Available online at www.sciencedirect.com
ScienceDirect
J. Math. Anal. Appl. 336 (2007) 140–155
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Localized null controllability and corresponding minimal norm control blowup rates of thermoelastic systems ☆

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Received 3 November 2006
Available online 22 February 2007
Submitted by D. Russell

Abstract

In this paper, we consider the problem of null controllability for an elastic operator under square root damping. Such partial differential equation models can be described by analytic semigroups on the basic space of finite energy. Thus by inherent smoothing coming from the parabolic-like behavior of the dynamics, the problem of null controllability is appropriate for consideration. In particular, we will show that the solution variables can be steered to the zero state by means of iterations of locally supported steering controls acting on appropriate finite dimensional systems. The hinged boundary conditions considered here admit of a diagonalization of the spatial operator. The control strategy implemented in [A. Benabdallah, M. Naso, Null controllability of a thermoelastic plate, Abstr. Appl. Anal. 7 (2002) 585–599] is used to construct a suboptimal control for the problem, but here we expand upon their results by providing a bound for the energy function $\mathcal{E}_{\min}(T)$, $T > 0$. Our results are valid for localized mechanical and thermal control. The strategy relies heavily on the availability of a Carleman's estimate for finite linear combinations of eigenfunctions of the Dirichlet Laplacian.

Keywords: Partial differential equations; Null controllability

☆ This material is based upon work partially supported by the National Science Foundation under Grant No. 0208121.
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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.02.031

1. Introduction and statement of main results

Our intent in this paper is twofold: (i) We wish to establish the null controllability property for the 2-D system of linear thermoelasticity, by means of locally distributed source control in either the "mechanical" or "thermal" component. We are considering the problem with the canonical hinged boundary conditions in place. It will ultimately be shown that the thermoelastic state variables can be driven to the zero state in arbitrary small time, $T > 0$. This circumstance is in line with the underlying infinite speed of propagation of signals (i.e., it is now known that the thermoelastic dynamics, under all possible boundary conditions, can be associated with the generator of an *analytic* semigroup [10]). (ii) Having established that the thermoelastic variables can be driven to the zero state in arbitrary short time, we can proceed to measure the rate of singularity of the associated *minimal energy function* $\mathcal{E}_{\min}(T)$ (as defined in (3) below), as $T \searrow 0$. As it is defined, $\mathcal{E}_{\min}(T)$ characterizes the "violence" of the fast null thermoelastic controllers. (Here we have adopted the now classic phrase from the fundamental work in [14].)

We now describe the PDE model under consideration: Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with C^∞ boundary $\partial\Omega$. Given terminal time $T > 0$ and constant $\rho > 0$, we will consider each of the following controlled PDEs:

$$\begin{array}{c}\text{Controlled}\\ \text{"Mechanical"}\\ \text{Component}\end{array}\quad \begin{cases} y_{tt} + \Delta^2 y + \rho\Delta\theta = \chi_\omega(x)u & \text{in } Q \equiv (0,T)\times\Omega,\\ \theta_t - \Delta\theta - \rho\Delta y_t = 0 & \text{in } Q,\\ y|_{\partial\Omega} = \Delta y|_{\partial\Omega} = \theta|_{\partial\Omega} = 0 & \text{on } \Sigma \equiv (0,T)\times\partial\Omega,\\ [y, y_t, \theta](0) = [y_0, y_1, \theta_0] \in H, \end{cases} \tag{1}$$

$$\begin{array}{c}\text{Controlled}\\ \text{"Thermal"}\\ \text{Component}\end{array}\quad \begin{cases} y_{tt} + \Delta^2 y + \rho\Delta\theta = 0 & \text{in } Q,\\ \theta_t - \Delta\theta - \rho\Delta y_t = \chi_\omega(x)u & \text{in } Q,\\ y|_{\partial\Omega} = \Delta y|_{\partial\Omega} = \theta|_{\partial\Omega} = 0 & \text{on } \Sigma,\\ [y, y_t, \theta](0) = [y_0, y_1, \theta_0] \in H. \end{cases} \tag{2}$$

Here, the controllability region $\omega \subseteq \Omega$ is an open subdomain of Ω. Thus, the PDE (1) reflects the imposition of mechanical control; the PDE (2) that of thermal control. If $\omega = \Omega$ and if for every $[y_0, y_1, \theta_0] \in H$ there exists $u \in L^2((0,T); L^2(\Omega))$ so that the solution at terminal time $[y(T), y_t(T), \theta(T)]$ is the zero state, then the system is said to be *null controllable*. If $\omega \subsetneq \Omega$ is nonempty and for every $[y_0, y_1, \theta_0] \in H$, there exists $u \in L^2((0,T); L^2(\omega))$ so that the solution $[y(T), y_t(T), \theta(T)]$ is the zero state, then the system is said to be *null controllable by locally distributed controls*.

Once the existence of locally distributed controls in $\mathcal{U} = L^2((0,T)\times\omega)$ has been established for given $\omega \subseteq \Omega$, we will consider the minimal norm controller. That is, given $y^0 = [y_0, y_1, \theta_0] \in H$ and $T > 0$, $u^*(T, y^0)$ is a minimal norm controller if

$$\left\|u^*\left(T, y^0\right)\right\|_{\mathcal{U}} = \min\left\{\|u\|_{\mathcal{U}}\colon e^{\mathcal{A}T}y^0 + \mathcal{L}_T u = 0\right\}.$$

Here the control $\to$ terminal state map $\mathcal{L}_T : \mathcal{U} \to H$, corresponding to *mechanical* control is given by

$$\mathcal{L}_T u \equiv \int_0^T e^{\mathcal{A}(T-s)} \begin{bmatrix} 0 \\ \chi_\omega u \\ 0 \end{bmatrix} ds,$$

and the control → terminal state map $\mathcal{L}_T : \mathcal{U} \to H$, corresponding to *thermal* control is given by

$$\mathcal{L}_T u \equiv \int_0^T e^{\mathcal{A}(T-s)} \begin{bmatrix} 0 \\ 0 \\ \chi_\omega u \end{bmatrix} ds.$$

With $u^*(T,\cdot)$ as defined, we define the *minimal energy function* $\mathcal{E}_{\min}(T)$ by

$$\mathcal{E}_{\min}(T) = \sup_{\|y^0\|_H = 1} \left\| u^*\left(T, y^0\right) \right\|_H. \tag{3}$$

Given this function, we wish to precisely quantify its behavior as $T \searrow 0$.

The issue of null controllability ($\omega = \Omega$), for the present hinged case, was originally established by Lasiecka and Triggiani in [9]. In [19] Triggiani continues the analysis of this problem, and provides optimal estimates for the singularity of the minimal energy function $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-5/2})$ for the fully distributed case. By altogether different methods, Avalos and Lasiecka provide the optimal asymptotics $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-5/2})$ for fully distributed internal control of thermoelastic and structurally damped systems in [1] (for the free case), [3] (for the canonical hinged case), and [4] (for the clamped case). (These papers consider mechanical or thermal controls.) If *both* fully distributed thermal and mechanical controls are in place, Avalos and Lasiecka show that the thermoelastic minimal energy blowup rate can be improved to $\mathcal{E}_{\min}(T) = \mathcal{O}(T^{-3/2})$.

The task here will be to compute $\mathcal{E}_{\min}(T)$ when *localized* distributed control is in play. In contrast to the asymptotics computed in [1,4,19] and [3] for fully distributed control, wherein one has the minimal energy obeying a *rational* rate of blowup, we expect here that the rate of singularity should be of exponential type. That is, we "expect" $\mathcal{E}_{\min}(T) = \exp(\mathcal{O}([1/T]))$. Indeed, such a rate would be in line with those seen for other PDE systems under localized or boundary control; see e.g. [8,13–16] and [17].

The issue of null controllability for the thermoelastic system with *locally* supported *thermal* control was shown by Benabdallah and Naso in [5]. In contrast with their paper, the proof presented in this paper will establish null controllability via locally supported *mechanical* controls. And further, we will provide a (suboptimal) bound for the associated minimal energy function. In this context the proof of this result follows exactly the same strategy and technical details presented in Cokeley and Avalos [2]. The proof of this result depends on two key ingredients: (i) a critical observability estimate from Avalos and Lasiecka in [1,3] and [4]; (ii) a control iterative scheme of Benabdallah and Naso in [5], this scheme being a very nontrivial adaptation of the methodology developed by Lebeau and Robbiano in [11].

After the submission of this paper, we became aware of the work of Miller in [12]. His paper employs the same aforesaid ingredients as that in the present paper; namely, the Avalos and Lasiecka estimate in [1,3] and [4]; and the control methodology of Benabdallah and Naso in [5].

This paper and [5] only consider the canonical hinged boundary conditions. The method of proof here does not lend itself to the case of clamped or free boundary conditions, as they are considered in [1] and [4], in the context of fully distributed control. The optimal asymptotics for the clamped and free models with interior control of full support ($\Omega = \omega$) are given in [4] and [1]. The complexity of proof in these cases (with respect to hinged boundary conditions) suggests that treatment of localized controls may require very different and more complex mathematical technology (e.g. Carleman's estimates for systems—rather than scalar equations). This problem is currently under investigation by the author.

The main result of this paper follows.

Theorem 1. *The systems* (1) *and* (2) *are null controllable through a locally distributed mechanical source and through a locally distributed thermal source respectively, within the class of controls* $\mathcal{U} = L^2((0,T) \times \omega)$. *Moreover, for any fixed* $\epsilon > 0$, $\mathcal{E}_{\min}(T) = \exp(\mathcal{O}([1/T]^{1+\epsilon}))$. *In particular, there is a positive constant* C_ϵ, *which depends on* ϵ *but not on* T, *such that* $\mathcal{E}_{\min}(T) \leqslant C_\epsilon \exp(\frac{C}{T^{1+\epsilon}})$.

The proof of Theorem 1 will use estimates from the following lemma.

Lemma 2. *The systems* (1) *and* (2) *are null controllable through a fully distributed mechanical source and through a fully distributed thermal source respectively, within a class of controls in* $L^2((0,T) \times \Omega)$. *Moreover, there exists* $C > 0$ *such that for each* $T > 0$ *and each* $y^0 \in H$, *there is a control* u *steering the solution state from* y^0 *to the zero state in* H *with* u *satisfying* $\|u\|_{L^2(0,T)\times\Omega} \leqslant CT^{-5/2}\|y^0\|_H$.

As mentioned earlier, a proof of this lemma can be found in [9].

Remark 3. Note that in view of the known asymptotics for other localized controllability problems, our result and the result of Miller in [12] are seemingly “unsharp by ϵ.” This is a consequence of (suboptimal) control strategy, which is in part, a suitable adaptation of the methodology implemented in [5]. Concerning boundary controllability of this problem, the optimal result ($\epsilon = 0$) is shown by Lasiecka and Seidman in [8] for special geometries. Note that the “localized interior controllability” result shown here will not yield the boundary controllability in [8], unless one is willing to allow for more controls on the boundary.

To prove Theorem 1, we will follow the outline below.

1. We begin by expressing the original PDE as a first order ODE system (4) on $(0,T)$. We then find the homogeneous adjoint system (5) (adjoint with respect to (4)) and express the solution $[\phi, \phi_t, \psi]$ in terms of the thermoelastic operator $\mathcal{A}$, where $\mathcal{A}: D(\mathcal{A}) \subset H \to H$ again satisfies hinged boundary conditions. By a similarity transformation, we diagonalize $\mathcal{A}$ to find its eigenfunctions $\{\Phi_j\}$. These eigenfunctions naturally depend on the eigenpairs of the Dirichlet Laplacian operator. With the eigenfunctions in hand, we can then express the solution of (5) as a series of the form $\sum a_j \Phi_j$.
2. We then consider the null controllability of the system (1) or (2) for initial data given in a finite dimensional space $H_\ell = \{\Phi_1, \Phi_2, \dots, \Phi_\ell\}$ and proceed to obtain the necessary observability inequality in terminal time T_ℓ, where $\sum T_\ell = T$. The observability inequality is first given for the fully distributed control problem (6) as was done in [19] and [1]. After noting the invariance of $\mathcal{A}$ on H_ℓ, the necessary observability inequality for locally supported null control is obtained by majorizing (6). In these steps we are using the main idea of the paper [5]. In [5] the improvement of fully distributed control ($\omega = \Omega$) to locally distributed control ($\omega \subset \Omega$) is made by invocation of a Carleman estimate given in [7]. This estimate is applicable to *finite* linear combinations of eigenfunctions of the Dirichlet Laplacian. We then construct a locally supported control u from controls $\{u^\ell\}$ acting on finite dimensional systems as prescribed in (21) that will steer initial data $[y_0, y_1, \theta_0]$ to zero in time $T > 0$.
3. Subsequently, we proceed to measure the singularity of the null controller u devised in Step 2. This is done by appealing to the rate of blowup of each controller u^ℓ. That is, $\|u\|_{\mathcal{U}} = \sum \|u^\ell\|_{\mathcal{U}} \leqslant \|y^0\| \sum C(T_\ell)$. During the course of this estimation, we will see that

the exponential rate of singularity for $\mathcal{E}_{\min}(T)$, given in Theorem 1, is due to the first ℓ^* terms of the series, where $\ell^* = \ell^*(T)$ is as given in (32). The contribution of the "tail end" of the series $\sum_{\ell^*+1}^{\infty} C(T_\ell)$ will be found to be essentially benign.

2. Abstract formulation for the dynamics

For the sake of clarity, the work to follow will focus on the thermoelastic system with a locally distributed *mechanical* source. Let $S : D(S) \subset L^2(\Omega) \to L^2(\Omega)$ be given by $Sy = -\Delta y$ for $y \in D(S) = H^2(\Omega) \cap H_0^1(\Omega)$. Then PDE (1) can be written as the first order ODE system on $(0, T)$,

$$\begin{cases} \begin{bmatrix} y \\ y_t \\ \theta \end{bmatrix}_t = \begin{bmatrix} 0 & I & 0 \\ -S^2 & 0 & \rho S \\ 0 & -\rho S & -S \end{bmatrix} \begin{bmatrix} y \\ y_t \\ \theta \end{bmatrix} + \begin{bmatrix} 0 \\ \chi_\omega u \\ 0 \end{bmatrix}, \\ [y, y_t, \theta](0) = [y_0, y_1, \theta_0] \quad \text{on } \Omega. \end{cases} \tag{4}$$

We set

$$\mathcal{A} = \begin{bmatrix} 0 & I & 0 \\ -S^2 & 0 & \rho S \\ 0 & -\rho S & -S \end{bmatrix} \quad \text{and} \quad B_\omega u = \begin{bmatrix} 0 \\ \chi_\omega u \\ 0 \end{bmatrix}$$

to determine the corresponding adjoint system. With the inner product on H given by

$$\left\langle \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right\rangle_H = (Sx_1, Sy_1)_{L^2(\Omega)} + (x_2, y_2)_{L^2(\Omega)} + (x_3, y_3)_{L^2(\Omega)}$$

for $[x_1, x_2, x_3], [y_1, y_2, y_3] \in H$,

we find

$$\mathcal{A}^* = \begin{bmatrix} 0 & -I & 0 \\ S^2 & 0 & -\rho S \\ 0 & \rho S & -S \end{bmatrix}.$$

The backward adjoint system is given by

$$\begin{cases} \begin{bmatrix} \phi \\ \phi_t \\ \psi \end{bmatrix}_t = -\mathcal{A}^* \begin{bmatrix} \phi \\ \phi_t \\ \psi \end{bmatrix} = \begin{bmatrix} \phi_t \\ -S^2\phi + \rho S\psi \\ -\rho S\phi_t + S\psi \end{bmatrix} & \text{in } Q, \\ [\phi, \phi_t, \psi](T) = [\phi_0, \phi_1, \psi_0] & \text{in } \Omega^3 \end{cases}$$

and has solution

$$\begin{bmatrix} \phi(t) \\ \phi_t(t) \\ \psi(t) \end{bmatrix} = e^{\mathcal{A}^*(T-t)} \begin{bmatrix} \phi_0 \\ \phi_1 \\ \psi_0 \end{bmatrix}.$$

After the change of variables $t := T - t$, the forward adjoint system is given by

$$\begin{cases} \begin{bmatrix} \phi \\ \phi_t \\ \psi \end{bmatrix}_t = \mathcal{A} \begin{bmatrix} \phi \\ \phi_t \\ \psi \end{bmatrix} := \begin{bmatrix} 0 & I & 0 \\ -S^2 & 0 & \rho S \\ 0 & -\rho S & -S \end{bmatrix} \begin{bmatrix} \phi \\ \phi_t \\ \psi \end{bmatrix} & \text{in } Q, \\ [\phi, \phi_t, \psi](0) = [\phi_0, -\phi_1, \theta_0] & \text{in } \Omega^3. \end{cases} \tag{5}$$

With $B_\omega = \begin{bmatrix} 0 \\ \chi_\omega \\ 0 \end{bmatrix}$ as before, $B_\omega^* = [0 \quad \chi_\omega \quad 0]$. Then the solution to the system (5) can be written as

$$\begin{bmatrix} \phi(t) \\ \phi_t(t) \\ \psi(t) \end{bmatrix} = e^{\mathcal{A}t} \begin{bmatrix} \phi_0 \\ -\phi_1 \\ \psi_0 \end{bmatrix}, \quad \text{thus } B_\omega^* e^{\mathcal{A}t} \begin{bmatrix} \phi_0 \\ -\phi_1 \\ \psi_0 \end{bmatrix} = \chi_\omega \phi_t(t).$$

Null controllability of the fully distributed system ($\omega = \Omega$) follows if there exists a constant $C_T > 0$ such that

$$\left\| \begin{bmatrix} \phi(T) \\ \phi_t(T) \\ \theta(T) \end{bmatrix} \right\|_H^2 \leqslant C_T \int_0^T \left(\|\phi_t(t)\|_{L^2(\Omega)}^2 \right) dt \tag{6}$$

for any solution $[\phi, \phi_t, \psi]$ of the system (5). In fact, this inequality follows by [19] with

$$C_T = \mathcal{O}\left(T^{-5}\right) \quad \text{as } T \searrow 0. \tag{7}$$

To obtain null controllability with locally distributed controls, we first look at the truncation of $\mathcal{A}$ on the span of finitely many eigenfunctions.

In the following work, we will also make use of the eigenpairs of $\mathcal{A}$. Let $\{\mu_n, e_n\}_{n=1}^\infty$ be the eigenpairs of S with $0 < \mu_n \leqslant \mu_{n+1}$ for all $n \in \mathbb{N}$. That is,

$$Se_n = \mu_n e_n \quad \text{for all } n \in \mathbb{N}. \tag{8}$$

Let $M = \begin{bmatrix} 0 & I & 0 \\ -I & 0 & \rho I \\ 0 & -\rho I & -I \end{bmatrix}$ so that

$$\begin{bmatrix} S & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \mathcal{A} \begin{bmatrix} S^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} 0 & S & 0 \\ -S & 0 & \rho S \\ 0 & -\rho S & -S \end{bmatrix} = \begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & S \end{bmatrix} M.$$

Also let λ_1, λ_2, and λ_3 be the roots of $p(x) = x^3 + x^2 + (\rho^2 + 1)x + 1$, the characteristic polynomial for M. p is a stable polynomial by Routh's theorem (see [20]). For estimates later, we will use that λ_1 is chosen so that

$$0 > \operatorname{Re}(\lambda_1) \geqslant \max\left\{\operatorname{Re}(\lambda_2), \operatorname{Re}(\lambda_3)\right\}. \tag{9}$$

Having in mind the diagonalization of $\mathcal{A}$ used in [9], let Π be the linear mapping on $L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ given by $\Pi = \begin{bmatrix} I & I & I \\ \lambda_1 I & \lambda_2 I & \lambda_3 I \\ \Lambda_1 I & \Lambda_2 I & \Lambda_3 I \end{bmatrix}$, where $\Lambda_j = -\frac{\rho\lambda_j}{1+\lambda_j}$. $\mathcal{A}$ can be diagonalized by

$$\Pi^{-1} \begin{bmatrix} S & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \mathcal{A} \begin{bmatrix} S^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \Pi = \begin{bmatrix} \lambda_1 S & 0 & 0 \\ 0 & \lambda_2 S & 0 \\ 0 & 0 & \lambda_3 S \end{bmatrix}.$$

The eigenpairs of the operator on the right-hand side of the above equation are easily found. In fact $\begin{bmatrix} \lambda_1 S & 0 & 0 \\ 0 & \lambda_2 S & 0 \\ 0 & 0 & \lambda_3 S \end{bmatrix}$ has eigenpairs

$$\left\{\left(\lambda_1\mu_n, \begin{bmatrix}\lambda_1\mu_n e_n\\0\\0\end{bmatrix} = \lambda_1\mu_n b_1 e_n\right), \left(\lambda_2\mu_n, \begin{bmatrix}0\\\lambda_2\mu_n e_n\\0\end{bmatrix} = \lambda_2\mu_n b_2 e_n\right),\right.$$
$$\left.\left(\lambda_3\mu_n, \begin{bmatrix}0\\0\\\lambda_3\mu_n e_n\end{bmatrix} = \lambda_3\mu_n b_3 e_n\right)\right\}_{n\in\mathbb{N}}.$$

With this setup, we find that $\mathcal{A}$ has eigenpairs

$$\left\{\left(\lambda_j\mu_n, \begin{bmatrix}S^{-1} & 0 & 0\\0 & I & 0\\0 & 0 & I\end{bmatrix}\Pi b_j e_n\right)\right\}_{n\in\mathbb{N},\ j=1,2,3}.$$

This can be written more concisely as

$$\left\{\lambda_j\mu_n, \Phi_{n,j} := \begin{bmatrix}1/\mu_n\\\lambda_j\\\Lambda_j\end{bmatrix} e_n\right\}_{n\in\mathbb{N},\ j=1,2,3}.$$

Define

$$H_m = \operatorname{span}\{\Phi_{n,j}\colon\ 1 \leqslant n \leqslant m,\ j = 1,2,3\} \tag{10}$$

and $\mathcal{P}_m$ to be the orthogonal projection from H onto H_m.

3. A technical lemma

Let the open, proper subset $\omega \subset \Omega$ be fixed for the remainder of the paper. Throughout this section, we will consider the system (1) with initial data in H_m. That is,

$$\begin{cases} y_{tt} + S^2 y - \rho S\theta = \chi_\omega(x)u & \text{in } Q,\\ \theta_t + S\theta + \rho S y_t = 0 & \text{in } Q,\\ [y, y_t, \theta](0) = y^0 & \text{in } H_m. \end{cases} \tag{11}$$

The lemma below will be implemented using different vales of m and T. The goal here is to show that the state $[y, y_t, \theta]$ can be steered from H_m to $H_m^\perp$ in time $T > 0$ by a localized control u. Further, we investigate the continuous dependence of $\|u\|_{L^2((0,T)\times\omega)}$ by the initial data, in terms of $T > 0$ and $m \in \mathbb{N}$. An argument for (2) follows by the same method.

Lemma 4. *There exists $C > 0$ such that for each $m \in \mathbb{N}$, each $T > 0$, and each $y^0 \in H_m$, there exists a control u supported in $[0, T] \times \omega$ with $\|u\|_{L^2((0,T)\times\omega)} \leqslant CT^{-5/2}e^{C\sqrt{\mu_m}}\|y^0\|_H$ so that the solution to* (11) *satisfies $[y(T), y_t(T), \theta(T)] \in H_m^\perp$.*

Proof. The controllability mentioned in Lemma 4 will be verified if we can show the containment

$$\operatorname{Range}\big(\mathcal{P}_m e^{\mathcal{A}T}\big) \subseteq \operatorname{Range}\left(\mathcal{P}_m \int_0^T e^{\mathcal{A}(T-s)}\,[0 \quad \chi_\omega \quad 0]\,ds\right). \tag{12}$$

(Note that in applying this containment to the truncated problem (1) with initial data in H_m, we are implicitly using the invariance of H_m under $e^{\mathcal{A}T}$.) We will show (12) by showing that for appropriate initial data in H_m, solutions to (5) satisfy the observability inequality below. I.e. for all solutions $[\phi, \phi_t, \psi]$ to (5), there exists $C_{T,m} > 0$ so that

$$\left\| \begin{bmatrix} \phi(T) \\ \phi_t(T) \\ \psi(T) \end{bmatrix} \right\|_H^2 \leqslant C_{T,m} \int_0^T \left(\left\| \phi_t(t) \right\|_{L^2(\omega)}^2 \right) dt. \tag{13}$$

A proof of Lemma 2 is given in [9]. Null controllability for the fully distributed system was shown there by establishing the following inequality:

$$\left\| \begin{bmatrix} \phi(T) \\ \phi_t(T) \\ \psi(T) \end{bmatrix} \right\|_H^2 \leqslant CT^{-5} \int_0^T \left(\left\| \phi_t(t) \right\|_{L^2(\Omega)}^2 \right) dt. \tag{14}$$

It is important to note that C above is independent of m and T.

Since H_m is invariant under $\mathcal{A}$, $\begin{bmatrix} \phi \\ \phi_t \\ \psi \end{bmatrix}(t) \in H_m$ for $t \in (0, T)$. By Parseval's relation we have now,

$$\left\| \phi_t(t) \right\|_{L^2(\Omega)}^2 = \sum_{n=1}^{m} \left| a_n(t) \right|^2 \quad \text{where } \phi_t(t) = \sum_{n=1}^{m} a_n(t) e_n. \tag{15}$$

Using a critical inequality from [5], we have an estimate for sums of eigenfunctions of S using a Carleman inequality. Namely,

$$\begin{aligned} \left\| \phi_t(t) \right\|_{L^2(\Omega)}^2 = \sum_{n=1}^{m} \left| a_n(t) \right|^2 &\leqslant Ce^{C\sqrt{\mu_m}} \int_\omega \left| \sum_{n=1}^{m} a_n(t) e_n(x) \right|^2 dx \\ &= Ce^{C\sqrt{\mu_m}} \left\| \phi_t(t) \right\|_{L^2(\omega)}^2. \end{aligned} \tag{16}$$

If mechanical controls are desired, we can combine relations (14) and (16) to have that

$$\left\| \begin{bmatrix} \phi(T) \\ \phi_t(T) \\ \psi(T) \end{bmatrix} \right\|_H^2 \leqslant CT^{-5} e^{C\sqrt{\mu_m}} \int_0^T \left\| \phi_t(t) \right\|_{L^2(\omega)}^2 dt. \tag{17}$$

It is important to note that C above is independent of T and m. This inequality and classical convex optimization (see e.g. [6]) gives the existence of a control $u = u(T, y^0)$, steering the solution to (11) from $y^0 \in H_m$ to $y(T) \in H_m^\perp$, satisfying

$$\left\| u\left(T, y^0\right) \right\|_{L^2((0,T)\times\omega)}^2 \leqslant CT^{-5} e^{C\sqrt{\mu_m}} \left\| y^0 \right\|_H^2. \qquad \square \tag{18}$$

4. Proof proper of Theorem 1

We have just proved that data in H_m can be steered to $H_m^\perp$ for any $m \in \mathbb{N}$. We will use Lemma 4 to establish a strategy, as in [5], to steer arbitrary initial data in H to zero. When it is convenient, we will take $m = m_\ell = 2^\ell$, $m_{\ell+1} = 2^{\ell+1}$, and $m_{\ell-1} = 2^{\ell-1}$. Let $\alpha \in (0, 1/2)$ and

$$T_\ell = K2^{-\ell\alpha} \quad \text{where } K = \frac{T(2^\alpha - 1)}{2}. \tag{19}$$

K is chosen so that $2\sum_{\ell=1}^{\infty} T_\ell = T$. Also let $a_0 = 0$ and $a_\ell = a_{\ell-1} + 2T_\ell$ for $l \in \mathbb{N}$.

Define the control $\to$ state map $\mathcal{L}_{t_0,t}(z, f)$ by having for all $\{z, f\} \in H \times L^2(t_0, t; L^2(\omega))$,

$$\mathcal{L}_{t_0,t}(z, f) = e^{\mathcal{A}(t-t_0)}z + \int_{t_0}^{t} e^{\mathcal{A}(t-s)} B_\omega f(s)\, ds.$$

Recall that $B_\omega = \begin{bmatrix} 0 \\ \chi_\omega \\ 0 \end{bmatrix}$. Moreover, for any index ℓ and corresponding $m = 2^\ell$, let $u_m(T_\ell, \mathcal{P}_m(z))$ denote the control, as given in Lemma 4, which steers initial data $\mathcal{P}_m(z) \in H_m$ to $H_m^\perp$ at time T_ℓ.

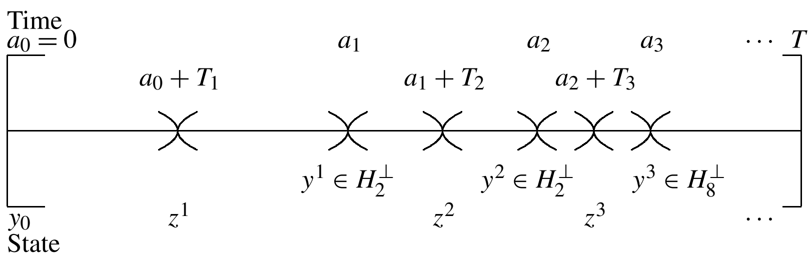
With quantities $\mathcal{L}_{t_0,t}(\cdot,\cdot)$ and $u_{2^\cdot}(\cdot,\cdot)$ in hand, we now define the iteration scheme upon which we will build our null steering control: Set $y^0 = [y_0, y_1, \theta_0] \in H$ to be the given initial data in (1). For $\ell = 1, 2, \ldots,$

$$z^\ell = e^{\mathcal{A}T_\ell} y^{\ell-1}, \quad \text{and then} \quad y^\ell = \mathcal{L}_{a_{\ell-1}+T_\ell, a_\ell}\big(z^\ell, u_m\big(\cdot - a_{\ell-1} - T_\ell; T_\ell, \mathcal{P}_m\big(z^\ell\big)\big)\big). \tag{20}$$

This constructs a control u given by

$$u(t) = \begin{cases} 0, & a_{\ell-1} \leqslant t < a_{\ell-1} + T_\ell,\ \ell \in \mathbb{N}, \\ u_m(t - (a_{\ell-1} + T_\ell); T_\ell, \mathcal{P}_m(z^\ell)), & a_{\ell-1} + T_\ell \leqslant t < a_{\ell-1} + 2T_\ell = a_\ell,\ \ell \in \mathbb{N}. \end{cases} \tag{21}$$

The following diagram should help illustrate the strategy:



To show that the state $[y, y_t, \theta]$ goes to zero in H, note that for each ℓ, we first estimate $\|y^\ell\|_H$ in terms of $\|z^\ell\|_H$. Using that $\mathcal{A}$ generates a semigroup of contractions and inequality (18), we have that

$$\begin{aligned} \|y^\ell\|_H &\leqslant \|e^{AT_\ell}z^\ell\|_H + \left\| \int_{a_{\ell-1}+T_\ell}^{a_\ell} e^{A(a_\ell - s)} B_\omega u_m\big(s - (a_{\ell-1} + T_\ell); T_\ell, \mathcal{P}_m\big(z^\ell\big)\big)\, ds \right\|_H \\ &\leqslant \|e^{AT_\ell}z^\ell\|_H + \int_0^{T_\ell} \big\|e^{A(T_\ell - s)} B_\omega u_m\big(s; T_\ell, \mathcal{P}_m\big(z^\ell\big)\big)\big\|_H\, ds \\ &\leqslant \|z^\ell\|_H + \int_0^{T_\ell} \big\|B_\omega u_m\big(s; T_\ell, \mathcal{P}_m\big(z^\ell\big)\big)\big\|_H\, ds \end{aligned}$$

$$
\begin{aligned}
&\leqslant \|z^\ell\|_H + \int_0^{T_\ell} 1 \cdot \|u_m(s; T_\ell, \mathcal{P}_m(z^\ell))\|_{L^2(\omega)}\, ds \\
&\leqslant \|z^\ell\|_H + \|1\|_{L^2((0,T)\times\omega)} \|u_m(\cdot; T_\ell, \mathcal{P}_m(z^\ell))\|_{L^2((0,T_\ell)\times\omega)}\, ds \\
&\leqslant \|z^\ell\|_H + C\|u_m(\cdot; T_\ell, \mathcal{P}_m(z^\ell))\|_{L^2((0,T_\ell)\times\omega)}\, ds \quad (\text{for } T<1, \text{ say}) \\
&\leqslant \|z^\ell\|_H + C T_\ell^{-5/2} e^{C\sqrt{\mu_m}} \|\mathcal{P}_m z^\ell\|_H
\end{aligned} \tag{22}
$$

where recall from Lemma 4, $\|u(T, y^0)\|_{L^2((0,T)\times\omega)} \leqslant CT^{-5/2}e^{C\sqrt{\mu_m}}\|y^0\|_H$ for $y^0 \in H_m$.

We now provide the crucial estimate for $\|z^\ell\|_H$ in terms of $\|y^{\ell-1}\|_H$. Recall that $z^\ell = e^{\mathcal{A}T_\ell} y^{\ell-1}$. Rather than using the contraction property of $\mathcal{A}$, we take advantage of the fact that $y^{\ell-1} \in H^\perp_{m_{\ell-1}}$. Before diagonalizing again, consider the following argument:

Note that for $\tilde{y} \in H^\perp_m$,

$$
\Pi^{-1} \begin{bmatrix} S & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tilde{y} \in \operatorname{Span}\left\{ \begin{bmatrix} e_n \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ e_n \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ e_n \end{bmatrix} \right\} \subseteq [L^2(\Omega)]^3.
$$

Write

$$
\begin{aligned}
&\Pi^{-1} \begin{bmatrix} S & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tilde{y} = \sum_{n=m+1}^\infty \begin{bmatrix} \xi_n^{(1)} e_n \\ \xi_n^{(2)} e_n \\ \xi_n^{(3)} e_n \end{bmatrix} \\
&\Rightarrow \quad \left\| \begin{bmatrix} e^{\lambda_1 S T_\ell} & 0 & 0 \\ 0 & e^{\lambda_2 S T_\ell} & 0 \\ 0 & 0 & e^{\lambda_3 S T_\ell} \end{bmatrix} \Pi^{-1} \begin{bmatrix} S & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tilde{y} \right\|^2_{[L^2(\Omega)]^3} \\
&\qquad = \left\| \sum_{n=m+1}^\infty \begin{bmatrix} e^{\lambda_1\mu_n T_\ell} \xi_n^{(1)} e_n \\ e^{\lambda_2\mu_n T_\ell} \xi_n^{(2)} e_n \\ e^{\lambda_3\mu_n T_\ell} \xi_n^{(3)} e_n \end{bmatrix} \right\|^2_{[L^2(\Omega)]^3}
\end{aligned}
$$

using the Spectral Theorem for self-adjoint operators. Continuing this estimate, we further have

$$
\begin{aligned}
&\left\| \begin{bmatrix} e^{\lambda_1 S T_\ell} & 0 & 0 \\ 0 & e^{\lambda_2 S T_\ell} & 0 \\ 0 & 0 & e^{\lambda_3 S T_\ell} \end{bmatrix} \Pi^{-1} \begin{bmatrix} S & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tilde{y} \right\|^2_{[L^2(\Omega)]^3} \\
&= \left\| \sum_{n=m+1}^\infty e^{\lambda_1\mu_n T_\ell} \xi_n^{(1)} e_n \right\|^2_{L^2(\Omega)} + \left\| \sum_{n=2^l+1}^\infty e^{\lambda_2\mu_n T_\ell} \xi_n^{(2)} e_n \right\|^2_{L^2(\Omega)} \\
&\quad + \left\| \sum_{n=m+1}^\infty e^{\lambda_3\mu_n T_\ell} \xi_n^{(3)} e_n \right\|^2_{L^2(\Omega)} \\
&= \sum_{n=m+1}^\infty \left(\|e^{\lambda_1\mu_n T_\ell} \xi_n^{(1)} e_n\|^2_{L^2(\Omega)} + \|e^{\lambda_2\mu_n T_\ell} \xi_n^{(2)} e_n\|^2_{L^2(\Omega)} + \|e^{\lambda_3\mu_n T_\ell} \xi_n^{(3)} e_n\|^2_{L^2(\Omega)} \right)
\end{aligned}
$$

$$\leqslant e^{2\operatorname{Re}(\lambda_1)\mu_{m+1}T_\ell} \sum_{n=m+1}^{\infty} \left(\left\| \xi_n^{(1)} e_n \right\|_{L^2(\Omega)}^2 + \left\| \xi_n^{(2)} e_n \right\|_{L^2(\Omega)}^2 + \left\| \xi_n^{(3)} e_n \right\|_{L^2(\Omega)}^2 \right)$$

after using size and order of λ_j given in (9). Continuing further with this estimate, we have

$$\begin{aligned}
&\left\| \begin{bmatrix} e^{\lambda_1 S T_\ell} & 0 & 0 \\ 0 & e^{\lambda_2 S T_\ell} & 0 \\ 0 & 0 & e^{\lambda_3 S T_\ell} \end{bmatrix} \Pi^{-1} \begin{bmatrix} S & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \tilde{y} \right\|_{[L^2(\Omega)]^3}^2 \\
&\quad = e^{2\operatorname{Re}(\lambda_1)\mu_{m+1}T_\ell} \left(\left\| \sum_{n=m+1}^{\infty} \xi_n^{(1)} e_n \right\|_{L^2(\Omega)}^2 + \left\| \sum_{n=m+1}^{\infty} \xi_n^{(2)} e_n \right\|_{L^2(\Omega)}^2 \right. \\
&\qquad \left. + \left\| \sum_{n=m+1}^{\infty} \xi_n^{(3)} e_n \right\|_{L^2(\Omega)}^2 \right) \\
&\quad = e^{2\operatorname{Re}(\lambda_1)\mu_{m+1}T_\ell} \left\| \sum_{n=m+1}^{\infty} \begin{bmatrix} \xi_n^{(1)} e_n \\ \xi_n^{(2)} e_n \\ \xi_n^{(3)} e_n \end{bmatrix} \right\|_{[L^2(\Omega)]^3}^2 \\
&\quad \leqslant e^{2\operatorname{Re}(\lambda_1)\mu_{m+1}T_\ell} \left\| \sum_{n=m+1}^{\infty} \begin{bmatrix} \xi_n^{(1)} e_n \\ \xi_n^{(2)} e_n \\ \xi_n^{(3)} e_n \end{bmatrix} \right\|_{[L^2(\Omega)]^3}^2 . \qquad (23)
\end{aligned}$$

Since $y^{\ell-1} \in H_{m_{l-1}}^{\perp}$, we can use the inequality above (23) to estimate $\|z^\ell\|_H$,

$$\begin{aligned}
\|z^\ell\|_H &= \left\| \begin{bmatrix} S^{-1} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \Pi \begin{bmatrix} e^{\lambda_1 S T_{\ell-1}} & 0 & 0 \\ 0 & e^{\lambda_2 S T_{\ell-1}} & 0 \\ 0 & 0 & e^{\lambda_3 S T_{\ell-1}} \end{bmatrix} \Pi^{-1} \begin{bmatrix} S & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} y^{l-1} \right\|_H \\
&= \left\| \Pi \begin{bmatrix} e^{\lambda_1 S T_{\ell-1}} & 0 & 0 \\ 0 & e^{\lambda_2 S T_{\ell-1}} & 0 \\ 0 & 0 & e^{\lambda_3 S T_{\ell-1}} \end{bmatrix} \Pi^{-1} \begin{bmatrix} S & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} y^{l-1} \right\|_{[L^2(\Omega)]^3} \\
&\leqslant \|\Pi\| e^{\operatorname{Re}(\lambda_1)\mu_{(m_{\ell-1}+1)}T_{\ell-1}} \|\Pi^{-1}\| \left\| \begin{bmatrix} S & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} y^{\ell-1} \right\|_{[L^2(\Omega)]^3} \\
&= C e^{\operatorname{Re}(\lambda_1)\mu_{(m_{\ell-1}+1)}T_{\ell-1}} \|y^{\ell-1}\|_H . \qquad (24)
\end{aligned}$$

Combining inequalities (22) and (24), we have

$$\|y^\ell\|_H \leqslant e^{\operatorname{Re}(\lambda_1)\mu_{(m_{\ell-1}+1)}T_{\ell-1}} \left(1 + C T_\ell^{-5/2} e^{C\sqrt{\mu_m}}\right) \|y^{\ell-1}\|_H . \qquad (25)$$

Weyl's formula states for this 2-dimensional setting that we can estimate large eigenvalues of S by $\mu_\ell \sim C(\Omega) l$ as $\ell \to \infty$, see [18, p. 395]. This implies that for positive constants C' and C, we have the estimates

$$\operatorname{Re}(\lambda_1)\mu_{m_{\ell-1}+1} \leqslant -C'\left(2^\ell\right) \quad \text{and} \quad \sqrt{\mu_m} \leqslant C 2^{\ell/2}. \qquad (26)$$

Applying estimates (22)–(25), we have that

$$\begin{aligned}\|y^\ell\|_H &\leqslant \big(e^{-C'(2^\ell)}\big)^{T_\ell}\big(1+CT_\ell^{-5/2}e^{C\sqrt{\mu_m}}\big)\|y^{\ell-1}\|_H\\ &\leqslant e^{-C'2^\ell T2^{-\alpha\ell}}C\big(T2^{-\alpha\ell}\big)^{-5/2}e^{C\sqrt{2^\ell}}\|y^{\ell-1}\|_H \quad \text{after using (19)}\\ &= CT^{-5/2}2^{5\alpha\ell/2}\exp\big(-C'T2^{(1-\alpha)\ell}+C2^{\ell/2}\big)\|y^{\ell-1}\|_H.\end{aligned}$$

Iterating this estimate, we obtain now that

$$\begin{aligned}\|y^\ell\|_H &\leqslant C^\ell T^{-5\ell/2}2^{5\alpha/2\sum_{j=0}^{\ell}j}\exp\Bigg(-C'T\sum_{j=0}^{\ell}2^{(1-\alpha)j}+C\sum_{j=0}^{\ell}2^{j/2}\Bigg)\|y^0\|_H\\ &= C^\ell T^{-5\ell/2}2^{5\alpha\ell(\ell+1)/4}\exp\bigg(-C'T\frac{2^{(1-\alpha)(\ell+1)}-1}{2^{1-\alpha}-1}+C\frac{2^{(\ell+1)/2}-1}{\sqrt{2}-1}\bigg)\|y^0\|_H\\ &\leqslant \exp\bigg(-C'T2^{(1-\alpha)\ell}+C2^{\ell/2}+\ell\ln C-\frac{5\ell\ln T}{2}+\frac{5\alpha\ell(\ell+1)\ln 2}{4}\bigg)\|y^0\|_H. \qquad (27)\end{aligned}$$

For each fixed T, $0<\alpha<1/2$ ensures that the dominant term in the exponent is $-C'T2^{(1-\alpha)\ell}$ as $\ell\to\infty$. Taking the limit as $\ell\to\infty$, we now have that

$$\lim_{\ell\to\infty}\|y^\ell\|_H=0. \qquad (28)$$

Since $[y, y_t, \theta]\in C([0,T], H)$, (28) proves the null controllability statement in Theorem 1.

To estimate u, we have

$$\begin{aligned}\|u\|_{L^2((0,T)\times\omega)} &= \sum_{\ell=1}^{\infty}\big\|u_m\big(\cdot-(a_{\ell-1}+T_\ell); T_\ell, z^\ell\big)\big\|_{L^2((a_{\ell-1}+T_\ell, a_\ell)\times\omega)}\\ &= \sum_{\ell=1}^{\infty}\big\|u_m\big(T_\ell, z^\ell\big)\big\|_{L^2((0,T_\ell)\times\omega)}\\ &\leqslant \sum_{\ell=1}^{\infty}CT_\ell^{-5/2}e^{\sqrt{\mu_m}}\|y^\ell\|_H\\ &\leqslant C_\alpha\sum_{\ell=1}^{\infty}T^{-5/2}2^{5\alpha\ell/2}e^{C2^{\ell/2}}\|y^\ell\|_H\end{aligned}$$

after using (7) and (26). Using (27), we continue on

$$\begin{aligned}\|u\|_{L^2((0,T)\times\omega)} &\leqslant C_\alpha\sum_{\ell=1}^{\infty}T^{-5/2}2^{5\alpha\ell/2}e^{C2^{\ell/2}}\exp\bigg(-C'T2^{(1-\alpha)\ell}+C2^{\ell/2}+\ell\ln C\\ &\qquad -\frac{5\ell\ln T}{2}+\frac{5\alpha\ell(\ell+1)\ln 2}{4}\bigg)\|y^0\|_H\\ &= C_\alpha T^{-5/2}\sum_{\ell=1}^{\infty}\exp\bigg(-C'T2^{(1-\alpha)\ell}+C2^{\ell/2}+\ell\ln C-\frac{5\ell\ln T}{2}\\ &\qquad +\frac{5\alpha\ell(\ell+1)\ln 2}{4}+\frac{5\alpha\ell\ln 2}{2}\bigg)\|y^0\|_H\end{aligned}$$

$$\leqslant C_\alpha T^{-5/2}\sum_{\ell=1}^{\infty}\exp\left(-C'T2^{(1-\alpha)\ell}+C2^{\ell/2}-\frac{5\ell\ln T}{2}\right)\|y^0\|_H. \tag{29}$$

The last inequality above follows since, as $\ell\to\infty$, $C2^{\ell/2}$ dominates terms in the exponential that are positive and independent of T.

To obtain a bound for $\|u\|$ as $T\searrow 0$, we further estimate (29),

$$\begin{aligned}
&\|u\|_{L^2((0,T)\times\omega)}\\
&\quad\leqslant C_\alpha T^{-5/2}\sum_{\ell=1}^{\infty}\exp\left(-C'T2^{(1-\alpha)\ell}+C2^{\ell/2}-\frac{5\ell\ln T}{2}\right)\|y^0\|_H\\
&\quad= C_\alpha T^{-5/2}\sum_{\ell=1}^{\infty}\exp\left(-C'T2^{(1-\alpha)\ell}\left(1-\frac{C2^{(\alpha-1/2)\ell}}{C'T}+\frac{5\ell2^{(\alpha-1)\ell}\ln T}{2C'T}\right)\right)\|y^0\|_H.
\end{aligned}\tag{30}$$

To bound this series, we first break the sum into two parts where the tail is composed of terms where

$$1-\frac{C2^{(\alpha-1/2)\ell}}{C'T}+\frac{5\ell2^{(\alpha-1)\ell}\ln T}{2C'T}\geqslant\frac{1}{2}. \tag{31}$$

That is, when ℓ is large enough so that

$$\frac{C'T}{2}\geqslant C2^{(\alpha-1/2)\ell}-\frac{5\ell2^{(\alpha-1)\ell}\ln T}{2}.$$

To do this we first consider the function $g(x)=\frac{5x2^{-x/2}}{2}$ for $x>0$. g is bounded above by $\delta:=\frac{5}{e\ln 2}$. Thus, for $T>0$ small,

$$\begin{aligned}
&-\delta\ln T\geqslant\frac{-5\ell2^{-\ell/2}\ln T}{2}\\
&\quad\Leftrightarrow\quad (C-\delta\ln T)2^{(\alpha-1/2)\ell}\geqslant C2^{(\alpha-1/2)\ell}-\frac{5\ell2^{(\alpha-1)\ell}\ln T}{2}.
\end{aligned}$$

Then for

$$\ell>\ell^*=\frac{\ln(2C-2\delta\ln T)-\ln(C'T)}{(1/2-\alpha)\ln 2}, \tag{32}$$

we have that

$$\frac{C'T}{2}\geqslant(C-\delta\ln T)2^{(\alpha-1/2)\ell}\geqslant C2^{(\alpha-1/2)\ell}-\frac{5\ell2^{(\alpha-1)\ell}\ln T}{2}. \tag{33}$$

Thus we have (31) for ℓ as in (32). In arriving at estimate (33), we are using the fact that $(C-\delta\ln T)2^{(\alpha-1/2)\ell}$ is a decreasing function in ℓ. We will now consider the tail $\sum_{\ell>\ell^*}$ of the sum (30). We have $\ln s<s$ when $s>1$, so

$$(1-\alpha)\ell\ln 2<2^{(1-\alpha)\ell}\quad\text{taking } s=2^{(1-\alpha)\ell}. \tag{34}$$

An application of the mean value theorem to the function $g_\beta(T)=e^{-\beta T}$ gives that for $\beta>0$ and $0<T<1$,

$$\frac{1}{1-\exp(-\beta T)}<\frac{\exp\beta}{\beta T}. \tag{35}$$

We now apply the estimates (31), (34) and (35) to the tail of the series in (30),

$$\begin{aligned}
&\sum_{\ell=\ell^*+1}^{\infty} \exp\left(-C'T2^{(1-\alpha)\ell} + C2^{\ell/2} - \frac{5\ell\ln T}{4}\right)\\
&\quad\leqslant \sum_{\ell=\ell^*+1}^{\infty} \exp\left(-\frac{C'T}{2}2^{(1-\alpha)\ell}\right)\\
&\quad\leqslant \sum_{\ell=\ell^*+1}^{\infty} \exp\left(-\frac{C'T(1-\alpha)\ell\ln 2}{2}\right) \quad \text{by (34)}\\
&\quad= \frac{\exp(-\frac{C'T(1-\alpha)\ln 2(\ell^*+1)}{2})}{1-\exp(-\frac{C'T(1-\alpha)\ln 2}{2})}\\
&\quad< \frac{1}{1-\exp(-\frac{C'T(1-\alpha)\ln 2}{2})}\\
&\quad< \frac{2\exp(\frac{C'(1-\alpha)\ln 2}{2})}{C'T(1-\alpha)\ln 2} \quad \text{by (35)}\\
&\quad\leqslant CT^{-1}.
\end{aligned} \tag{36}$$

We break the remaining finite sum into two pieces using the inequality $ab \leqslant \frac{a^2+b^2}{2}$,

$$\begin{aligned}
&\sum_{\ell=1}^{\ell^*} \exp\left(-C'T2^{(1-\alpha)\ell} + C2^{\ell/2} - \frac{5\ell\ln T}{2}\right)\\
&\quad\leqslant \frac{1}{2}\left(\sum_{\ell=1}^{\ell^*} \exp\left(-2C'T2^{(1-\alpha)\ell} + 2C2^{\ell/2}\right) + \sum_{\ell=1}^{\ell^*} \exp(-5\ell\ln T)\right).
\end{aligned} \tag{37}$$

To deal with the first term, consider the following function on $\mathbb{R}$,

$$G_T(x) = -2C'T2^{(1-\alpha)x} + 2C2^{x/2}.$$

As α is fixed in $(0, 1/2)$, G_T enjoys the properties that (i) $\lim_{x\to-\infty} G_T(x) = 0$, (ii) $\lim_{x\to\infty} G_T(x) = -\infty$, and (iii) $\frac{dG_T}{dx}$ has exactly one zero. Therefore G_T has a global maximum at

$$x^* = \frac{1}{(1/2-\alpha)\ln 2}\ln\left(\frac{C}{2C'T(1-\alpha)}\right).$$

Further,

$$\begin{aligned}
G_T(x^*) &= -2C'T\exp\left(\frac{1-\alpha}{1/2-\alpha}\ln\left(\frac{C}{2C'T(1-\alpha)}\right)\right) + 2C\exp\left(\frac{\ln(\frac{C}{2C'T(1-\alpha)})}{1-2\alpha}\right)\\
&\leqslant 2C\left(\frac{C}{2C'T(1-\alpha)}\right)^{(1-2\alpha)^{-1}}\\
&\leqslant CT^{(-1+2\alpha)^{-1}},
\end{aligned}$$

giving the following estimate:

$$\sum_{\ell=1}^{\ell^*} \exp\bigl(-2C'T2^{(1-\alpha)\ell} + 2C2^{\ell/2}\bigr) \leqslant \ell^* \exp\bigl(G_T\bigl(x^*\bigr)\bigr) \\ \leqslant \ell^* \exp\bigl(CT^{(-1+2\alpha)^{-1}}\bigr). \tag{38}$$

The other finite sum can be dealt with in a similar way. For this estimate we need that

$$\ell^* = \frac{\ln(2C - 2\delta \ln T) - \ln(C'T)}{(1/2-\alpha)\ln 2} \leqslant \frac{2C - 2\delta \ln T - \ln(C'T)}{(1/2-\alpha)\ln 2} \leqslant -C\ln T. \tag{39}$$

Also since $s^2 < \exp(\frac{s}{1-2\alpha})$ for $s \gg 1$, we have $(\ln T)^2 < T^{(-1+2\alpha)^{-1}}$ where $0 < T \ll 1$, by taking $s = -\ln T$. Therewith, we have

$$\sum_{\ell=1}^{\ell^*} \exp(-5\ell \ln T) \leqslant \ell^* \exp\bigl(-5\ell^* \ln T\bigr) \\ \leqslant \ell^* \exp\bigl(C(\ln T)^2\bigr) \\ \leqslant \ell^* \exp\bigl(CT^{(-1+2\alpha)^{-1}}\bigr). \tag{40}$$

Combining (30), (36)–(38), and (40), we have that

$$\|u\|_{L^2(0,T)\times\omega} \leqslant C_\alpha T^{-5/2}\left(\sum_{\ell=1}^{\ell^*} \exp\left[-C'T2^{(1-\alpha)\ell} + C2^{\ell/2} - \frac{5\ell \ln T}{2}\right] \right. \\ \left. + \sum_{\ell=\ell^*+1}^{\infty} \exp\left[-C'T2^{(1-\alpha)\ell} + C2^{\ell/2} - \frac{5\ell \ln T}{2}\right]\right)\|y^0\|_H \\ \leqslant C_\alpha T^{-5/2}\bigl[-C\ln T \exp\bigl(CT^{(-1+2\alpha)^{-1}}\bigr) + CT^{-1}\bigr]\|y^0\|_H.$$

Noting that the dominant term on the right-hand side is $\exp(CT^{(-1+2\alpha)^{-1}})$, we have that

$$\|u\|_{L^2((0,T)\times\omega)} \leqslant C_\alpha \exp\bigl(CT^{(-1+2\alpha)^{-1}}\bigr)\|y^0\|_H.$$

Taking $\alpha = \frac{\epsilon}{2(1+\epsilon)}$ and taking the supremum over $y^0 \in H$ with $\|y^0\|_H = 1$, gives that

$$\mathcal{E}(T) = C_\epsilon \exp\bigl(C/T^{1+\epsilon}\bigr) \quad \text{as } T \searrow 0$$

where the definition of the minimal energy is as given in (3). The same argument works to show that Theorem 1 is valid for the thermoelastic system under the influence of a locally distributed thermal source. This completes the proof of Theorem 1.

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