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Dual and Feller–Reuter–Riley transition functions

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Abstract

In this paper, we investigate duality and Feller–Reuter–Riley (FRR) property of continuous-time Markov chains (CTMCs). A criterion of dual q-functions is given in terms of their q-matrices. For a dual q-matrix Q, a necessary and sufficient conditions for the minimal Q-function to be a FRR transition function are also given. Finally, by using dual technique, we give a criterion of FRR Q-functions when Q is monotone.

Keywords: Continuous-time Markov chains; Stochastic monotonicity; Duality; FRR; q-Functions; q-Matrices; Zero-exit; Zero-entrance

1. Introduction and preliminaries

In this paper, we study duality and Feller–Reuter–Riley property of continuous time Markov chains (CTMCs) (see [1–7]). We only consider CTMCs on a linear ordering set, that is, the state space $E = Z_+ = \{0, 1, 2, \ldots\}$, and assume always that all transition functions are standard and all q-matrices are stable, as in Anderson [1].

Definition 1.1 [10]. A transition function $P(t) = (p_{ij}(t);\ i, j \in E)$ is monotone if $\sum_{j \geqslant k} p_{ij}(t)$ is a non-decreasing function of i, for fixed $j \in E$ and $t \geqslant 0$.

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Definition 1.2 [11]. A transition function $P(t)$ is a dual if $\sum_{k=0}^{j} p_{ik}(t) \downarrow 0$ as $i \to \infty$ for $j \in E$ and $t \geqslant 0$.

Definition 1.3 [9]. A transition function $P(t)$ is a Feller–Reuter–Riley transition function, (briefly, FRR) if $\lim_{i\to\infty} p_{ij}(t) = 0$ for $j \in E$ and $t \geqslant 0$.

Obviously, a dual transition function is FRR. Moreover, duality and monotonicity have the following relationship.

Proposition 1.4 (Siegmund's theorem). *A transition function $P(t)$ is monotone if and only if there exists a dual $\tilde{P}(t)$ for $P(t)$ (namely, if and only if there exists another transition function $\tilde{P}(t)$) such that*

$$\sum_{k=0}^{j} \tilde{p}_{ik}(t) = \sum_{k=i}^{\infty} p_{jk}(t) \quad (\forall i, j \in E,\ t \geqslant 0). \tag{1.1}$$

Siegmund's theorem can be stated in an equivalent form: a transition function $\tilde{P}(t)$ is a dual if and only if there exists a monotone $P(t)$ satisfying (1.1).

An infinite matrix $Q = (q_{ij};\ i, j \in E)$ is called to be a (stable) q-matrix, if

$$0 \leqslant q_{ij} < +\infty, \tag{1.2}$$

$$\sum_{j\neq i} q_{ij} \leqslant -q_{ii} \equiv q_i < +\infty. \tag{1.3}$$

A transition function $P(t)$ is called to be a Q-function if

$$P'(0) = Q \quad \text{(componentwise)}. \tag{1.4}$$

It is well known that for a given q-matrix Q, there exists a minimal Q-function $F(t)$, and that if $P(t)$ is an FRR q-function then it must be the minimal one (see [1]).

Two questions are considered in this paper.

Question 1 [11]. For a given q-matrix Q, what are the necessary and sufficient conditions for the minimal Q-function $F(t)$ to be a dual Q-function?

Question 2 [9]. For a given q-matrix Q, what are the necessary and sufficient conditions for the minimal Q-functions to be a FRR Q-functions?

Zhang and Chen [11] gave answer to Question 1. Unfortunately, one do not know whether their results [11, Theorem 4.6] are correct. Because they seem ignore the possible difference between zero-entrance in l_1 and in l_1^+, and thus use incorrectly Reuter and Riley's result. Question 2 is raised by Reuter and Riley [9] and partially answered by many author (see [1,8,9,11], etc.). For instance, Zhang and Chen [11, Theorem 5.1] gave a criteria of FRR Q-function when Q is dual. However, this result is also not exactly correct with the same reason as above.

In present paper, we give a complete answer to Question 1 (see Theorem 3.2). As to Question 2, the discussion is concentrated on two classes of important q-matrices: dual and monotone q-matrices. A criteria of FRR q-functions for dual q-matrices is given in Theorem 4.1, another criteria of FRR q-functions for monotone q-matrices is given in Theorem 4.3.

2. Zero-exit and zero-entrance

Definition 2.1. A q-matrix Q is zero-exit in l_∞ or in l_∞^+ if $l_\infty(\lambda)=0$ or $l_\infty^+(\lambda)=0$, respectively, and is zero-entrance in l_1 or in l_1^+ if $l_1(\lambda)=0$ or $l_1^+(\lambda)=0$, respectively, where

$$\begin{aligned} &l_\infty(\lambda)=\{x\in l_\infty \mid (\lambda I-Q)x=0\}, && l_\infty^+(\lambda)=\{x\in l_\infty(\lambda)\mid x\geqslant 0\};\\ &l_1(\lambda)=\{y\in l_1 \mid y(\lambda I-Q)=0\}, && l_1^+(\lambda)=\{y\in l_1(\lambda)\mid y\geqslant 0\}. \end{aligned} \tag{2.1}$$

It is well known that zero-exit in l_∞ and in l_∞^+ are equivalent each to other, so ones briefly called it zero-exit. However, whether are zero-entrance in l_1 and in l_1^+ equivalent? This question is raised by Reuter–Riley [9] and remains open. For birth–death matrix and branching matrix, we have an affirmative answer based on the following proposition.

Proposition 2.2. *If a q-matrix $Q=(q_{ij};\, i,j\in E)$ satisfies*

$$q_{ij}=0,\quad \text{for } i\geqslant j+2, \tag{2.2}$$

then Q is zero-entrance in l_1 if and only if Q is zero-entrance in l_1^+.

Proof. Necessity is obvious. To prove sufficiency, we assume $y=(y_k;\ k\in E)\in l_1$ such that $y(\lambda I-Q)=0$. We show that either $y\in l_1^+$ or $-y\in l_1^+$. To this end, we assume without lose of generality that $y_0>0$ (if $y_0<0$, considering $y=(-y_j)$, and if $y_0=0$, passing to the first non-zero element), and claim that $y_j>0$ for all $j\in E$. Indeed, $y\in l_1(\lambda)$ can be written as

$$\lambda y_j=\sum_{i\in E} y_i q_{ij}\quad \text{for } j\in E. \tag{2.3}$$

Sum the above equality for $j=0$ to $j=m$, and use (2.2), we obtain

$$\begin{aligned}\sum_{j=0}^m \lambda y_j &= \sum_{j=0}^m\sum_{i=0}^{j+1} y_i q_{ij} = \sum_{j=0}^m\sum_{i=1}^{j+1} y_i q_{ij} + \sum_{j=0}^m y_0 q_{0j} = \sum_{i=1}^{m+1}\sum_{j=i-1}^{m} y_i q_{ij} + \sum_{j=0}^m y_0 q_{0j}\\ &= y_{m+1}q_{m+1,m} + \sum_{i=1}^m\sum_{j=i-1}^m y_i q_{ij} + \sum_{j=0}^m y_0 q_{0j}.\end{aligned}$$

Thus

$$y_{m+1}q_{m+1,m}=\sum_{i=1}^m y_i\left(\lambda-\sum_{j=i-1}^m q_{ij}\right)+y_0\left(\lambda-\sum_{j=0}^m q_{0j}\right),\quad m\in E. \tag{2.4}$$

Since $\sum_{j=i-1}^{m} q_{ij} \leqslant 0$ for $1 \leqslant i \leqslant m$, it follows that $\lambda - \sum_{j=i-1}^{m} q_{ij} > 0$ for $1 \leqslant i \leqslant m$ and $\lambda - \sum_{j=0}^{m} q_{0j} > 0$. Thus (2.4), together with an induction argument, show that $y_j > 0$ for all $j \in E$, this means $y \in l_1^+(\lambda)$, which implies that $y = 0$ if Q is zero-entrance in l_1^+. Therefore we have proved that Q is zero-entrance in l_1 if Q is zero-entrance in l_1^+. $\square$

Remark. The birth–death matrix and branching matrix satisfy (2.2). Moreover, we get from the above proof that if Q satisfies (2.2) and if $y \in l_1$ is a solution of the equation $y(\lambda - Q) = 0$, then either $y \in l_1^+$ or $-y \in l_1^+$. But this is not always true. For example, let

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & -(6-1) & 0 & 0 & 0 & \ldots \\ 6 & 6 & -(6^2-1) & 0 & 0 & \ldots \\ 0 & 6^2 & 6^2 & -(6^3-1) & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

It is easy to verify that $y = (1, -\frac{1}{2}, \frac{1}{2^2}, -\frac{1}{2^3}, \ldots, (-1)^n \frac{1}{2^n}, \ldots) \in l_1$ is a solution of the equation $y(I - Q) = 0$.

3. Dual Q-functions

In this section, we give the characterization of dual q-functions in terms of q-matrices. We first give some notations.

Definition 3.1. A q-matrix $Q = (q_{ij})$ is called to be dual if

$$\sum_{k=0}^{j} q_{ik} \geqslant \sum_{k=0}^{j} q_{i+1,k}, \quad j \neq i; \tag{3.1}$$

Q is monotone if

$$\sum_{k \geqslant j} q_{ik} \leqslant \sum_{k \geqslant j} q_{i+1,k}, \quad j \neq i+1; \tag{3.2}$$

Q is Feller–Reuter–Riley (FRR) if

$$q_{ij} \to 0 \quad \text{as } i \to \infty \quad \text{for every } j \in E. \tag{3.3}$$

We then state our result.

Theorem 3.2. *For a given q-matrix $Q = (q_{ij})$, the minimal Q-function $F(t)$ is a dual* (*of some monotone one*) *if and only if*

(i) *Q is dual, and*
(ii) *either*
   (a) *Q is FRR and zero-entrance in l_1, or*

(b) *for some* $\lambda > 0$ (*and hence for all* $\lambda > 0$) *the equations*

$$\lambda x_i = d_i + \sum_{k=0}^{\infty} q_{ik} x_k, \quad 0 \leqslant x_i \leqslant 1, \ i \in E, \tag{3.4}$$

has a solution $x = (x_i)$ *satisfying* $\sup_{i \in E} x_i = 1$. *Here* $d = (d_i) = (-\sum_j q_{ij})$ *is the nonconservative quantity of* Q.

Remark. Theorem 3.2 is slightly different from [11, Theorem 4.6]. The only difference is where l_1^+ instead of l_1 (in condition (ii)(a)). They [11] seem ignore the difference, and incorrectly used Reuter–Riley's result [9, Theorem 8] in their proof of sufficiency and get an "unsolved" conclusion.

Now sufficiency in above theorem is easy to obtain by using Reuter–Riley's result and Zhang and Chen's method. However, we need prove necessity. To this end, we need improve lemmas in [1,3,11]. The following lemma can be seen from [1,3] for the case of that $Q^{(1)}$ is conservative.

Lemma 3.3 [1]. *Let* $Q^{(1)}$ *be a monotone* q*-matrix* (*that is,* $Q^{(1)}$ *satisfies* (3.2)) *and define the matrix* $Q^{(2)}$ *by*

$$q_{ij}^{(2)} = \sum_{k=i}^{\infty} \big(q_{jk}^{(1)} - q_{j-1,k}^{(1)}\big), \quad i, j \in E, \tag{3.5}$$

where $q_{-1,k}^{(1)} \equiv 0$. *Then*:

(1) $Q^{(2)}$ *is a FRR* q*-matrix.*
(2) *For* $i, j \in E$, *we have*

$$\sum_{k=0}^{j} q_{ik}^{(2)} = \sum_{m=i}^{\infty} q_{jm}^{(1)}, \tag{3.6}$$

$$q_{i+1,j}^{(2)} - q_{i,j}^{(2)} = q_{j-1,i}^{(1)} - q_{ji}^{(1)}. \tag{3.7}$$

(3) $Q^{(2)}$ *is dual, namely*

$$\sum_{k=0}^{j} q_{ik}^{(2)} \geqslant \sum_{k=0}^{j} q_{i+1,k}^{(2)}, \quad j \neq i. \tag{3.8}$$

(4) $Q^{(2)}$ *is conservative if and only if* $Q^{(1)}$ *is Reuter, that is,* $\sum_{k=j}^{\infty} q_{ik}^{(1)} \to 0$ *as* $i \to \infty$ *for every* j.

Proof. (1) and (4) can be seen from [1]. Sum (3.5) we get

$$\sum_{k=0}^{j} q_{ik}^{(2)} = \sum_{k=0}^{j} \sum_{m=i}^{\infty} \big(q_{km}^{(1)} - q_{k-1,m}^{(1)}\big) = \sum_{m=i}^{\infty} \sum_{k=0}^{j} \big(q_{km}^{(1)} - q_{k-1,m}^{(1)}\big) = \sum_{m=i}^{\infty} q_{jm}^{(1)}$$

which means that (3.6) holds. (3.7) is easy deduced from (3.5). By (3.6),

$$\sum_{k=0}^{j}\big(q_{ik}^{(2)}-q_{i+1,k}^{(2)}\big)=\sum_{m=i}^{\infty}q_{jm}^{(1)}-\sum_{m=i+1}^{\infty}q_{jm}^{(1)}=q_{ji}\geqslant 0$$

for $j\neq i$, which implies (3.8). □

The following lemma improve [3, Lemma 3.10].

Lemma 3.4. *Let $Q^{(1)}$ be a monotone q-matrix, $Q^{(2)}$ defined as in* (3.5). *If $Q^{(1)}$ is zero-exit, then $Q^{(2)}$ is zero-entrance in l_1.*

Proof. If $Q^{(2)}$ is not zero-entrance in l_1, then there is a $y\in l_1$ with $y=(y_j)\neq 0$ such that $y(\lambda I-Q^{(2)})=0$, that is,

$$\lambda y_j=\sum_{k=0}^{\infty}y_k q_{kj}^{(2)},\quad j\in E. \tag{3.9}$$

Define $x=(x_i)$ by

$$x_i=\sum_{k=0}^{i}y_k,\quad i\in E.$$

Then $0\neq x\in l_\infty$ with $\|x\|_\infty=\sup_{i\in E}|x_i|\leqslant\sum_{k=0}^{\infty}|y_k|=\|y\|_1$. We claim that $(\lambda I-Q^{(1)})x=0$. Indeed, using (3.9) and (3.6) we can calculate as follows:

$$\lambda\sum_{j=0}^{i}y_j=\sum_{j=0}^{i}\sum_{k=0}^{\infty}y_k q_{kj}^{(2)}=\sum_{k=0}^{\infty}y_k\sum_{j=0}^{i}q_{kj}^{(2)}=\sum_{k=0}^{\infty}y_k\sum_{m=k}^{\infty}q_{im}^{(1)}=\sum_{m=0}^{\infty}q_{im}^{(1)}\sum_{k=0}^{m}y_k$$

that is, $\lambda x_i=\sum_{m=0}^{\infty}q_{im}^{(1)}x_m$ for every $i\in E$, and thus $x=(x_i)$ is a nonzero solution of $(\lambda I-Q^{(1)})x=0$. Therefore $Q^{(1)}$ is nonzero-exit in l_∞, which implies by [1, Theorem 2.2.7] that $Q^{(1)}$ is nonzero-exit (in l_∞^+). This contradicts to the assumption. □

Proof of Theorem 3.2. *Sufficiency*. If condition (a) in (ii) holds, then by Reuter and Riley's result [9, Theorem 8], the minimal Q-function $F(t)$ is FRR. The other proof is the same as in [11, Theorem 4.6].

Necessity. Let the minimal Q-function $F(t)$ be a dual of a monotone $Q^{(1)}$-function $P^{(1)}(t)$. Then the condition (i) can be seen from [11, Theorem 4.6]. To get (ii), we suppose (ii)(b) is not true, then it follows from the proof of necessity in [11, Theorem 4.6] that Q is FRR and zero-entrance in l_1^+, and that $P^{(1)}(t)$ is the minimal $Q^{(1)}$-function. Hence $P^{(1)}(t)$ must satisfy the Kolmogorov backward equations. Thus by [3, Theorem 2.5],

$$q_{ij}=\sum_{k=0}^{\infty}\big(q_{jk}^{(1)}-q_{j-1,k}^{(1)}\big). \tag{3.10}$$

Now, since $P^{(1)}(t)$ is monotone, it follows from [11, Theorem 3.1] that $Q^{(1)}$ is zero-exit. This, together with (3.10), implies by Lemma 3.4 (where $Q^{(1)}=Q^{(1)}$, $Q^{(2)}=Q$) that Q is zero-entrance in l_1. Thus (iia) holds. We have proved (ii). □

4. Feller–Reuter–Riley Q-functions

In this section, we consider Question 2 announced in the introduction. Our main interest is two classes of q-matrices: dual and monotone q-matrices. For dual case, we have the following result to remedy some inconsistencies in [11, Theorem 5.1] (where l_1^+ instead of l_1).

Theorem 4.1. *Let $Q = (q_{ij})$ be a dual q-matrix. Then the minimal Q-function $F(t) = (f_{ij}(t))$ is FRR if and only if either*:

(i) *Q is FRR and zero-entrance in l_1, or*
(ii) *for some $\lambda > 0$ (and hence for all $\lambda > 0$), the equations*

$$\lambda x_i = d_i + \sum_{k=0}^{\infty} q_{ik} x_k, \quad 0 \leqslant x_i \leqslant 1,\ i \in E, \tag{4.1}$$

has a solution $x = (x_i)$ satisfying $\sup_{i \in E} x_i = 1$.

Proof. Since Q is dual, it follows from [11, Proposition 2.4] that

$$\sum_{k=0}^{j} f_{ik}(t) \geqslant \sum_{k=0}^{j} f_{i+1,k}(t), \quad i, j \in E,$$

which implies that $F(t)$ is FRR if and only if $F(t)$ is dual. Thus the needed conclusion follows from Theorem 3.2. □

Corollary 4.2. *Assume Q be a dual q-matrix and the nonconservative quantity $\{d_i\}$ is bounded. Then the minimal Q-function is FRR if and only if either* (i) *Q is FRR and zero-entrance in l_1, or* (ii) *Q is nonzero-exit.*

Proof. By [1, Proposition 4.3.3], the condition (ii) in Theorem 4.1 is equivalent to that Q is nonzero-exit if $\{d_i\}$ is bounded. □

We then turn to the case of monotone q-matrix. It is worth point that the monotone case is more fundamental and more difficult.

Theorem 4.3. *Given a monotone q-matrix Q, the minimal Q-function is FRR if and only if either*:

(i) *Q is FRR and zero-entrance in l_1, or*
(ii) *Q is nonzero-exit.*

To prove this result, we need some lemmas.

Lemma 4.4. *Let $P(t)$ be a monotone transition function and $\tilde{P}(t)$ the dual of $P(t)$. Then $P(t)$ is FRR if and only if $\tilde{P}(t)$ is monotone.*

Proof. *Necessity*. Since $P(t)$ is monotone, it follows from Definition 1.1 that the limit

$$c_i(t) = \lim_{j\to\infty} \sum_{k=i}^{\infty} p_{jk}(t), \quad i \in E, \tag{4.2}$$

exists for $i \in E$ and $t \geqslant 0$. Since $P(t)$ is also FRR, it follows that

$$c_i(t) = \lim_{j\to\infty} \sum_{k=0}^{\infty} p_{jk}(t) - \lim_{j\to\infty} \sum_{k=0}^{i-1} p_{jk}(t) = c_0(t) \tag{4.3}$$

which is independent of $i \in E$ for $t \geqslant 0$. Letting $\tilde{c}_i(t) = \sum_{k=0}^{\infty} \tilde{p}_{ik}(t)$ for $i \in E$ and $t \geqslant 0$, and using (1.1) we get

$$\tilde{c}_i(t) = \lim_{j\to\infty} \sum_{k=0}^{j} \tilde{p}_{ik}(t) = \lim_{j\to\infty} \sum_{k=i}^{\infty} p_{jk}(t) = c_i(t) = c_0(t) \tag{4.4}$$

which is independent of i for $t \geqslant 0$. Thus

$$\sum_{k=j}^{\infty} \tilde{p}_{ik}(t) = \sum_{k=0}^{\infty} \tilde{p}_{ik}(t) - \sum_{k=0}^{j-1} \tilde{p}_{ik}(t) = c_0(t) - \sum_{k=0}^{j-1} \tilde{p}_{ik}(t). \tag{4.5}$$

Since $\sum_{k=0}^{j-1} \tilde{p}_{ik}(t)\downarrow$ as $i \to \infty$ for $j \in E$ and $t \geqslant 0$, it follows from above equality that $\sum_{k=j}^{\infty} \tilde{p}_{ik}(t)\uparrow$ as $i \to \infty$ for $j \in E$ and $t \geqslant 0$, which means $\tilde{P}(t)$ is monotone.

Sufficiency. Since $\tilde{P}(t)$ is dual, we have

$$\sum_{j=0}^{k} \tilde{p}_{ij}(t) \geqslant \sum_{j=0}^{k} \tilde{p}_{i+1,j}(t) \quad \text{for } k, i \in E \text{ and } t \geqslant 0. \tag{4.6}$$

Letting $k \to \infty$ we get

$$\sum_{j=0}^{\infty} \tilde{p}_{ij}(t) \geqslant \sum_{j=0}^{\infty} \tilde{p}_{i+1,j}(t) \quad \text{for } i \in E \text{ and } t \geqslant 0. \tag{4.7}$$

On the other hand, monotonicity of $\tilde{P}(t)$ implies that

$$\sum_{j=0}^{\infty} \tilde{p}_{ij}(t) \leqslant \sum_{j=0}^{\infty} \tilde{p}_{i+1,j}(t). \tag{4.8}$$

Thus $\sum_{j=0}^{\infty} \tilde{p}_{ij}(t) = c(t)$ is independent of i for $t \geqslant 0$. This, together with (1.1), implies that the limit

$$\lim_{i\to\infty} \sum_{k=j}^{\infty} p_{ik}(t) = \lim_{i\to\infty} \sum_{k=0}^{i} \tilde{p}_{jk}(t) = c(t) \tag{4.9}$$

exists and is independent of $j \in E$ for $t \geqslant 0$. Therefore,

$$\lim_{i\to\infty} p_{ij}(t) = \lim_{i\to\infty} \left(\sum_{k=j}^{\infty} p_{ik}(t) - \sum_{k=j+1}^{\infty} p_{ik}(t) \right) = c(t) - c(t) = 0 \tag{4.10}$$

for every $j \in E$, which means that $P(t)$ is FRR. $\square$

According to above lemma, if $P(t)$ is monotone and FRR, then its dual $\tilde{P}(t)$ is also monotone. Thus there exist another dual function $\tilde{\tilde{P}}(t)$ of $\tilde{P}(t)$, which is the twice dual of $P(t)$. Of course, $\tilde{\tilde{P}}(t)$ is monotone, dual and FRR. Moreover, it also have the following properties which is useful to prove Theorem 4.3.

Lemma 4.5. *Let $P(t)$ be a FRR and monotone transition function with the q-matrix Q, $\tilde{P}(t)$, $\tilde{\tilde{P}}(t)$ be the dual and twice dual of $P(t)$ with the q-matrix $\tilde{Q}$ and $\tilde{\tilde{Q}}$, respectively. Then*:

(i) *the nonconservative quantity $\tilde{d} = (\tilde{d}_i)$, $\tilde{\tilde{d}} = (\tilde{\tilde{d}}_i)$ are constant, namely,*

$$\tilde{d}_i = \tilde{\tilde{d}}_i = \alpha \geqslant 0 \quad \text{for every } i \in E; \tag{4.11}$$

(ii) *the dual and twice dual function satisfy*

$$\sum_{j=0}^{\infty} \tilde{p}_{ij}(t) = \sum_{j=0}^{\infty} \tilde{\tilde{p}}_{ij}(t) = e^{-\alpha t} \tag{4.12}$$

which is independent of $i \in E$, for $t \geqslant 0$;

(iii) *$\tilde{\tilde{Q}} = (\tilde{\tilde{q}}_{ij})$ can be denoted by Q,*

$$\tilde{\tilde{q}}_{ij} = \begin{cases} q_{i-1,i-1} & \text{for } i, j \geqslant 1, \\ -\alpha\delta_{0j} & \text{for } i = 0,\ j \in E, \\ d_{i-1} - \alpha & \text{for } i \geqslant 1,\ j = 0. \end{cases} \tag{4.13}$$

Proof. (i) and (ii). By the proof of Lemma 4.4, we have

$$\sum_{j=0}^{\infty} \tilde{p}_{ij}(t) = \lim_{k\to\infty} \sum_{j=0}^{\infty} p_{kj}(t) = c(t) \tag{4.14}$$

which is independent of $i \in E$ for $t \geqslant 0$. We claim that $c(t)$ satisfies:

(a) $c(t)$ is continuously differentiable for $t \geqslant 0$, with $c(0) = 1$ and $c'(0) = -\alpha \leqslant 0$; and
(b) $c(t+s) = c(t)c(s)$.

Indeed, for $i \in E$, the ith deficit function is

$$\tilde{d}_i(t) = 1 - \sum_{j=0}^{\infty} \tilde{p}_{ij}(t) = 1 - c(t). \tag{4.15}$$

By Anderson [1], $\tilde{d}_i(t)$ is continuously differentiable for $t \geqslant 0$ and

$$\frac{d}{dt}\tilde{d}_i(t)\Big|_{t=0} = \tilde{d}_i \quad \text{for } i \in E, \tag{4.16}$$

where (4.16) valid since $\tilde{P}(t)$ satisfy the backward equation. Now conclusion (a) follows from (4.15) and (4.16). By (4.14), we calculate as follows:

$$c(t+s)=\sum_{j=0}^{\infty}\tilde{p}_{ij}(t+s)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\tilde{p}_{ik}(t)\tilde{p}_{kj}(s)$$
$$=\sum_{k=0}^{\infty}\tilde{p}_{ik}(t)\sum_{j=0}^{\infty}\tilde{p}_{kj}(s)=c(s)\sum_{k=0}^{\infty}\tilde{p}_{ik}(t)=c(s)c(t),$$

which proves (b). It is easy from (a) and (b) to get

$$c(t)=e^{-\alpha t}. \tag{4.17}$$

Thus it follows from (4.14)–(4.17) that

$$\tilde{d}_i=\alpha\geqslant 0 \quad \text{and} \quad \sum_{j=0}^{\infty}\tilde{p}_{ij}(t)=e^{-\alpha t}$$

which is independent of $i\in E$ for $t\geqslant 0$. Since $\tilde{P}(t)$ is also monotone and FRR, it follows from (4.14) that

$$\sum_{j=0}^{\infty}\tilde{\tilde{p}}_{ij}(t)=\lim_{k\to\infty}\sum_{j=0}^{\infty}\tilde{p}_{kj}(t)=c(t)=e^{-\alpha t}$$

and

$$\tilde{\tilde{d}}_i=\frac{d}{dt}\left(1-\sum_{j=0}^{\infty}\tilde{\tilde{p}}_{ij}(t)\right)\bigg|_{t=0}=\frac{d}{dt}\left(1-e^{-\alpha t}\right)\big|_{t=0}=\alpha$$

which completes the proof of (4.11) and (4.12).

(iii) Using (1.1) and (4.12),we calculate

$$\tilde{\tilde{p}}_{ij}(t)=\sum_{k=i}^{\infty}\tilde{p}_{jk}(t)-\sum_{k=i}^{\infty}\tilde{p}_{j-1,k}(t)=e^{-\alpha t}-\sum_{k=0}^{i-1}\tilde{p}_{jk}(t)-\left(e^{-\alpha t}-\sum_{k=0}^{i-1}\tilde{p}_{j-1,k}(t)\right)$$
$$=\sum_{k=j-1}^{\infty}p_{i-1,k}(t)-\sum_{k=j}^{\infty}p_{i-1,k}(t)=p_{i-1,j-1}(t),$$

for $i,j\geqslant 1$. Differentiating above equality on two side at $t=0$, we get

$$\tilde{\tilde{q}}_{ij}=q_{i-1,j-1} \quad \text{for } i\geqslant 1,\ j\geqslant 1,$$

which proved (4.13) for the case of $i,j\geqslant 1$. Similarly, (4.13) holds for the case of $i=0$ or $j=0$. □

Lemma 4.6. *Let $P(t)$ be a monotone and FRR q-function with the q-matrix $Q=(q_{ij})$. Then Q is FRR and zero-entrance in l_1.*

Proof. Let $\tilde{P}(t)$, $\tilde{\tilde{P}}(t)$ be the dual and twice dual of $P(t)$ with q-matrices $\tilde{Q}$ and $\tilde{\tilde{Q}}$, respectively. It follows from Lemma 4.5(ii) that

$$\inf_i\sum_{j=0}^{\infty}\tilde{\tilde{p}}_{ij}(t)=e^{-\alpha t}>0, \tag{4.18}$$

which implies, by [1] or [11], that the condition (ii)(b) in Theorem 3.2 does not hold. Thus by Theorem 3.2 (where $\tilde{\tilde{Q}}$ instead of Q), we obtain that $\tilde{\tilde{Q}}$ must be FRR and zero-entrance in l_1, which implies, by Lemma 4.5(iii), that $q_{ij} = \tilde{\tilde{q}}_{i+1,j+1} \to 0$ as $i \to \infty$ for every $j \in E$. Namely, Q is FRR.

To prove that Q is zero-entrance in l_1, we suppose $y = (y_k) \in l_1$ satisfy $y(\lambda I - Q) = 0$ for some $\lambda > 0$. We show that $y = 0$. Indeed, define $z = (z_k) \in l_1$ by

$$z_k = y_{k-1} \quad \text{for } k \geqslant 1, \quad \text{and} \quad z_0 = \frac{1}{\lambda + \tilde{\tilde{q}}_0} \sum_{k=1}^{\infty} z_k \tilde{\tilde{q}}_{k0}. \tag{4.19}$$

Noting that $\tilde{\tilde{q}}_{k0} \geqslant \tilde{\tilde{q}}_{k+1,0}$ for $k \geqslant 1$ (because $\tilde{\tilde{Q}}$ is dual), we obtain that

$$|z_0| \leqslant \frac{\tilde{\tilde{q}}_{1,0}}{\lambda + \tilde{\tilde{q}}_0} \|y\|_1 < +\infty,$$

which means z_0 is well defined and $z = (z_k) \in l_1$. Since $\sum_k y_k(\lambda\delta_{kj} - q_{kj}) = 0$ for $j \in E$, it follows from Lemma 4.5(iii) and (4.19) that, for $j \geqslant 1$,

$$\begin{aligned}
\sum_{k=0}^{\infty} z_k\big(\lambda\delta_{kj} - \tilde{\tilde{q}}_{kj}\big) &= z_0(\lambda + \alpha)\delta_{0j} + \sum_{k=1}^{\infty} y_{k-1}(\lambda\delta_{kj} - q_{k-1,j-1}) \\
&= \sum_{k=0}^{\infty} y_k(\lambda\delta_{k+1,j} - q_{k,j-1}) = \sum_{k=0}^{\infty} y_k(\lambda\delta_{k,j-1} - q_{k,j-1}) = 0
\end{aligned}$$

and

$$\sum_{k=0}^{\infty} z_k\big(\lambda\delta_{k0} - \tilde{\tilde{q}}_{k0}\big) = z_0\big(\lambda + \tilde{\tilde{q}}_0\big) - \sum_{k=1}^{\infty} z_k \tilde{\tilde{q}}_{k0} = 0.$$

Thus we have proved that $z(\lambda - \tilde{\tilde{Q}}) = 0$, which implies, by the zero-entrance of $\tilde{\tilde{Q}}$ in l_1, that $z = 0$. This, together with (4.19), implies that $y = 0$, and thus Q is zero-entrance in l_1. $\square$

Now we can prove Theorem 4.3 by using above lemmas.

Proof of Theorem 4.3. *Necessity*. Assume that the minimal Q-function $F(t)$ is FRR. If condition (ii) does not hold, then Q is zero-exit. Since Q is also monotone, it follows from [11, Theorem 3.1] that $F(t)$ is monotone. Therefore it follows from Lemma 4.6 that Q is FRR and zero-entrance in l_1.

Sufficiency. If condition (i) holds, namely, Q is FRR and zero-entrance in l_1, then, by Reuter and Riley's result [9, Theorem 8] $F(t)$ is FRR.

Assume condition (ii) hold, namely, Q is nonzero-exit. Add a state $\Delta \notin E$ to form E_Δ with order relation: $\Delta < 0 < 1 < \cdots$, and define a q-matrix ${}_\Delta Q = ({}_\Delta q_{ij})$ on E_Δ by

$${}_\Delta q_{ij} = \begin{cases} q_{ij}, & i, j \in E, \\ d_i, & i \in E,\ j = \Delta, \\ 0, & i = \Delta,\ j \in E_\Delta, \end{cases}$$

where d_i is the nonconservative quantity of Q. Then ${}_\Delta Q$ is monotone (it is easy to verify that ${}_\Delta Q$ is monotone if and only if Q is) and conservative. Thus ${}_\Delta Q$ is dual and is nonzero-exit (in fact, ${}_\Delta Q$ is zero-exit if and only if Q is). Thus it follows from Corollary 4.2 that the minimal ${}_\Delta Q$-function ${}_\Delta F(t)$ is FRR. But Δ is an absorbing state for ${}_\Delta Q$ and thus

$$ {}_\Delta f_{ij}(t) = f_{ij}(t) \quad \text{for } i, j \in E. $$

Therefore $F(t)$ is FRR. □

5. Questions and examples

The condition that Q is zero-entrance in l_1 in our result is important. Can it be instead of the condition that Q is zero-entrance in l_1^+? That is, the following question remains open.

Question 5.1. Are our main result in Sections 3, 4 (i.e. Theorems 3.2, 4.1, 4.3) true if l_1^+ instead of l_1?

If Q satisfies (2.2), that is, if Q is a downward skip-free Q-matrix, (which contains the birth–death matrix and Markov branching matrix), then, by Proposition 2.2, the answer is affirmative. For wider case, Question 5.1 remains open.

Now we use two examples to illustrate our results.

Example 5.2 (birth–death process). Let $Q = (q_{ij})$ be a birth–death q-matrix, that is

$$ q_{ij} = \begin{cases} \lambda_i & \text{if } j = i+1,\ i \geqslant 0, \\ \mu_i & \text{if } j = i-1,\ i \geqslant 1, \\ -(\lambda_i + \mu_i) & \text{if } j = i,\ i \geqslant 0, \\ 0 & \text{otherwise,} \end{cases} $$

where $\lambda_i, \mu_i \geqslant 0$. Then Q is monotone, and if $\mu_0 = 0$, then Q is dual (see [11]). Applying our result (Theorems 3.2, 4.1, 4.3) and noting Proposition 2.2, we get the following.

Proposition 5.3. *Let Q be a birth–death matrix and $F(t)$ the minimal Q-function. Then*

(i) *$F(t)$ is a dual function if and only if $S = \infty$ or $R < \infty$; and $\mu_0 = 0$;*
(ii) *$F(t)$ is FRR if and only if $S = \infty$ or $R < \infty$,*

where

$$ S = \sum_{n=1}^{\infty} \frac{1}{\mu_{n+1}} \left(1 + \frac{\lambda_n}{\mu_n} + \frac{\lambda_n \lambda_{n-1}}{\mu_n \mu_{n-1}} + \cdots + \frac{\lambda_n \ldots \lambda_2 \lambda_1}{\mu_n \ldots \mu_2 \mu_1} \right), $$

$$ R = \sum_{n=1}^{\infty} \left(\frac{1}{\lambda_n} + \frac{u_n}{\lambda_n \lambda_{n-1}} + \cdots + \frac{\mu_n \ldots \mu_2}{\lambda_n \ldots \lambda_2 \lambda_1} \right). $$

Remark. Above result (for the case of that $\mu_0 = 0$) are also obtained by [11]. But without our Proposition 2.2, their proofs are incomplete.

Example 5.4 (branching process). Recall that a branching q-matrix Q is defined by

$$q_{ij} = \begin{cases} ib_{j-i+1} & \text{if } j \geqslant i-1, \\ 0 & \text{otherwise,} \end{cases}$$

where $b_k \geqslant 0$ ($k \neq 1$) and $\sum_{k=0}^{\infty} b_k = 0$. Q is conservative, FRR, monotone and thus dual. Further, Q is always zero-entrance in l_1^+ (see [1]), and thus, by Proposition 2.2, Q is zero-entrance in l_1. But Q is not always zero-exit (regular), the regular criteria can be seen from [1]. Applying our result we obtain the following.

Proposition 5.5. *Let $F(t)$ be the minimal branching Q-function. Then*

(i) *$F(t)$ is always FRR*;
(ii) *$F(t)$ is always a dual Q-function.*

Note that $F(t)$ must be a dual of some monotone $Q^{(1)}$-function $P^{(1)}(t)$ (here we might call $P^{(1)}(t)$ to be pre-dual of $F(t)$), we can calculate to get

$$q_{ij}^{(1)} = \begin{cases} 0 & \text{if } i < j-1, \\ jb_{i-j+1} - \sum_{m=0}^{i-j} b_m & \text{if } i \geqslant j-1. \end{cases}$$

Thus $Q^{(1)}$ is a upwardly skip-free q-matrix; that is, the branching process $F(t)$ must be a dual of some upwardly skip-free process $P^{(1)}(t)$. Although $F(t)$ is FRR, $P^{(1)}(t)$ is not necessarily FRR. In fact, we have

Proposition 5.6. *$P^{(1)}(t)$ is FRR if and only if Q is regular.*

Proof. By Lemma 4.4, $P^{(1)}(t)$ is FRR if and only if its dual $P^{(1)}(t)$ is monotone. This is equivalent to that Q is zero-exit and thus regular. □

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References

[1] W.J. Anderson, Continuous Time Markov Chains, Springer-Verlag, New York, 1991.
[2] W.J. Anderson, Some remarks on the duality of continuous time Markov chains, Comm. Statist. Theory Methods 18 (12) (1989) 4533–4538.
[3] A.Y. Chen, H.J. Zhang, Existence, uniqueness, and constructions for stochastically monotone q-process, Southeast Asian Bull. Math. 23 (1999) 559–583.
[4] K.L. Chung, Markov Chains with Stationary Transition Probabilities, Springer-Verlag, Berlin, 1967.
[5] D.J. Daley, Stochastically monotone Markov chains, Z. Wahrsch. Verw. Geb. 10 (1983) 305–317.
[6] Z.T. Hou, F. Guo, Time-Homogeneous Markov Processes with Countable State Space, Springer-Verlag, Berlin, 1988.

[7] B.M. Kirstein, Monotonicity and comparability of time-homogeneous Markov processes with discrete state space, Math. Operationsforsch. Statist. 7 (1976) 151–168.
[8] Y.R. Li, Contraction integrated semigroups and their applications to continuous-time Markov chains, Acta Math. Sinica (English Ser.) 19 (2003) 605–618.
[9] G.H.E. Reuter, P.W. Riley, The Feller property for Markov semigroups on a countable state space, J. London Math. Soc. 5 (1972) 267–275.
[10] D. Siegmund, The equivalence of absorbing and reflecting barrier problems for stochastically monotone Markov process, Ann. Probab. 4 (1976) 914–924.
[11] H.J. Zhang, A.Y. Chen, Stochastic comparability and dual Q-functions, J. Math. Anal. Appl. 234 (1999) 482–499.