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Well-posedness and stability for abstract spline problems

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Abstract

In this work well-posedness and stability properties of the abstract spline problem are studied in the framework of reflexive spaces. Tykhonov well-posedness is proved without restrictive assumptions. In the context of Hilbert spaces, also the stronger notion of Levitin–Polyak well-posedness is established. A sequence of parametric problems converging to the given abstract spline problem is considered in order to study stability. Under natural assumptions, convergence results for sequences of solutions of the perturbed problems are obtained.

Keywords: Abstract splines; Well-posedness; Set-convergence; Stability

1. Introduction

Spline functions play a fundamental role in many fields of numerical analysis and statistics and they are involved in many important applications, for instance in engineering, economics, biology and medicine.

The variational approach to interpolation techniques has been fruitfully employed to obtain existence and uniqueness results for interpolating spline functions (for a comprehensive exposition see, e.g., [4] and the references therein). An abstract unifying framework for this topic

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is classically developed in the setting of Hilbert spaces [2]. Here, following the approach introduced in [7], we consider the problem in the more general setting of reflexive spaces. The aim of this work is to study well-posedness and stability properties of the abstract spline problem.

The property of well-posedness of an optimization problem essentially requires that the minimizing sequences are well-behaved. This is a relevant feature in order to individuate those problems that can be studied through direct methods and to develop efficient numerical procedures. Here, we show that the same assumptions that usually ensure existence and uniqueness of a solution for an abstract spline problem also imply the stronger property of well-posedness.

The study of the stability properties of some special parametric families of abstract spline problems has already been developed, e.g., in [3,5,7,12,13]. We use some variational techniques to obtain stability results for a general version of the abstract spline problem. These results are refined in the special case where the admissible region is an affine set and, in particular, it is determined by a set of evaluation functionals, as in the classical interpolation problem.

The structure of the paper is the following. Section 2 is devoted to introduce some notations and to recall some known results used in the sequel. In Section 3, we study the well-posedness of the abstract spline problem both in the sense of Tykhonov and of Levitin–Polyak. In Section 4, we prove some technical results about the convergence of the images of a sequence of linear operators. Finally, in Section 5, we use the results of the previous section to study the convergence properties of the solutions of a sequence of perturbed abstract spline problems.

2. Notations and preliminaries

Let X, Y be Banach spaces, endowed respectively with the norms $\|.\|_X$ and $\|.\|_Y$. By X^* and Y^* we denote the topological dual spaces of X and Y endowed with their corresponding dual norms $\|.\|_{X^*}$ and $\|.\|_{Y^*}$. We denote by $\langle f, x\rangle$ the value $f(x)$ of the functional $f \in X^*$ at $x \in X$. Let $\{x_n\} \subset X$ be a sequence, we denote by $x_n \to x$ or $x = \lim_{n\to\infty} x_n$ the convergence of $\{x_n\}$ to x with respect to the norm-topology and by $x_n \rightharpoonup x$ or $x = \text{w-}\lim_{n\to\infty} x_n$ the convergence of $\{x_n\}$ to x with respect to the weak topology. We denote by $\mathcal{L}(X, Y)$ the space of the bounded linear operators between X and Y endowed with the usual norm of the operators $\|.\|_{\mathcal{L}(X,Y)}$.

In this paper some geometrical properties of the spaces under consideration play a crucial role. We recall that a Banach space is *strictly convex* if for each $x_1, x_2 \in X$, $\|x_1\|_X = \|x_2\|_X = 1$, $x_1 \neq x_2$ and for each $0 < t < 1$, we have

$$\big\|tx_1 + (1-t)x_2\big\|_X < 1.$$

Moreover X has the *Kadeč–Klee property* when $x_n \rightharpoonup x$ and $\|x_n\|_X \to \|x\|_X$ imply $x_n \to x$.

A Banach space X is said to be an E-space if it is reflexive, strictly convex and has the Kadeč–Klee property.

The notion of E-space has an important role in the study of the convex best approximation problem, i.e. of problem

$$\min_{c\in C} \|x_0 - c\|_X \tag{BA}$$

where $x_0 \in X$ is a fixed point and $C \subset X$ is a closed convex set. We denote by $S(C|x_0)$ the set of the solutions of the problem (BA). When the best approximation problem (BA) has a unique solution, i.e. when $S(C|x_0)$ is a singleton, we set $S(C|x_0) = s_C(x_0)$. We recall that a solution of problem (BA) exists whenever X is a reflexive space; moreover, if X is also strictly convex, problem (BA) has a unique solution.

One of the aims of this paper is to study the properties of well-posedness of the abstract spline problem.

Here we introduce the notion of well-posedness for the general optimization problem

$$\min_{x\in A} f(x) \tag{P}$$

where A is a subset of X and $f : X \to (-\infty, \infty]$ is a lower semicontinuous function.

Problem (P) is said to be Tykhonov well-posed when

1. there exists a unique $\bar{x} \in A$ such that $\bar{x}$ is a solution of problem (P),
2. every sequence $\{x_n\} \subset A$ such that $f(x_n) \to \inf_{x\in A} f(x)$ is such that $x_n \to \bar{x}$.

In the case of constrained problems, it is also interesting to consider minimizing sequences that are not necessarily included in the admissible region. Hence we consider the notion of Levitin–Polyak well-posedness.

Problem (P) is said to be Levitin–Polyak well-posed when

1. there exists a unique $\bar{x} \in A$ such that $\bar{x}$ is a solution of problem (P),
2. every sequence $\{x_n\} \subset X$ such that $d(x_n, A) \to 0$ and $f(x_n) \to \inf_{x\in A} f(x)$ is such that $x_n \to \bar{x}$.

For an extensive exposition on this topic, see e.g. the monographs [6,9]. There is a strong relationship between the well-posedness of the convex best approximation problem and the structure of the space. Indeed, the E-spaces are characterized through the well-posedness of every convex best approximation problem.

Theorem 2.1. *(See [9].) Let X be a Banach space. X is an E-space if and only if for every $x_0 \in X$ and for every closed convex set $C \subset X$, the problem* (BA) *is Tykhonov well-posed.*

In order to study the stability of the abstract spline problem we use the notion of set-convergence introduced by U. Mosco in [11].

Given a sequence of sets $\{A_n\}$ in X, we say that A_n converges to A in the sense of Mosco ($A_n \xrightarrow{M} A$) when

$$\text{w-Ls}\, A_n \subset A \subset \text{Li}\, A_n$$

where

$$\text{Li}\, A_n = \left\{x \in X\colon\ x = \lim_{n\to\infty} x_n,\ x_n \in A_n,\ \text{for all large } n\right\},$$

$$\text{w-Ls}\, A_n = \left\{x \in X\colon\ x = \text{w-}\lim_{s\to\infty} x_s,\ x_s \in A_{n_s},\ \{n_s\} \text{ subsequence of } \{n\}\right\}.$$

We remark that the limit set of a Mosco converging sequence of sets is closed. Moreover, the limit of a sequence of convex sets is convex.

Finally, we quote a result (see, e.g., [9]) concerning the convergence of the solutions of a sequence of convex best approximation problems.

Theorem 2.2. *Let X be an E-space and $C_n, C \subset X$ be closed convex sets. If $C_n \xrightarrow{M} C$ then $s_{C_n}(x) \to s_C(x)$ for every $x \in X$.*

3. Well-posedness of abstract spline problems

This section is devoted to the study of the well-posedness properties of the abstract spline problem.

Let X, Y be Banach spaces and let $R \in \mathcal{L}(X, Y)$.

We consider the following optimization problem

$$\min_{x \in K} \left\| R(x) \right\|_Y \tag{ASP}$$

where K is a nonempty closed convex subset of X. This problem is known as the *abstract spline problem* (see, e.g., [8]).

The following result introduces some sufficient conditions for the Tykhonov well-posedness of the abstract spline problem.

In the statement of the theorem, we denote the affine hull of a set $K \subset X$ by $\operatorname{Aff} K$.

Theorem 3.1. *Let X be a Banach space and Y be an E-space. Let $R \in \mathcal{L}(X, Y)$. If*

1. *R has closed range,*
2. *$R(K)$ is closed in Y,*
3. $\operatorname{Ker} R \cap V = \{0\}$*, where V is a closed subspace of X such that*

   $$\operatorname{cl}(\operatorname{Aff} K) = k + V$$

   with $k \in K$,
4. *$R(V)$ is closed in Y,*

then the problem (ASP) *is Tykhonov well-posed.*

Proof. The existence of a solution for problem (ASP) follows immediately from the reflexivity of Y and from the convexity and the closedness of $R(K)$. The strict convexity of Y implies also that the solution $s_{R(K)}(0)$ of the problem

$$\min_{y \in R(K)} \|y\|_Y \tag{1}$$

is unique and we denote it by y_0. The solutions of problem (ASP) are the elements of the set $R^{-1}(y_0)$. By assumption 3. it is easy to see that $R^{-1}(y_0)$ is a singleton. We denote the unique solution of problem (ASP) by x_0. Since Y is an E-space, by Theorem 2.1, problem (1) is Tykhonov well-posed in Y.

Let $\{x_n\} \subset K$ be a minimizing sequence for problem (ASP), i.e.

$$\left\| R(x_n) \right\|_Y \to \left\| R(x_0) \right\|_Y.$$

Since the sequence $\{R(x_n)\} \subset R(K)$ is a minimizing sequence for problem (1), we have $R(x_n) \to R(x_0)$, hence $R(x_n - x_0) \to 0$, where $x_n - x_0 \in V$, for every n.

By assumption 4., the restriction of R to the subspace V has closed range. Moreover, assumption 3. implies that R is injective on V. Hence, Theorem 2.5 in [1] ensures that the restriction of R to V is a bounded below operator, i.e. there exists a real number $\alpha > 0$ such that

$$\left\| R(v) \right\|_Y \geqslant \alpha \|v\|_X \quad \text{for every } v \in V.$$

Therefore, from $R(x_n - x_0) \to 0$, we obtain $x_n \to x_0$. $\square$

In the special case where the kernel of the spline operator R is finite dimensional, which is commonly verified in a number of concrete spline problems, the assumptions of the previous theorem can be considerably simplified.

Remark 3.2. If $\operatorname{Ker} R$ is a finite dimensional subspace of X, the assumptions 2. and 4. in the previous theorem can be omitted. Indeed, we can observe that $K_\infty \subset V$, where $K_\infty = \{x \in X: k + tx \in K,\ \forall t \geqslant 0,\ \forall k \in K\}$. Hence, by assumption 3., $K_\infty \cap \operatorname{Ker} R = \{0\}$. From the local compactness of $\operatorname{Ker} R$, we obtain that $K + \operatorname{Ker} R$ is closed in X (see Lemma in 15D [8]). Since the closedness of $R(K)$ in Y is equivalent to the closedness of $K + \operatorname{Ker} R$ in X (see Lemma in 17H [8]), assumption 2. holds. Moreover, by the same argument, we can conclude that $R(V)$ is also closed.

In the classical formulation of abstract spline problem [2] the set K is an affine subspace of X, determined as the counterimage of a fixed element z_0 in a normed space Z through a continuous linear operator L. Hence we consider also the following problem

$$\min_{x \in L^{-1}(z_0)} \left\| R(x) \right\|_Y \tag{LSP}$$

where $L \in \mathcal{L}(X, Z)$ and $z_0 \in Z$ fixed. In this case the previous theorem can be restated as follows.

Corollary 3.3. *Let X be Banach space, Y an E-space and Z a normed space. Let $R \in \mathcal{L}(X, Y)$ and $L \in \mathcal{L}(X, Z)$. If*

1. *R has closed range and L is surjective,*
2. *$R(\operatorname{Ker} L)$ is closed in Y,*
3. *$\operatorname{Ker} R \cap \operatorname{Ker} L = \{0\}$,*

then problem (LSP) *is Tykhonov well-posed.*

Proof. The proof follows directly from Theorem 3.1 with $K = L^{-1}(z_0) = x_0 + \operatorname{Ker} L$ and $V = \operatorname{Ker} L$ where x_0 is the solution of (LSP). □

We conclude this section with a result on the stronger notion of Levitin–Polyak well-posedness, that is especially fit for constrained problems. Since a key tool in our proof is the Projection Theorem, we restrict to the classical setting where X is a Hilbert space.

Theorem 3.4. *Let X be a Hilbert space, Y be an E-space and Z be a normed space. Let $R \in \mathcal{L}(X, Y)$ and $L \in \mathcal{L}(X, Z)$. If*

1. *R has closed range and L is surjective,*
2. *$R(\operatorname{Ker} L)$ is closed in Y,*
3. *$\operatorname{Ker} R \cap \operatorname{Ker} L = \{0\}$,*

then the problem (LSP) *is Levitin–Polyak well-posed.*

Proof. Following the same arguments of the proof of Theorem 3.1, we can show that problem (LSP) has a unique solution $x_0 \in L^{-1}(z)$. Let $\{x_n\} \subset X$ be an LP-minimizing sequence, i.e. $\{x_n\}$ is such that

$$\left\| R(x_n) \right\|_Y \to \left\| R(x_0) \right\|_Y \tag{2}$$

and

$$\inf_{a \in L^{-1}(z)} \|x_n - a\|_X \to 0. \tag{3}$$

By the classical Projection Theorem in the Hilbert spaces, we have

$$x_n = s_{\operatorname{Ker} L}(x_n) + s_{(\operatorname{Ker} L)^\perp}(x_n) \quad \text{for every } n,$$

where $(\operatorname{Ker} L)^\perp = \{x \in X\colon \langle x, v\rangle = 0 \text{ for every } v \in \operatorname{Ker} L\}$. Hence we have

$$x_n = s_{(\operatorname{Ker} L)^\perp}(x_0) + s_{\operatorname{Ker} L}(x_n) + s_{(\operatorname{Ker} L)^\perp}(x_n - x_0).$$

Now, we can easily observe that

$$s_{(x_0+\operatorname{Ker} L)}(x_n) = s_{\operatorname{Ker} L}(x_n) + s_{(\operatorname{Ker} L)^\perp}(x_0) \quad \text{for every } n.$$

Therefore, we obtain

$$x_n = s_{(x_0+\operatorname{Ker} L)}(x_n) + s_{(\operatorname{Ker} L)^\perp}(x_n - x_0). \tag{4}$$

Since $L^{-1}(z) = x_0 + \operatorname{Ker} L$, we can rewrite relation (3) as

$$\left\| x_n - s_{(x_0+\operatorname{Ker} L)}(x_n) \right\|_X \to 0.$$

Hence we obtain that

$$s_{(\operatorname{Ker} L)^\perp}(x_n - x_0) \to 0. \tag{5}$$

Now we can prove that $\{s_{(x_0+\operatorname{Ker} L)}(x_n)\}$ is a minimizing sequence in the sense of Tykhonov for problem (LSP). Indeed it holds that

$$R\big(s_{(x_0+\operatorname{Ker} L)}(x_n)\big) = R(x_n) - R\big(s_{(\operatorname{Ker} L)^\perp}(x_n - x_0)\big) \quad \text{for every } n.$$

Hence, recalling that x_0 is the solution of the problem, we have

$$\left\| R(x_0) \right\|_Y \leqslant \left\| R\big(s_{(x_0+\operatorname{Ker} L)}(x_n)\big) \right\|_Y \leqslant \left\| R(x_n) \right\|_Y + \left\| R\big(s_{(\operatorname{Ker} L)^\perp}(x_n - x_0)\big) \right\|_Y.$$

Therefore, by (2) and (5), we obtain that

$$\left\| R\big(s_{(x_0+\operatorname{Ker} L)}(x_n)\big) \right\|_Y \to \left\| R(x_0) \right\|_Y.$$

By Corollary 3.3, we can conclude that $s_{(x_0+\operatorname{Ker} L)}(x_n) \to x_0$. Finally, by (4) and (5) we obtain $x_n \to x_0$. $\square$

We underline that the assumptions of Corollary 3.3 and Theorem 3.4 coincide with the conditions that guarantee the existence and the uniqueness of the solution of problem (LSP) (see [2]). Here, without additional hypotheses, we obtain the stronger property of well-posedness.

4. A result on linear operators

This section is devoted to prove a technical result concerning the convergence of the image of a converging sequence of convex sets through a converging sequence of continuous linear operators. More precisely, we state and prove a new sufficient condition to ensure the convergence of the mentioned sequence of sets. This condition seems to be more suitable to study the stability properties of the abstract spline problem than others known in the literature (see, e.g., [14]), where the injectivity of the operator is required.

For the convenience of the reader, we recall here two notions that will be used in the sequel. Let $A \subset X$ be a convex set, the *recession cone* A_∞ is defined as follows:

$$A_\infty = \{x \in X \colon\ a + tx \in A,\ \forall t \geqslant 0,\ \forall a \in A\}.$$

A closed subspace $W \subset X$ is *complementable* whenever there exists a closed subspace W such that $X = W \oplus T$. It is well known that every finite dimensional subspace of X is complementable.

Lemma 4.1. *Let X, Y be Banach spaces. Let us suppose that*

1. *$T, T_n \in \mathcal{L}(X, Y)$ such that $\|T_n - T\|_{\mathcal{L}(X,Y)} \to 0$;*
2. *T is a closed range operator with a finite dimensional kernel*;
3. *$A_n \subset X$ are closed and convex sets such that $A_n \xrightarrow{M} A$;*
4. $\operatorname{Ker} T \cap (A)_\infty = \{0\}$.

If $\{x_n\}$ is a sequence such that $x_n \in A_n$ and $\{T_n(x_n)\} \subset Y$ is a bounded sequence then $\{x_n\}$ is a bounded sequence.

Proof. Let $\{x_n\}$ be a sequence such that $x_n \in A_n$ and $\{T_n(x_n)\} \subset Y$ is a bounded sequence. Since $\operatorname{Ker} T$ is a finite dimensional subspace, there exists a closed subspace V of X such that $X = \operatorname{Ker} T \oplus V$. Hence, there exist two sequences $\{v_n\} \subset V$ and $\{u_n\} \subset \operatorname{Ker} T$ such that $x_n = v_n + u_n$, for every integer n.

Since $T_n(u) \to T(u) = 0$ for every $u \in \operatorname{Ker} T$, we have that $T_n(u_n) \to 0$. From the boundedness of $\{T_n(x_n)\}$, it follows that also $\{T_n(v_n)\}$ is a bounded sequence.

By Theorem 2.5 in [1], there exists a positive real number α such that

$$\|T(v)\|_Y \geqslant 2\alpha \|v\|_X \quad \text{for every } v \in V.$$

Hence, the following inequalities hold

$$\|T_n(v)\|_Y \geqslant \|T(v)\|_Y - \|T(v) - T_n(v)\|_Y \geqslant 2\alpha\|v\|_X - \|T - T_n\|_{\mathcal{L}(X,Y)}\|v\|_X$$

for every $v \in V$. By assumption 1., there exists $n_0 \in \mathbb{N}$ such that for every integer $n \geqslant n$ it holds:

$$\|T_n(v)\|_Y \geqslant \alpha\|v\|_X \quad \text{for every } v \in V. \tag{6}$$

The inequality (6) and the boundedness of $\{T_n(v_n)\}$ imply that $\{v_n\}$ is a bounded sequence.

Now we show that $\{u_n\}$ is also a bounded sequence. Indeed, by contradiction, we suppose that $\|u_n\| \to +\infty$. Since $\dim \operatorname{Ker} T < \infty$, the sequence $\{\frac{u_n}{\|u_n\|}\} \subset \operatorname{Ker} T$ converges (up to a subsequence) to an element $u \in \operatorname{Ker} T$, $u \neq 0$. Moreover, we can show that $u \in (A)_\infty$, i.e. $a + \lambda u \in A$ for every $a \in A$ and $\lambda > 0$. Indeed, we can always find a sequence $\{a_n\}$ such that $a_n \in A_n$ and $a_n \to a$. By the convexity of the sets A_n,

$$a_n + \alpha(x_n - a_n) \in A_n \quad \text{for every } \alpha \in [0, 1].$$

Choosing $\alpha = \frac{\lambda}{\|u_n\|}$ (for n large enough), it holds

$$a_n + \lambda\left(\frac{x_n - a_n}{\|u_n\|}\right) \in A_n.$$

Since

$$a_n + \lambda\left(\frac{x_n - a_n}{\|u_n\|}\right) = a_n + \lambda\left(\frac{v_n - a_n}{\|u_n\|} + \frac{u_n}{\|u_n\|}\right) \to a + \lambda u \in A,$$

we have $u \in \operatorname{Ker} T \cap (A)_\infty$, against assumption 4. The thesis follows immediately, since $x_n = v_n + u_n$. □

In [14] the same thesis is proved under a different assumption (named condition (H5)) that implies the injectivity of the operators T_n, eventually. This condition is not satisfied in most of the families of spline problems existing in the literature. Even in the classical problem of the determination of the natural cubic spline, where the spline operator R is the second derivative, the assumption of injectivity is not satisfied.

Now we can prove a result concerning the Mosco convergence of the images of a sequence of linear operators. A finite dimensional version of the same result can be found in the proof of Theorem 4.1 in [10], in a completely different context. Once the thesis of Lemma 4.1 is obtained, the proof of this theorem can follow the lines of the proof of Theorem 3.4 in [14]. Here we give an explicit proof for the convenience of the reader.

Theorem 4.2. *Let X be a reflexive Banach space and Y be a Banach space. Let us suppose that*:

1. *T, $T_n \in T(X,Y)$ such that $T_n \xrightarrow{\mathcal{L}(X,Y)} T$*;
2. *T is a closed range operator with a finite dimensional kernel*;
3. *$A_n \subset X$ are closed and convex sets such that $A_n \xrightarrow{M} A$*;
4. $\operatorname{Ker} T \cap (A)_\infty = \{0\}$;

then $T_n(A_n) \xrightarrow{M} T(A)$.

Proof. First we prove the upper part of the Mosco convergence of $T_n(A_n)$ to $T(A)$. By contradiction, there exists a sequence $\{y_k\}$ such that $y_k \in T_{n_k}(A_{n_k})$ and $y_k \rightharpoonup y \notin T(A)$. Now let $x_k \in A_{n_k}$ such that $T_{n_k}(x_k) = y_k$.

By Lemma 4.1 we can conclude that $\{x_k\}$ is a bounded sequence, hence there exists a subsequence $\{x_{k_s}\}$ weakly converging to an element $x \in A$. Assumption 1. implies that

$$T_{n_{k_s}}(x_{k_s}) \rightharpoonup T(x) \in T(A)$$

(see, e.g., [14]). Since $T(x) = y$, we have a contradiction.

Now we prove the lower part of the Mosco convergence of $T_n(A_n)$ to $T(A)$. Let $y \in T(A)$, then there exists $x \in A$ such that $T(x) = y$. By assumption 3. there exists a sequence $\{x_n\}$ convergent to x where $x_n \in A_n$ for every n. Then

$$T_n(x_n) \to T(x). \qquad \square$$

We conclude this section with a result on the special case where the sequence of operators $\{T_n\}$ reduces to a constant sequence.

Proposition 4.3. *Let X, Y be Banach spaces, $T \in \mathcal{L}(X,Y)$ be a closed range operator with a finite dimensional kernel. Let $A_n \subset X$ be closed and convex sets such that $A_n \xrightarrow{M} A$ and $\operatorname{Ker} T \cap (A)_\infty = \{0\}$. If $\{x_n\}$ is a sequence such that $x_n \in A_n$ and there exists an element $\bar{y} \in Y$ such that $T(x_n) \to \bar{y}$, then $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to \bar{x} \in T^{-1}(\bar{y}) \cap A$.*

Proof. Let $\{x_n\}$ be a sequence such that $x_n \in A_n$ and $T(x_n) \to \bar{y}$. Since $\operatorname{Ker} T$ is a finite dimensional subspace, there exists a closed subspace V of X such that $X = \operatorname{Ker} T \oplus V$. Hence, there exist two sequences $\{v_n\} \subset V$ and $\{u_n\} \subset \operatorname{Ker} T$ such that $x_n = v_n + u_n$, for every integer n. It follows that $T(v_n) \to \bar{y}$. Since $T(X) = T(V)$ is a closed subspace of Y, there exists an element $\bar{v} \in V$ such that $T(\bar{v}) = \bar{y}$.

By Theorem 2.5 in [1], there exists a positive real number α such that

$$\|T(v)\|_Y \geqslant \alpha \|v\|_X$$

for every $v \in V$. Hence $\|T(v_n - \bar{v})\|_Y \geqslant \alpha \|v_n - \bar{v}\|_X$. Since $T(v_n) \to T(\bar{v})$ we have $v_n \to \bar{v}$.

Following the same arguments as in the proof of Lemma 4.1, we can prove that $\{u_n\}$ is a bounded sequence. Since $\operatorname{Ker} T$ is a finite dimensional subspace of X, we can extract a subsequence $\{u_{n_k}\}$ with $u_{n_k} \to \bar{u} \in \operatorname{Ker} T$. Now let $x_{n_k} = v_{n_k} + u_{n_k}$ and $\bar{x} = \bar{v} + \bar{u}$. Trivially, it holds $x_{n_k} \to \bar{x} \in T^{-1}(\bar{y}) \cap A$. $\square$

5. Stability of the abstract spline problem

In this section we apply the results obtained in Section 4 to the study of the convergence properties of the solutions of a sequence of perturbed abstract spline problems. It is remarkable that the assumptions on the limit problem that allow us to prove our results are the natural requirements to ensure the existence of a solution of the limit problem itself (see, e.g., [8, 21B]).

We begin to study the general formulation of the abstract spline problem (ASP). We obtain here a result on the weak convergence of a sequence of solutions of the perturbed problem to a solution of the original problem (ASP).

Theorem 5.1. *Let X be a reflexive space and Y be an E-space. Let R, $R_n \in \mathcal{L}(X, Y)$ such that $\|R_n - R\|_{\mathcal{L}(X,Y)} \to 0$ and R be a closed range operator with a finite dimensional kernel. Let $\{K_n\}$ be a sequence of nonempty closed convex subsets of X such that $K_n \xrightarrow{M} K$ and* $\operatorname{Ker} R \cap (K)_\infty = \{0\}$. *If the problem*

$$\min_{x \in K_n} \|R_n(x)\|_Y \tag{ASP$_n$}$$

has a solution $\hat{x}_n \in K_n$ for every n, then the sequence $\{\hat{x}_n\}$ admits a subsequence weakly converging to a solution of problem (ASP).

Proof. Let $\hat{x}_n$ be a solution of (ASP$_n$) problem. Clearly it is

$$R_n(\hat{x}_n) = s_{R_n(K_n)}(0) = s_{\mathrm{cl}(R_n(K_n))}(0) \quad \text{for every } n.$$

Now Theorem 4.2 implies that $R_n(K_n) \xrightarrow{M} R(K)$ or, equivalently,

$$\mathrm{cl}(R_n(K_n)) \xrightarrow{M} \mathrm{cl}(R(K)).$$

Hence $R_n(\hat{x}_n) \to s_{R(K)}(0)$ by Theorem 2.2. By Lemma 4.1, we show that $\{\hat{x}_n\}$ is a bounded sequence. Hence $\{\hat{x}_n\}$ admits a subsequence $\{\hat{x}_{n_s}\}$ such that $\hat{x}_{n_s} \rightharpoonup \hat{x}$. Finally, since the assumption $\|R_n - R\|_{\mathcal{L}(X,Y)} \to 0$ implies that $R_{n_s}(\hat{x}_{n_s}) \rightharpoonup R(\hat{x})$, it is easy to verify that

$$R(\hat{x}) = s_{R(K)}(0),$$

i.e. $\hat{x}$ is a solution of the problem (ASP). $\square$

Remark 5.2. If the problem (ASP) has a unique solution $\hat{x}$, we can easily guarantee the weak convergence of the whole sequence $\{\hat{x}_n\}$ to $\hat{x}$. Indeed, let us suppose, to the contrary, that there exists a subsequence of $\{\hat{x}_n\}$ that does not converge to $\hat{x}$. Let us denote by $\{\hat{x}_{n_s}\}$ the subsequence of $\{\hat{x}_n\}$ given by the union of all the subsequences of $\{\hat{x}_n\}$ that do not converge to $\hat{x}$. By

Lemma 4.1, $\{\hat{x}_{n_s}\}$ is a bounded sequence, hence it contains $\{\hat{x}_{n_{s_i}}\}$ such that $\hat{x}_{n_{s_i}} \rightharpoonup \bar{x} \neq \hat{x}$. Since $R_{n_{s_i}}(\hat{x}_{n_{s_i}}) \rightharpoonup R(\bar{x})$, we have that $R(\bar{x}) = s_{R(K)}(0) = R(\hat{x})$, a contradiction against the uniqueness of the solution of the problem (ASP).

Now, as in Section 3, we study the relevant special case where the admissible region of the spline problem is an affine subspace of X. In this particular case, we do not need to make additional assumptions on the existence of the solutions of the perturbed problems, since the same conditions that allow us to prove the stability properties also ensure the existence of solutions for the perturbed problems (LSP_n).

Theorem 5.3. *Let X be a reflexive space, Y be an E-space and Z be a Banach space. Let R, $R_n \in \mathcal{L}(X,Y)$ such that $\|R_n - R\|_{\mathcal{L}(X,Y)} \to 0$ and R be a closed range operator with a finite dimensional kernel. Let L, $L_n \in \mathcal{L}(X,Z)$ such that $\|L_n - L\|_{\mathcal{L}(X,Y)} \to 0$ and L be a surjective operator with* $\operatorname{Ker} L$ *complementable and* $\operatorname{Ker} R \cap \operatorname{Ker} L = \{0\}$. *Let $\{z_n\} \subset Z$ such that $z_n \to \bar{z}$. Then there exists $n_0 \in \mathbb{N}$ such that the problem*

$$\min_{x \in L_n^{-1}(z_n)} \|R_n(x)\|_Y \tag{LSP_n}$$

has a solution $\hat{x}_n$ for every $n \geqslant n_0$. Moreover the sequence $\{\hat{x}_n\}$ is weakly convergent to the solution of problem (LSP).

Proof. There exists $n_0 \in \mathbb{N}$ such that, for every $n \geqslant n_0$, $\dim \operatorname{Ker} R_n < \infty$ and R_n is a closed range operator (see [1, Corollary 2.6]). Hence, the problem (LSP_n) has a solution $\hat{x}_n$ (see [8, 21B]). From $\|L_n - L\|_{\mathcal{L}(X,Y)} \to 0$ and $z_n \to \bar{z}$, it follows that $L_n^{-1}(z_n) \xrightarrow{M} L^{-1}(\bar{z})$ (see [14, Corollary 6.3 and Remark 2.6]). Moreover $\operatorname{Ker} R \cap (L^{-1}(\bar{z}))_\infty = \operatorname{Ker} R \cap \operatorname{Ker} L = \{0\}$. Hence the assumptions of Theorem 5.1 hold. Since the assumption $\operatorname{Ker} R \cap \operatorname{Ker} L = \{0\}$ implies the uniqueness of the solution of problem (LSP), the thesis follows from Remark 5.2. □

In the special case where we do not consider perturbations of the spline operator R, we obtain stronger stability results. We omit the proofs of the following results since they are simple adaptations of the proofs of Theorems 5.1 and 5.3 where Proposition 4.3 is used instead of Lemma 4.1.

Theorem 5.4. *Let X be a reflexive Banach space and Y be an E-space. Let $R \in \mathcal{L}(X,Y)$ be a closed range operator with a finite dimensional kernel. Let $\{K_n\}$ be a sequence of nonempty closed convex subsets of X such that $K_n \xrightarrow{M} K$ and* $\operatorname{Ker} R \cap (K)_\infty = \{0\}$. *If the problem*

$$\min_{x \in K_n} \|R(x)\|_Y \tag{7}$$

has a solution $\hat{x}_n \in K_n$ for every n, then the sequence $\{\hat{x}_n\}$ admits a subsequence converging to a solution of the problem (ASP).

Theorem 5.5. *Let X be a reflexive Banach space, Y be an E-space and Z be a Banach space. Let $R \in \mathcal{L}(X,Y)$ be a closed range operator with a finite dimensional kernel. Let L, $L_n \in \mathcal{L}(X,Z)$ such that $\|L_n - L\|_{\mathcal{L}(X,Y)} \to 0$ and L be a surjective operator with* $\operatorname{Ker} L$ *complementable and* $\operatorname{Ker} R \cap \operatorname{Ker} L = \{0\}$. *Let $\{z_n\} \subset Z$ such that $z_n \to \bar{z}$. Then there exists $n_0 \in \mathbb{N}$ such that the problem*

$$\min_{x \in L_n^{-1}(z_n)} \|R(x)\|_Y \tag{8}$$

has a solution $\hat{x}_n$ for every $n \geqslant n_0$. Moreover the sequence $\{\hat{x}_n\}$ converges to the solution of the problem (LSP).

We underline that in this section many of the assumptions are satisfied in a number of common concrete R-spline problem. Indeed, we consider, for instance, the problem where X and Y are respectively the Sobolev space $W^{m,p}((a,b))$ and $L^p((a,b),\mu,\mathbb{R})$, where μ is the Lebesgue measure and $1 < p < \infty$ and the operator R is defined as follows:

$$R = \sum_{k=0}^{m} a_k(t)\frac{d^k}{dt^k}, \quad a_m(t) \neq 0,\ a_k \in C^k\big([a,b],\mathbb{R}\big),\ [a,b] \subset \mathbb{R}.$$

In this case, X is a reflexive space and Y is an E-space. Moreover, it can be verified that R is a surjective (hence closed range) operator with a finite dimensional kernel. Moreover, we recall that the assumption $\operatorname{Ker} R \cap (K)_\infty = \{0\}$ in Theorem 5.1, is quite natural. Indeed, in the relevant case of problem (LSP), it becomes the classical condition $\operatorname{Ker} R \cap \operatorname{Ker} L = \{0\}$ that guarantees the uniqueness of the solution of LSP (see [2]).

Now we apply our results to some particular situations that already deserved some attention in the literature. We begin to consider the sequence of problems (ASP_n) where $R_n = R$ is a fixed spline operator and $\{K_n\}$ is a decreasing sequence of closed convex nested sets (i.e. $K_{n+1} \subset K_n$) such that $\bigcap_{n=1}^{\infty} K_n = K$ is a nonempty set. It can be easily proved that here $K_n \xrightarrow{M} K$, hence we can use our approach in terms of set-convergences in order to obtain results similar to those contained in [3,7,12].

A special case considered in [5,13], deals with the stability of the (ASP) problem with respect to the perturbed problems (ASP_n), where the constraint sets have a particular form. Here we apply our results in order to obtain some stability properties for this case. Preliminary, we recall some notations that will be used in the proof of the following theorem.

Let A be a nonempty subset of X, the *indicator function* of the set A is defined by

$$I_A(x) = \begin{cases} 0 & \text{if } x \in A, \\ \infty & \text{elsewhere.} \end{cases}$$

Let $g : X \to (-\infty, \infty]$ be an arbitrary function. The *Fenchel conjugate* of g is the function $g^* : X^* \to [-\infty, \infty]$ defined as

$$g^*\big(x^*\big) = \sup_{x \in X}\big\{\big\langle x^*, x\big\rangle - g(x)\big\}.$$

Whenever $V \subset X$ is a subspace we have that $(I_V)^* = I_{(V)^\perp}$ where

$$(V)^\perp = \big\{x^* \in X^*\colon \big\langle x^*, v\big\rangle = 0,\ \forall v \in V\big\}$$

(see, e.g., [9]).

Theorem 5.6. *Let X be a reflexive Banach space and Y be an E-space. Let $R \in L(X,Y)$ be a closed range operator with a finite dimensional kernel. Let K, K_n be defined by*

$$K = x + \bigcap_{f \in F} \operatorname{Ker} f, \qquad K_n = x + \bigcap_{f \in F_n} \operatorname{Ker} f \tag{9}$$

where x is a fixed element in X and F, F_n are closed subspaces of X^* such that $F_n \xrightarrow{M} F$. If $\operatorname{Ker} R \cap (\bigcap_{f \in F} \operatorname{Ker} f) = \{0\}$, *then there exists $n_0 \in \mathbb{N}$ such that the problem* (ASP_n) *has a solution $\hat{x}_n$ for every n. Moreover the sequence $\{\hat{x}_n\}$ converges to the solution of the problem* (ASP).

Proof. Since K_n and K are affine sets, the problems (ASP$_n$) and (ASP) have a solution (see [8, 21B]). Moreover, since $\operatorname{Ker} R \cap (\bigcap_{f\in F} \operatorname{Ker} f) = \{0\}$, the limit problem has a unique solution $\hat{x}$. In order to prove the thesis, in force of Theorem 5.4, we only need to prove that $K_n \xrightarrow{M} K$.

We begin to remark that

$$\bigcap_{f\in F} \operatorname{Ker} f = (F)^{\perp} = \bigl\{x \in X\colon\ \langle f, x\rangle = 0,\ \forall f \in F\bigr\},$$

$$\bigcap_{f\in F_n} \operatorname{Ker} f = (F_n)^{\perp} = \bigl\{x \in X\colon\ \langle f, x\rangle = 0,\ \forall f \in F_n\bigr\}.$$

Since $F_n \xrightarrow{M} F$, we have that $\operatorname{epi} I_{F_n} \xrightarrow{M} \operatorname{epi} I_F$, where $\operatorname{epi} I_{F_n}$ and $\operatorname{epi} I_F$ denote respectively the epigraph of the indicator functions of F_n and F (see [9]). By the continuity of the Fenchel conjugate with respect to Mosco epiconvergence (see e.g. [9]), we obtain $\operatorname{epi}(I_{F_n})^* \xrightarrow{M} \operatorname{epi}(I_F)^*$. Since X is a reflexive space, it holds $(I_{F_n})^* = I_{(F_n)^{\perp}}$ and $(I_F)^* = I_{(F)^{\perp}}$. Hence, we obtain $I_{(F_n)^{\perp}} \xrightarrow{M} I_{(F)^{\perp}}$ or, equivalently, $(F_n)^{\perp} \xrightarrow{M} (F)^{\perp}$ and $K_n \xrightarrow{M} K$. $\square$

In the theory of interpolation, the relevance of this special class of abstract spline problems is apparent when we consider the families F, F_n as families of evaluation functionals defined on a space X of functions.

Theorem 5.6 allows us to compare our results with those contained in [5]. The spline problem considered there is a particular version of the problem considered in Theorem 5.6, where X is a Hilbert space and $F = X^*$ (or equivalently $K = \{x\}$). We study the problem in a more general framework, under the additional assumption that the spline operator R has a finite dimensional kernel.

References

[1] Y.A. Abramovich, C.D. Aliprantis, An Invitation to Operator Theory, Grad. Stud. Math., vol. 50, Amer. Math. Soc., Providence, RI, 2002.

[2] M. Atteia, Hilbertian Kernels and Spline Functions, Stud. Comput. Math., vol. 4, North-Holland, Amsterdam, 1992.

[3] N.M. Benbourhim, J. Gaches, T_f-splines et approximation par T_f-prolongement, Studia Math. 106 (1993) 204–211.

[4] R. Champion, C.T. Lenard, T.M. Mills, A variational approach to splines, ANZIAM J. 42 (2000) 1119–1135.

[5] C. de Boor, Convergence of abstract splines, J. Approx. Theory 31 (1981) 80–89.

[6] A. Dontchev, T. Zolezzi, Well-Posed Optimization Problems, Lecture Notes in Math., vol. 1543, Springer-Verlag, Berlin, 1993.

[7] R.B. Holmes, R-splines in Banach spaces: I. Interpolation of linear manifolds, J. Math. Anal. Appl. 40 (1972) 574–593.

[8] R.B. Holmes, Geometric Functional Analysis and Its Applications, Grad. Texts in Math., vol. 24, Springer-Verlag, New York, 1975.

[9] R. Lucchetti, Convexity and Well-Posed Problems, CMS Books Math., Springer-Verlag, New York, 2006.

[10] R. Lucchetti, E. Miglierina, Stability for convex vector optimization problems, Optimization 53 (2004) 517–528.

[11] U. Mosco, Convergence of convex sets and of solutions of variational inequalities, Adv. Math. 3 (1969) 510–585.

[12] A.Y. Bezhaev, V.A. Vasilenko, Variational Theory of Splines, Kluwer Academic/Plenum Publishers, New York, 2001.

[13] B. Shektman, Unconditional convergence of abstract splines, J. Approx. Theory 30 (1980) 237–246.

[14] Y. Sonntag, C. Zălinescu, Convergence for sequences of sets and linear mappings, J. Math. Anal. Appl. 188 (1994) 616–640.