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Vanishing theorems for L^2 harmonic p-forms on Riemannian manifolds with a weighted p-Poincaré inequality

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A B S T R A C T

This paper mainly deals with several vanishing results for L^2 harmonic p-forms on complete Riemannian manifolds with a weighted p-Poincaré inequality and some lower bound of the curvature. Some results are in the spirit of Li-Wang, Lam, and Dung-Sung, but without assumptions of sign and growth rate of the weight function as Vieira did for manifolds with weighted Poincaré inequality, and some are vanishing results without curvature restrictions. Moreover, a vanishing and splitting theorem is established with a much weaker curvature condition and a lower bound of the first eigenvalue of the Laplacian.

1. Introduction

One of the interesting problem in differential geometry is when the space of L^2 harmonic p-forms will be trivial. Recall that on an n dimensional complete Riemannian manifold M, the Hodge Laplacian is defined as

$$\Delta = -(\mathrm{d}\delta + \delta\mathrm{d}),$$

$\Omega^p(M)$ is the set of smooth p-forms on M, $\Omega_0^p(M)$ means with compact support, and the space of L^2 harmonic p-forms is denoted by $\mathcal{H}^p(L^2(M))$. The first eigenvalue of the Hodge Laplacian on $\Omega_0^p(M)$ is denoted by $\lambda_{1,p}(M)$, which is characterized by

$$\lambda_{1,p}(M) = \inf_{\alpha\in\Omega_0^p(M)} \frac{\int_M (|\mathrm{d}\alpha|^2 + |\delta\alpha|^2)}{\int_M |\alpha|^2},$$

called the first p-spectrum, and when $p = 0$, we represent $\lambda_1(M) := \lambda_{1,0}(M)$. As we all know, the first eigenvalue of the Laplacian is given by

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$$\lambda_1(M) = \inf_{\psi \in C_0^\infty(M)} \frac{\int_M |\nabla \psi|^2}{\int_M \psi^2}.$$

Li and Wang [4] obtained the following outstanding vanishing theorem for L^2 harmonic 1-forms on complete Riemannian manifolds with $\lambda_1(M) > 0$,

Theorem 1.1. ([4, Theorem 4.2]) *If a complete Riemannian manifold M^n has positive spectrum $\lambda_1(M) > 0$ and the Ricci curvature satisfies*

$$\mathrm{Ric} \geq -a\lambda_1(M)$$

for some $0 < a < \frac{n}{n-1}$, then the space of L^2 harmonic one-forms on M^n is trivial.

By variational principle, the first spectrum $\lambda_1(M) > 0$ implies the following Poincaré inequality, i.e.,

$$\lambda_1(M) \int_M \psi^2 \leq \int_M |\nabla \psi|^2 \quad \text{for } \psi \in C_0^\infty(M).$$

Hence, a natural generalization is the weighted Poincaré inequality [5],

$$\int_M \rho(x)\psi^2 \leq \int_M |\nabla \psi|^2 \quad \text{for } \psi \in C_0^\infty(M),$$

with a nonnegative continuous weight function $\rho \in C(M)$. Lam [3] generalized Li and Wang's Theorem 1.1, replacing the positive spectrum condition by the weighted Poincaré inequality, but with a growth rate condition on the weight function ρ. Later, Vieira [6] removed the growth rate condition on the weighted function, and more surprisingly, ρ can be any continuous function, need not to be nonnegative.

In another direction, Chen and Sung [1], Dung and Sung [2] continued to study vanishing and splitting theorems for complete Riemannian manifolds with weighted p-Poincaré inequality,

$$\int_M \rho(x)|\alpha|^2 \leq \int_M |\mathrm{d}\alpha|^2 + |\delta\alpha|^2 \quad \text{for } \alpha \in \Omega_0^p(M), \tag{1.1}$$

and they called the Riemannian manifold (M, g) satisfies property $(\mathcal{P}_{p,\rho})$ if M satisfies the weighted p-Poincaré inequality and the conformal metric ρg is complete. For a local orthonormal frame $\{e_1, \cdots, e_n\}$ on M with dual coframe $\{\theta^1, \cdots, \theta^n\}$ the curvature operator acting on p-forms is defined by

$$\mathcal{R}_p = \sum_{i,j} \theta^i \wedge \iota_{e_j} R(e_j, e_i)$$

with $R(e_i, e_j) = \nabla_i\nabla_j - \nabla_j\nabla_i$. Furthermore, $\mathcal{R}_p \geq -a\rho$ means

$$\langle \mathcal{R}_p\omega, \omega\rangle \geq -a\rho|\omega|^2, \text{ for any } \omega \in \Omega^p(M).$$

More precisely, they get the following two theorems,

Theorem 1.2. ([2, Theorem 1.6]) *Let (M, g) be an n-dimensional complete Riemannian manifold satisfying a $(\mathcal{P}_{1,\rho})$ property with a nonnegative weight function $\rho(x)$. Assume that the Ricci curvature satisfies*

$$\mathrm{Ric}_M(x) \geq -\frac{n}{n-1}\rho(x)$$

and $\rho \leq O(e^{2r_\rho})$, where $r_\rho(x)$ is the geodesic distance from some fixed point to x with respect to the conformal metric ρg, then either
(1) $\mathcal{H}^1(L^2(M)) = 0$; *or*
(2) $\widetilde{M} = \mathbb{R} \times N$, *where $\widetilde{M}$ is the universal cover of M and N is a manifold of dimension $(n-1)$.*

Theorem 1.3. ([2, Theorem 1.7]) *Let M be an n-dimensional complete Riemannian manifold satisfying a $(\mathcal{P}_{p,\rho})$ property with a nonnegative weight function $\rho(x)$. Assume that the curvature operator $\mathcal{R}_p$ on M has the lower bound*

$$\mathcal{R}_p \geq -\frac{n-p+1}{n-p}\rho(x)$$

and $\rho \leq O(e^{2r_\rho})$. Suppose that ω is a harmonic p-form in $L^2(M)(p \geq 2)$. Then either
(1) $\mathcal{H}^p(L^2(M)) = 0$; *or*
(2) *there exists a 1-form β with $\beta \wedge \omega = 0$ such that*

$$\nabla\omega = \beta \otimes \omega - \frac{1}{n-p+1}\sum_{j=1}^{n}\theta^j \otimes \left(\theta^j \wedge \iota_{\beta^\#}\omega\right),$$

where $\beta^\#$ is the dual vector of β.

The curvature restriction in Theorem 1.2, 1.3 are in the same style of that in Theorem 1.1. It is known that $\lambda_1(M) \geq \lambda_{1,1}(M)$, since the exterior differential operator d commutes with the Hodge Laplacian Δ. Hence, one could guess that the weighted 1-Poincaré inequality condition might be stronger than the weighted Poincaré inequality condition. In order to improve the above vanishing results on L^2 harmonic p-forms, we must further explore the weighted p-Poincaré inequality condition, and the curvature condition might be weakened.

In this paper, we further investigate vanishing results on complete manifolds with a weighted p-Poincaré inequality and in the sequel, $1 \leq p \leq n$. One of the main results is the following, which shows that Dung and Sung's Theorem 1.2 and 1.3 can be strengthened so that only the vanishing case can happen, but with a more general lower bound of the curvature and no restriction on the sign or growth rate on the weight function ρ.

Theorem 1.4. *Let M be a complete n-dimensional Riemannian manifold satisfying the weighted p-Poincaré inequality with weight function ρ, and $a > 0$ be any positive real number, if the curvature operator satisfies*

$$\mathcal{R}_p \geq -a\rho,$$

then

$$\mathcal{H}^p(L^2(M)) = \{0\}.$$

Actually, there is a simple but interesting observation on the relationship between the first p spectrum of Hodge Laplacian and the space of L^2 harmonic p-forms, without any curvature conditions.

Theorem 1.5. *If the greatest lower bound $\lambda_{1,p}$ of the p spectrum of M is positive, then*

$$\mathcal{H}^p(L^2(M)) = \{0\}.$$

Conversely, if $\mathcal{H}^p(L^2(M))$ is non-trivial, the Hodge Laplacian Δ does not have positive eigenvalues on $\Omega^p(M)$. This kind of result might be known, but we do not find it in other places, so present here.

Under a more general lower bound of the curvature operator when the first eigenvalue of the Laplacian has a certain lower bound, we also have a vanishing result on $\mathcal{H}^p(L^2(M))$, which is Theorem 4.1 in this paper. In particular, when $p = 1$, we have the following vanishing and splitting theorem,

Theorem 1.6. *Let M be a complete n-dimensional Riemannian manifold satisfying a weighted 1-Poincaré inequality with nonnegative weight function ρ, and $a, b > 0$ be any positive real number, suppose the Ricci curvature satisfies*

$$\mathrm{Ric} \geq -a\rho - b.$$

If the first eigenvalue of the Laplacian satisfies

$$\lambda_1(M) \geq \frac{n-1}{n} b,$$

then either
(1) $\mathcal{H}^1(L^2(M)) = \{0\}$*; or*
(2) *the universal covering of M splits as $\widetilde{M} = \mathbb{R} \times N$ with the warped product metric*

$$g_{\widetilde{M}} = \mathrm{d}t^2 + \eta(t)^2 g_N.$$

It is interesting to find manifolds who satisfy a weighted p-Poincaré inequality, and manifolds with a weighted Poincaré inequality but without any weighted p-Poincaré inequality. The stable submanifolds might be a potential candidate.

The organization of the paper is as follows: in section 2, we recall the definition of weighted p-Poincaré inequality and present some lemmas to be used later, and get some vanishing results for L^2 harmonic p-form without curvature conditions; section 3 is about the vanishing results involving weighted p-Poincaré inequality and curvature lower bound condition; the last section is vanishing and splitting results under further weaker curvature conditions and an additional condition on $\lambda_1(M)$.

2. Preliminaries

For reader's convenience, let's recall the definition of weighted p-Poincaré inequality,

Definition 2.1. Let M be an n-dimensional complete Riemannian manifold. We say that M satisfies a weighted p-Poincaré inequality with weight function $\rho \in C^\infty(M)$, if

$$\int_M \rho(x)|\alpha|^2 \, \mathrm{d}v \leq \int_M (|\mathrm{d}\alpha|^2 + |\delta\alpha|^2) \, \mathrm{d}v, \quad \text{for } \alpha \in \Omega_0^p(M). \tag{2.1}$$

If the weight function ρ in Definition 2.1 equals to 0, then inequality (2.1) tells nothing. Hence, in the sequel, we always assume the weight function $\rho \not\equiv 0$. Similar to weighted Poincaré inequality, the weighted p-Poincaré inequality can be seen as a generalization of the assumption that $\lambda_{1,p} > 0$.

The following lemma is simple, but will play a key role in this paper,

Lemma 2.2. *If an n-dimensional complete Riemannian manifold M satisfies a weighted p-Poincaré inequality, then for any L^2-harmonic p-form $\omega \in \mathcal{H}^p(L^2(M))$,*

$$\int_M \rho|\omega|^2 \, \mathrm{d}v \le 0.$$

Proof. For any L^2-harmonic p-form $\omega \in \mathcal{H}^p(L^2(M))$, we have

$$\mathrm{d}\omega = 0, \ \delta\omega = 0, \ \text{and} \int_M |\omega|^2 \, \mathrm{d}v < \infty.$$

Choose a cut-off function

$$\phi = \begin{cases} 1, & \text{on } B(R), \\ 0, & \text{on } M \backslash B(2R), \end{cases} \tag{2.2}$$

such that $|\nabla\phi|^2 \le \dfrac{C}{R^2}$ on $B(2R)\backslash B(R)$. Then

$$\begin{aligned} |\mathrm{d}(\phi\omega)|^2 &= |\mathrm{d}\phi \wedge \omega|^2 = |\mathrm{d}\phi|^2|\omega|^2 - \langle \mathrm{d}\phi, \omega\rangle^2, \\ |\delta(\phi\omega)|^2 &= |-\iota_{e_j}\nabla_{e_j}(\phi\omega)|^2 = |-\langle \mathrm{d}\phi, \omega\rangle + \phi\delta\omega|^2 = \langle \mathrm{d}\phi, \omega\rangle^2. \end{aligned}$$

Since M satisfies the weighted p-Poincaré inequality,

$$\begin{aligned} \int_M \rho|\phi\omega|^2 \, \mathrm{d}v &\le \int_M (|\mathrm{d}(\phi\omega)|^2 + |\delta(\phi\omega)|^2) \, \mathrm{d}v \\ &= \int_M |\mathrm{d}\phi|^2|\omega|^2 \, \mathrm{d}v \\ &\le \frac{C}{R^2}\int_M |\omega|^2 \, \mathrm{d}v. \end{aligned}$$

Hence, letting $R \to \infty$, we obtain

$$\int_M \rho|\omega|^2 \, \mathrm{d}v \le 0. \quad \square$$

As an immediate deduction,

Corollary 2.3. *If the weighted p-Poincaré inequality holds on M with the weight function $\rho > 0$,*

$$\mathcal{H}^p(L^2(M)) = \{0\}.$$

Proof. Since $\rho > 0$, by Lemma 2.2, for any L^2-harmonic p-form $\omega \in \mathcal{H}^2(L^2(M))$, we have $\rho|\omega|^2 = 0$. Hence, $\omega = 0$. $\square$

Note that Corollary 2.3 holds without any curvature assumption. More specifically, we can obtain some relationship between the space of L^2 harmonic p-forms and the first p spectrum of the Hodge Laplacian, which is Theorem 1.5.

In order to study the case when the weight function ρ may change sign, we will need the following lemma,

Lemma 2.4. *Let M be a complete n-dimensional Riemannian manifold and let $a, b, c > 0$ be any positive real numbers. Suppose a function $h \in C^\infty(M)$ satisfies*

$$h\Delta h \geq -a\rho h^2 - bh^2 + c|\nabla h|^2, \tag{2.3}$$

and $\int_{B(2R)} h^2 \, \mathrm{d}v = o(R^2)$ as $R \to \infty$, then

$$\int_M |\nabla h|^2 \, \mathrm{d}v \leq \frac{a}{1+c} \int_M \rho h^2 \, \mathrm{d}v + \frac{b}{1+c} \int_M h^2 \, \mathrm{d}v.$$

Proof. Choose the cut-off function ϕ, (2.2) in Lemma 2.2. Multiplying both sides of (2.3) by ϕ^2 to get

$$\int_M \phi^2 h\Delta h \, \mathrm{d}v \geq -a \int_M \phi^2 \rho h^2 \, \mathrm{d}v - b \int_M \phi^2 h^2 \, \mathrm{d}v + c \int_M \phi^2 |\nabla h|^2 \, \mathrm{d}v.$$

The left hand side

$$\begin{aligned}
&\int_M \phi^2 h\Delta h \, \mathrm{d}v \\
&= -\int_M \langle \nabla(\phi^2 h), \nabla h \rangle \, \mathrm{d}v \\
&= -\int_M \phi^2 |\nabla h|^2 \, \mathrm{d}v - 2\int_M \langle h\nabla\phi, \phi\nabla h \rangle \, \mathrm{d}v \\
&\leq -\int_M \phi^2 |\nabla h|^2 \, \mathrm{d}v + \varepsilon \int_M \phi^2 |\nabla h|^2 \, \mathrm{d}v + \frac{1}{\varepsilon} \int_M h^2 |\nabla \phi|^2 \, \mathrm{d}v.
\end{aligned}$$

Hence,

$$\begin{aligned}
&(c + 1 - \varepsilon) \int_M \phi^2 |\nabla h|^2 \, \mathrm{d}v \\
&\leq a \int_M \phi^2 \rho h^2 \, \mathrm{d}v + b \int_M \phi^2 h^2 \, \mathrm{d}v + \frac{1}{\varepsilon} \int_M h^2 |\nabla \phi|^2 \, \mathrm{d}v \\
&\leq a \int_M \phi^2 \rho h^2 \, \mathrm{d}v + b \int_M \phi^2 h^2 \, \mathrm{d}v + \frac{C}{\varepsilon R^2} \int_M h^2 \, \mathrm{d}v.
\end{aligned}$$

Let $R \to \infty$, and then let $\varepsilon \to 0$ to obtain

$$(c+1) \int_M |\nabla h|^2 \, \mathrm{d}v \leq a \int_M \rho h^2 \, \mathrm{d}v + b \int_M h^2 \, \mathrm{d}v. \quad \square$$

For harmonic p-form ω, combining the Bochner formula

$$|\omega|\Delta|\omega| = |\nabla\omega|^2 - |\nabla|\omega||^2 + \langle \mathcal{R}_p \omega, \omega \rangle,$$

and the refined Kato inequality

$$|\nabla\omega|^2 \geq C_{n,p}|\nabla|\omega||^2$$

with

$$C_{n,p} = \begin{cases} 1 + \frac{1}{n-p}, & 1 \leq p \leq n/2 \\ 1 + \frac{1}{p}, & n/2 \leq p \leq n-1 \end{cases},$$

we get

$$|\omega|\Delta|\omega| \geq (C_{n,p} - 1)|\nabla|\omega||^2 + \langle \mathcal{R}_p\omega, \omega\rangle. \tag{2.4}$$

In next two sections, we establish the vanishing theorems and splitting theorems. We need the following lemma for the proof of Theorem 1.6.

Lemma 2.5. ([5, Lemma 4.1]) *Let M^n be a complete Riemannian manifold of dimension $n \geq 2$. Assume that the Ricci curvature of M satisfies the lower bound*

$$\mathrm{Ric}_M(x) \geq -(n-1)\tau(x)$$

for all $x \in M$. Suppose f is a nonconstant harmonic function defined on M. Then the function ∇f must satisfy the differential inequality

$$\Delta|\nabla f| \geq -(n-1)\tau|\nabla f| + \frac{|\nabla|\nabla f||^2}{(n-1)|\nabla f|}$$

in the weak sense. Moreover, if equality holds, then M is given by $M = \mathbb{R} \times N^{n-1}$ with the warped product metric

$$\mathrm{d}s_M^2 = \mathrm{d}t^2 + \eta^2(t)\mathrm{d}s_N^2$$

for some positive function $\eta(t)$, and some manifold N^{n-1}. In this case, $\tau(t)$ is a function of t alone satisfying

$$\eta''(t)\eta^{-1}(t) = \tau(t).$$

3. Vanishing theorems

Based on preparations in section 2, we establish vanishing theorem for manifolds with a weighted p-Poincaré inequality, in which the weight function ρ does not have sign or growth rate restrictions, and the curvature condition is much weaker than the corresponding theorems in [2].

Proof of Theorem 1.4. For any L^2-harmonic p-form $\omega \in \mathcal{H}^p(L^2(M))$, by the condition on the curvature operator and (2.4), we have

$$|\omega|\Delta|\omega| \geq -a\rho|\omega|^2 + (C_{n,p} - 1)|\nabla|\omega||^2. \tag{3.1}$$

Then, we let $h = |\omega|$, and apply Lemma 2.4 with $b = 0,\ c = C_{n,p} - 1$ to obtain

$$\int_M |\nabla h|^2 \,\mathrm{d}v \leq \frac{a}{C_{n,p}} \int_M \rho h^2 \,\mathrm{d}v.$$

Since M satisfies the weighted p-Poincaré inequality, by Lemma 2.2, we get $\int_M |\nabla h|^2 \, \mathrm{d}v = 0$. Hence, $h =$ constant. Then (3.1) implies

$$a\rho h^2 \geq 0,$$

so $\rho \geq 0$. Therefore, $\int_M \rho \, \mathrm{d}v > 0$, otherwise $\rho \equiv 0$. Still by Lemma 2.2, $h = 0$. $\square$

A special case is when $p = 1$, which can be compared to Theorem 1.2,

Corollary 3.1. *Let M be a complete n-dimensional Riemannian manifold satisfying the weighted 1-Poincaré inequality with weight function ρ, and $a > 0$ be any positive real number, if the Ricci curvature satisfies*

$$\mathrm{Ric} \geq -a\rho,$$

then

$$\mathcal{H}^1(L^2(M)) = \{0\}.$$

4. Vanishing and splitting theorems

If we want to further weaken the curvature assumption, we may need extra restrictions. Like Vieira did in Theorem 6 in [6], here we show that with a much weaker curvature assumption, if the first eigenvalue of the Laplacian has certain lower bound, we still obtain the vanishing theorem. Note that, although the weight function ρ need to be nonnegative, the lower bound of the curvature operator and lower bound of the first eigenvalue is much weaker.

Theorem 4.1. *Let M be a complete n-dimensional Riemannian manifold satisfying the weighted p-Poincaré inequality with nonnegative weight function ρ, and $a, b > 0$ be any positive real number, suppose the curvature operator satisfies*

$$\mathcal{R}_p \geq -a\rho - b.$$

If the first eigenvalue of the Laplacian satisfies

$$\lambda_1(M) > \frac{b}{C_{n,p}},$$

then

$$\mathcal{H}^p(L^2(M)) = \{0\}.$$

Proof. For any L^2-harmonic p-form $\omega \in \mathcal{H}^p(L^2(M))$. Let $h = |\omega|$. By the condition on the curvature operator and (2.4), we have

$$h\Delta h \geq -a\rho h^2 - bh^2 + (C_{n,p} - 1)|\nabla h|^2. \tag{4.1}$$

Then, we apply Lemma 2.4 with $c = C_{n,p} - 1$ to obtain

$$\int_M |\nabla h|^2 \, \mathrm{d}v \leq \frac{a}{C_{n,p}} \int_M \rho h^2 \, \mathrm{d}v + \frac{b}{C_{n,p}} \int_M h^2 \, \mathrm{d}v.$$

Since M satisfies the weighted p-Poincaré inequality with weight function $\rho \geq 0$, by Lemma 2.2, we get $\int_M \rho h^2 \, \mathrm{d}v = 0$. Hence,

$$\int_M |\nabla h|^2 \, \mathrm{d}v \leq \frac{b}{C_{n,p}} \int_M h^2 \, \mathrm{d}v. \tag{4.2}$$

If $\lambda_1(M) > \frac{b}{C_{n,p}} (> 0)$, by variational principle, we have

$$\lambda_1(M) \int_M \psi^2 \, \mathrm{d}v \leq \int_M |\nabla \psi|^2 \, \mathrm{d}v, \text{ for any } \psi \in C_0^\infty(M).$$

Take the cut-off function ϕ, (2.2) in Lemma 2.2, we have

$$\begin{aligned}
&\lambda_1(M) \int_M (\phi h)^2 \, \mathrm{d}v \\
&\leq \int_M |\nabla(\phi h)|^2 \, \mathrm{d}v \\
&= \int_M \phi^2 |\nabla h|^2 \, \mathrm{d}v + \int_M h^2 |\nabla \phi|^2 \, \mathrm{d}v + 2\int_M \langle \phi \nabla h, h \nabla \phi \rangle \, \mathrm{d}v \\
&\leq (1+\varepsilon) \int_M \phi^2 |\nabla h|^2 \, \mathrm{d}v + (1+\frac{1}{\varepsilon}) \int_M h^2 |\nabla \phi|^2 \, \mathrm{d}v \\
&\leq (1+\varepsilon) \int_M \phi^2 |\nabla h|^2 \, \mathrm{d}v + (1+\frac{1}{\varepsilon}) \frac{C}{R^2} \int_M h^2 \, \mathrm{d}v.
\end{aligned}$$

Take $R \to \infty$, and then $\varepsilon \to 0$, we obtain

$$\lambda_1(M) \int_M h^2 \, \mathrm{d}v \leq \int_M |\nabla h|^2 \, \mathrm{d}v. \tag{4.3}$$

If $h \not\equiv 0$, combining (4.2) and (4.3), we get that $\lambda_1(M) \leq \frac{b}{C_{n,p}}$, which contradicts with the condition that $\lambda_1(M) > \frac{b}{C_{n,p}}$. Hence, $\omega \equiv 0$. $\square$

For the special case when $p = 1$, $C_{n,1} = \frac{n}{n-1}$, we get the following vanishing and splitting results,

Proof of Theorem 1.6. If $\mathcal{H}^1(L^2(M)) = \{0\}$, it's done.

Otherwise, we choose a non-vanishing L^2 harmonic 1-form ω and set $h = |\omega|$. By $\lambda_1(M) \geq \frac{n-1}{n} b$, (4.2) and (4.3) for $p = 1$,

$$\lambda_1(M) \int_M h^2 \, \mathrm{d}v = \int_M |\nabla h|^2 \, \mathrm{d}v = \frac{n-1}{n} b \int_M h^2 \, \mathrm{d}v,$$

i.e., the equality holds in (4.2). Tracing back, we see equality holds in (4.1) for $p = 1$. That is

$$h\Delta h = -a\rho h^2 - bh^2 + \frac{1}{n-1}|\nabla h|^2.$$

Lift the metric of M to the universal cover $\widetilde{M}$ and the harmonic 1-form is lifted to a harmonic 1-form $\widetilde{\omega}$ on $\widetilde{M}$. Since $\widetilde{M}$ is simply connected, $\widetilde{\omega}$ is exact, i.e., there exists a smooth function f such that $\widetilde{\omega} = \mathrm{d}f$. Hence, f is a non-constant harmonic function on $\widetilde{M}$ such that

$$|\nabla f|\Delta|\nabla f| = -(a\rho + b)|\nabla f|^2 + \frac{1}{n-1}|\nabla|\nabla f||^2.$$

Applying Lemma 4.1 in [5] (see Lemma 2.5 in this paper) with $\tau = \dfrac{1}{n-1}(a\rho + b)$, we get the splitting of $\widetilde{M}$. $\square$

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