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Large time behaviour of solutions to the 3D-NSE in $\mathcal{X}^\sigma$ spaces

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A B S T R A C T

In this paper we study the incompressible Navier-Stokes equations in $L^2(\mathbb{R}^3) \cap \mathcal{X}^{-1}(\mathbb{R}^3)$. In the global existence case, we establish that if the solution u is in the space $C(\mathbb{R}^+, L^2 \cap \mathcal{X}^{-1})$, then for $\sigma > -3/2$ the decay of $\|u(t)\|_{\mathcal{X}^\sigma}$ is at least of the order of $t^{-(2\sigma+3)/4}$. Fourier analysis and standard techniques are used.

Contents

1. Introduction

In this paper we study the large time behaviour in Fourier norms of the solution to the incompressible Navier-Stokes equations in three spatial dimensions

$$(NS)\begin{cases}\partial_t u - \Delta u + u.\nabla u = & -\nabla p \text{ in } \mathbb{R}^+ \times \mathbb{R}^3\\ \operatorname{div} u = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3\\ u(0,x) = u^0(x) \quad \text{in } \mathbb{R}^3,\end{cases}$$

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where $u = u(t,x) = (u_1, u_2, u_3)$ and $p = p(t,x)$ denote respectively the unknown velocity and the unknown pressure of the fluid at the point $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$, and $(u.\nabla u) := u_1\partial_1 u + u_2\partial_2 u + u_3\partial_3 u$, while $u^0 = (u_1^o(x), u_2^o(x), u_3^o(x))$ is an initial given velocity. If u^0 is quite regular, the divergence free condition determines the pressure p. We recall in our case it was assumed the viscosity is unitary ($\nu = 1$) in order to simplify the calculations and the proofs of our results.

The Navier-Stokes system has the following scaling property: If $u = u(t,x)$ is a solution of (NS) with initial date $u^0 = u^0(x)$ on the interval $[0,T]$, then for all $\lambda > 0, u_\lambda = \lambda u(\lambda^2 t, \lambda x)$ is a solution of (NS) with initial data $u_\lambda(0,x) = \lambda u^0(\lambda x)$ on the interval $[0, \frac{T}{\lambda^2}]$. A functional space $(X, \|.\|_X)$ is called critical space of (NS) system if $\|f_\lambda\|_X = \|f\|_X; \quad \forall \lambda > 0, \quad \forall f \in X$, where $f_\lambda(x) = \lambda f(\lambda x)$. Particularly, L^3, $\dot{H}^{1/2}$ and $\mathcal{X}^{-1}$ are critical spaces for the system (NS), where $\mathcal{X}^\sigma$ is defined as follows

$$\mathcal{X}^\sigma(\mathbb{R}^3) = \{f \in \mathcal{S}'(\mathbb{R}^3)/\ \widehat{f} \in L^1_{loc} \text{ and } \int_{\mathbb{R}^3} |\xi|^\sigma |\widehat{f}(\xi)| d\xi < \infty\}.$$

In order to explain the idea of studying the (NS) system in the space $L^2 \cap \mathcal{X}^{-1}$, by making the intersection with the energy space L^2, we obtain similar results with that the Sobolev spaces $H^s = L^2 \cap \dot{H}^s$ for $s \geq 0$, (see [5] and [9]).

It is well-known that $\mathcal{X}^{-1}$ is a subspace of homogeneous Besov space $\dot{B}^{-1}_{\infty,1}$. Using this inclusion and $\dot{B}^{-1}_{\infty,1} \hookrightarrow \dot{B}^{-1}_{\infty,q}$ for $1 \leq q \leq \infty$ (see [2]) we get $\mathcal{X}^{-1} \hookrightarrow \dot{B}^{-1}_{\infty,q}$ which is critical in the scaling sense for (NS) $(\lambda = 2^k,\ k \in \mathbb{Z})$. Unfortunately, the (NS) is ill posed in $\dot{B}^{-1}_{\infty,q}$ for all $1 \leq q \leq \infty$ (see [10]). So, it is interesting to consider reasonable substitution of $\dot{B}^{-1}_{\infty,q}$, where $\mathcal{X}^{-1}$ is a good choice for the global well-posedness of (NS). Notice that $\mathcal{X}^{-1}$ is also a subspace of BMO^{-1}, where the global well-posedness of (NS) in BMO^{-1} holds (see [7]). On the technical side, the choice of the space $\mathcal{X}^{-1}$ is a bit special because it is not comparable with homogenous Sobolev space $\dot{H}^{1/2}$ (see [3]). But it is also defined by the Fourier transformation which helps us to prove the principle result, by dividing the global solution into low and high frequencies.

The goal of this paper is to establish uniform decay rates in $\mathcal{X}^\sigma(\mathbb{R}^3)$ spaces and for the L^2 in space. The decay proof idea is new for the $\mathcal{X}^\sigma(\mathbb{R}^3)$ norms, but the decay for the L^2 norm will follow using idea of [5], (see also [4]). The decay study in L^2 is inspired by [5], precisely we cut u into high and low frequencies: $w_\delta = \mathcal{F}^{-1}(\mathbf{1}_{\{|\xi|<\delta\}}\widehat{u}), \quad v_\delta = \mathcal{F}^{-1}(\mathbf{1}_{\{|\xi|>\delta\}}\widehat{u})$. We start by showing that $\limsup_{\delta\to 0} \sup_{t\to\infty} \|w_\delta(t)\|_{L^2} = 0$. For high frequencies we prove that $(t \longmapsto \|v_\delta(t)\|_{L^2}) \in L^1(\mathbb{R}^+) \cap C(\mathbb{R}^+)$, which gives the existence of a time $t_0 = t_0(\varepsilon)$ and $\delta_0 = \delta_0(\varepsilon) > 0$ such that $\|u(t_0)\|_{L^2} \leq \|w_{\delta_0}(t_0)\|_{L^2} + \|v_{\delta_0}(t_0)\|_{L^2} < \varepsilon$, which ends the case L^2. The decay study for the $\mathcal{X}^\sigma$ norms returns to the study of the case $\sigma = -1$ which is based on the fundamental lemma (see Lemma 2.5 and Remark 2.6): if f is a continuous function of $\mathbb{R}^+$ in $\mathbb{R}^+$ verifying $f(t) \leq M_0 + \theta_1 f(\theta_2 t)$, with $\theta_1, \theta_2 \in (0,1)$, then $\limsup_{t\to\infty} f(t) \leq M_0/(1-\theta_1)$. Precisely the case $-3/2 < \sigma < -1$ is deduced in a direct way by interpolation of the spaces L^2 and $\mathcal{X}^{-1}$. The case $\sigma > -1$ is due to the analytic property of the (NS) solution with small initial data (see [1]) and the decay result in $\mathcal{X}^{-1}$.

Before treating the decay rates of global solutions, we show a result of local existence and global existence if the initial data is small in $\mathcal{X}^{-1}(\mathbb{R}^3)$. Our first result is the following.

Theorem 1.1. *Let $u^0 \in \mathcal{X}^{-1}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ be a divergence free vector fields, then there is a time $T > 0$ and unique solution $u \in C([0,T], \mathcal{X}^{-1}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))$. Moreover $u \in L^1([0,T], \mathcal{X}^1(\mathbb{R}^3))$. If $\|u^0\|_{\mathcal{X}^{-1}} < 1$, then u is global.*

Remark 1.2. **(i)** If the maximal time T^* is finite then $\int_0^{T^*} \|u(t)\|_{\mathcal{X}^1} = +\infty$. Indeed: The integral form of the system (NS):

$$u(t) = e^{t\Delta}u^0 - \int_0^t e^{(t-z)\Delta}\mathbb{P}(u.\nabla u)dz$$

implies

$$\begin{aligned}\|u(t)\|_{L^2} &\leq \|e^{t\Delta}u^0\|_{L^2} + \textstyle\int_0^t \|e^{(t-z)\Delta}\mathbb{P}(u.\nabla u)\|_{L^2}dz \\ &\leq \|u^0\|_{L^2} + \textstyle\int_0^t \|u\nabla u\|_{L^2}dz \\ &\leq \|u^0\|_{L^2} + \textstyle\int_0^t \|u\|_{L^2}\|\nabla u\|_{L^\infty}dz.\end{aligned}$$

Using the fact $\|\nabla u\|_{L^\infty} \leq (2\pi)^{-3}\|u\|_{\mathcal{X}^1}$ and Gronwall's lemma we get

$$\|u(t)\|_{L^2} \leq \|u^0\|_{L^2} \exp\big((2\pi)^{-3}\int_0^t \|u\|_{\mathcal{X}^1}\big). \tag{1.1}$$

Then, if $\int_0^{T^*} \|u\|_{\mathcal{X}^1}$ is finite we get $u \in C([0,T^*), L^2 \cap \mathcal{X}^{-1}) \cap L^\infty([0,T^*), L^2 \cap \mathcal{X}^{-1})$. Then the solution lives beyond the time T^* which contradicts the fact that T^* is the maximum time of existence.

(ii) If $\|u^0\|_{\mathcal{X}^{-1}} < 1/2$, the above remark and [3] imply the global existence of solution u of (NS) with $u \in C_b(\mathbb{R}^+, \mathcal{X}^{-1}) \cap L^1(\mathbb{R}^+, \mathcal{X}^1) \cap C(\mathbb{R}^+, L^2)$. Moreover,

$$\|u(t)\|_{\mathcal{X}^{-1}} + \frac{1}{2}\int_0^t \|u\|_{\mathcal{X}^1} \leq \|u^0\|_{\mathcal{X}^{-1}};\ \ \forall t \geq 0. \tag{1.2}$$

(iii) Using (i)-(ii) and [3], we get if $u \in C(\mathbb{R}^+, L^2 \cap \mathcal{X}^{-1})$ is a global solution of (NS), then $u \in L^1(\mathbb{R}^+, \mathcal{X}^1(\mathbb{R}^3))$.

(iv) Using (i)-(ii)-(iii) and [3], we get if $u \in C(\mathbb{R}^+, L^2 \cap \mathcal{X}^{-1})$ is a global solution of (NS), then $u \in \mathcal{C}_b(\mathbb{R}^+, L^2(\mathbb{R}^3))$. Indeed: By [3] there is a time $t_0 \geq 0$ such that $\|u(t_0)\|_{\mathcal{X}^{-1}} < 1/2$. Then (i)-(ii) imply for $t \geq t_0$

$$\begin{aligned}\|u(t)\|_{L^2} &\leq \|u(t_0)\|_{L^2} \exp\big((2\pi)^{-3}\int_{t_0}^\infty \|u(z)\|_{\mathcal{X}^1}dz\big) \\ &\leq \|u(t_0)\|_{L^2} \exp\big((2\pi)^{-3}2\|u(t_0)\|_{\mathcal{X}^{-1}}\big) \\ &\leq \|u(t_0)\|_{L^2} \exp\big((2\pi)^{-3}\big) \\ &\leq 2\|u(t_0)\|_{L^2},\end{aligned}$$

which implies

$$\|u(t)\| \leq 2\max_{0\leq z\leq t_0} \|u(z)\|_{L^2},\ \forall t \geq 0.$$

Particularly, if $\|u^0\|_{\mathcal{X}^{-1}} < 1/2$ we get

$$\|u(t)\|_{L^2} \leq 2\|u^0\|_{L^2},\ \forall t \geq 0.$$

Before stating the result of decay for the global solution of (NS), we recall the following results which will be useful in the following:

Theorem 1.3. *([1]) There exists a positive constant $\epsilon_0 > 0$ such that for any initial data u^0 in $\mathcal{X}^{-1}(\mathbb{R}^3)$ with $\|u^0\|_{\mathcal{X}^{-1}} < \epsilon_0$, the solution of Navier-Stokes system is analytic in the sense that*

$$\|\exp(\sqrt{t}|D|)u(t)\|_{\mathcal{X}^{-1}} + \frac{1}{2}\int_0^t \|\exp(\sqrt{z}|D|)u(z)\|_{\mathcal{X}^1} dz \leq 2\|u^0\|_{\mathcal{X}^{-1}},\ \forall t \geq 0.$$

Theorem 1.4. *([3]) Let $u \in C(\mathbb{R}^+, \mathcal{X}^{-1}(\mathbb{R}^3))$ be a global solution of Navier-Stokes system. Then*

$$\lim_{t\to\infty} \|u(t)\|_{\mathcal{X}^{-1}} = 0.$$

Theorem 1.5. *([6]) For any initial data $u^0 \in H^s(\mathbb{R}^3)$, $s > 5/2$, with $\mathrm{div}\,\mathrm{u}^0 = 0$, there exists a unique solution $u \in C([0, T_0], H^s(\mathbb{R}^3))$ such that $T_0 = T_0(s, \|u^0\|_{H^s})$.*

Our second result is the following.

Theorem 1.6. *Let $u \in \mathcal{C}(\mathbb{R}^+, \mathcal{X}^{-1}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3))$ be a global solution of Navier-Stokes system. Then*

(i) $\lim_{t\to+\infty} \|u(t)\|_{\mathcal{X}^{-1}} = 0$,
(ii) $\lim_{t\to+\infty} \|u(t)\|_{L^2} = 0$,
(iii) $\|u(t)\|_{\mathcal{X}^{-1}} = o(t^{-1/4})$; $t \to +\infty$,
(iv) *For all $\sigma > -3/2$, we have $\|u(t)\|_{\mathcal{X}^\sigma} = o(t^{-(2\sigma+3)/4})$; $t \to +\infty$.*

Remark 1.7. The new parts of our theorem are **(iii)-(iv)**, the part **(i)** is treated by [3] and the part **(ii)** is treated by many authors but with other hypothesis.

The remainder of our paper is organized as follows. In the second section we give some notations, definitions and preliminary results. Section 3 is devoted to prove the well posedness of (NS) in $L^2\cap\mathcal{X}^{-1}$ space, this proof used the Fixed Point Theorem with a good choice of space $X = \mathcal{C}([0,T], L^2\cap\mathcal{X}^{-1})\cap L^1([0,T], \mathcal{X}^1)$. In section 4 we prove the decay of global solutions in $L^2\cap\mathcal{X}^{-1}$, this proof used a Fourier analysis and standard techniques. The last part of section 4 is devoted to prove the decay results of the global solution in $\mathcal{X}^\sigma$, this proof uses in a fundamental way the decay in $L^2 \cap \mathcal{X}^{-1}$.

2. Notations and preliminary results

2.1. Notations

In this section, we collect some notations and definitions that will be used later.

- The Fourier transformation is normalized as

$$\mathcal{F}(f)(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^3} \exp(-ix.\xi) f(x)dx,\ \ \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.$$

- The inverse Fourier formula is

$$\mathcal{F}^{-1}(g)(x) = (2\pi)^{-3} \int_{\mathbb{R}^3} \exp(i\xi.x) g(\xi)d\xi,\ \ x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

- The convolution product of a suitable pair of function f and g on $\mathbb{R}^3$ is given by

$$(f*g)(x) := \int_{\mathbb{R}^3} f(y)g(x-y)dy.$$

- If $f=(f_1,f_2,f_3)$ and $g=(g_1,g_2,g_3)$ are two vector fields, we set

$$f\otimes g := (g_1 f, g_2 f, g_3 f),$$

and

$$\operatorname{div}(f\otimes g) := (\operatorname{div}(g_1 f), \operatorname{div}(g_2 f), \operatorname{div}(g_3 f)).$$

Moreover, if $\operatorname{div} \mathrm{g}=0$ we obtain

$$\operatorname{div}(f\otimes g) := g_1\partial_1 f + g_2\partial_2 f + g_3\partial_3 f := g.\nabla f.$$

- Let $(B, ||.||)$, be a Banach space, $1\le p\le\infty$ and $T>0$. We define $L^p_T(B)$ the space of all measurable functions $[0,t]\ni t\mapsto f(t)\in B$ such that $t\mapsto ||f(t)||\in L^p([0,T])$.
- The Sobolev space $H^s(\mathbb{R}^3)=\{f\in\mathcal{S}'(\mathbb{R}^3);\ (1+|\xi|^2)^{s/2}\widehat{f}\in L^2(\mathbb{R}^3)\}$.
- The homogeneous Sobolev space $\dot H^s(\mathbb{R}^3)=\{f\in\mathcal{S}'(\mathbb{R}^3);\ \widehat{f}\in L^1_{loc}$ and $|\xi|^s\widehat{f}\in L^2(\mathbb{R}^3)\}$.
- The Lei-Lin space $\mathcal{X}^\sigma(\mathbb{R}^3)=\{f\in\mathcal{S}'(\mathbb{R}^3);\ \widehat{f}\in L^1_{loc}$ and $|\xi|^\sigma\widehat{f}\in L^1(\mathbb{R}^3)\}$.

2.2. Preliminary results

In this section, we recall some classical results and we give new technical lemmas.

Lemma 2.1. *We have $\mathcal{X}^{-1}(\mathbb{R}^3)\cap\mathcal{X}^1(\mathbb{R}^3)\hookrightarrow\mathcal{X}^0(\mathbb{R}^3)$. Precisely, we have*

$$\|f\|_{\mathcal{X}^0(\mathbb{R}^3)}\le\|f\|^{1/2}_{\mathcal{X}^{-1}(\mathbb{R}^3)}\|f\|^{1/2}_{\mathcal{X}^1(\mathbb{R}^3)},\ \forall f\in\mathcal{X}^{-1}(\mathbb{R}^3)\cap\mathcal{X}^1(\mathbb{R}^3). \tag{2.1}$$

Proof. We can write

$$\|f\|_{\mathcal{X}^0}=\int_{\mathbb{R}^3}|\xi|^{-1/2}|\widehat{f}(\xi)|^{1/2}\frac{|\widehat{f}(\xi)|^{1/2}}{|\xi|^{1/2}}d\xi.$$

Cauchy-Schwartz inequality gives the result. □

Lemma 2.2. *Let $\sigma,s\in\mathbb{R}$ such that $0<\sigma+\frac{3}{2}<s$. Then $H^s(\mathbb{R}^3)\hookrightarrow\mathcal{X}^\sigma(\mathbb{R}^3)$. Precisely, there is a constant $C=C(s,\sigma)$ such that*

$$\|f\|_{\mathcal{X}^\sigma(\mathbb{R}^3)}\le C\|f\|^{1-\frac{\sigma+\frac{3}{2}}{s}}_{L^2(\mathbb{R}^3)}\|f\|^{\frac{\sigma+\frac{3}{2}}{s}}_{\dot H^s(\mathbb{R}^3)},\ \ \forall f\in H^s(\mathbb{R}^3). \tag{2.2}$$

Proof. For $\lambda>0$, we have

$$\|f\|_{\mathcal{X}^\sigma}=I_\lambda+J_\lambda,$$

with

$$I_\lambda = \int\limits_{\{\xi\in\mathbb{R}^3/|\xi|<\lambda\}} |\xi|^\sigma |\widehat{f}(\xi)| d\xi, \;\; J_\lambda = \int\limits_{\{\xi\in\mathbb{R}^3/|\xi|>\lambda\}} |\xi|^\sigma |\widehat{f}(\xi)| d\xi.$$

We have

$$\begin{aligned} I_\lambda \leq & \Big(\int_{\{\xi\in\mathbb{R}^3/|\xi|<\lambda\}} |\xi|^{2\sigma} d\xi\Big)^{1/2} \|f\|_{L^2} \leq \tfrac{c}{\sqrt{2\sigma+3}} \lambda^{\sigma+\frac{3}{2}} \|f\|_{L^2}, \\ J_\lambda \leq & \Big(\int_{\{\xi\in\mathbb{R}^3/|\xi|>\lambda\}} |\xi|^{2(\sigma-s)} d\xi\Big)^{1/2} \|f\|_{\dot{H}^s} \leq \tfrac{C}{\sqrt{s-\sigma-\frac{3}{2}}} \lambda^{\sigma+\frac{3}{2}-s} \|f\|_{\dot{H}^s}. \end{aligned}$$

For $\lambda = (\|f\|_{\dot{H}^s}/\|f\|_{L^2})^{1/s}$, we obtain the desired result. □

Lemma 2.3. *Let $\sigma_0 > -3/2$. If we have*

$$\mathcal{X}^{\sigma_0}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3) \hookrightarrow \mathcal{X}^\sigma(\mathbb{R}^3); \quad \forall -3/2 < \sigma \leq \sigma_0.$$

Precisely

$$\|f\|_{\mathcal{X}^\sigma} \leq c_0 \|f\|_{L^2}^{1-\theta} \|f\|_{\mathcal{X}^{\sigma_0}}^\theta, \quad \forall c_0 = c(\sigma_0,\sigma), \;\; \theta = \frac{\frac{3}{2}+\sigma}{\frac{3}{2}+\sigma_0}. \tag{2.3}$$

Proof. For $\lambda > 0$, we have

$$\|f\|_{\mathcal{X}^\sigma} = A(\lambda) + B(\lambda),$$

with

$$A(\lambda) = \int\limits_{\{\xi\in\mathbb{R}^3/|\xi|<\lambda\}} |\xi|^\sigma |\widehat{f}(\xi)| d\xi, \;\; B(\lambda) = \int\limits_{\{\xi\in\mathbb{R}^3/|\xi|>\lambda\}} |\xi|^\sigma |\widehat{f}(\xi)| d\xi.$$

We have

$$\begin{aligned} A(\lambda) \leq & \int_{\{\xi\in\mathbb{R}^3/|\xi|<\lambda\}} |\xi|^{2\sigma} d\xi\Big)^{1/2} \|f\|_{L^2} \leq \tfrac{c}{\sqrt{2\sigma+3}} \lambda^{\sigma+\frac{3}{2}} \|f\|_{L^2} \\ B(\lambda) \leq & \int_{\{\xi\in\mathbb{R}^3/|\xi|>\lambda\}} |\xi|^{(\sigma-\sigma_0)} |\xi|^{\sigma_0} |\widehat{f}(\xi)| d\xi \leq \lambda^{\sigma-\sigma_0} \|f\|_{\mathcal{X}^{\sigma_0}}, \end{aligned}$$

which imply

$$\|f\|_{\mathcal{X}^\sigma} \leq \frac{c}{\sqrt{2\sigma+3}} \lambda^{\sigma+\frac{3}{2}} \|f\|_{L^2} + \lambda^{\sigma-\sigma_0} \|f\|_{\mathcal{X}^{\sigma_0}}.$$

For $\lambda = (\|f\|_{\mathcal{X}^{\sigma_0}}/\|f\|_{L^2})^{1/(3/2+\sigma_0)}$, we obtain

$$\|f\|_{\mathcal{X}^\sigma} \leq c_{\sigma,\sigma_0} \|f\|_{L^2}^{\frac{\sigma_0-\sigma}{\frac{3}{2}+\sigma_0}} \|f\|_{\mathcal{X}^{\sigma_0}}^{\frac{\sigma+\frac{3}{2}}{\frac{3}{2}+\sigma_0}}. \quad \square$$

Lemma 2.4. *Let $f, g \in L^\infty_T(\mathcal{X}^{-1}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)) \cap L^1_T(\mathcal{X}^1(\mathbb{R}^3))$ such that* $\operatorname{div} f = 0$ *almost everywhere. Then*

$$\sup_{0\leq t\leq T} \|\int_0^t e^{(t-z)\Delta} \mathbb{P}(f.\nabla g) dz\|_{\mathcal{X}^{-1}} \leq \|f\|_{L^\infty_T(\mathcal{X}^{-1})}^{1/2} \|f\|_{L^1_T(\mathcal{X}^1)}^{1/2} \|g\|_{L^\infty_T(\mathcal{X}^{-1})}^{1/2} \|g\|_{L^1_T(\mathcal{X}^1)}^{1/2}, \tag{2.4}$$

$$\sup_{0\leq t\leq T}\|\int_0^t e^{(t-z)\Delta}\mathbb{P}(f.\nabla g)dz\|_{L^2}\leq (2\pi)^{-3}\|f\|_{L^\infty_T(L^2)}\|g\|_{L^1_T(\mathcal{X}^1)},\tag{2.5}$$

$$\int_0^T\|\int_0^t e^{(t-z)\Delta}\mathbb{P}(f.\nabla g)dz\|_{\mathcal{X}^1}dt\leq \|f\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|f\|^{1/2}_{L^1_T(\mathcal{X}^1)}\|g\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|g\|^{1/2}_{L^1_T(\mathcal{X}^1)}.\tag{2.6}$$

Proof. • Proof of (2.4): We can write

$$\begin{aligned}
\|\int_0^t e^{(t-z)\Delta}\mathbb{P}(f.\nabla g)dz\|_{\mathcal{X}^{-1}} &\leq \int_0^t \|e^{(t-z)\Delta}\mathbb{P}(f.\nabla g)\|_{\mathcal{X}^{-1}}dz\\
&\leq \int_0^t \|f.\nabla g\|_{\mathcal{X}^{-1}}dz\\
&\leq \int_0^T \|\mathrm{div}\,(f\otimes g)\|_{\mathcal{X}^{-1}}dz\\
&\leq \int_0^T \|f\otimes g\|_{\mathcal{X}^0}dz\\
&\leq \int_0^T \|f\|_{\mathcal{X}^0}\|g\|_{\mathcal{X}^0}dz\\
&\leq \int_0^T \|f\|^{1/2}_{\mathcal{X}^{-1}}\|f\|^{1/2}_{\mathcal{X}^1}\|g\|^{1/2}_{\mathcal{X}^{-1}}\|g\|^{1/2}_{\mathcal{X}^1}dz\\
&\leq \|f\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|g\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\int_0^T \|f\|^{1/2}_{\mathcal{X}^1}\|g\|^{1/2}_{\mathcal{X}^1}dz\\
&\leq \|f\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|f\|^{1/2}_{L^1_T(\mathcal{X}^1)}\|g\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|g\|^{1/2}_{L^1_T(\mathcal{X}^1)}.
\end{aligned}$$

• Proof of (2.5): We can write

$$\begin{aligned}
\|\int_0^t e^{(t-z)\Delta}\mathbb{P}(f.\nabla g)dz\|_{L^2} &\leq \int_0^t \|e^{(t-z)\Delta}\mathbb{P}(f.\nabla g)\|_{L^2}dz\\
&\leq \int_0^t \|f.\nabla g\|_{L^2}dz\\
&\leq \int_0^T \|f.\nabla g\|_{L^2}dz\\
&\leq \int_0^T \|f\|_{L^2}\|\nabla g\|_{L^\infty}dz\\
&\leq (2\pi)^{-3}\|f\|_{L^\infty_T(L^2)}\int_0^T \|g\|_{\mathcal{X}^1}dz\\
&\leq (2\pi)^{-3}\|f\|_{L^\infty_T(L^2)}\|g\|_{L^1_T(\mathcal{X}^1)}dz.
\end{aligned}$$

• Proof of (2.6): We can write

$$\begin{aligned}
\int_0^T\|\int_0^t e^{(t-z)\Delta}\mathbb{P}(f.\nabla g)dz\|_{\mathcal{X}^1}dt &\leq \int_0^T\int_0^t \|e^{(t-z)\Delta}\mathbb{P}(f.\nabla g)\|_{\mathcal{X}^1}dzdt\\
&\leq \int_0^T\int_0^t\int_{\mathbb{R}^3} e^{-(t-z)|\xi|^2}|\xi|.|\mathcal{F}(\mathrm{div}\,(f\otimes g))(z,\xi)|d\xi dzdt\\
&\leq \int_0^T\int_0^t\int_{\mathbb{R}^3} e^{-(t-z)|\xi|^2}|\xi|^2.|\mathcal{F}(f\otimes g)(z,\xi)|d\xi dzdt\\
&\leq \int_{\mathbb{R}^3}|\xi|^2\Big(\int_0^T\int_0^t e^{-(t-z)|\xi|^2}|\mathcal{F}(f\otimes g)(z,\xi)|dzdt\Big)d\xi\\
&\leq \int_{\mathbb{R}^3}|\xi|^2\Big(\|e^{-t|\xi|^2}*_t|\mathcal{F}(f\otimes g)(t,\xi)\|_{L^1([0,T])}\Big)d\xi\\
&\leq \int_{\mathbb{R}^3}|\xi|^2\Big(\|e^{-|\xi|^2}\|_{L^1([0,T])}\|\mathcal{F}(f\otimes g)(.,\xi)\|_{L^1([0,T])}\Big)d\xi\\
&\leq \int_{\mathbb{R}^3}|\xi|^2\Big(\tfrac{1-e^{-T|\xi|^2}}{\nu|\xi|^2}\int_0^T|\mathcal{F}(f\otimes g)(t,\xi)|dt\Big)d\xi\\
&\leq \int_0^T\int_{\mathbb{R}^3}|\mathcal{F}(f\otimes g)(t,\xi)|d\xi dt\\
&\leq \int_0^T\|f\otimes g(t)\|_{\mathcal{X}^0}dt\\
&\leq \int_0^T\|f\otimes g\|_{\mathcal{X}^0}dz\\
&\leq \int_0^T\|f\|_{\mathcal{X}^0}\|g\|_{\mathcal{X}^0}dz\\
&\leq \int_0^T \|f\|^{1/2}_{\mathcal{X}^{-1}}\|f\|^{1/2}_{\mathcal{X}^1}\|g\|^{1/2}_{\mathcal{X}^{-1}}\|g\|^{1/2}_{\mathcal{X}^1}dz\\
&\leq \|f\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|g\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\int_0^T \|f\|^{1/2}_{\mathcal{X}^1}\|g\|^{1/2}_{\mathcal{X}^1}dz\\
&\leq \|f\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|f\|^{1/2}_{L^1_T(\mathcal{X}^1)}\|g\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|g\|^{1/2}_{L^1_T(\mathcal{X}^1)}.\quad\square
\end{aligned}$$

Lemma 2.5. *Let $T > 0$ and $f : [0,T] \to \mathbb{R}_+$ be continuous function such that*

$$f(t) \leq M_0 + \theta_1 f(\theta_2 t); \quad \forall\, 0 \leq t \leq T \tag{2.7}$$

with $M_0 \geq 0$ and $\theta_1, \theta_2 \in (0,1)$. Then

$$f(t) \leq \frac{M_0}{1-\theta_1}; \quad \forall 0 \leq t \leq T.$$

Proof. As f is a positive and continuous function, then there is a time $t_0 \in [0,T]$ such that

$$0 \leq f(t_0) = \max_{0 \leq t \leq T} f(t).$$

Applying (2.7) at $t = t_0$ we get

$$f(t_0) \leq M_0 + \theta_1 f(\theta_2 t_0) \leq M_0 + \theta_1 f(t_0)$$

which implies $f(t_0) \leq \frac{M_0}{1-\theta_1}$. As $f(t_0) = \max_{0 \leq t \leq T} f(t)$, we get the desired result. $\square$

Remark 2.6. Applying Lemma 2.5 to a positive continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$f(t) \leq M_0 + \theta_1 f(\theta_2 t); \quad \forall t \geq 0$$

with $M_0 \geq 0$ and $\theta_1, \theta_2 \in (0,1)$, we obtain

$$\limsup_{t \to \infty} f(t) \leq \frac{M_0}{1-\theta_1}.$$

3. Well posedness results in $L^2(\mathbb{R}^3) \cap \mathcal{X}^{-1}(\mathbb{R}^3)$

In this section we prove Theorem 1.1. To prove the existence result we need the following remark: For $f \in L^2(\mathbb{R}^3) \cap \mathcal{X}^{-1}(\mathbb{R}^3)$ and $\varepsilon_0 > 0$ there is $\lambda > 0$ such that

$$\|\lambda f(\lambda .)\|_{\mathcal{X}^{-1}} = \|f\|_{\mathcal{X}^{-1}} \;\text{ and }\; \|\lambda f(\lambda .)\|_{L^2} < \varepsilon_0.$$

Precisely, just take $\lambda = \frac{\varepsilon_0^2}{4\|f\|_{L^2}^2 + 1}$. Then we can choose $\lambda_0 > 0$ such that

$$\|\lambda_0 u^0(\lambda_0 .)\|_{\mathcal{X}^{-1}} = \|u^0\|_{\mathcal{X}^{-1}} \;\text{ and }\; \|\lambda_0 u^0(\lambda_0 .)\|_{L^2} < \frac{1}{48}.$$

Consider then the Navier-Stokes system

$$(NS_{\lambda_0}) \begin{cases} \partial_t v - \Delta v + v.\nabla v = \; -\nabla q \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} v = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\ v(0,x) = \lambda_0 u^0(\lambda_0 x) \quad \text{in } \mathbb{R}^3. \end{cases}$$

If the system (NS_{λ_0}) has a unique solution v in $C([0,T], L^2 \cap \mathcal{X}^{-1})$, then $u = \lambda_0^{-1} v(\lambda_0^{-2} t, \lambda_0^{-1} x)$ is a solution of Navier-Stokes system starting by u^0. Therefore, we can assume in the following that

$$\|u^0\|_{L^2} < \frac{1}{48}. \tag{3.1}$$

Let's go back to the proof of Theorem 1.1. A uniqueness in $L^2 \cap \mathcal{X}^{-1}$ is given by the uniqueness in $\mathcal{X}^{-1}$, (see [8]). It remains a proven existence, for this let $k \in \mathbb{N}^*$ such that

$$\int_{\{\xi \in \mathbb{R}^3 / |\xi| > k\}} \frac{|\widehat{u^0}(\xi)|}{|\xi|} d\xi < \frac{1}{16}.$$

Put

$$a^0 = \mathcal{F}^{-1}(\mathbf{1}_{|\xi|<k}\widehat{u^0}(\xi)), \;\; b^0 = u^0 - a^0 = \mathcal{F}^{-1}(\mathbf{1}_{|\xi|\geq k}\widehat{u^0}(\xi)).$$

We have

$$a^0 \in H^s(\mathbb{R}^3), \;\; \forall s \geq 0, \tag{3.2}$$

and

$$\|b^0\|_{\mathcal{X}^{-1}} < \frac{1}{16}. \tag{3.3}$$

Moreover

$$\|a^0\|_{L^2} \leq \|u^0\|_{L^2} \;\text{ and }\; \|b^0\|_{L^2} \leq \|u^0\|_{L^2}. \tag{3.4}$$

There is a time $T_0 > 0$ such that the system (NS) has a unique solution a in $C([0,T_0], H^4(\mathbb{R}^3))$ with initial condition a^0 (see [6]). Using the fact (see Lemma 2.2)

$$H^4(\mathbb{R}^3) \hookrightarrow L^2(\mathbb{R}^3) \cap \mathcal{X}^{-1}(\mathbb{R}^3) \cap \mathcal{X}^1(\mathbb{R}^3),$$

we get

$$a \in C([0,T_0], L^2(\mathbb{R}^3) \cap \mathcal{X}^{-1}(\mathbb{R}^3) \cap \mathcal{X}^1(\mathbb{R}^3)). \tag{3.5}$$

Using the regularity of the function a and inequality (3.4), we obtain

$$\|a(t)\|_{L^2}^2 + 2\int_0^t \|\nabla a(z)\|_{L^2}^2 = \|a^0\|_{L^2}^2 \leq \|u^0\|_{L^2}^2, \; \forall t \in [0,T_0]. \tag{3.6}$$

Put $b = u - a$, b satisfies the following system

$$(RNS) \begin{cases} \partial_t b - \Delta b + b.\nabla b + b.\nabla a + a.\nabla b = & -\nabla q \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} b = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\ b(0,x) = b^0(x) \quad \text{in } \mathbb{R}^3. \end{cases}$$

The integral form of (RNS) is

$$b = \psi(b) = e^{t\Delta} b^0 - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}(a.\nabla b) - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}(b.\nabla a) - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}(b.\nabla b).$$

Put

$$\begin{array}{lll} f_0 & = e^{t\Delta}b^0: & \text{the constant part} \\ L(b) & = -\int_0^t e^{(t-\tau)\Delta}\mathbb{P}(a.\nabla b) - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}(b.\nabla a): & \text{the linear part} \\ Q(b) & = -\int_0^t e^{(t-\tau)\Delta}\mathbb{P}(b.\nabla b): & \text{the nonlinear part.} \end{array}$$

For $T > 0$ put the space

$$X_T = C([0,T], L^2(\mathbb{R}^3) \cap \mathcal{X}^{-1}(\mathbb{R}^3)) \cap L^1([0,T], \mathcal{X}^1(\mathbb{R}^3)).$$

This vector space is equipped with the norm

$$\|f\|_T = \|f\|_{L^\infty_T(L^2)} + \|f\|_{L^\infty_T(\mathcal{X}^{-1})} + \|f\|_{L^1_T(\mathcal{X}^1)}.$$

For $\varepsilon, T > 0$ (to fixed later), such that $T \le T_0$, put the closed subset of X_T defined by

$$B(\varepsilon, T) = \left\{ f \in X_T; \begin{cases} \|f\|_{L^\infty_T(L^2)} & \le 2\|b^0\|_{L^2} \\ \|f\|_{L^\infty_T(\mathcal{X}^{-1})} & \le 2\|b^0\|_{\mathcal{X}^{-1}} \\ \|f\|_{L^1_T(\mathcal{X}^1)} & \le \varepsilon \end{cases} \right\}$$

Explanation of the choice of ε and T: We have

$$\begin{array}{ll} \|f_0\|_{L^\infty_T(\mathcal{X}^{-1})} & \le \|b^0\|_{\mathcal{X}^{-1}} \\ \|f_0\|_{L^\infty_T(L^2)} & \le \|b^0\|_{L^2} \\ \|f_0\|_{L^1_T(\mathcal{X}^1)} & = \int_0^T \int_{\mathbb{R}^3} e^{-t|\xi|^2}|\xi|.|\widehat{b^0}(\xi)|d\xi dt \\ & = \int_{\mathbb{R}^3} \left(\int_0^T e^{-t|\xi|^2}dt\right)|\xi|.|\widehat{b^0}(\xi)|d\xi \\ & = \int_{\mathbb{R}^3} \left(\frac{1-e^{-T|\xi|^2}}{|\xi|^2}\right)|\xi|.|\widehat{b^0}(\xi)|d\xi \\ & = \int_{\mathbb{R}^3}(1-e^{-T|\xi|^2})\frac{|\widehat{b^0}(\xi)|}{|\xi|}d\xi. \end{array}$$

Dominated Convergence Theorem implies

$$\lim_{t\to 0^+} \|f_0\|_{L^1_T(\mathcal{X}^1)} = 0. \tag{3.7}$$

Let $0 < \varepsilon < 1/24$ and $0 < T \le T_0$ such that

$(H1)$ $\|a\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|a\|^{1/2}_{L^1_T(\mathcal{X}^1)}\sqrt{2}\sqrt{\varepsilon}\|b^0\|^{1/2}_{\mathcal{X}^{-1}} \le \frac{\|b^0\|_{\mathcal{X}^{-1}}}{4}$

$(H2)$ $\|a^0\|_{L^2}\varepsilon + 2\|a\|_{L^1_T(\mathcal{X}^1)}\|b^0\|_{L^2} \le \frac{\|b^0\|_{L^2}}{4}$

$(H3)$ $\|a\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|a\|^{1/2}_{L^1_T(\mathcal{X}^1)}\sqrt{2}\sqrt{\varepsilon}\|b^0\|^{1/2}_{\mathcal{X}^{-1}} \le \varepsilon/3$

$(H4)$ $\varepsilon + 2\|b^0\|_{L^2} \le 1/12$

$(H5)$ $\sqrt{2}\sqrt{\varepsilon}\|b^0\|^{1/2}_{\mathcal{X}^{-1}} \le 1/12$

$(H6)$ $\|f_0\|_{L^1_T(\mathcal{X}^1)} \le \varepsilon/3$

$(H7)$ $2\sqrt{2\varepsilon}\|b_0\|_{\mathcal{X}^{-1}} \le 1/12$

$(H8)$ $\|a\|_{L^\infty_T(\mathcal{X}^{-1})}\|a\|_{L^1_T(\mathcal{X}^1)} \le 1/12$

$(H9)$ $\|a\|_{L^1_T(\mathcal{X}^1)} \le 1/24.$

These choices are possible just use the equations (3.3)-(3.5)-(3.7). Now we want to prepare to apply the Fixed Point Theorem, for this we prove the following

$$\psi(B(\varepsilon,T)) \subset B(\varepsilon,T). \tag{3.8}$$

$$\|\psi(\alpha_1)-\psi(\alpha_2)\|_T \le \frac{1}{2}\|\alpha_1-\alpha_2\|_T,\ \forall \alpha_1,\alpha_2 \in B(\varepsilon,T). \tag{3.9}$$

Proof of (3.8): Using inequality (2.4), we obtain

$$\begin{aligned}
\|L(b)\|_{L^\infty_T(\mathcal{X}^{-1})} &\le \|a\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|a\|^{1/2}_{L^1_T(\mathcal{X}^1)}\|b\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|b\|^{1/2}_{L^1_T(\mathcal{X}^1)}\\
&\le \|a\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|a\|^{1/2}_{L^1_T(\mathcal{X}^1)}\sqrt{2}\sqrt{\varepsilon}\|b^0\|^{1/2}_{\mathcal{X}^{-1}}\\
&\le \tfrac{\|b^0\|_{\mathcal{X}^{-1}}}{4},\ (by\,(H1))
\end{aligned}$$

$$\begin{aligned}
\|Q(b)\|_{L^\infty_T(\mathcal{X}^{-1})} &\le \|b\|_{L^\infty_T(\mathcal{X}^{-1})}\|b\|_{L^1_T(\mathcal{X}^1)}\\
&\le 2\varepsilon\|b^0\|_{\mathcal{X}^{-1}}\\
&\le \tfrac{\|b^0\|_{\mathcal{X}^{-1}}}{4}.
\end{aligned}$$

Then

$$\|\psi(b)\|_{L^\infty_T(\mathcal{X}^{-1})} \le 2\|b^0\|_{\mathcal{X}^{-1}},\ \forall b \in B(\varepsilon,T). \tag{3.10}$$

Similarly, inequality (2.5) gives

$$\begin{aligned}
\|L(b)\|_{L^\infty_T(L^2)} &\le \|a\|_{L^\infty_T(L^2)}\|b\|_{L^1_T(\mathcal{X}^1)} + \|b\|_{L^\infty_T(L^2)}\|a\|_{L^1_T(\mathcal{X}^1)}\\
&\le \|a^0\|_{L^2}\varepsilon + 2\|a\|_{L^1_T(\mathcal{X}^1)}\|b^0\|_{L^2}\\
&\le \tfrac{\|b^0\|_{L^2}}{4},\ (by\,(H2))
\end{aligned}$$

$$\begin{aligned}
\|Q(b)\|_{L^\infty_T(L^2)} &\le \|b\|_{L^\infty_T(L^2)}\|b\|_{L^1_T(\mathcal{X}^1)}\\
&\le 2\varepsilon\|b^0\|_{L^2}\\
&\le \tfrac{\|b^0\|_{L^2}}{4}.
\end{aligned}$$

Then

$$\|\psi(b)\|_{L^\infty_T(L^2)} \le 2\|b^0\|_{L^2},\ \forall b \in B(\varepsilon,T). \tag{3.11}$$

Finally, inequality (2.6) gives

$$\begin{aligned}
\|L(b)\|_{L^1_T(\mathcal{X}^1)} &\le \|a\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|a\|^{1/2}_{L^1_T(\mathcal{X}^1)}\|b\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|b\|^{1/2}_{L^1_T(\mathcal{X}^1)}\\
&\le \|a\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|a\|^{1/2}_{L^1_T(\mathcal{X}^1)}\sqrt{2}\sqrt{\varepsilon}\|b^0\|^{1/2}_{\mathcal{X}^{-1}}\\
&\le \varepsilon/3,\ (by\,(H3))
\end{aligned}$$

$$\begin{aligned}
\|Q(b)\|_{L^1_T(\mathcal{X}^1)} &\le \|b\|_{L^\infty_T(\mathcal{X}^{-1})}\|b\|_{L^1_T(\mathcal{X}^1)}\\
&\le 2\varepsilon\|b^0\|_{\mathcal{X}^{-1}}\\
&\le \varepsilon/3,\ (\text{by } (3.3)).
\end{aligned}$$

Then,

$$\|\psi(b)\|_{L^1_T(\mathcal{X}^1)} \le \varepsilon,\ \forall b \in B(\varepsilon,T). \tag{3.12}$$

Therefore inequalities (3.10)-(3.11)-(3.12) imply (3.8).

Proof of (3.9): Using inequality (2.4), we obtain

$$\begin{aligned}
\|L(\alpha_1)-L(\alpha_2)\|_{L^\infty_T(\mathcal{X}^{-1})} &\leq \|L(\alpha_1-\alpha_2)\|_{L^\infty_T(\mathcal{X}^{-1})}\\
&\leq \|a\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|a\|^{1/2}_{L^1_T(\mathcal{X}^{1})}\|\alpha_1-\alpha_2\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|\alpha_1-\alpha_2\|^{1/2}_{L^1_T(\mathcal{X}^{1})}\\
&\leq \|a\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|a\|^{1/2}_{L^1_T(\mathcal{X}^{1})}\|\alpha_1-\alpha_2\|_{\varepsilon,T}\\
&\leq \tfrac{1}{12}\|\alpha_1-\alpha_2\|_{\varepsilon,T},\ (by\,(H8))
\end{aligned}$$

$$\begin{aligned}
\|Q(\alpha_1)-Q(\alpha_2)\|_{L^\infty_T(\mathcal{X}^{-1})} &= \|\textstyle\int_0^t e^{-(t-z)\Delta}\mathbb{P}((\alpha_1-\alpha_2).\nabla\alpha_1+\alpha_2.\nabla(\alpha_1-\alpha_2))\|_{L^\infty_T(\mathcal{X}^{-1})}\\
&\leq \Big(\sum_{i=1}^{2}\|\alpha_i\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|\alpha_i\|^{1/2}_{L^1_T(\mathcal{X}^{1})}\Big)\|\alpha_1-\alpha_2\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|\alpha_1-\alpha_2\|^{1/2}_{L^1_T(\mathcal{X}^{1})}\\
&\leq 2\sqrt{2}\sqrt{\varepsilon}\|b^0\|^{1/2}_{\mathcal{X}^{-1}}\|\alpha_1-\alpha_2\|_T\\
&\leq \tfrac{1}{12}\|\alpha_1-\alpha_2\|_T,\ (by\,(H7)).
\end{aligned}$$

Then

$$\|\psi(\alpha_1)-\psi(\alpha_2)\|_{L^\infty_T(\mathcal{X}^{-1})} \leq \frac{1}{6}\|\alpha_1-\alpha_2\|_T,\ \forall\alpha_1,\alpha_2\in B(\varepsilon,T). \tag{3.13}$$

Similarly, inequality (2.5) gives

$$\begin{aligned}
\|L(\alpha_1)-L(\alpha_2)\|_{L^\infty_T(L^2)} &\leq \|L(\alpha_1-\alpha_2)\|_{L^\infty_T(L^2)}\\
&\leq \|a\|_{L^\infty_T(L^2)}\|\alpha_1-\alpha_2\|_{L^1_T(\mathcal{X}^1)}+\|a\|_{L^1_T(\mathcal{X}^1)}\|\alpha_1-\alpha_2\|_{L^\infty_T(L^2)}\\
&\leq \Big(\|a\|_{L^\infty_T(L^2)}+\|a\|_{L^1_T(\mathcal{X}^1)}\Big)\|\alpha_1-\alpha_2\|_T\\
&\leq \Big(\|a^0\|_{L^2}+\|a\|_{L^1_T(\mathcal{X}^1)}\Big)\|\alpha_1-\alpha_2\|_T\\
&\leq \Big(\|u^0\|_{L^2}+\|a\|_{L^1_T(\mathcal{X}^1)}\Big)\|\alpha_1-\alpha_2\|_T\\
&\leq \Big(\tfrac{1}{24}+\|a\|_{L^1_T(\mathcal{X}^1)}\Big)\|\alpha_1-\alpha_2\|_T,\ (\text{by } (3.1))\\
&\leq \tfrac{1}{12}\|\alpha_1-\alpha_2\|_T,\ (by\,(H9))
\end{aligned}$$

$$\begin{aligned}
\|Q(\alpha_1)-Q(\alpha_2)\|_{L^\infty_T(L^2)} &= \|\textstyle\int_0^t e^{-(t-z)\Delta}\mathbb{P}((\alpha_1-\alpha_2).\nabla\alpha_1+\alpha_2.\nabla(\alpha_1-\alpha_2))\|_{L^\infty_T(L^2)}\\
&\leq \|\alpha_1-\alpha_2\|_{L^\infty_T(L^2)}\|\alpha_1\|_{L^1_T(\mathcal{X}^1)}+\|\alpha_2\|_{L^\infty_T(L^2)}\|\alpha_1-\alpha_2\|_{L^1_T(\mathcal{X}^1)}\\
&\leq \Big(\|\alpha_1\|_{L^1_T(\mathcal{X}^1)}+\|\alpha_2\|_{L^\infty_T(L^2)}\Big)\|\alpha_1-\alpha_2\|_T\\
&\leq (\varepsilon+2\|b^0\|_{L^2})\|\alpha_1-\alpha_2\|_T\\
&\leq (\varepsilon+2\|u^0\|_{L^2})\|\alpha_1-\alpha_2\|_T\\
&\leq \tfrac{1}{12}\|\alpha_1-\alpha_2\|_T,\ (by\,(H4)).
\end{aligned}$$

Then

$$\|\psi(\alpha_1)-\psi(\alpha_2)\|_{L^\infty_T(L^2)} \leq \frac{1}{6}\|\alpha_1-\alpha_2\|_T,\ \forall\alpha_1,\alpha_2\in B(\varepsilon,T). \tag{3.14}$$

Finally, inequality (2.6) gives

$$\begin{aligned}\|L(\alpha_1)-L(\alpha_2)\|_{L^1_T(\mathcal{X}^1)} &= \|L(\alpha_1-\alpha_2)\|_{L^1_T(\mathcal{X}^1)}\\ &\le \|a\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|a\|^{1/2}_{L^1_T(\mathcal{X}^1)}\|\alpha_1-\alpha_2\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|\alpha_1-\alpha_2\|^{1/2}_{L^1_T(\mathcal{X}^1)}\\ &\le \|a\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|a\|^{1/2}_{L^1_T(\mathcal{X}^1)}\|\alpha_1-\alpha_2\|_T\\ &\le \tfrac{1}{12}\|\alpha_1-\alpha_2\|_T,\ (by\,(H8))\end{aligned}$$

$$\begin{aligned}\|Q(\alpha_1)-Q(\alpha_2)\|_{L^1_T(\mathcal{X}^1)} &= \|\textstyle\int_0^t e^{(t-z)\Delta}\mathbb{P}((\alpha_1-\alpha_2).\nabla\alpha_1+\alpha_2.\nabla(\alpha_1-\alpha_2))\|_{L^1_T(\mathcal{X}^1)}\\ &\le \Big(\sum_{i=1}^2\|\alpha_i\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|\alpha_i\|^{1/2}_{L^1_T(\mathcal{X}^1)}\Big)\|\alpha_1-\alpha_2\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|\alpha_1-\alpha_2\|^{1/2}_{L^1_T(\mathcal{X}^1)}\\ &\le \Big(\sum_{i=1}^2\|\alpha_i\|^{1/2}_{L^\infty_T(\mathcal{X}^{-1})}\|\alpha_i\|^{1/2}_{L^1_T(\mathcal{X}^1)}\Big)\|\alpha_1-\alpha_2\|_T\\ &\le 2\sqrt{2}\sqrt{\varepsilon}\|b^0\|^{1/2}_{\mathcal{X}^{-1}}\|\alpha_1-\alpha_2\|_T\\ &\le \tfrac{1}{12}\|\alpha_1-\alpha_2\|_T,\ (by\,(H5)).\end{aligned}$$

Then,

$$\|\psi(\alpha_1)-\psi(\alpha_2)\|_{L^1_T(\mathcal{X}^1)} \le \frac{1}{2}\|\alpha_1-\alpha_2\|_T,\ \forall\alpha_1,\alpha_2\in B(\varepsilon,T). \tag{3.15}$$

Therefore inequalities (3.13)-(3.14)-(3.15) give

$$\|\psi(\alpha_1)-\psi(\alpha_2)\|_T \le \frac{1}{2}\|\alpha_1-\alpha_2\|_T,\ \forall\alpha_1,\alpha_2\in B(\varepsilon,T). \tag{3.16}$$

Fixed Point Theorem gives the existence and uniqueness of solution of (RNS) in $C_T(L^2\cap\mathcal{X}^{-1})\cap L^1_T(\mathcal{X}^1)$. Therefore, we can deduce the existence and uniqueness of a local solution for Navier-Stokes system.

4. Proof of Theorem 1.6

Proof of (i): Let $u\in C(\mathbb{R}^+,L^2\cap\mathcal{X}^{-1})$ be global solution of (NS). By [3] we have

$$\limsup_{t\to\infty}\|u(t)\|_{\mathcal{X}^{-1}}=0.$$

Proof of (ii): To prove the long time decay in L^2 we use Benameur-Selmi method (see [5]). Let $u\in C(\mathbb{R}^+,L^2\cap\mathcal{X}^{-1})$ be global solution of (NS). Now we want to prove that $\limsup_{t\to\infty}\|u(t)\|_{L^2}=0$. For a strictly positive real number δ and a given distribution f, we define the operators $A_\delta(D)$ and $B_\delta(D)$, respectively, by the following:

$$\begin{aligned}A_\delta(D)f &= \mathcal{F}^{-1}\big(\mathbf{1}_{\{|\xi|<\delta\}}\widehat{f}\big),\\ B_\delta(D)f &= \mathcal{F}^{-1}\big(\mathbf{1}_{\{|\xi|>\delta\}}\widehat{f}\big).\end{aligned}$$

Let u be a solution of (NS). Denote by $w_\delta=A_\delta(D)u$ and $v_\delta=B_\delta(D)u$, respectively, the low-frequency part and the high-frequency part of u and so on w^0_δ and v^0_δ for the initial data u^0. Applying the pseudo-differential operator $A_\delta(D)$ to the (NS), we get

$$\partial_t w_\delta-\Delta w_\delta+A_\delta(D)\mathbb{P}(u.\nabla u)=0 \tag{4.1}$$

Taking the $L^2(\mathbb{R}^3)$-inner product and using the fact $A_\delta(D)^2=A_\delta(D)$, we obtain

$$\begin{aligned}
\tfrac{1}{2}\tfrac{d}{dt}\|w_\delta(t)\|_{L^2}^2+\|\nabla w_\delta(t)\|_{L^2}^2 &\le |\langle A_\delta(D)\mathbb{P}(u.\nabla u)(t)/w_\delta(t)\rangle_{L^2}|\\
&\le |\langle \mathbb{P}(u.\nabla u)(t)/A_\delta(D)w_\delta(t)\rangle_{L^2}|\\
&\le |\langle \mathbb{P}(u.\nabla u)(t)/w_\delta(t)\rangle_{L^2}|\\
&\le |\langle u.\nabla u(t)/\mathbb{P}(w_\delta(t))\rangle_{L^2}|\\
&\le |\langle u.\nabla u(t)/w_\delta(t)\rangle_{L^2}|\\
&\le |\langle (\operatorname{div}(u\otimes u))(t)/w_\delta(t)\rangle_{L^2}|\\
&\le |\langle u\otimes u(t)/\nabla w_\delta(t)\rangle_{L^2}|\\
&\le \|u\otimes u(t)\|_{L^1}\|\nabla w_\delta(t)\|_{L^\infty}\\
&\le (2\pi)^{-3}\|u(t)\|_{L^2}^2\|w_\delta(t)\|_{\mathcal{X}^1}
\end{aligned}$$

Integrating with respect to time and using Remark 1.2-(iv), we obtain

$$\|w_\delta(t)\|_{L^2}^2\le \|w_\delta^0\|_{L^2}^2+m_0\int_0^t\|w_\delta(s)\|_{\mathcal{X}^1}ds,$$

where $m_0=(2\pi)^{-3}\|u\|_{L^\infty(\mathbb{R}^+,L^2)}$. Also using Remark 1.2-(iii) we get $\|w_\delta(t)\|_{L^2}^2\le M_\delta$, where

$$M_\delta=\|w_\delta^0\|_{L^2}^2+m_0\int_0^\infty\|w_\delta(s)\|_{\mathcal{X}^1}ds.$$

On the one hand, it is clear that $\lim_{\delta\to0}\|w_\delta^0\|_{L^2}^2=0$. On the other, we have $\lim_{\delta\to0}\|w_\delta(t)\|_{\mathcal{X}^1}=0$ and $\|w_\delta(t)\|_{\mathcal{X}^1}\le\|u(t)\|_{\mathcal{X}^1}\in L^1([0,\infty))$. Then Dominated Convergence Theorem implies that

$$\lim_{\delta\to0}\int_0^\infty\|w_\delta(s)\|_{\mathcal{X}^1}ds=0.$$

Hence, $\lim_{\delta\to0}M_\delta=0$ and thus

$$\lim_{\delta\to0}\sup_{t\ge0}\|w_\delta(t)\|_{L^2}^2\to0. \tag{4.2}$$

Let us investigate the high-frequency part. To do so, one applies the pseudo-differential operator $B_\delta(D)$ to the (NS) to get

$$\partial_t v_\delta-\Delta v_\delta+B_\delta(D)\mathbb{P}(u.\nabla u)=0. \tag{4.3}$$

The integral form of v_δ is

$$v_\delta(t)=e^{t\Delta}v_\delta^0-\int_0^t e^{(t-\tau)\Delta}B_\delta(D)\mathbb{P}(u.\nabla u)d\tau.$$

Taking the $L^2(\mathbb{R}^3)$ norm, we obtain

$$\begin{aligned}
\|v_\delta(t)\|_{L^2}&\le \|e^{t\Delta}v_\delta^0\|_{L^2}+\textstyle\int_0^t\|e^{(t-\tau)\Delta}B_\delta(D)\mathbb{P}(u.\nabla u)\|_{L^2}d\tau\\
&\le e^{-t\delta^2}\|v_\delta^0\|_{L^2}+\textstyle\int_0^t e^{-(t-\tau)\delta^2}\|u\nabla u\|_{L^2}d\tau\\
&\le e^{-t\delta^2}\|u^0\|_{L^2}+\textstyle\int_0^t e^{-(t-\tau)\delta^2}\|u\|_{L^2}\|\nabla u\|_{L^\infty}d\tau.
\end{aligned}$$

Then

$$\|v_\delta(t)\|_{L^2} \leq e^{-t\delta^2}\|u^0\|_{L^2} + m_0 \int_0^t e^{-(t-\tau)\delta^2}\|u(\tau)\|_{\mathcal{X}^1} d\tau := G_\delta(t).$$

We have

$$\int_0^\infty G_\delta(t)dt \leq \frac{\|u^0\|_{L^2}}{\delta^2} + \frac{m_0}{\delta^2}\|u\|_{L^1(\mathbb{R}^+,\mathcal{X}^1)} < \infty.$$

This leads to the fact that the function $(t \to \|v_\delta(t)\|_{L^2})$ is both continuous and Lebesgue integrable over $\mathbb{R}^+$. Let $\varepsilon > 0$ be positive real number small enough. Firstly, equation (4.2) implies that some $\delta_\varepsilon > 0$ exists such that

$$\|w_{\delta_\varepsilon}(t)\|_{L^2} \leq \varepsilon/2, \quad \forall t \geq 0. \tag{4.4}$$

Secondly, consider the set R_{δ_ε} defined by

$$R_{\delta_\varepsilon} := \{t > 0, \|v_{\delta_\varepsilon}(t)\|_{L^2} > \varepsilon/2\}. \tag{4.5}$$

If we denote by $\lambda_1(R_{\delta_\varepsilon})$ the Lebesgue measure of R_{δ_ε}, we have

$$\int_0^\infty \|v_{\delta_\varepsilon}(t)\|_{L^2} dt \geq \int_{R_{\delta_\varepsilon}} \|v_{\delta_\varepsilon}(t)\|_{L^2(\mathbb{R}^3)} dt \geq \frac{\varepsilon}{2}\lambda_1(R_{\delta_\varepsilon}).$$

By this, we can deduce that $\lambda_1(R_{\delta_\varepsilon}) \leq T_\varepsilon$, where $T_\varepsilon = (2/\varepsilon)\int_0^\infty \|v_{\delta_\varepsilon}(t)\|_{L^2(\mathbb{R}^3)} dt$. Then, there is $t_\varepsilon \in [0, T_\varepsilon + 1]$ such that t_ε does not belong to R_{δ_ε}. This implies that

$$\|v_{\delta_\varepsilon}(t_\varepsilon)\|_{L^2(\mathbb{R}^3)} \leq \varepsilon/2. \tag{4.6}$$

Equations (4.4) and (4.6) together with triangular inequality imply that $\|u(t_\varepsilon)\|_{L^2(\mathbb{R}^3)} < \varepsilon$. For $t \geq t_\varepsilon$, we have

$$\begin{aligned} \|u(t)\|_{L^2} &\leq \|u(t_\varepsilon)\|_{L^2} \exp((2\pi)^{-3} \textstyle\int_{t_0}^\infty \|u(z)\|_{\mathcal{X}^1} dz) \\ &\leq \varepsilon \exp((2\pi)^{-3} \textstyle\int_0^\infty \|u\|_{\mathcal{X}^1}). \end{aligned}$$

It suffices to replace ε by $\varepsilon \exp(-(2\pi)^{-3}\int_0^\infty \|u\|_{\mathcal{X}^1})$ in (4.4)-(4.5)-(4.6) we get the desired result.
Proof of (iii): In this subsection we want to give a precision for the decay of $\|u(t)\|_{\mathcal{X}^{-1}}$ at ∞. Let $\varepsilon > 0$ such that $\varepsilon < \epsilon_0$(ϵ_0 is given by Theorem 1.3), by Theorem 1.4 and Theorem 1.6-**(ii)** we can suppose that,

$$\|u^0\|_{\mathcal{X}^{-1}} < \min(\varepsilon, \frac{1}{2}) \ \text{ and } \ \|u^0\|_{L^2} < \varepsilon/2.$$

Then, by Remark 1.2-(ii)-(iv) we get $\|u(t)\|_{L^2} \leq 2\|u^0\|_{L^2} < \varepsilon$ for all $t \geq 0$ and

$$\|u(t)\|_{\mathcal{X}^{-1}} + \frac{1}{2}\int_0^t \|u(z)\|_{\mathcal{X}^1} dz \leq \|u^0\|_{\mathcal{X}^{-1}} < \frac{1}{2}, \ \ \forall t \geq 0. \tag{4.7}$$

For $\lambda > 0$ and $t > 0$, we have

$$\|u(t)\|_{\mathcal{X}^{-1}} = I_\lambda(t) + J_\lambda(t),$$

with

$$I_\lambda(t) = \int\limits_{\{\xi\in\mathbb{R}^3/|\xi|<\lambda\}} \frac{|\widehat{u}(t,\xi)|}{|\xi|}d\xi, \text{ and } J_\lambda(t) = \int\limits_{\{\xi\in\mathbb{R}^3/|\xi|>\lambda\}} \frac{|\widehat{u}(t,\xi)|}{|\xi|}d\xi.$$

We have

$$I_\lambda(t) \leq \Big(\int\limits_{\{\xi\in\mathbb{R}^3/|\xi|<\lambda\}} \frac{1}{|\xi|^2}d\xi\Big)^{1/2}\|\widehat{u}\|_{L^2} \leq c_0\sqrt{\lambda}\|\widehat{u}(t)\|_{L^2} \leq c_1\sqrt{\lambda}\|u^0\|_{L^2}$$

and

$$J_\lambda(t) \leq \int\limits_{\{\xi\in\mathbb{R}^3/|\xi|>\lambda\}} e^{-\sqrt{t/2}|\xi|}e^{\sqrt{t/2}|\xi|}\frac{|\widehat{u}(t,\xi)|}{|\xi|}d\xi \leq e^{-\sqrt{t/2}\lambda}\int\limits_{\mathbb{R}^3} e^{\sqrt{t/2}|\xi|}\frac{|\widehat{u}(t,\xi)|}{|\xi|}d\xi.$$

For a fixed time $t > 0$ the $v : (z,x) \to u(\frac{t}{2}+z,x)$ satisfies $\|v(0)\|_{\mathcal{X}^{-1}} < \epsilon_0$ and it is the unique global solution of the following system,

$$\begin{cases} \partial_t v - \Delta v + v.\nabla v = & -\nabla q \\ \operatorname{div} v = 0 \\ v(0,x) = u(\frac{t}{2},x). \end{cases}$$

By Theorem 1.3, we get

$$\int\limits_{\mathbb{R}^3} e^{\sqrt{z}|\xi|}\frac{|\widehat{v}(z,\xi)|}{|\xi|}d\xi + \frac{1}{2}\int\limits_0^z\int\limits_{\mathbb{R}^3} e^{\sqrt{s}|\xi|}\frac{|\widehat{v}(s,\xi)|}{|\xi|}d\xi ds \leq 2\|v(0)\|_{\mathcal{X}^{-1}}$$

or

$$\int\limits_{\mathbb{R}^3} e^{\sqrt{z}|\xi|}\frac{|\widehat{u}(\frac{t}{2}+z,\xi)|}{|\xi|}d\xi + \frac{1}{2}\int\limits_0^z\int\limits_{\mathbb{R}^3} e^{\sqrt{s}|\xi|}\frac{|\widehat{u}(\frac{t}{2}+s,\xi)|}{|\xi|}d\xi ds \leq 2\int\limits_{\mathbb{R}^3}\frac{|\widehat{u}(\frac{t}{2},\xi)|}{|\xi|}d\xi.$$

For $z = \frac{t}{2}$, we get $\int_\xi e^{\sqrt{t/2}|\xi|}\frac{|\widehat{u}(t,\xi)|}{|\xi|}d\xi \leq \|u(t/2)\|_{\mathcal{X}^{-1}}$, which implies

$$J_\lambda(t) \leq e^{-\sqrt{t/2}\lambda}\|u(t/2)\|_{\mathcal{X}^{-1}}.$$

Then

$$\|u(t)\|_{\mathcal{X}^{-1}} \leq c_1\sqrt{\lambda}\|u^0\|_{L^2} + e^{-\sqrt{t/2}\lambda}\|u(t/2)\|_{\mathcal{X}^{-1}}.$$

Multiplying this inequality by $t^{1/4}$

$$t^{1/4}\|u(t)\|_{\mathcal{X}^{-1}} \leq t^{1/4}c_1\sqrt{\lambda}\|u^0\|_{L^2} + 2^{1/4}e^{-\sqrt{t/2}\lambda}(\frac{t}{2})^{1/4}\|u(t/2)\|_{\mathcal{X}^{-1}}$$

and choose $\lambda > 0$ such that

$$2^{1/4}e^{-\sqrt{t/2}\lambda} = 1/2 \Rightarrow \sqrt{t/2}\lambda = 5/4\ln 2 \Rightarrow \lambda = \frac{5\sqrt{2}\ln 2}{4\sqrt{t}},$$

we obtain

$$t^{1/4}\|u(t)\|_{\mathcal{X}^{-1}} \leq M_0 + \frac{1}{2}(\frac{t}{2})^{1/4}\|u(t/2)\|_{\mathcal{X}^{-1}}$$

with

$$M_0 = c_0(\frac{5\sqrt{2}\ln 2}{4})^{1/2}\|u^0\|_{L^2}.$$

Applying Lemma 2.5 and Remark 2.6 with

$$f(t) = t^{1/4}\|u(t)\|_{\mathcal{X}^{-1}}, \quad \theta_1 = \theta_2 = 1/2,$$

we get

$$\limsup_{t\to+\infty} t^{1/4}\|u(t)\|_{\mathcal{X}^{-1}} \leq 2M_0.$$

Applying this result to the solution of the following system, for $a \geq 0$

$$\begin{cases} \partial_t w - \Delta w + w.\nabla w = \ -\nabla h \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} v = 0 \text{ in } \mathbb{R}^+ \times \mathbb{R}^3 \\ w(0,x) = u(a,x) \text{ in } \mathbb{R}^3, \end{cases}$$

we obtain

$$\limsup_{t\to\infty} t^{1/4}\|u(t)\|_{\mathcal{X}^{-1}} \leq c_0(\frac{5\sqrt{2}\ln 2}{4})^{1/2}\|u(a)\|_{L^2}.$$

Then the fact $\lim_{a\to\infty}\|u(a)\|_{L^2} = 0$ implies the desired result.
Proof of (iv): In this section we want to prove the long time decay in $\mathcal{X}^\sigma$, for $\sigma > -3/2$. To do that we distinguish two cases.

First case : $-3/2 < \sigma < -1$. For $\lambda > 0$ and $t > 0$, we have

$$\|u(t)\|_{\mathcal{X}^\sigma} = I_1(t,\lambda) + I_2(t,\lambda)$$

with

$$I_1(t,\lambda) = \int_{\{\xi\in\mathbb{R}^3/|\xi|<\lambda\}} |\xi|^\sigma|\widehat{u}(t,\xi)|d\xi \text{ and } I_2(t,\lambda) = \int_{\{\xi\in\mathbb{R}^3/|\xi|>\lambda\}} |\xi|^\sigma|\widehat{u}(t,\xi)|d\xi.$$

We have

$$I_1(t,\lambda) \leq \Big(\int_{|\xi|<\lambda} |\xi|^{2\sigma}d\xi\Big)^{1/2}\|\widehat{u}(t)\|_{L^2} \leq c_1\lambda^{\sigma+3/2}\|u(t)\|_{L^2}$$

and

$$I_2(t,\lambda) \leq \int\limits_{\{\xi\in\mathbb{R}^3/|\xi|>\lambda\}} |\xi|^{\sigma+1}\frac{|\widehat{u}(t,\xi)|}{|\xi|}d\xi \leq \lambda^{\sigma+1}\|u(t)\|_{\mathcal{X}^{-1}}.$$

We get $\|u(t)\|_{\mathcal{X}^\sigma} = A\lambda^{\sigma+3/2} + B\lambda^{\sigma+1} := \varphi(\lambda)$, with

$$A = c_0\|u(t)\|_{L^2} \ \text{ and } \ B = \|u(t)\|_{\mathcal{X}^{-1}}.$$

The study of the function φ gives

$$\varphi'(\lambda) = (\sigma+3/2)A\lambda^{\sigma+1/2} + (\sigma+1)B\lambda^\sigma,$$

then

$$\varphi'(\lambda) = 0 \Rightarrow \lambda = \lambda_0 = \Big(\frac{-(1+\sigma)B}{(\sigma+3/2)A}\Big)^2.$$

For $\lambda = \lambda_0$, we get

$$\begin{aligned}\|u(t)\|_{\mathcal{X}^\sigma} &\leq A\big(\tfrac{-(1+\sigma)B}{(\sigma+3/2)A}\big)^{2\sigma+3} + B\big(\tfrac{-(1+\sigma)B}{(\sigma+3/2)A}\big)^{2\sigma+2}\\ &\leq c_\sigma A^{-2\sigma-2}B^{3+2\sigma}.\end{aligned}$$

Then

$$\|u(t)\|_{\mathcal{X}^\sigma} \leq c'_\sigma(\|u(t)\|_{L^2})^{-2\sigma-2}(\|u(t)\|_{\mathcal{X}^{-1}})^{3+2\sigma}.$$

Using the fact $\|u(t)\|_{\mathcal{X}^{-1}} = o(t^{-1/4})$ and $\|u(t)\|_{L^2} \to 0$, which given by Theorem 1.6-**(ii)**-**(iii)**, we get the desired result.

Second case : $-1 < \sigma$. By Theorem 1.4 we can assume that $\|u^0\|_{\mathcal{X}^{-1}} < \epsilon_0/2$ and Theorem 1.3 gives,

$$\begin{aligned}\|u(t)\|_{\mathcal{X}^\sigma} &= \int_{\mathbb{R}^3} e^{-\sqrt{t/2}|\xi|}|\xi|^{\sigma+1}e^{\sqrt{t/2}|\xi|}\tfrac{|\widehat{u}(t,\xi)|}{|\xi|}d\xi\\ &= \tfrac{1}{(\sqrt{t/2})^{\sigma+1}}\int(\sqrt{t/2}|\xi|)^{\sigma+1}e^{-\sqrt{t/2}|\xi|}e^{\sqrt{t/2}|\xi|}\tfrac{|\widehat{u}(t,\xi)|}{|\xi|}d\xi\\ &\leq Ct^{-\frac{\sigma+1}{2}}\int e^{\sqrt{t/2}|\xi|}\tfrac{|\widehat{u}(\frac{t}{2}+\frac{t}{2},\xi)|}{|\xi|}d\xi\\ &\leq 2Ct^{-\frac{\sigma+1}{2}}\|u(t/2)\|_{\mathcal{X}^{-1}},\end{aligned}$$

with $C = \sup_{z\geq 0} z^{\sigma+1}e^{-z}$. Combining this result with the fact $\|u(t/2)\|_{\mathcal{X}^{-1}} = o(t^{-1/4})$ we get the desired result.

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