the standard inner product is denoted by $\ell^2(V)$. For $u \in V$, we define $e_u \in \ell^2(V)$ to be the characteristic function of the set $\{u\}$. It turns out that the set $\{e_v\}_{v\in V}$ is an orthonormal basis of $\ell^2(V)$. We put $V_\prec := \{v \in V : \mathrm{card}(\mathrm{Chi}(v)) \geqslant 2\}$ and call the member of this set a *branching vertex* of $\mathscr{T}$.

Given a system $\lambda = \{\lambda_v\}_{v\in V^\circ}$ of complex numbers, we define the operator S_λ in $\ell^2(V)$, which is called a *weighted shift* on $\mathscr{T}$ with weights λ, as follows

$$\mathcal{D}(S_\lambda) := \{f \in \ell^2(V) : \Lambda_{\mathscr{T}} f \in \ell^2(V)\} \quad \text{and} \quad S_\lambda f := \Lambda_{\mathscr{T}} f \quad \text{for} \quad f \in \mathcal{D}(S_\lambda),$$

where

$$(\Lambda_{\mathscr{T}} f)(v) := \begin{cases} \lambda_v f(\mathrm{par}(v)) & \text{if } v \in V^\circ, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.3 *(Proposition 3.5.1 [27]). If S_λ is a densely defined weighted shift on a directed tree $\mathscr{T}$ with weights $\lambda = \{\lambda_v\}_{v\in V^\circ}$, then*

$$\mathcal{N}(S^*_\lambda) = \begin{cases} \langle e_{\mathrm{root}}\rangle \oplus \bigoplus_{u\in V_\prec}(\ell^2(\mathrm{Chi}(u)) \ominus \langle\lambda^u\rangle) & \text{if } \mathscr{T} \text{ has a root,} \\ \bigoplus_{u\in V_\prec}(\ell^2(\mathrm{Chi}(u)) \ominus \langle\lambda^u\rangle) & \text{otherwise,} \end{cases}$$

where $\lambda^u \in \ell^2(\mathrm{Chi}(u))$ is given by $\lambda^u : \ell^2(\mathrm{Chi}(u)) \ni v \to \lambda_v \in \mathbb{C}$.

A subgraph of a directed tree $\mathscr{T}$ which itself is a directed tree will be called a subtree of $\mathscr{T}$. We refer the reader to [27] for more details on weighted shifts on directed trees.

3. Analytic model

This section provides an analytic model for left-invertible operators. We show that a left-invertible operator, which satisfies certain conditions can be modeled as a multiplication operator on a reproducing kernel Hilbert space of vector-valued analytic functions on an annulus or a disc.

Let $T \in \boldsymbol{B}(\mathcal{H})$ be a left-invertible operator and E be a subspace of $\mathcal{H}$ denote by $[E]_{T^*,T'}$ the following subspace of $\mathcal{H}$:

$$[E]_{T^*,T'} := \bigvee \big(\{T^{*n}x \colon x \in E, n \in \mathbb{N}\} \cup \{T'^{n}x \colon x \in E, n \in \mathbb{N}\}\big),$$

where T' is the Cauchy dual of T.

To avoid the repetition, we state the following assumption which will be used frequently in this section.

$$\text{The operator } T \in \boldsymbol{B}(\mathcal{H}) \text{ is left-invertible and } E \text{ is a closed subspace of } \mathcal{H} \text{ such that } [E]_{T^*,T'} = \mathcal{H}. \tag{$\clubsuit$}$$

Suppose ($\clubsuit$) holds. In this case, we may construct a Hilbert space $\mathscr{H}$ associated with T, of formal Laurent series with vector coefficients. We proceed as follows. For each $x \in \mathcal{H}$, define a formal Laurent series U_x with vector coefficients as

$$U_x(z) = \sum_{n=1}^{\infty} (P_E T^n x)\frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E T'^{*n} x) z^n. \tag{3.1}$$

Let $\mathscr{H}$ denote the vector space of formal Laurent series with vector coefficients of the form U_x, $x \in \mathcal{H}$. Consider the map $U : \mathcal{H} \to \mathscr{H}$ defined by $Ux := U_x$. As shown in Lemma 3.1 below, by the assumption U is injective. In particular, we may equip the space $\mathscr{H}$ with the inner product induced from $\mathcal{H}$, so that U is unitary isomorphism. Observe that every $f \in \mathscr{H}$ can be represented as follows

$$f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n) z^n,$$

where

$$\hat{f}(n) = \begin{cases} P_E T'^{*n} U^* f & \text{if } n \in \mathbb{N}, \\ P_E T^{-n} U^* f & \text{if } n \in \mathbb{Z} \setminus \mathbb{N}. \end{cases}$$

Lemma 3.1. *Suppose ($\clubsuit$) holds and $\mathscr{H}$, U are as above. Then $\mathcal{N}(U) = \{0\}$.*

Proof. Suppose that $x \in \mathcal{H}$ is such that

$$P_E T^n x = 0 \quad \text{and} \quad P_E T'^{*n} x = 0, \quad n \in \mathbb{N}.$$

Then for every $y \in E$

$$\langle T^n x, y \rangle = 0 \quad \text{and} \quad \langle T'^{*n} x, y \rangle = 0, \quad n \in \mathbb{N}.$$

This implies

$$\langle x, T^{*n} y \rangle = 0 \quad \text{and} \quad \langle x, T'^{n} y \rangle = 0, \quad n \in \mathbb{N}.$$

We see that the above condition is equivalent to the following one

$$x \perp [E]_{T^*,T'} (= \mathcal{H}).$$

This completes the proof. □

As shown below, the operator T is unitary equivalent to the operator $\mathscr{M}_z : \mathscr{H} \to \mathscr{H}$ of multiplication by z on $\mathscr{H}$ given by

$$(\mathscr{M}_z f)(z) = z f(z), \quad f \in \mathscr{H},$$

and operator T'^* is unitary equivalent to the operator $\mathscr{L} : \mathscr{H} \to \mathscr{H}$ given by

$$(\mathscr{L} f)(z) = \frac{f(z) - (P_{\mathcal{N}(\mathscr{M}_z^*)} f)(z)}{z}, \quad f \in \mathscr{H}.$$

Theorem 3.2. *Suppose (♣) holds. Then the following assertions are valid:*

(i) $UT = \mathscr{M}_z U$,
(ii) $UT'^* = \mathscr{L} U$.

Proof. (i) Let $x \in \mathcal{H}$. Applying (3.1) to operator T and vector Tx, we see that

$$\begin{aligned}(UTx)(z) &= \sum_{n=1}^{\infty} (P_E T^n T x) \frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E T'^{*n} T x) z^n \\ &= \sum_{n=1}^{\infty} (P_E T^{n+1} x) \frac{1}{z^n} + (P_E T x) + \sum_{n=1}^{\infty} (P_E T'^{*n-1} x) z^n \\ &= z (Ux)(z).\end{aligned}$$

(ii) Since

$$\begin{aligned}(UT'^* x)(z) &= \sum_{n=1}^{\infty} (P_E T^n T'^* x) \frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E T'^{*n+1} x) z^n \\ &= \sum_{n=1}^{\infty} (P_E T^{n-1} (I - P_{\mathcal{N}(T^*)}) x) \frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E T'^{*n+1} x) z^n \\ &= \frac{(Ux)(z) - (U P_{\mathcal{N}(T^*)} x)(z)}{z} = (\mathscr{L} U x)(z)\end{aligned}$$

the proof is complete. □

Now we show that in the case of a left-invertible and analytic operators our analytic model with $E := \mathcal{N}(T^*)$ coincides with the Shimorin's analytic model.

Theorem 3.3. *Let $T \in \boldsymbol{B}(\mathcal{H})$ be left-invertible and analytic, $\mathscr{H}_1$, U_1 be the Hilbert space and the unitary isomorphism constructed in (3.1) with $E := \mathcal{N}(T^*)$ and $\mathscr{H}_2$, U_2 be the Hilbert space and the unitary isomorphism obtained in Shimorin's construction. Then $\mathscr{H}_1 = \mathscr{H}_2$ and $U_1 = U_2$.*

Proof. Set $\mathcal{H}_\infty := \bigcap_{i=0}^\infty T^i\mathcal{H}$. By [52, Proposition 2.7], $\mathcal{H}_\infty^\perp = [E]_{T'}$. Since T is analytic $\mathcal{H}_\infty = \{0\}$, we see that $[E]_{T'} = \mathcal{H}$. Therefore, condition (♣) is satisfied. By kernel-range decomposition, $P_{\mathcal{N}(T^*)}T^n = 0$ for $n \in \mathbb{Z}_+$. Hence, the first sum in (3.1) vanishes. This completes the proof. □

Now we describe how to obtain a collection of subspaces of $\mathcal{H}$ with property (♣) from a single subspace with this property.

Theorem 3.4. *Suppose (♣) holds. Then for every $m \in \mathbb{N}$ the following assertions hold:*

(i) *$T'^m E$ is a closed supspace and $[T'^m E]_{T^*,T'} = \mathcal{H}$,*
(ii) *the mapping $\Phi_m : \mathscr{H}_0 \to \mathscr{H}_m$ defined by*

$$\Phi_m\Big(\sum_{n=-\infty}^{\infty} a_n z^n\Big) := \sum_{n=-\infty}^{\infty} (V_m a_{m+n}) z^n, \quad \sum_{n=-\infty}^{\infty} a_n z^n \in \mathscr{H}_0$$

is a unitary isomorphism, where $\mathscr{H}_k$ for $k \in \mathbb{N}$ is the Hilbert space constructed in (3.1) with subspace $T'^k E$ and $V_k : E \to T'^k E$ for $k \in \mathbb{N}$ is defined by,

$$Ve = P_{T'^k E} T^k e, \qquad e \in E.$$

Proof. (i) Since $T^*T' = I$, we get that $[T'^m E]_{T^*,T'} = [E]_{T^*,T'}$, $m \in \mathbb{N}$. This in turn implies that $[T'^m E]_{T^*,T'} = \mathcal{H}$. The operator T'^m is left-infertible and hence bounded below. This implies that the subspace $T'^m E$ is closed.

(ii) We will denote by U_k, $k \in \mathbb{N}$ the unitary operator of the form (3.1) between $\mathcal{H}$ and $\mathscr{H}_k$. Fix $m \in \mathbb{N}$. First, we note that for every $e \in E$ and $n \in \mathbb{N}$, we have

$$\langle T^n x, T'^m e\rangle = \begin{cases} \langle T^{n-m}x, e\rangle & \text{if } n \geqslant m, \\ \langle T'^{*m-n}x, e\rangle & \text{if } n < m. \end{cases} \tag{3.2}$$

Take $y \in T'^m E$. Then there exists $e \in E$ such that $y = T'^m e$. Employing (3.2), we verify that

$$\begin{aligned} \langle \widehat{(U_m x)}(-n), y\rangle &= \langle P_{T'^m E} T^n x, y\rangle = \langle T^n x, T'^m e\rangle \overset{(3.2)}{=} \langle \widehat{(U_0 x)}(m-n), e\rangle \\ &= \langle \widehat{(U_0 x)}(m-n), T^{*m}T'^m e\rangle = \langle T^m \widehat{(U_0 x)}(m-n), T'^m e\rangle \\ &= \langle P_{T'^m E} T^m \widehat{(U_0 x)}(m-n), y\rangle, \end{aligned}$$

for $n \in \mathbb{N}$. This implies that

$$\widehat{(U_m x)}(-n) = P_{T'^m E} T^m \widehat{(U_0 x)}(m-n), \qquad n \in \mathbb{N}. \tag{3.3}$$

Arguing as above, we deduce that

$$\begin{aligned} \langle \widehat{(U_m x)}(n), y\rangle &= \langle P_{T'^m E} T'^{*n} x, y\rangle = \langle T'^{*n} x, T'^m e\rangle = \langle T'^{*n+m} x, e\rangle \\ &= \langle P_E T'^{*n+m} x, T^{*m} T'^m e\rangle = \langle T^m P_E T'^{*n+m} x, T'^m e\rangle \\ &= \langle P_{T'^m E} T^m \widehat{(U_0 x)}(m+n), y\rangle, \end{aligned}$$

for $n \in \mathbb{N}$. As a consequence, we see that

$$\widehat{(U_m x)}(n) = P_{T'^m E} T^m \widehat{(U_0 x)}(m+n), \qquad n \in \mathbb{N}.$$

This and (3.3) imply that Φ_m is an isomorphism. Since the Hilbert space structure on $\mathscr{H}_k$ for $k \in \mathbb{N}$ is induced from $\mathcal{H}$, we deduce that Φ_m is unitary. This completes the proof. $\square$

For left-invertible operator $T \in \boldsymbol{B}(\mathcal{H})$, among all subspaces satisfying condition (♣) we distinguish those subspaces E which satisfy the following condition

$$E \perp T^n E \qquad \text{and} \qquad E \perp T'^n E, \qquad n \in \mathbb{Z}_+. \tag{♠}$$

A similar condition was studied in the context of 2-isometries in [1] where analog of Wold decompositions was obtained.

Theorem 3.5. *Suppose (♣) holds. Then the following assertions hold:*

(i) *if additionally (♠) holds, then $U(E)$ is a copy of E in $\mathscr{H}$, the subspace consisting of constant functions; moreover, E-valued polynomials in z are included in $\mathscr{H}$,*
(ii) $(\mathscr{M}_z^* \mathscr{M}_z)^{-1} \mathscr{M}_z^* = \mathscr{L}$.

Proof. (i) This is obvious.

(ii) Fix any $x \in \mathcal{H}$. Combining Theorem 3.2 and the kernel-range decomposition, we deduce that

$$\begin{aligned}(\mathscr{L}\mathscr{M}_z Ux)(z) &= \frac{(\mathscr{M}_z Ux)(z) - (UP_{\mathcal{N}(T^*)}U^{-1}\mathscr{M}_z Ux)(z)}{z} \\ &= \frac{z(Ux)(z) - (UP_{\mathcal{N}(T^*)}Tx)(z)}{z} = (Ux)(z),\end{aligned}$$

which means that $\mathscr{L}$ is a left-inverse of $\mathscr{M}_z$. Since

$$\mathscr{L}\mathscr{M}_z = I \quad \text{and} \quad (\mathscr{M}_z^* \mathscr{M}_z)^{-1} \mathscr{M}_z^* \mathscr{M}_z = I,$$

we see that $\mathscr{L}|_{\mathcal{R}(\mathscr{M}_z)} = (\mathscr{M}_z^* \mathscr{M}_z)^{-1} \mathscr{M}_z^*|_{\mathcal{R}(\mathscr{M}_z)}$. One can verify that

$$\mathscr{L}|_{\mathcal{N}(\mathscr{M}_z)} = (\mathscr{M}_z^* \mathscr{M}_z)^{-1} \mathscr{M}_z^*|_{\mathcal{N}(\mathscr{M}_z)},$$

which completes the proof. $\square$

Now we shall discuss the extent to which our formal Laurent series actually represent analytic functions. If the series (3.1) is convergent in E on an open and nonempty subset $\Omega \subset \mathbb{C}$ for every $x \in \mathcal{H}$, then based upon Lemma 3.6 below we regard the Hilbert space $\mathscr{H}$ as a space of vector-valued holomorphic functions on Ω by identifying each formal Laurent series (3.1) with the function

$$\tilde{U}_x \colon \Omega \ni z \to \sum_{n=1}^{\infty} (P_E T^n x) \frac{1}{z^n} + \sum_{n=0}^{\infty} (P_E T'^{*n} x) z^n \in E.$$

Lemma 3.6. *Let $\sum_{n=-\infty}^{\infty} a_n z^n$ be the formal Laurent series which represent constant zero function on an open and nonempty subset $\Omega \subset \mathbb{C}$. Then $a_n = 0$, $n \in \mathbb{Z}$.*

Proof. Take $e \in E$. Then the function $\Omega \ni z \to \sum_{n=-\infty}^{\infty} \langle a_n, e\rangle z^n \in \mathbb{C}$ is holomorphic, on the one hand, and identically equal to zero, on the other. By Identity theorem $\langle a_n, e\rangle = 0$, $n \in \mathbb{Z}$. This shows that $a_n = 0$, $n \in \mathbb{Z}$ and hence completes the proof. $\square$

Theorem 3.7. *Suppose (♣) holds. Let*

$$r^+ := \inf_{x\in\mathcal{H}} \liminf_{n\to\infty} \|P_E T'^{*n} x\|^{-\frac{1}{n}},$$

$$r^- := \sup_{x\in\mathcal{H}} \limsup_{n\to\infty} \|P_E T^n x\|^{\frac{1}{n}}.$$

If $r^+ > r^-$, then formal Laurent series (3.1) represent analytic functions on annulus $\mathbb{A}(r^-, r^+)$.

Proof. Fix $x \in \mathcal{H}$. An application of the root test [49, page 198] shows that the radius of convergence of the regular part of the series (3.1) is

$$R(x) = \liminf_{n\to\infty} \|P_E T'^{*n} x\|^{-\frac{1}{n}},$$

and the radius of convergence of the principal part of this series is

$$r(x) = \limsup_{n\to\infty} \|P_E T^n x\|^{\frac{1}{n}}.$$

This implies that the regular part and principal part are convergent for every $x \in \mathcal{H}$ in the disc $\mathbb{D}(r^+)$ and in the set $\mathbb{C} \setminus \mathbb{D}(r^-)$, respectively. This completes the proof. □

As will be shown below, if the series (3.1) is convergent in E on $\Omega \subset \mathbb{C}$ for every $x \in \mathcal{H}$, then $\mathscr{H}$ is a reproducing kernel Hilbert space of vector-valued holomorphic functions on Ω.

Theorem 3.8. *Suppose (♣) holds and the series (3.1) is convergent in E on an annulus $\mathbb{A}(r^-, r^+)$ with $r^- < r^+$ and $r^-, r^+ \in [0, \infty)$ for every $x \in \mathcal{H}$. Then $\mathscr{H}$ is a reproducing kernel Hilbert space of E-valued holomorphic functions on $\mathbb{A}(r^-, r^+)$. The reproducing kernel $\kappa_{\mathscr{H}} : \mathbb{A}(r^-, r^+) \times \mathbb{A}(r^-, r^+) \to \boldsymbol{B}(E)$ associated with $\mathscr{H}$ is given by*

$$\begin{aligned}\kappa_{\mathscr{H}}(z,\lambda) = &\sum_{i,j\geqslant 1} P_E T^i T^{*j}|_E \frac{1}{z^i}\frac{1}{\bar{\lambda}^j} + \sum_{i\geqslant 1, j\geqslant 0} P_E T^i T'^j|_E \frac{1}{z^i}\bar{\lambda}^j \\ &+ \sum_{i\geqslant 0, j\geqslant 1} P_E T'^{*i} T^{*j}|_E z^i \frac{1}{\bar{\lambda}^j} + \sum_{i,j\geqslant 0} P_E T'^{*i} T'^j|_E z^i \bar{\lambda}^j,\end{aligned} \tag{3.4}$$

for any $z, \lambda \in \mathbb{A}(r^-, r^+)$. Moreover, the following assertions hold.

(i) *For any $\lambda \in \mathbb{A}(r^-, r^+)$*

$$\sum_{n=1}^{\infty} (P_E T^n) \frac{1}{\lambda^n} + \sum_{n=0}^{\infty} (P_E T^{*\prime n}) \lambda^n \in \boldsymbol{B}(\mathcal{H}, E), \tag{3.5}$$

$$\sum_{n=1}^{\infty} T^{*n} \frac{1}{\lambda^n} + \sum_{n=0}^{\infty} T'^n \lambda^n \in \boldsymbol{B}(E, \mathcal{H}). \tag{3.6}$$

(ii) *The series (3.4), (3.5) and (3.6) converges absolutely and uniformly in operator norm on any compact set contained in $\mathbb{A}(r^-, r^+) \times \mathbb{A}(r^-, r^+)$, $\mathbb{A}(r^-, r^+)$ and $\mathbb{A}(r^-, r^+)$, respectively.*

(iii) *The function $\mathbb{A}(r^-, r^+) \ni \lambda \to \kappa_{\mathscr{H}}(\cdot, \bar{\lambda})e \in \mathscr{H}$, $e \in E$ is holomorphic and given by*

$$\kappa_{\mathscr{H}}(\cdot, \bar{\lambda})e = \sum_{n=1}^{\infty} U T^{*n} e \frac{1}{\lambda^n} + \sum_{n=0}^{\infty} U T'^n e \lambda^n, \qquad \lambda \in \mathbb{A}(r^-, r^+).$$

Proof. We claim that series

$$\sum_{n=0}^{\infty}(P_E T'^{*n})\lambda^n,$$

converges absolutely and uniformly in the norm of $\boldsymbol{B}(\mathcal{H}, E)$ on any compact set contained in $\mathbb{A}(r^-, r^+)$. Fix $r < r^+$. It follows from our assumptions on the series in (3.1) that series

$$\sum_{n=0}^{\infty}(P_E T'^{*n} x) r^n,$$

converges for every $x \in \mathcal{H}$. Thus, there exists a constant $C(r, x) > 0$ such that

$$\|(P_E T'^{*n} x) r^n\| < C(r, x), \qquad n \in \mathbb{N}.$$

By uniform boundedness principle (see [48, Theorem 2.6]) we obtain that there exists a constant $M(r) > 0$ such that

$$\|(P_E T'^{*n}) r^n\| < M(r), \qquad n \in \mathbb{N}.$$

If $|\lambda| < r$, then applying the above, we see that

$$\begin{aligned}\|\sum_{n=0}^{\infty}(P_E T'^{*n})\lambda^n\| &\leqslant \sum_{n=0}^{\infty}\|(P_E T'^{*n})\lambda^n\| \leqslant \sum_{n=0}^{\infty}\|(P_E T'^{*n}) r^n\| \Big[\frac{|\lambda|}{r}\Big]^n \\ &\leqslant M(r)\sum_{n=0}^{\infty}\Big[\frac{|\lambda|}{r}\Big]^n.\end{aligned}$$

This proves our claim. Following steps analogous to those above, we obtain that

$$\sum_{n=1}^{\infty}(P_E T^n)\frac{1}{\lambda^n}$$

also converges absolutely and uniformly in the norm of $\boldsymbol{B}(\mathcal{H}, E)$ on any compact set contained in $\mathbb{A}(r^-, r^+)$. It follows from what has already been proved that the same conclusion holds also for the series in (3.5). This implies that the series (3.6) converges absolutely and uniformly in the norm of $\boldsymbol{B}(E, \mathcal{H})$ on any compact set contained in $\mathbb{A}(r^-, r^+)$. Since the operator (3.4) is a composition of the operators in (3.5) and (3.6), the assertions (i) and (ii) are justified.

Let $\lambda \in \mathbb{A}(r^-, r^+)$ and $e \in E$. Then

$$\begin{aligned}\langle f(\lambda), e\rangle_E &= \langle \sum_{n=1}^{\infty}(P_E T^n U^{-1} f)\frac{1}{\lambda^n} + \sum_{n=0}^{\infty}(P_E T'^{*n} U^{-1} f)\lambda^n, e\rangle_E \\ &= \langle U^{-1} f, \sum_{n=1}^{\infty} T^{*n} e\frac{1}{\bar{\lambda}^n} + \sum_{n=0}^{\infty} T'^n e\bar{\lambda}^n\rangle_{\mathcal{H}}\end{aligned}$$

for any $f \in \mathscr{H}$. As a consequence, we obtain

$$\kappa_{\mathscr{H}}(\cdot, \lambda) = U\big(\sum_{n=1}^{\infty} T^{*n}\frac{1}{\bar{\lambda}^n} + \sum_{n=0}^{\infty} T'^n\bar{\lambda}^n\big). \tag{3.7}$$

This implies that

$$\begin{aligned}\langle \kappa_{\mathscr{H}}(z,\lambda)e_0, e_1\rangle_E &= \langle \kappa_{\mathscr{H}}(\cdot,\lambda)e_0, \kappa_{\mathscr{H}}(\cdot,z)e_1\rangle_{\mathscr{H}} \\ &= \Big\langle \sum_{n=1}^{\infty} T^{*n}e_0\frac{1}{\lambda^n} + \sum_{n=0}^{\infty} T'^n\lambda^n e_0, \sum_{n=1}^{\infty} T^{*n}e_1\frac{1}{z^n} + \sum_{n=0}^{\infty} T'^n e_1 z^n\Big\rangle_E,\end{aligned}$$

for $e_0, e_1 \in E$ and thus $\mathscr{H}$ is a reproducing kernel Hilbert space of E-valued holomorphic functions on $\mathbb{A}(r^-, r^+)$ and the reproducing kernel is given by (3.4).

The assertion (iii) is a direct consequence of (3.7) and (ii). This completes the proof. □

Now, we turn to the properties of the Cauchy dual operator T'. The Cauchy dual operator T' of a left-invertible operator T is itself left-invertible. Assume now that there exist a subspace $E \subset \mathcal{H}$ such that $[E]_{T^*,T'} = \mathcal{H}$ and $[E]_{T',T} = \mathcal{H}$ hold. Then for both operators T and T' we can construct Hilbert spaces $\mathscr{H}$ and $\mathscr{H}'$ of E-valued Laurent series. Therefore, by (3.1) the formal Laurent series U'_x takes the form

$$U'_x(z) := \sum_{n=1}^{\infty}(P_E T'^n x)\frac{1}{z^n} + \sum_{n=0}^{\infty}(P_E T^{*n}x)z^n$$

and $\mathscr{H}'$ is the space of Laurent series of the form U'_x, $x \in \mathcal{H}$.

Theorem 3.9. *Let $T \in \boldsymbol{B}(\mathcal{H})$ be left-invertible, $E \subset \mathcal{H}$ be a closed subspace and U, $\mathscr{H}$, U', $\mathscr{H}'$ are as above. Suppose that $[E]_{T^*,T'} = \mathcal{H}$, $[E]_{T'^*,T} = \mathcal{H}$ and (♠) holds. Let $f \in \mathscr{H}$ and $g \in \mathscr{H}'$ be E-valued series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n.$$

Then

$$\langle U^{-1}f, U'^{=1}g\rangle = \sum_{n=0}^{\infty}\langle a_n, b_n\rangle.$$

Proof. It suffices to consider the case $f(z) = e_0 z^n$ and $g(z) = e_1 z^m$, $m, n \in \mathbb{N}$, $e_0, e_1 \in E$. Observe that

$$\langle U^{-1}f, U'^{-1}g\rangle = \begin{cases} \langle T^{n-m}e_0, e_1\rangle & \text{if } n \geqslant m, \\ \langle T'^{m-n}e_0, e_1\rangle & \text{otherwise.}\end{cases}$$

Since (♠), we deduce that $\langle U^{-1}f, U'^{-1}g\rangle = \delta_{m-n}\langle e_0, e_1\rangle$. This finishes the proof. □

Now we use analytic model constructed in this section to discuss spectral theory of left-invertible operators and its adjoints. Perhaps it is appropriate at this point to note that the condition (iv) below appeared in [18, Definition 1.2].

Theorem 3.10. *Suppose (♣) holds and the series (3.1) is convergent in E for every $x \in \mathcal{H}$ on open nonempty subset $\Omega \subset \mathbb{C}$. Then the following assertions hold:*

(i) *the point spectrum of T is empty, that is $\sigma_p(T) = \emptyset$,*
(ii) *$\mathscr{M}_z^*\kappa_{\mathscr{H}}(\cdot,\mu)g = \bar{\mu}\kappa_{\mathscr{H}}(\cdot,\mu)g$, for every $\mu \in \Omega$, $g \in E$,*
(iii) *$\bar{\Omega} \subset \sigma_p(T^*)$,*
(iv) *$\bigvee\{\mathcal{N}(T^* - \bar{\mu}) \colon \mu \in U\} = \mathcal{H}$, where $U \subset \Omega$ and $\operatorname{int} U \neq \emptyset$.*

Proof. (i) Suppose, to derive a contradiction, that $\mu \in \sigma_p(T)$. Then by Theorem 3.2, $\mu \in \sigma_p(\mathscr{M}_z)$. Let $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in \mathscr{H}$ be such that $(\mathscr{M}_z - \mu)f = 0$. Using the properties of reproducing kernel, one gets the following

$$\langle(\mathscr{M}_z - \mu)f, \kappa_{\mathscr{H}}(\cdot, \lambda)e\rangle = 0, \qquad \lambda \in \Omega,\ e \in E.$$

Since the series $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ is convergent on Ω, we see that the above equality is equivalent to the following one

$$(\lambda - \mu) \sum_{n=-\infty}^{\infty} \langle a_n, e\rangle \lambda^n = 0, \qquad \lambda \in \Omega,\ e \in E.$$

If $\mu = 0$, then by Identity theorem

$$\langle a_n, e\rangle = 0, \qquad e \in E,\ n \in \mathbb{Z}.$$

As a consequence, we get $a_n = 0$ for $n \in \mathbb{Z}$. This shows that $f = 0$, which gives (i). We now consider the other case when $\mu \neq 0$. Using Identity theorem again, we see that

$$\langle a_n, e\rangle = \mu^{-n}\langle a_0, e\rangle, \qquad n \in \mathbb{Z}. \tag{3.8}$$

Suppose that there exist some $e \in E$ such that $\langle a_0, e\rangle \neq 0$. Therefore, by (3.8)

$$\sum_{n=-\infty}^{\infty} \langle a_n, e\rangle \lambda^n = \langle a_0, e\rangle \sum_{n=-\infty}^{\infty} \left(\frac{\lambda}{\mu}\right)^n, \qquad \lambda \in \Omega,\ e \in E.$$

Clearly, $\sum_{n=-\infty}^{\infty} (\frac{z}{\lambda})^n$ is divergent, which contradicts our assumption that $\langle a_0, e\rangle \neq 0$. This and (3.8) shows that $a_n = 0$ for $n \in \mathbb{Z}$. As a consequence, we get $f = 0$. This completes the proof of (i).

(ii) By Theorem 3.8, we have

$$\begin{aligned}\langle U_x, \mathscr{M}_z^* \kappa_{\mathscr{H}}(\cdot, \lambda)g\rangle &= \langle \mathscr{M}_z U_x, \kappa_{\mathscr{H}}(\cdot, \lambda)g\rangle = \langle \lambda U_x(\lambda), g\rangle \\ &= \langle U_x(\lambda), \bar{\lambda} g\rangle = \langle U_x, \bar{\lambda}\kappa_{\mathscr{H}}(\cdot, \lambda)g\rangle,\end{aligned}$$

for $x \in \mathcal{H}$, $\lambda \in \Omega$ and $g \in E$. This gives the equality

$$\mathscr{M}_z^* \kappa_{\mathscr{H}}(\cdot, \lambda)g = \bar{\lambda}\kappa_{\mathscr{H}}(\cdot, \lambda)g.$$

(iii) This is a direct consequence of (ii).

(iv) Suppose that $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \in \mathscr{H}$ is orthogonal to the subspace $\bigvee \{\mathcal{N}(\mathscr{M}_z^* - \bar{\mu}) \colon \mu \in U\}$. Since, by (ii), $\kappa_{\mathscr{H}}(\cdot, \mu)e \in \mathcal{N}(\mathscr{M}_z^* - \bar{\mu})$ for every $e \in E$, the following equalities hold

$$\sum_{n=-\infty}^{\infty} \langle a_n, e\rangle \lambda^n = \langle f(\lambda), e\rangle = \langle f, \kappa_{\mathscr{H}}(\cdot, \lambda)e\rangle = 0, \qquad \lambda \in U,\ e \in E.$$

By Identity theorem this implies that $a_n = 0$ for every $n \in \mathbb{Z}$. Thus $f = 0$ and $\bigvee \{\mathcal{N}(\mathscr{M}_z^* - \bar{\mu}) \colon \mu \in U\} = \mathscr{H}$. □

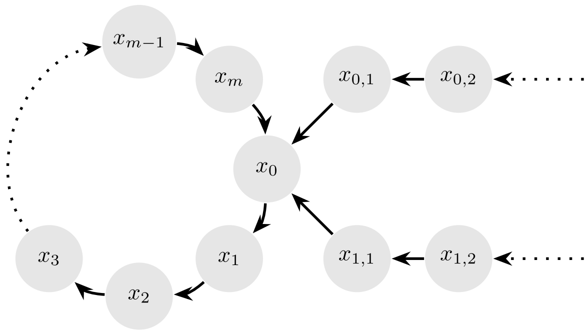

Fig. 1. An example of directed graph (X, E^{φ}) induced by self-map φ.

4. Weighted composition operators

In this section as an application of the model presented in Section 3, we obtain significantly improved model for weighted composition operator upon provided the symbol of this operator has finite branching index.

We begin by recalling the definition of finite branching index. Let $\mathscr{T} = (V, E)$ be a rootless directed tree. Following [15], we say that $\mathscr{T}$ has *finite branching index* if there exist $m \in \mathbb{N}$ such that

$$\mathrm{Chi}^k(V_{\prec}) \cap V_{\prec} = \emptyset, \quad k \geqslant m,\ k \in \mathbb{N}.$$

The next lemma shows that in the case of rootless directed tree with finite branching index there exist some special vertex.

Lemma 4.1 *([15]). Let $\mathscr{T} = (V, E)$ be a rootless directed tree with finite branching index $m \in \mathbb{N}$. Then there exists a vertex $\omega \in V_{\prec}$ such that*

$$\mathrm{card}(\mathrm{Chi}(\mathrm{par}^{(n)}(\omega))) = 1, \qquad n \in \mathbb{Z}_+. \tag{4.1}$$

Moreover, if $V_{\prec}$ is non-empty, then there exists a unique $\omega \in V_{\prec}$ satisfying (4.1).

The vertex $\omega \in V_{\prec}$ appearing in the statement of Lemma 4.1 is called *generalized root*. We put $x^* := \mathrm{par}(\omega)$ in the definition of function $[\varphi] : X \to \mathbb{Z}$ for orbit F of φ not containing a cycle (see Section 2). Since any self-map $\varphi : X \to X$ induces a directed graph (X, E^{φ}) (see Fig. 1) given by

$$E^{\varphi} = \{(x, y) \in X \times X : x = \varphi(y)\} \tag{4.2}$$

it is natural to extend the notion of finite branching index to self-maps. We say that φ has *finite branching index* if

$$\sup\left\{|[\varphi](x)| : \mathrm{card}(\varphi^{-1}(x)) \geqslant 2,\ x \in X\right\} < \infty.$$

Perhaps it is appropriate at this point to note that a self-map with one orbit can have at most one cycle.

Recall that a weighted shift on a rootless directed tree can be identified with composition operator in L^2-spaces (see [28, Lemma 4.3.1]).

Let X be a countable set, $w : X \to \mathbb{C}$ be a complex function on X, $\varphi : X \to X$ be a transformation of X and $C_{\varphi,w}$ be a weighted composition operator in $\ell^2(X)$. We will need only consider composition functions with one orbit, since an orbit induces a reducing subspace to which the restriction of the weighted composition operator is again a weighted composition operator.

The following lemma describes a subspace $E \subset \ell^2(X)$ of the operator $C_{\varphi,w}$ which satisfies condition (♠) with $C_{\varphi,w}$ and $C_{\varphi',w}$ in place of T. It requires considering two distinct cases.

Lemma 4.2. *Let X be a countable set, $w : X \to \mathbb{C}$ be a complex function on X and $\varphi : X \to X$ be a transformation of X, which has finite branching index. Let $C_{\varphi,w}$ be a weighted composition operator in $\ell^2(X)$ and*

$$E := \begin{cases} \bigoplus_{x\in \mathrm{Gen}_\varphi(1,1)} \langle e_x\rangle \oplus \mathcal{N}((C_{\varphi,w}|_{\ell^2(\mathrm{Des}(x))})^*) & \textit{when } \varphi \textit{ has a cycle},\\ \langle e_\omega\rangle \oplus \mathcal{N}(C^*_{\varphi,w}) & \textit{otherwise}, \end{cases} \tag{4.3}$$

where $\mathrm{Des}(x) := \bigcup_{n=0}^\infty \varphi^{(-n)}(x)$ and ω is a generalized root of the tree defined by (4.2). Then the subspace E has the following properties:

(i) *$[E]_{C^*_{\varphi,w},C_{\varphi',w}} = \ell^2(X)$ and $[E]_{C_{\varphi,w},C^*_{\varphi',w}} = \ell^2(X)$,*
(ii) *$E \perp C^n_{\varphi,w}E$ and $E \perp C^n_{\varphi',w}E$, $n \in \mathbb{Z}_+$.*

Proof. (i) First, we consider the case when φ does not have a cycle. Clearly, the weighted composition operator $C_{\varphi,w}$ can be identified with a weighted shift S_λ on a rootless directed tree given by (4.2). We show that the subspace $E := \langle e_\omega\rangle \oplus \mathcal{N}(S_\lambda{}^*)$ satisfies (♠) and $[E]_{S^*_\lambda,S_{\lambda'}} = \ell^2(X)$. Note that the space $\ell^2(\mathrm{Des}(\omega))$ is invariant for S_λ. We will denote by $S_{\lambda\to(\omega)}$ the operator $S_\lambda|_{\ell^2(\mathrm{Des}(\omega))}$. The subtree $\mathscr{T}_{\mathrm{Des}(\omega)}$ of $\mathscr{T}$ is a directed tree with root ω and by Lemma 2.3, $\mathcal{N}(S_{\lambda\to(\omega)}) = \langle e_\omega\rangle \oplus \mathcal{N}(S_\lambda{}^*)$. Since by [15, Lemma 3.3] $S_{\lambda\to(\omega)}$ is analytic, it follows from Shimorin's analytic model that $[E]_{S_{\lambda\to(\omega)}} = \ell^2(\mathrm{Des}(\omega))$. Hence,

$$\bigvee\{S^n_\lambda x\colon x\in E, n\in\mathbb{N}\} = [E]_{S_{\lambda\to(\omega)}} = \ell^2(\mathrm{Des}(\omega)). \tag{4.4}$$

By [52, Proposition 2.7], we have $[E]_{(S_{\lambda\to(\omega)})'} = \ell^2(\mathrm{Des}(\omega))$. Note that the subspace $\ell^2(\mathrm{Des}(\omega))$ is invariant for S_λ and $S^*_\lambda S_\lambda$ is diagonal. Recall that if $T \in \boldsymbol{B}(\mathcal{H})$ and closed subspace $\mathcal{G}$ is invariant for T, then $(T|_\mathcal{G})^* = P_\mathcal{G}T^*|_\mathcal{G}$. Therefore, we have

$$\begin{aligned}(S_{\lambda\to(\omega)})' &= S_\lambda|_{\ell^2(\mathrm{Des}(\omega))}((S_\lambda|_{\ell^2(\mathrm{Des}(\omega))})^*S_\lambda|_{\ell^2(\mathrm{Des}(\omega))})^{-1}\\ &= S_\lambda|_{\ell^2(\mathrm{Des}(\omega))}(P_{\ell^2(\mathrm{Des}(\omega))}S^*_\lambda|_{\ell^2(\mathrm{Des}(\omega))}S_\lambda|_{\ell^2(\mathrm{Des}(\omega))})^{-1}\\ &= S_\lambda(S^*_\lambda S_\lambda)^{-1}|_{\ell^2(\mathrm{Des}(\omega))} = S_{\lambda'}|_{\ell^2(\mathrm{Des}(\omega))}.\end{aligned}$$

This implies that

$$\bigvee\{S^n_{\lambda'} x\colon x\in E, n\in\mathbb{N}\} = [E]_{(S_\lambda|_{\ell^2(\mathrm{Des}(\omega))})'} = \ell^2(\mathrm{Des}(\omega)). \tag{4.5}$$

The assertion (i) of Lemma 2.1, shows that

$$\bigvee\{S^{*n}_\lambda x\colon x\in E, n\in\mathbb{N}\} = \ell^2(X\setminus \mathrm{Des}(\omega)),$$

$$\bigvee\{S^{*n}_{\lambda'} x\colon x\in E, n\in\mathbb{N}\} = \ell^2(X\setminus \mathrm{Des}(\omega)).$$

This together with (4.4) and (4.5) yields $[E]_{S^*_\lambda,S_{\lambda'}} = \ell^2(X)$ and $[E]_{S_\lambda,S^*_{\lambda'}} = \ell^2(X)$.

If φ has a cycle, then the operator

$$C_{\varphi,w}|_{\ell^2(\mathrm{Des}(x))} \quad \text{for} \quad x \in \mathrm{Gen}_\varphi(1,1)$$

is a weighted shift on directed tree with root x. Using [15, Lemma 3.4] again and arguing as in the previous case we obtain

$$\bigvee\{C^n_{\varphi',w}y\colon y\in\langle e_x\rangle\oplus\mathcal{N}((C_{\varphi,w}|_{\ell^2(\mathrm{Des}(x))})^*), n\in\mathbb{N}\}=\ell^2(\mathrm{Des}(x)),\ \ x\in\mathrm{Gen}_\varphi(1,1), \tag{4.6}$$
$$\bigvee\{C^n_{\varphi,w}y\colon y\in\langle e_x\rangle\oplus\mathcal{N}((C_{\varphi,w}|_{\ell^2(\mathrm{Des}(x))})^*), n\in\mathbb{N}\}=\ell^2(\mathrm{Des}(x)),\ \ x\in\mathrm{Gen}_\varphi(1,1).$$

Applying the assertion (i) of Lemma 2.1 again, we see that

$$\bigvee\{C^n_{\varphi,w}x\colon x\in E, n\in\mathbb{N}\}=\ell^2(X\setminus\bigcup_{x\in\mathrm{Gen}_\varphi(1,1)}\mathrm{Des}(x)),$$
$$\bigvee\{C^n_{\varphi',w}x\colon x\in E, n\in\mathbb{N}\}=\ell^2(X\setminus\bigcup_{x\in\mathrm{Gen}_\varphi(1,1)}\mathrm{Des}(x)).$$

This and (4.6), implies that $[E]_{C^*_{\varphi,w},C_{\varphi',w}}=\ell^2(X)$ and $[E]_{C_{\varphi,w},C^*_{\varphi',w}}=\ell^2(X)$.

(ii) First, we consider the case when φ does not have a cycle. According to Lemma 2.3, $E=\langle e_\omega\rangle\oplus\mathcal{N}(C^*_{\varphi,w})$. If $e,f\in E$ and $n\in\mathbb{N}$, then

$$\langle C^{*n}_{\varphi,w}e,f\rangle=\langle C^{*n}_{\varphi,w}P_{\langle e_\omega\rangle}e,f\rangle=0,$$
$$\langle C^{*n}_{\varphi',w}e,f\rangle=\langle C^{*n}_{\varphi',w}P_{\langle e_\omega\rangle}e,f\rangle=0,$$

where in the last step we used assertion (iii) of Lemma 2.1. This immediately yields that condition (♠) holds, which completes the proof of the case when φ does not have a cycle.

If φ has a cycle, then similar reasoning leads to the equalities

$$\langle C^{*n}_{\varphi,w}e,f\rangle=\langle C^{*n}_{\varphi,w}P_{\tilde E}e,f\rangle=0,$$
$$\langle C^{*n}_{\varphi',w}e,f\rangle=\langle C^{*n}_{\varphi',w}P_{\tilde E}e,f\rangle=0,$$

where $\tilde E:=\bigvee\{e_x\colon x\in\mathrm{Gen}_\varphi(1,1)\}$. This completes the proof. □

Before we turn to the main theorem of this section, we need to give some definitions. Suppose (♣) holds with $C_{\varphi,w}$ in place of T. Let φ be a self-map of X and E be a subspace of $\ell^2(X)$. Define

$$k_\varphi(E):=\min\{n\in\mathbb{N}\colon E\subset\bigvee\{e_x\colon\ \mathrm{Gen}_\varphi(1,n)\}\}.$$

A number $k_\varphi(E)$ will be called an *index* of E with respect to φ. Now, we can define some subsets of X by

$$W_0^{E,\varphi}:=\mathrm{Gen}_\varphi(1,k_\varphi(E)),$$

and then

$$W_n^{E,\varphi}:=\begin{cases}\varphi^{(-n)}(W_0^{E,\varphi}) & n\in\mathbb{N}\text{ when }\varphi\text{ has a cycle}\\ \varphi^{(-n)}(W_0^{E,\varphi}) & n\in\mathbb{Z}\text{ otherwise.}\end{cases}$$

Finally, we are ready to define radii of convergence for $C_{\varphi,w}$. The non-negative number

$$r^+_{w,\varphi}:=\liminf_{n\to\infty}\Big(\sum_{\substack{x\in W_n^{E,\varphi}\\ n\geqslant 0}}|w'(x)w'(\varphi(x))\cdots w'(\varphi^{(n-1)}(x))|^2\Big)^{-\frac{1}{2n}} \tag{4.7}$$

will be called the *outer radius of convergence* for $C_{\varphi,w}$, and similarly the non-negative number

$$r^-_{w,\varphi} := \begin{cases} \sqrt[\tau]{\prod_{x\in\mathscr{C}_\varphi}|w(x)|} & \text{if } \varphi \text{ has a cycle,} \\ \limsup_{n\to\infty} \sqrt[n]{|w(\varphi^1(\omega))w(\varphi^2(\omega))\dots w(\varphi^n(\omega))|} & \text{otherwise,} \end{cases} \tag{4.8}$$

where $\tau := \operatorname{card}\mathscr{C}_\varphi$ will be called the *inner radius of convergence* for $C_{\varphi,w}$.

Now we are in a position to prove the main result of this section (compare with [15, Theorem 2.2]).

Theorem 4.3. *Let X be a countable set, $w : X \to \mathbb{C}$ be a complex function on X and $\varphi : X \to X$ be a transformation of X, which has finite branching index. Let $C_{\varphi,w}$ be a left-invertible weighted composition operator in $\ell^2(X)$. If $r^+_{w,\varphi} > r^-_{w,\varphi}$, then there exist a z-invariant reproducing kernel Hilbert space $\mathscr{H}$ of E-valued holomorphic functions defined on the annulus $\mathbb{A}(r^-_{w,\varphi}, r^+_{w,\varphi})$ and a unitary isomorphism $U : \ell^2(X) \to \mathscr{H}$ such that $\mathscr{M}_z U = UC_{\varphi,w}$, where $\mathscr{M}_z$ denotes the operator of multiplication by z on $\mathscr{H}$ and E is as in (4.3). Moreover, the following assertions hold:*

(i) *the reproducing kernel $\kappa_{\mathscr{H}} : \mathbb{A}(r^-_{w,\varphi}, r^+_{w,\varphi}) \times \mathbb{A}(r^-_{w,\varphi}, r^+_{w,\varphi}) \to \boldsymbol{B}(E)$ associated with $\mathscr{H}$ has the property that $\kappa_{\mathscr{H}}(\cdot, w)g \in \mathscr{H}$ and $\langle Uf, \kappa_{\mathscr{H}}(\cdot, w)g\rangle = \langle (Uf)(w), g\rangle$ for $f, g \in \ell^2(X)$,*

(ii) *the reproducing kernel $\kappa_{\mathscr{H}}$ has the following form:*

$$\kappa_{\mathscr{H}}(z,\lambda) = \sum_{i,j\geqslant 1} A_{i,j}\frac{1}{z^i}\frac{1}{\bar\lambda^j} + \sum_{i\geqslant 1, j\geqslant 0} B_{i,j}\frac{1}{z^i}\bar\lambda^j + \sum_{i\geqslant 0, j\geqslant 1} C_{i,j} z^i \frac{1}{\bar\lambda^j} + \sum_{i,j\geqslant 0} D_{i,j} z^i \bar\lambda^j,$$

where $A_{i,j}, B_{i,j}, C_{i,j}, D_{i,j} \in \boldsymbol{B}(E)$; if additionally φ has no cycle, then

$$\begin{aligned} A_{i,j} &= 0 \quad \text{if} \quad |i-j| > k_{\varphi(E)}, \\ B_{i,j} &= 0 \quad \text{if} \quad i+j > k_{\varphi(E)}, \\ C_{i,j} &= 0 \quad \text{if} \quad i+j > k_{\varphi(E)}, \\ D_{i,j} &= 0 \quad \text{if} \quad |i-j| > k_{\varphi(E)}, \end{aligned}$$

(iii) *if φ does not have a cycle, then the linear subspace generated by E-valued polynomials in z and $\tilde{E}$-valued polynomials involving only negative powers of z is dense in $\mathscr{H}$, that is*

$$\bigvee\left(\{z^n E\colon n\in\mathbb{N}\} \cup \{\frac{1}{z^n}\tilde{E}\colon n \in \mathbb{Z}_+\}\right) = \mathscr{H},$$

where $\tilde{E} := \bigvee\{e_x\colon x \in \operatorname{Gen}_\varphi(1,1)\}$; if φ has a cycle $\mathscr{C}_\varphi$ with $\tau := \operatorname{card}\mathscr{C}_\varphi$, then there exist τ functions $f_1, \dots, f_\tau$ on $\mathbb{A}(r^-_{w,\varphi}, r^+_{w,\varphi})$ given by the following Laurent series

$$f_i(z) := \sum_{k=0}^{\infty}\sum_{i=1}^{\tau} \Lambda^k A_i \frac{1}{z^{k\tau+i}}, \quad i = 1, ..., \tau,$$

where $\Lambda := \prod_{x\in\mathscr{C}_\varphi} w(x)$ and $A_i \in \tilde{E}$, $i = 1, \dots, \tau$ such that the linear subspace generated by E-valued polynomials in z and the above functions is dense in $\mathscr{H}$, that is

$$\bigvee\left(\{z^n E\colon n\in\mathbb{N}\} \cup \{f_i\colon i \in \{1, \dots \tau\}\}\right) = \mathscr{H}.$$

Proof. We begin by showing that the E-valued series

$$\sum_{n=0}^{\infty} P_E C^{*n}_{\varphi',w} f z^n$$

converges absolutely in E on the disc $\mathbb{D}(r^+_{w,\varphi})$. Let $f = \sum_{x\in X} f(x)e_x$. Applying Lemma 2.1, we obtain

$$\begin{aligned} P_E C^{*n}_{\varphi',w} f &= \sum_{x\in X} f(x)w'(x)w'(\varphi(x))\cdots w'(\varphi^{(n-1)}(x))P_E e_{\varphi^{(n)}(x)} \\ &= \sum_{x\in W^{E,\varphi}_n} f(x)w'(x)w'(\varphi(x))\cdots w'(\varphi^{(n-1)}(x))P_E e_{\varphi^{(n)}(x)}. \end{aligned}$$

Observe that $W^{E,\varphi}_n \cap W^{E,\varphi}_m = \emptyset$ for $|m-n| > k_{\varphi(E)}$, $m,n\in\mathbb{N}$. As a consequence, we have

$$\sum_{\substack{x\in W^{E,\varphi}_n\\ n\geqslant 0}} |f(x)|^2 \leqslant (k_{\varphi(E)}+1)\sum_{x\in X}|f(x)|^2 = (k_{\varphi(E)}+1)\|f\|^2. \tag{4.9}$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \Big\|\sum_{n=0}^{k} P_E C^{*n}_{\varphi',w} f z^n\Big\| &\leqslant \sum_{\substack{x\in W^{E,\varphi}_n\\ n\geqslant 0}} |f(x)w'(x)w'(\varphi(x))\cdots w'(\varphi^{(n-1)}(x))z^n| \\ &\leqslant \Big(\sum_{\substack{x\in W^{E,\varphi}_n\\ n\geqslant 0}} |f(x)|^2\Big)^{\frac12}\Big(\sum_{\substack{x\in W^{E,\varphi}_n\\ n\geqslant 0}} |w'(x)w'(\varphi(x))\cdots w'(\varphi^{(n-1)}(x))z^n|^2\Big)^{\frac12} \\ &\overset{(4.9)}{\leqslant} \sqrt{k_\varphi(E)+1}\,\|f\|\Big(\sum_{\substack{x\in W^{E,\varphi}_n\\ n\geqslant 0}} |w'(x)w'(\varphi(x))\cdots w'(\varphi^{(n-1)}(x))z^n|^2\Big)^{\frac12}. \end{aligned}$$

An application of the root test [49, page 198] shows that the above series converges on the disc $\mathbb{D}(r^+_{w,\varphi})$.

Now we show that the E-valued series

$$\sum_{n=0}^{\infty} P_E C^{n}_{\varphi,w} f \frac{1}{z^n}$$

converges absolutely in E on $\mathbb{C}\setminus\mathbb{D}(r^-_{w,\varphi})$. First, we consider the case when φ does not have a cycle. For this, note that using Lemma 2.1 again,

$$\begin{aligned} P_E C^{n}_{\varphi,w} f &= \sum_{x\in X} f(x)\sum_{y\in\varphi^{-n}(x)} w(y)w(\varphi(y))\cdots w(\varphi^{(n-1)}(y))P_E e_y \\ &= \sum_{x\in W^{E,\varphi}_{-n}} f(x)\sum_{y\in\varphi^{-n}(x)} w(y)w(\varphi(y))\cdots w(\varphi^{(n-1)}(y))P_E e_y \\ &= \sum_{x\in W^{E,\varphi}_{-n}} f(x)w(\varphi^{-1}(x))w(\varphi^{-2}(x))\cdots w(\varphi^{[\varphi](x)}(x))P_E C^{n+[\varphi](x)}_{\varphi,w} e_{\varphi^{[\varphi](x)}(x)}, \end{aligned}$$

for $n\geqslant k_{\varphi(E)}$. Put $M := \max\{1, \|C_{\varphi,w}\|^{k_{\varphi(E)}}\}$. Note that $n+[\varphi](x)\leqslant k_{\varphi(E)}$ and thus, $\|P_E C^{n+[\varphi](x)}_{\varphi,w}\|\leqslant M$. Repeating the argument in (4.9), we see that

$$\sum_{\substack{x\in W_{-n}^{E,\varphi}\\ n\geqslant k_{\varphi(E)}}} |f(x)|^2 \leqslant (k_{\varphi(E)}+1)\|f\|^2.$$

Hence, by the Cauchy-Schwarz inequality again, we have

$$\begin{aligned}
\|\sum_{n=k_{\varphi(E)}}^{k} P_E C_{\varphi,w}^n f\frac{1}{z^n}\| &\leqslant M \sum_{\substack{x\in W_{-n}^{E,\varphi}\\ n\geqslant k_{\varphi(E)}}} |f(x)w(\varphi^{-1}(x))w(\varphi^{-2}(x))\cdots w(\varphi^{[\varphi](x)}(x))\frac{1}{z^n}| \\
&\leqslant M\Big(\sum_{\substack{x\in W_{-n}^{E,\varphi}\\ n\geqslant k_{\varphi(E)}}} |f(x)|^2\Big)^{\frac{1}{2}}\Big(\sum_{\substack{x\in W_{-n}^{E,\varphi}\\ n\geqslant k_{\varphi(E)}}} |w(\varphi^{-1}(x))w(\varphi^{-2}(x))\cdots w(\varphi^{[\varphi](x)}(x))\frac{1}{z^n}|^2\Big)^{\frac{1}{2}} \\
&\leqslant M\sqrt{k_\varphi(E)+1}\,\|f\|\,\Big(\sum_{\substack{x\in W_{-n}^{E,\varphi}\\ n\geqslant k_{\varphi(E)}}} w(\varphi^{-1}(x))w(\varphi^{-2}(x))\cdots w(\varphi^{[\varphi](x)}(x)\frac{1}{z^n}|^2\Big)^{\frac{1}{2}},
\end{aligned}$$

for $k \geqslant k_{\varphi(E)}$. Since the series on the right-hand side converges absolutely on $\mathbb{C}\setminus\mathbb{D}(r_{w,\varphi}^-)$, we are done. It remains to consider the other case when φ has a cycle. It is easily seen that

$$\sum_{n=0}^{\infty} P_E C_{\varphi,w}^n f z^n = \sum_{n=0}^{\infty} P_E C_{\varphi,w}^n P_{\mathcal{H}_\varphi} f z^n + \sum_{n=0}^{\infty} P_E C_{\varphi,w}^n P_{\mathcal{H}_\varphi^\perp} f z^n,$$

where $\mathcal{H}_\varphi := \operatorname{lin}\{e_x : x\in\mathscr{C}_\varphi\}$. We show that both above series converge. Observe that if $h \in \mathcal{H}_\varphi^\perp \cap \bigvee\{e_x : x\in \mathrm{Gen}_\varphi(m,n)\}$ for $m,n\in\mathbb{N}$, then $C_{\varphi,w}h \in \bigvee\{e_x : x\in\mathrm{Gen}_\varphi(m+1,n+1)\}$. This, together with the fact that $E\subset\bigvee\{e_x : x\in\mathrm{Gen}_\varphi(1,k_{\varphi(E)})\}$ yields

$$\Big\|\sum_{n=0}^{\infty} P_E C_{\varphi,w}^n P_{\mathcal{H}_\varphi^\perp} f z^n\Big\| = \Big\|\sum_{n=0}^{k_\varphi(E)} P_E C_{\varphi,w}^n P_{\mathcal{H}_\varphi^\perp} f z^n\Big\| \leqslant \sum_{n=0}^{k_\varphi(E)} \|C_{\varphi,w}\|^n\,\|f\|\,z^n.$$

Let us now observe that

$$P_{W_i} C_{\varphi,w}^{n+\tau} e_x = \Lambda P_{W_i} C_{\varphi,w}^n e_x, \quad x\in\mathscr{C}_\varphi, \tag{4.10}$$

where $W_i = \bigvee\{e_x : x\in\mathrm{Gen}_\varphi(i,i)\}$, $i\in\mathbb{N}$. We now apply this observation to estimate the following sum.

$$\begin{aligned}
\sum_{n=0}^{\infty} P_E C_{\varphi,w}^n P_{\mathcal{H}_\varphi} f\frac{1}{z^n} &= \sum_{n=0}^{\infty} P_E C_{\varphi,w}^n\Big(\sum_{x\in\mathscr{C}_\varphi} f(x)e_x\Big)\frac{1}{z^n} = \sum_{x\in\mathscr{C}_\varphi} f(x)\sum_{n=0}^{\infty} P_E C_{\varphi,w}^n e_x\frac{1}{z^n} \\
&= \sum_{x\in\mathscr{C}_\varphi}\sum_{i=0}^{k_{\varphi(E)}} f(x)\sum_{n=0}^{\infty} P_E P_{W_i} C_{\varphi,w}^n e_x\frac{1}{z^n} \\
&= \sum_{x\in\mathscr{C}_\varphi}\sum_{i=0}^{k_{\varphi(E)}}\sum_{j=0}^{\tau} f(x)\sum_{n=0}^{\infty} P_E P_{W_i} C_{\varphi,w}^{n\tau+j} e_x\frac{1}{z^{n\tau+j}} \\
&= \sum_{x\in\mathscr{C}_\varphi}\sum_{i=0}^{k_{\varphi(E)}}\sum_{j=0}^{\tau} f(x)\sum_{n=0}^{\infty} \Lambda^n P_E P_{W_i} C_{\varphi,w}^{j} e_x\frac{1}{z^{n\tau+j}}.
\end{aligned}$$

The above series is a finite sum of a geometric series with ratio $\frac{\Lambda}{z^{\tau}}$ and hence converges absolutely on $\mathbb{C} \setminus \mathbb{D}(r^-_{w,\varphi})$. Putting these results together, we conclude that if $r^+_{w,\varphi} > r^-_{w,\varphi}$, then the series (3.1) with $C_{\varphi,w}$ in place of T converges absolutely on $\mathbb{A}(r^-_{w,\varphi}, r^+_{w,\varphi})$. Moreover, combining Theorems 3.2 and 3.8 with Lemma 4.2, we deduce that there exist a z-invariant reproducing kernel Hilbert space $\mathscr{H}$ of E-valued holomorphic functions defined on the annulus $\mathbb{A}(r^-_{w,\varphi}, r^+_{w,\varphi})$ and a unitary mapping $U : \ell^2(X) \to \mathscr{H}$ such that $\mathscr{M}_z U = UC_{\varphi,w}$.

Now we turn to the proof of the "moreover" part.

(i) This assertion is a direct consequence of Theorem 3.8.

(ii) Recall that by (3.4), the kernel has the following form

$$\begin{aligned}\kappa_{\mathscr{H}}(z,\lambda) = &\sum_{i,j\geqslant 1} A_{i,j}\frac{1}{z^i}\frac{1}{\lambda^j} + \sum_{i\geqslant 1, j\geqslant 0} B_{i,j}\frac{1}{z^i}\lambda^j \\ &+ \sum_{i\geqslant 0, j\geqslant 1} C_{i,j} z^i \frac{1}{\lambda^j} + \sum_{i,j\geqslant 0} D_{i,j} z^i \lambda^j,\end{aligned}$$

where

$$\begin{aligned} A_{i,j} &= P_E C^i_{\varphi,w} C^{*j}_{\varphi,w}|_E, & B_{i,j} &= P_E C^i_{\varphi,w} C_{\varphi',w}{}^j|_E, \\ C_{i,j} &= P_E C^{*i}_{\varphi',w} C^{*j}_{\varphi,w}|_E, & D_{i,j} &= P_E C^{*i}_{\varphi',w} C^j_{\varphi',w}|_E. \end{aligned}$$

Observe that

$$\begin{aligned} C^{*m}_{\varphi',w} C^n_{\varphi',w} E &\subset \bigvee\{e_x \colon x \in W^{E,\varphi}_{n-m}\}, & C^m_{\varphi,w} C^n_{\varphi',w} E &\subset \bigvee\{e_x \colon x \in W^{E,\varphi}_{m+n}\}, \\ C^{*m}_{\varphi',w} C^{*n}_{\varphi,w} E &\subset \bigvee\{e_x \colon x \in W^{E,\varphi}_{-m-n}\}, & C^m_{\varphi,w} C^{*n}_{\varphi,w} E &\subset \bigvee\{e_x \colon x \in W^{E,\varphi}_{m-n}\}. \end{aligned}$$

Since $E \subset \bigvee\{e_x \colon x \in W^{E,\varphi}_0\}$, the subspace E is orthogonal to $\bigvee\{e_x \colon x \in W^{E,\varphi}_k\}$ if $|k| > k_{\varphi(E)}$. This completes the proof of (ii).

(iii) It follows from Lemma 4.2 that

$$\bigvee(\{C^n_{\varphi,w}E \colon n \in \mathbb{N}\} \cup \{C^{*n}_{\varphi',w}E \colon n \in \mathbb{N}\}) = \mathcal{H}.$$

We now consider two disjunctive cases which cover all possibilities. First we consider the case when φ does not have a cycle. Since $C_{\varphi,w}$ is unitarily equivalent to $\mathscr{M}_z$, we see that

$$U(\bigvee\{C^n_{\varphi,w}E \colon n \in \mathbb{N}\}) = \bigvee\{\mathscr{M}^n_z(E) \colon n \in \mathbb{N}\}. \tag{4.11}$$

Note that

$$U(C^{*n}_{\varphi',w}e_\omega) = \Big(\prod_{i=0}^{n-1} w(\varphi^{(i)}(\omega))\overline{w'(\varphi^{(i)}(\omega))}\Big) e_\omega \frac{1}{z^n}, \qquad n \in \mathbb{Z}_+. \tag{4.12}$$

It follows from (4.3) and equality $\mathcal{N}(C^*_{\varphi',w}) = \mathcal{N}(C^*_{\varphi,w})$ that

$$\bigvee\{C^{*n}_{\varphi',w}E \colon n \in \mathbb{N}\} = \bigvee\{C^{*n}_{\varphi',w}e_\omega \colon n \in \mathbb{N}\}.$$

Combining this with (4.11) and (4.12) completes the proof of the case when φ does not have a cycle. It remains to consider the other case when φ has a cycle. Looking at the formula (4.3), we deduce that

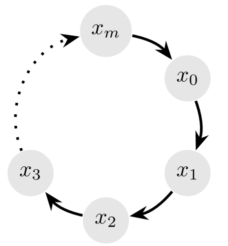

Fig. 2. The directed graph (X, E^{φ}) induced by self-map φ whose all vertices are contained in the cycle.

$$C^*_{\varphi',w} e = \begin{cases} \overline{w'(x)} e_{\varphi(x)} & \text{if } e = e_x,\ x \in \mathrm{Gen}_{\varphi}(1,1) \\ 0 & \text{if } e \in \bigoplus_{x \in \mathrm{Gen}_{\varphi}(1,1)} \mathcal{N}((C_{\varphi,w}|_{\ell^2(\mathrm{Des}(x))})^*) \end{cases} \tag{4.13}$$

Note that if $x \in \mathrm{Gen}_{\varphi}(1,1) \cup \mathscr{C}_{\varphi}$, then $\varphi(x) \in \mathscr{C}_{\varphi}$. This, combined with (4.13), yields

$$\bigvee \{C^{*n}_{\varphi',w} E \colon n \in \mathbb{Z}_+\} = \bigvee \{e_x \colon x \in \mathscr{C}_{\varphi}\}.$$

We now describe the value of the map $U : \mathcal{H} \to \mathscr{H}$ at e_x, $x \in \mathscr{C}_{\varphi}$. In view of (4.10), we can deduce from (3.1) that $U(e_x)$, $x \in \mathscr{C}_{\varphi}$ has the following form

$$U(e_x) = \sum_{k=0}^{\infty} \sum_{i=1}^{\tau} \Lambda^k A_i^x \frac{1}{z^{k\tau+i}},$$

for some $A_i^x \in \tilde{E}$, $i = 1, \ldots, \tau$. This completes the proof. □

5. Examples

In this section, we illustrate Theorem 4.3 by considering several interesting examples. We begin by giving an example of left-invertible weighted composition operator for which the series in (3.1) does not converge absolutely on any open subset of $\mathbb{C}$.

Example 5.1. Fix $m \in \mathbb{N}$ and set $X = \{0, 1, \ldots, m\}$. Let $w : X \to \mathbb{C}$ be a function and define a mapping $\varphi : X \to X$ by

$$\varphi(i) := \begin{cases} i+1 & \text{if } i < m \\ 0 & \text{if } i = m \end{cases}$$

(see Fig. 2). Set $\Lambda := w(0)w(1)\ldots w(m)$. Let $C_{\varphi,w}$ be the left-invertible composition operator in $\mathbb{C}^{m+1}$. The matrix of this operator is of the form

$$C_{\varphi,w} = \begin{bmatrix} 0 & w(0) & 0 & \cdots & 0 \\ 0 & 0 & w(1) & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w(m) & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Let $E := \mathrm{lin}\,\{e_1\}$. It is easy to see that $[E]_{C^*_{\varphi,w}, C_{\varphi',w}} = \ell^2(X)$. Using Lemma 2.1 and Lemma 2.2, one can o verify that

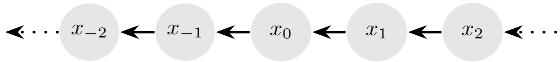

Fig. 3. The directed graph (X, E^{φ}) induced by self-map φ, which not have a cycle whose all vertices have valency one.

$$P_E C_{\varphi,w}^{mk+r} x = \Lambda^k \Big(\prod_{i=0}^{r-1} w(i) \Big) x_r e_0,$$

$$P_E C_{\varphi,w}^{\prime *(mk+r)} x = \frac{1}{\Lambda^k} \Big(\prod_{i=m+1-r}^{m} w(i) \Big)^{-1} x_{n+1-r} e_0,$$

for $r < n$, $r, k \in \mathbb{N}$. This shows that formal Laurent series in (3.1) takes the following form:

$$U_x(z) = \sum_{k=1}^{\infty} \sum_{r=0}^{n-1} \Big(\Lambda^k \Big(\prod_{i=0}^{r-1} w(i) \Big) x_r e_0 \Big) \frac{1}{z^{nk+r}}$$
$$+ \sum_{k=0}^{\infty} \Big(\sum_{r=0}^{n-1} \frac{1}{\Lambda^k} \Big(\prod_{i=m+1-r}^{m} w(i) \Big)^{-1} x_{n+1-r} e_0 \Big) z^{nk+r}.$$

Since $C_{\varphi,w}^*$ acts on the finite dimensional space, the spectrum of $C_{\varphi,w}^*$ is finite. Therefore, by assertion (iii) of Theorem 3.10 the above series does not converge absolutely on any open subset of $\mathbb{C}$. Alternatively, one can prove this fact directly by calculating convergences radii.

The next example shows that our analytic model generalizes the Gellar's analytic model for bilateral weighted shift [24].

Example 5.2 *(Bilateral weighted shift).* Let $S_\lambda : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be a bilateral weighted shift with weights $\{\lambda_n\}_{n\in\mathbb{Z}}$ and $\{e_n\}_{n\in\mathbb{Z}}$ be the standard orthonormal basis of $\ell^2(\mathbb{Z})$. Then

$$S_\lambda e_n := \lambda_{n+1} e_{n+1}, \qquad n \in \mathbb{Z}$$

(see Fig. 3). Let $E := \operatorname{lin}\{e_0\}$. It is easy to see that $[E]_{S_\lambda^*, S_\lambda'} = \mathcal{H}$. It is worth noting that $\mathcal{N}(S_\lambda^*) = \{0\}$ and thus $[\mathcal{N}(S_\lambda^*)]_{S_\lambda^*, S_\lambda'} = \{0\}$. This phenomenon is quite different comparing with the case of left-invertible and analytic operators in which $[\mathcal{N}(T^*)]_{T^*, T'} = \mathcal{H}$, where T is in this class.

It is a matter of routine to verify that the adjoint of the Cauchy dual $S_\lambda^{\prime *}$ of S_λ has the following form

$$S_\lambda^{\prime *} e_n = \frac{1}{\lambda_n} e_{n-1}, \qquad n \in \mathbb{Z}.$$

It is now easily seen that

$$P_E (S_\lambda^{\prime *})^n x = \Big(\prod_{i=1}^{n} \lambda_i \Big)^{-1} x_n e_0, \qquad n \in \mathbb{Z}_+,$$

and

$$P_E S_\lambda^n x = \Big(\prod_{i=-n+1}^{0} \lambda_i \Big) x_{-n} e_0, \qquad n \in \mathbb{Z}_+.$$

Therefore, by (3.1) the formal Laurent series takes the form

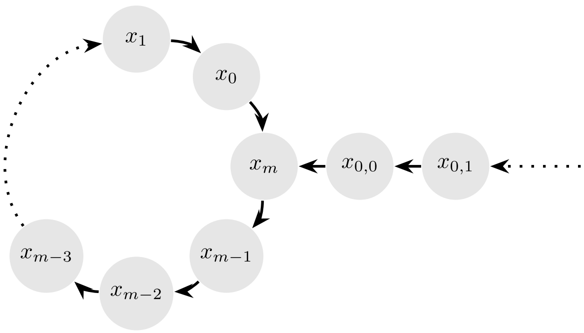

Fig. 4. The directed graph (X, E^{φ}) whose vertices, all but one, have valency one induced by self-map φ, which has a cycle.

$$U_x(z) = \sum_{n=1}^{\infty} \Big(\prod_{i=-n+1}^{0} \lambda_i\Big) x_{-n} \frac{1}{z^n} + \sum_{n=0}^{\infty} \Big(\prod_{i=1}^{n} \lambda_i\Big)^{-1} x_n z^n.$$

Comparing the above series with the formal Laurent series in [24, Section 2] one can realize that our analytic model and the Gellar analytic model coincide in the case of left-invertible bilateral weighted shifts. Noting that $W_n^{E,\varphi} = \{n\}$ for $n \in \mathbb{Z}$, we infer from (4.7) and (4.8) that

$$r^{+}_{w,\varphi} = \liminf_{n\to\infty} \sqrt[n]{\prod_{i=1}^{n} |\lambda_i|}$$

and

$$r^{-}_{w,\varphi} = \limsup_{n\to\infty} \sqrt[n]{\prod_{i=-n+1}^{0} |\lambda_i|}.$$

In this case, the reproducing kernel $\kappa_{\mathscr{H}} : \mathbb{A}(r^-, r^+) \times \mathbb{A}(r^-, r^+) \to \boldsymbol{B}(E)$ is diagonal and given by

$$\kappa_{\mathscr{H}}(z, \lambda) = \sum_{i=1}^{\infty} \prod_{i=-n+1}^{0} |\lambda_i|^2 \frac{1}{(z\bar{\lambda})^i} + \sum_{i=0}^{\infty} \Big(\prod_{i=1}^{n} |\lambda_i|^2\Big)^{-1} (z\bar{\lambda})^i.$$

Now we provide two more examples of left-invertible compositions operators over connected directed graphs induced by self-maps whose vertices, all but one, have valency one and the valency of the remaining vertex is nonzero.

Example 5.3. Set $m \in \mathbb{N}$ and $X = \{0, 1, \dots m\} \sqcup \{(0, i) \colon i \in \mathbb{N}\}$. Let $w : X \to \mathbb{C}$ be a measurable function and $\varphi : X \to X$ be transformation of X defined by

$$\varphi(x) := \begin{cases} (0, i-1) & \text{for } x = (0, i),\ i \in \mathbb{N} \setminus \{0\}, \\ m & \text{for } x = (0, 0), \\ i - 1 & \text{for } x = i \text{ and } i \in \{1, \dots, m\}, \\ m & \text{for } x = 0, \end{cases}$$

(see Fig. 4). Let $C_{\varphi,w} : \ell^2(X) \to \ell^2(X)$ be a left-invertible composition operator. It is easily seen that

$$C_{\varphi,w} e_x = \begin{cases} w(0, i+1) e_{(0,i+1)} & \text{for } x = (0, i),\ i \in \mathbb{Z}_+ \\ w(i+1) e_{i+1} & \text{for } x = i \text{ and } i \in \{0, 1, \dots, m\} \\ w(0) e_0 + w(0, 0) e_{(0,0)} & \text{for } x = m. \end{cases}$$

It is routine to verify that $\mathcal{N}(C^*_{\varphi,w}) = \operatorname{lin}\{\overline{w(0,0)}e_0 - \overline{w(0)}e_{(0,0)}\}$. Let $E := \operatorname{lin}\{e_{(0,0)}\}$. One can check that this one-dimensional subspace satisfies (♣). This implies that[1]

$$P_E(C^*_{\varphi',w})^n x = \Big(\prod_{i=1}^{n} w(0,i)\Big)^{-1} x_n e_{(0,0)},$$
$$P_E C^{nm+r+1}_{\varphi,w} x = \Lambda^n w(0,0)\Big(\prod_{i=0}^{r-1} w(m-i)\Big) x_{m-r} e_{(0,0)},$$

for $r < m$, $r, n \in \mathbb{N}$. Hence, by (3.1) the Hilbert space $\mathscr{H}$ consist of the functions of the form

$$U_x(z) = \sum_{n=1}^{\infty}\sum_{r=0}^{k} \Lambda^k w(0,0)\Big(\prod_{i=0}^{r-1} w(m-i)\Big) x_{m-r} \frac{1}{z^{nm+r+1}} + \sum_{n=0}^{\infty}\Big(\prod_{i=1}^{n} w(0,i)\Big)^{-1} x_n z^n.$$

The formulas for the inner and outer radius of convergence take the following form

$$r^+_{w,\varphi} = \liminf_{n\to\infty} \sqrt[n]{\prod_{i=1}^{n} |w(0,i)|}$$

and

$$r^-_{w,\varphi} = \sqrt[m+1]{\prod_{i=0}^{m} |w(i)|}.$$

The reproducing kernel $\kappa_{\mathscr{H}} : \mathbb{A}(r^-, r^+) \times \mathbb{A}(r^-, r^+) \to \boldsymbol{B}(E)$ by Theorem 3.8 takes the form

$$\kappa_{\mathscr{H}}(z,\lambda) = \sum_{i\geqslant 1, j\geqslant 1} \Lambda^i \bar{\Lambda}^j |w(1,0)|^2 \Big(\prod_{i=0}^{r-1} |w(m-i)|^2\Big) \frac{1}{z^{im+r+1}\bar{\lambda}^{jm+r+1}} + \sum_{i=0}^{\infty}\Big(\prod_{i=1}^{n} |w(0,i)|^2\Big)^{-1} (z\bar{\lambda})^i.$$

In Example 5.4 below, we demonstrate composition operator which can be identified with weighted shift on a rootless directed tree. A weighted shift on a directed tree is a circular operator [27, Theorem 3.3.1.]. Hence, without loss of generality we can assume that the weight is positive.

Example 5.4. Set $m \in \mathbb{N}$ and $X = \mathbb{N} \sqcup \{(i,j)\colon i \in \{0,1\}, j \in \mathbb{N}\}$. Let $w : X \to (0,\infty)$ be a measurable function and $\varphi : X \to X$ be transformation of X defined by

$$\varphi(x) = \begin{cases} (i, j-1) & \text{for } x = (i,j),\ i \in \mathbb{Z}_+,\ j \in \{0,1\}, \\ 0 & \text{for } x \in \{(0,0),(1,0)\}, \\ x+1 & \text{for } x \in \mathbb{N}, \end{cases}$$

[1] To make the notation more readable, we adopt the convention that $\prod_{i=n}^{m} a_i = 1$ if $n > m$.

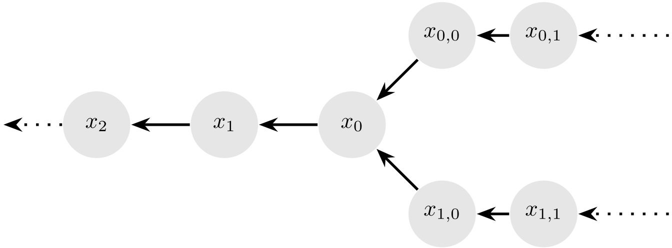

Fig. 5. The directed graph (X, E^{φ}) whose vertices, all but one, have valency one induced by self-map φ, which not have a cycle.

(see Fig. 5). Let $C_{\varphi,w} \in \boldsymbol{B}(\ell^2(X))$ be a left-invertible composition operator. As an immediate consequence of the definition, we obtain

$$C_{\varphi,w} e_x = \begin{cases} w(i,j+1)e_{(i,j+1)} & \text{for } x=(i,j),\ i \in \{0,1\},\ j \in \mathbb{N}, \\ w(i-1)e_{i-1} & \text{for } x = i \text{ and } i \in \mathbb{Z}_+, \\ w(1,0)e_{(1,0)} + w((1,0))e_{(1,0)} & \text{for } x = 0. \end{cases}$$

Applying Lemma 2.3, we get $\mathcal{N}(C^*_{\varphi,w}) = \operatorname{lin}\{w(0,0)e_{(1,0)} - w(1,0)e_{(0,0)}\}$. Therefore, by Lemma 4.2 $E = \operatorname{lin}\{e_0, w(2,0)e_{(1,0)} - w(2,0)e_{(1,0)}\}$. By more or less elementary calculations, one can verify that

$$\begin{aligned} P_E C^n_{\varphi,w} x &= \Big(\prod_{i=0}^{n-1} w(i)\Big) x_n e_0, \\ P_E (C^*_{\varphi',w})^n x &= W\Big[w(0,0)\Big(\prod_{i=1}^{n-1} w(0,i)\Big)^{-1} x_{(0,n-1)} + w(1,0)\Big(\prod_{i=1}^{n-1} w(1,i)\Big)^{-1} x_{(1,n-1)}\Big] e_0 \\ &\quad + \Big[\Big(\prod_{i=1}^{n} w(1,i)\Big)^{-1} w(1,0) x_{(0,n)} - \Big(\prod_{i=1}^{n} w(1,i)\Big)^{-1} w(0,0) x_{(1,n)}\Big]\tilde{e}, \end{aligned}$$

for $n \in \mathbb{Z}_+$, where

$$W = \frac{1}{w^2(0,0) + w^2(1,0)}, \qquad \tilde{e} = \frac{w(1,0)e_{(0,0)} - w(0,0)e_{(1,0)}}{w^2(0,0) + w^2(1,0)}.$$

Therefore, by (3.1) the formal Laurent series takes the form

$$\begin{aligned} U_x(z) &= \sum_{n=1}^{\infty} \Big(\prod_{i=0}^{n-1} w(i)\Big) x_n e_0 \frac{1}{z^n} \\ &\quad + \sum_{n=0}^{\infty} W\Big[w(0,0)\Big(\prod_{i=1}^{n-1} w(0,i)\Big)^{-1} x_{(0,n-1)} + w(1,0)\Big(\prod_{i=1}^{n-1} w(1,i)\Big)^{-1} x_{(1,n-1)}\Big] e_0 \\ &\quad + \Big[\Big(\prod_{i=1}^{n} w(1,i)\Big)^{-1} w(1,0) x_{(0,n)} - \Big(\prod_{i=1}^{n} w(0,i)\Big)^{-1} w(0,0) x_{(1,n)}\Big]\tilde{e}] z^n \end{aligned}$$

We infer from (4.7) and (4.8) that the formulas for the inner and outer radius of convergence take the following form

$$r^+_{w,\varphi} := \liminf_{n\to\infty} \Big(\sum_{i=0}^{1} \big(\frac{1}{w^2(i,0)} + \frac{1}{w^2(i,n+1)}\big) \prod_{j=1}^{n} \frac{1}{w^2(i,j)}\Big)^{-\frac{1}{2n}}$$

and

$$r^-_{w,\varphi} := \limsup_{n\to\infty} \sqrt[n]{\prod_{i=0}^{n-1} |w(i)|}.$$

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