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# Triple Solutions for Second-Order Three-Point Boundary Value Problems[1]

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We establish the existence of at least three positive solutions to the second-order three-point boundary value problem, $u'' + f(t, u) = 0, u(0) = 0, \alpha u(\eta) = u(1)$, where $\eta$: $0 < \eta < 1, 0 < \alpha < 1/\eta$, and $f$: $[0, 1] \times [0, \infty) \to [0, \infty)$ is continuous. We accomplish this by making growth assumptions on $f$ which can apply to many more cases than the sublinear and superlinear ones discussed in recent works.
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*Key Words:* three-point boundary value problem; multiple solutions; fixed points; cone.

## 1. INTRODUCTION

In the past few years, there has been increasing interest in studying certain three-point boundary value problems for nonlinear ordinary differential equations; to identify a few, we refer the reader to [1–6]. The methods and techniques employed in these papers are the Leray–Schauder continuation theorem, the nonlinear alternative of Leray–Schauder, or the coincidence degree theory. In a recent paper [1], Ma studied the following second-order three-point boundary value problem

$$u'' + \lambda h(t) f(u) = 0, \qquad t \in (0, 1), \tag{1}$$

$$u(0) = 0, \qquad \alpha u(\eta) = u(1), \tag{2}$$

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where

(A) $\lambda$ is a positive parameter; $\eta \in (0, 1)$ and $0 < \alpha\eta < 1$.

(B) $a : [0, 1] \to [0, \infty)$ is continuous and does not vanish identically on any subset of positive measure.

(C) $f : [0, \infty) \to [0, \infty)$ is continuous.

(D) $f_\infty := \lim_{u\to\infty} \frac{f(u)}{u} = \infty$.

The methods employed in [1] are fixed-point index theorems and Leray–Schauder degree and upper and lower solutions. In the case $\lambda = 1$, Ma [3] considered the existence of positive solutions of (1), (2) by using a fixed-point theorem in a cone under the assumptions (A)–(C) and $f$ is either superlinear or sublinear.

In this paper, we are concerned with the multiplicity of positive solutions of the problem

$$u'' + f(t, u) = 0, \qquad t \in (0, 1), \tag{3}$$

$$u(0) = 0, \qquad \alpha u(\eta) = u(1). \tag{4}$$

Throughout this paper, we assume that

(H1) $\eta \in (0, 1)$, $\alpha > 0$ and $\alpha\eta < 1$.

(H2) $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous, and $f(t, \cdot)$ does not vanish identically on any subset of $[0, 1]$ with positive measure.

To show the existence of three positive solutions to (3), (4), we will use the Leggett–Williams fixed-point theorem [8]. Recently, this theorem has been reported to be a successful technique for dealing with the existence of three solutions for the two-point boundary value problems of differential and difference equations; see [9, 10], for example. In [7], Anderson used this fixed-point theorem to obtain three solutions for the third-order three-point boundary value problem

$$-x'''(t) + f(x(t)) = 0,$$

$$x(0) = x'(t_2) = x''(1) = 0, \qquad 1/2 \le t_2 < 1.$$

In this paper, we shall apply the Leggett–Williams fixed-point theorem to yield multiple positive solutions for (3), (4). Our results are new and different from those of [1–3] and are easy to check. Particularly, we do not need the assumption that $f$ is sublinear or superlinear, which was required in [1–3].

## 2. BACKGROUND AND DEFINITIONS

The proof of our main result is based on the Leggett–Williams fixed-point theorem, which deals with fixed points of a cone-preserving operator defined on an ordered Banach space. For the convenience of the reader, we present here the necessary definitions from cone theory in Banach spaces; these definitions can be found in the recent literature.

DEFINITION 2.1. Let $E$ be a Banach space over $R$. A nonempty, closed set $P \subset E$ is said to be a cone provided that

(a) $\alpha u + \beta v \in P$ for all $u, v \in P$ and all $\alpha, \beta \geq 0$ and

(b) $u, -u \in P$ implies $u = 0$.

If $P \subset E$ is a cone, we denote the order induced by $P$ on $E$ by $\leq$. For $u, v \in P$, we write $u \leq v$ if and only if $v - u \in P$.

DEFINITION 2.2. The map $\psi$ is said to be a nonnegative continuous concave functional on $P$ provided that $\psi: P \to [0, \infty)$ is continuous and

$$\psi(tx + (1-t)y) \geq t\psi(x) + (1-t)\psi(y)$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

DEFINITION 2.3. Let $0 < a < b$ be given and let $\psi$ be a nonnegative continuous concave functional on the cone $P$. Define the convex sets $P_r$ and $P(\psi, a, b)$ by

$$P_r = \{y \in P | \|y\| < r\} \qquad \text{and}$$

$$P(\psi, a, b) = \{y \in P | a \leq \psi(y), \|y\| \leq b\}.$$

Next we state the Leggett–Williams fixed-point theorem. The proof can be found in Deimling's text [12].

THEOREM 2.1 (Leggett–Williams Fixed-Point Theorem). *Let $T: \overline{P}_c \to \overline{P}_c$ be a completely continuous operator and let $\psi$ be a nonnegative continuous concave functional on $P$ such that $\psi(y) \leq \|y\|$ for all $y \in \overline{P}_c$. Suppose that there exist $0 < a < b < d \leq c$ such that*

(C1) *$\{y \in P(\psi, b, d) | \psi(y) > b\} \neq \varnothing$ and $\psi(Ty) > b$ for $y \in P(\psi, b, d)$,*

(C2) *$\|Ty\| < a$ for $\|y\| \leq a$, and*

(C3) *$\psi(Ty) > b$ for $y \in P(\psi, b, c)$ with $\|Ty\| > d$.*

*Then $T$ has at least three fixed points $y_1, y_2$, and $y_3$ such that $\|y_1\| < a, b < \psi(y_2)$ and $\|y_3\| > a$ with $\psi(y_3) < b$.*

## 3. SOME PRELIMINARY RESULTS

In order to prove our main result, we need some preliminary results. Let us consider the following boundary value problem

$$u'' + y(t) = 0, \qquad t \in (0, 1), \tag{5}$$

$$u(0) = 0, \qquad \alpha u(\eta) = u(1). \tag{6}$$

For problem (5), (6), we have the following conclusions which are derived from [1, 2].

LEMMA 3.1 [1, 2]. *Let $\alpha\eta \neq 1$; then for $y \in C[0, 1]$, problem* (5), (6) *has the unique solution*

$$u(t) = -\int_0^t (t - s)y(s)\,ds - \frac{\alpha t}{1 - \alpha\eta}\int_0^\eta (\eta - s)y(s)\,ds + \frac{t}{1 - \alpha\eta}\int_0^1 (1 - s)y(s)\,ds.$$

LEMMA 3.2 [1, 2]. *Let $0 < \alpha < 1/\eta$. If $y \in C[0, 1]$ and $y \geq 0$, then the unique solution $u$ of problem* (5), (6) *satisfies*

$$u(t) \geq 0, \qquad t \in [0, 1].$$

LEMMA 3.3 [1, 2]. *Let $0 < \alpha < 1/\eta$. If $y \in C[0, 1]$ and $y \geq 0$, then the unique solution $u$ of problem* (5), (6) *satisfies*

$$\min_{t\in[\eta,1]} u(t) \geq r\|u\|,$$

*where* $r = \min\{\alpha\eta, \alpha(1 - \eta)/(1 - \alpha\eta), \eta\}$.

## 4. EXISTENCE OF TRIPLE SOLUTIONS

Let $E = C[0, 1]$ be endowed with the maximum norm, $\|y\| = \max_{0\leq t\leq 1} |y(t)|$, and the ordering $x \leq y$ if $x(t) \leq y(t)$ for all $t \in [0, 1]$. From the fact $u''(t) = -f(t, u) \leq 0$, we know that $u(t)$ is concave on $[0, 1]$. So, define the cone $P \subset E$ by

$$P = \{u \in E | u \text{ is concave and nonnegative valued on } [0, 1]\}.$$

Finally, let the nonnegative continuous concave functional $\psi: P \to [0, \infty)$ be defined by

$$\psi(u) = \min_{\eta\leq t\leq 1} u(t), \qquad u \in P. \tag{7}$$

We notice that, for each $u \in P$,

$$\psi(u) \leq \|u\|, \tag{8}$$

and also that by Lemma 3.1, $u \in P$ is a solution of (3), (4) if and only if

$$u(t) = -\int_0^t (t-s)f(s, u(s))\,ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)f(s, u(s))\,ds + \frac{t}{1-\alpha\eta}\int_0^1 (1-s)f(s, u(s))\,ds. \tag{9}$$

In the following, we denote

$$m = \left(\frac{2-\alpha\eta+\alpha\eta^2}{2(1-\alpha\eta)}\right)^{-1}, \qquad \delta = \min\left\{\frac{\alpha\eta(1-\eta)^2}{2(1-\alpha\eta)}, \frac{\eta(1-\eta)^2}{2(1-\alpha\eta)}\right\}.$$

THEOREM 4.1. *Suppose that there exist constants* $0 < a < b < b/r \leq c$ *such that*

(D1) $f(t, u) < ma$, *for* $0 \leq t \leq 1, 0 \leq u \leq a$,

(D2) $f(t, u) \geq b/\delta$, *for* $\eta \leq t \leq 1, b \leq u \leq b/r$, *and*

(D3) $f(t, u) \leq mc$, *for* $0 \leq t \leq 1, 0 \leq u \leq c$

Then, the boundary value problem (3), (4) has at least three positive solutions $u_1, u_2$, and $u_3$ satisfying $\|u_1\| < a, b < \psi(u_2)$ and $\|u_3\| > a$ with $\psi(u_3) < b$.

*Proof.* To begin with, we define an operator $T: P \to E$ by

$$Tu(t) = -\int_0^t (t-s)f(s, u(s))\,ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)f(s, u(s))\,ds + \frac{t}{1-\alpha\eta}\int_0^1 (1-s)f(s, u(s))\,ds.$$

We note that, if $u \in P$, then from (H2) and Lemma 3.2, it follows that $Tu(t) \geq 0, 0 \leq t \leq 1$. In view of $(Tu)''(t) = -f(t, u(t)) \leq 0, 0 \leq t \leq 1$, we see that $Tu \in P$; that is to say, $T: P \to P$. It is shown in [1, 2] that $T$ is completely continuous.

We now show that all the conditions of Theorem 2.1 are satisfied. From (8), we know that $\psi(u) \leq \|u\|$, for all $u \in P$. Now if $u \in \overline{P}_c$, then $\|u\| \leq c$

and assumption (D3) implies $f(t, u(t)) \le mc$, $0 \le t \le 1$. Consequently,

$$\begin{aligned}
\|Tu\| &= \max_{0\le t\le 1} -\int_0^t (t-s)f(s,u(s))\,ds \\
&\quad - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)f(s,u(s))\,ds \\
&\quad + \frac{t}{1-\alpha\eta}\int_0^1 (1-s)f(s,u(s))\,ds \\
&\le \max_{0\le t\le 1}\int_0^t (t-s)f(s,u(s))\,ds + \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s)f(s,u(s))\,ds \\
&\quad + \frac{t}{1-\alpha\eta}\int_0^1 (1-s)f(s,u(s))\,ds \\
&\le \max_{0\le t\le 1}\left(\int_0^t (t-s)\,ds + \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s)\,ds \right.\\
&\qquad \left. + \frac{t}{1-\alpha\eta}\int_0^1 (1-s)\,ds\right)\times mc \\
&= \frac{2-\alpha\eta+\alpha\eta^2}{2(1-\alpha\eta)}\times mc \\
&= c.
\end{aligned}$$

Hence, $T$: $\overline{P}_c \to \overline{P}_c$. In the same way, if $u \in \overline{P}_a$, then assumption (D1) yields $f(t, u(t)) < ma$, $0 \le t \le 1$. As in the argument above, we can obtain that $T$: $\overline{P}_a \to P_a$. Therefore, condition (C2) of Theorem 2.1 is satisfied.

To check condition (C1) of Theorem 2.1, we choose $u(t) = b/r$, $0 \le t \le 1$. It is easy to see that $u(t) = b/r \in P(\psi, b, b/r)$ and $\psi(u) = \psi(b/r) > b$, and so $\{u \in P(\psi, b, b/r) | \psi(u) > b\} \ne \varnothing$. Hence, if $u \in P(\psi, b, b/r)$, then $b \le u(t) \le b/r$, $\eta \le t \le 1$. From assumption (D2), we have $f(t, u(t)) \ge b/\delta$, $\eta \le t \le 1$, and by the definitions of $\psi$ and the cone $P$, we have to distinguish two cases, (i) $\psi(Tu) = Tu(\eta)$ and (ii) $\psi(Tu) = Tu(1)$.

In case (i), we have

$$\begin{aligned}
\psi(Tu) &= Tu(\eta) \\
&= -\int_0^\eta (\eta - s)f(s,u(s))\,ds - \frac{\alpha\eta}{1-\alpha\eta}\int_0^\eta (\eta-s)f(s,u(s))\,ds \\
&\quad + \frac{\eta}{1-\alpha\eta}\int_0^1 (1-s)f(s,u(s))\,ds
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{1}{1-\alpha\eta}\int_0^{\eta}(\eta - s)f(s,u(s))\,ds \\
&\quad + \frac{\eta}{1-\alpha\eta}\int_0^{1}(1-s)f(s,u(s))\,ds \\
&= -\frac{\eta}{1-\alpha\eta}\int_0^{\eta} f(s,u(s))\,ds + \frac{1}{1-\alpha\eta}\int_0^{\eta} sf(s,u(s))\,ds \\
&\quad + \frac{\eta}{1-\alpha\eta}\int_0^{1} f(s,u(s))\,ds - \frac{\eta}{1-\alpha\eta}\int_0^{1} sf(s,u(s))\,ds \\
&= \frac{\eta}{1-\alpha\eta}\int_{\eta}^{1} f(s,u(s))\,ds + \frac{1}{1-\alpha\eta}\int_0^{\eta} sf(s,u(s))\,ds \\
&\quad - \frac{\eta}{1-\alpha\eta}\int_0^{1} sf(s,u(s))\,ds \\
&> \frac{\eta}{1-\alpha\eta}\int_{\eta}^{1} f(s,u(s))\,ds - \frac{\eta}{1-\alpha\eta}\int_{\eta}^{1} sf(s,u(s))\,ds \\
&\geq \frac{\eta}{1-\alpha\eta}\int_{\eta}^{1}(1-s)\,ds \times \frac{b}{\delta} \\
&= \frac{\eta(1-\eta)^2}{2(1-\alpha\eta)} \times \frac{b}{\delta} \\
&\geq b.
\end{aligned}
$$

In case (ii), we have

$$
\begin{aligned}
\psi(Tu) &= Tu(1) \\
&= -\int_0^{1}(1-s)f(s,u(s))\,ds - \frac{\alpha}{1-\alpha\eta}\int_0^{\eta}(\eta - s)f(s,u(s))\,ds \\
&\quad + \frac{1}{1-\alpha\eta}\int_0^{1}(1-s)f(s,u(s))\,ds \\
&= \frac{\alpha\eta}{1-\alpha\eta}\int_0^{1}(1-s)f(s,u(s))\,ds - \frac{\alpha}{1-\alpha\eta}\int_0^{\eta}(\eta - s)f(s,u(s))\,ds \\
&= \frac{\alpha\eta}{1-\alpha\eta}\int_0^{1} f(s,u(s))\,ds - \frac{\alpha\eta}{1-\alpha\eta}\int_0^{1} sf(s,u(s))\,ds \\
&\quad - \frac{\alpha\eta}{1-\alpha\eta}\int_0^{\eta} f(s,u(s))\,ds + \frac{\alpha}{1-\alpha\eta}\int_0^{\eta} sf(s,u(s))\,ds \\
&= \frac{\alpha\eta}{1-\alpha\eta}\int_{\eta}^{1} f(s,u(s))\,ds - \frac{\alpha\eta}{1-\alpha\eta}\int_0^{1} sf(s,u(s))\,ds \\
&\quad + \frac{\alpha}{1-\alpha\eta}\int_0^{\eta} sf(s,u(s))\,ds
\end{aligned}
$$

$$
\begin{aligned}
&> \frac{\alpha\eta}{1-\alpha\eta}\int_{\eta}^{1} f(s,u(s))\,ds - \frac{\alpha\eta}{1-\alpha\eta}\int_{\eta}^{1} sf(s,u(s))\,ds \\
&= \frac{\alpha\eta}{1-\alpha\eta}\int_{\eta}^{1} (1-s)f(s,u(s))\,ds \\
&\geq \frac{\alpha\eta(1-\eta)^2}{2(1-\alpha\eta)} \times \frac{b}{\delta} \\
&\geq \delta \times \frac{b}{\delta} = b;
\end{aligned}
$$

i.e.,

$$\psi(Tu) > b,\ \forall\, u \in P(\psi, b, b/r).$$

This shows that condition (C1) of Theorem 2.1 is satisfied. We finally show that (C3) of Theorem 2.1 also holds. Suppose that $u \in P(\psi, b, c)$ with $\|Tu\| > b/r$. Then, from (7) and Lemma 3.3, we have

$$
\begin{aligned}
\psi(Tu) &= \min_{\eta \leq t \leq 1} Tu(t) \\
&\geq r \times \|Tu\| \\
&> r \times \frac{b}{r} = b.
\end{aligned}
$$

So, condition (C3) of Theorem 2.1 is satisfied. Therefore, an application of Theorem 2.1 completes the proof.

From Theorem 4.1, we see that, when assumptions like (D1)–(D3) are appropriately imposed on $f$, we can obtain any number of positive solutions of problem (3), (4). To be more precise, we have the following conclusion.

COROLLARY 4.2. *Suppose that there exist constants* $0 < a_1 < b_1 < b_1/r < a_2 < b_2 < b_2/r < \cdots < a_n, n \in N$, *such that the following conditions are satisfied*:

(E1) $f(t,u) < ma_i,\ 0 \leq t \leq 1, 0 \leq u \leq a_i, 1 \leq i \leq n$, *and*

(E2) $f(t,u) \geq b_i/\delta,\ \eta \leq t \leq 1, b_i \leq u \leq b_i/r, 1 \leq i \leq n-1$.

*Then problem* (3), (4) *admits at least* $2n-1$ *positive solutions*.

*Proof.* When $n = 1$, it follows from condition (E1) that $T$: $\overline{P}_{a_1} \to P_{a_1} \subset \overline{P}_{a_1}$, which means that $T$ has at least one fixed point $u_1 \in \overline{P}_{a_1}$ by the Schauder fixed-point theorem. When $n = 2$, it is clear that Theorem 4.1 holds (with $c_1 = a_2$). Then we can obtain at least three positive solutions $u_1, u_2$, and $u_3$ satisfying $\|u_1\| < a_1, \psi(u_2) > b_1$, and $\|u_3\| > a_1$ with $\psi(u_3) < b_1$. Following this way, we finish the proof by the induction method.

Finally, we present an example to check our result.

EXAMPLE. Consider the boundary value problem

$$\begin{aligned} &u'' + f(t, u) = 0, \qquad 0 < t < 1, \\ &u(0) = 0, \quad 2u(1/4) = u(1), \end{aligned} \tag{10}$$

where $f(t, u) = e^{-t}h(u)$ and

$$h(u) = \begin{cases} u, & 0 \le u \le 1; \\ 47u - 46, & 1 \le u \le 2; \\ 48, & 2 \le u \le 48; \\ u, & 48 \le u \le 50; \\ u\left(\dfrac{1 + 2u}{51 + u}\right), & u \ge 50. \end{cases}$$

In this example, we note that $\alpha = 2$, $\eta = 1/4$. After a simple calculation, we get $m = 8/7$, $\delta = 9/64$, and $r = 1/4$. We choose $a = 1/2$, $b = 2$, and $c = 48$; consequently,

$$f(t, u) = e^{-t}u \le u < \frac{8}{7} \times \frac{1}{2} = ma, \qquad 0 \le t \le 1, 0 \le u \le 1/2,$$

$$f(t, u) = e^{-t}48 \ge \frac{48}{e} > 2 \times \frac{64}{9} = b/\delta, \qquad 1/4 \le t \le 1, 2 \le u \le 8,$$

$$f(t, u) = e^{-t}h(u) \le 48 < \frac{8}{7} \times 48 = mc, \qquad 0 \le t \le 1, 0 \le u \le 48.$$

That is to say, all the conditions of Theorem 4.1 are satisfied. Then problem (10) has at least three positive solutions $u_1$, $u_2$, and $u_3$ satisfying

$$\|u_1\| < \frac{1}{2}, \psi(u_2) > 2, \qquad \|u_3\| > \frac{1}{2}, \qquad \text{with } \psi(u_3) < 2.$$

*Remark.* The results in [1–3], for example, are not applicable to the above problem. In conclusion, we see that the function $f$ in the above example is neither superlinear nor sublinear as mentioned in Section 1.

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