Positive sesquilinear form measures and generalized eigenvalue expansions

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Abstract

Positive operator measures (with values in the space of bounded operators on a Hilbert space) and their generalizations, mainly positive sesquilinear form measures, are considered with the aim of providing a framework for their generalized eigenvalue type expansions. Though there are formal similarities with earlier approaches to special cases of the problem, the paper differs e.g. from standard rigged Hilbert space constructions and avoids the introduction of nuclear spaces. The techniques are predominantly measure theoretic and hence the Hilbert spaces involved are separable. The results range from a Naimark type dilation result to direct integral representations and a fairly concrete generalized eigenvalue expansion for unbounded normal operators.
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1. Introduction

There is a long history of various approaches to the mathematical clarification and justification of the well-known and heuristically appealing formulation of Quantum Mechanics due to P.A.M. Dirac [8]. Rather than try to recount this history, we refer to the recent article [13] for references to, and a unification of, several classical approaches based on the notion of *rigged Hilbert spaces*, and to [11] for a completely different framework of *trajectory spaces*. In Dirac's work, the notions of an "eigenvalue" λ and a corresponding "eigenvector" |λ⟩ of an operator representing a physical observable are in a key role. In general these notions cannot be understood in their conventional mathematical sense, and it is the task of the ancillary mathematical theories to create a rigorous framework for their interpretation.

The present article contributes to this line of study by providing a relatively easily accessible setting for the analysis—spectral in a wide sense of the word—of a class of mathematical objects generalizing the *positive operator measures* which have been successfully used to represent physical observables in situations where the more traditional self-adjoint operators and their spectral measures have proved inadequate [2,3,6,14,15,18]. These generalizations, the so-called *sesquilinear form measures* or *generalized operator measures* were introduced in [23] in order to describe measurement situations where only a restricted class of state preparations are available. They also arise naturally in the quantization of classical phase variables [23].

Despite the physical background, our study is mathematically motivated and addresses e.g. some technical measurability issues interesting in their own right. The organization of the paper is as follows. In Section 2 we introduce sesquilinear form measures in an abstract ("test") vector space and give an illustrative concrete example. Section 3 contains a general representation theorem for positive sesquilinear form measures. Some further analysis leads to a pointwise representation theorem in Section 4, and we relate this result to so-called *direct integral* representations in Section 5. In Section 6 we show that the operations in the previous sections naturally extend from the original test space to a larger space with a Hilbert structure. Positive operator measures in a Hilbert space are then considered in Section 7. This section deals with a special case of the abstract results, but we also give an alternative approach. In Section 8 we further specialize our results to spectral measures of normal operators, showing that the functionals in our general direct integral expansion indeed admit the interpretation as generalized eigenvectors in this case. The concluding Section 9 addresses the question of the relationship between the spectrum of an operator and the set of complex numbers eligible as generalized eigenvalues in some natural sense. An example is provided showing that even in the case of bounded self-adjoint operators complications arise thus indicating the need for further analysis.

When compared with much of the earlier work in this field, our approach appears as predominantly measure theoretic and, in particular, avoids any consideration of nuclear spaces. Inherent in the measure theoretic setting and especially in the use of direct integrals is the separability of the Hilbert spaces involved. As a byproduct of the constructions of Section 3 we obtain a generalization of the (separable Hilbert space version of) the Naimark dilation theorem.

2. Preliminaries on sesquilinear form measures

Let V and W be vector spaces. The scalar field of all vector spaces will be $\mathbb{C}$. A mapping $\Phi : V \times V \to W$ is said to be *sesquilinear*, if it is antilinear (i.e., conjugate linear) in the first and linear in the second variable. If, in addition, $W = \mathbb{C}$, we call Φ a *sesquilinear form*. It is *positive*, if $V(\phi, \phi) \geqslant 0$ for all $\phi \in V$. We let $S(V)$ (respectively $PS(V)$) denote the set of sesquilinear

forms (respectively positive sesquilinear forms) on $V \times V$. It is sometimes useful to observe that $S(V)$ may be naturally identified with the space of all linear maps from V to $V^\times$ where $V^\times$ is the space of antilinear functionals on V. Any sesquilinear map Φ on $V \times V$ satisfies the *polarization identity* $\Phi(\phi, \psi) = \frac{1}{4} \sum_{k=0}^{3} i^k \Phi(\phi + i^k \psi, \phi + i^k \psi)$. A positive sesquilinear form $\Phi : V \times V \to \mathbb{C}$ also satisfies the equation $\Phi(\phi, \psi) = \overline{\Phi(\psi, \phi)}$ and the *Cauchy–Schwarz inequality* $|\Phi(\phi, \psi)|^2 \leqslant \Phi(\phi, \phi)\Phi(\psi, \psi)$ for all $\phi, \psi \in V$.

Let (Ω, Σ) be a measurable space, i.e., Σ is a σ-algebra of subsets of Ω.

Definition 2.1. Let $E : \Sigma \to S(V)$ be a mapping and denote $E(X) = E_X$ for $X \in \Sigma$. We call E a *sesquilinear form measure* if the mapping $X \mapsto E_X(\phi, \psi)$ is σ-additive, i.e. a complex measure, for all $\phi, \psi \in V$. If, in addition, $E(\Sigma) \subset PS(V)$, E is called a *positive sesquilinear form measure* (or a *PSF measure* or just a *PSFM* for short). A PSF measure $E : \Sigma \to PS(V)$ is called *strict*, if $E_\Omega(\phi, \phi) > 0$ for all $\phi \in V \setminus \{0\}$.

In view of the polarization identity, in the above definition it suffices to require that $X \mapsto E_X(\phi, \phi)$ be σ-additive for all $\phi \in V$. For any positive sesquilinear form measure $E : \Sigma \to PS(V)$, the set $N = \{\phi \in V \mid E_\Omega(\phi, \phi) = 0\}$ is, by virtue of the Cauchy–Schwarz inequality, a linear subspace of V, and the mapping $\tilde{E} : \Sigma \to PS(V/N)$ which is (unambiguously) defined by the formula $\tilde{E}_X(\phi + N, \psi + N) = E_X(\phi, \psi)$, is a strict PSFM. It is sometimes convenient to assume that a PSFM is strict. In view of the above observation, in many situations this assumption does not detract essentially from generality.

Positive sesquilinear form measures may be viewed as a generalization of some notions which we now recall. The inner product of any Hilbert space in this paper is linear in the second variable and usually denoted by $\langle \cdot | \cdot \rangle$. We let $\mathcal{L}(H)$ denote the space of bounded linear operators on H, and $\mathcal{L}_+(H) = \{T \in \mathcal{L}(H) \mid T \geqslant 0\}$. We shall later also encounter (the space of) the trace class (operators) on H, denoted by $\mathcal{L}^1(H)$, and its positive cone $\mathcal{L}^1_+(H)$. The Hilbert space of the Hilbert–Schmidt operators on H is denoted by $\mathcal{L}^2(H)$. For a possibly unbounded operator A in H, we denote by $\mathsf{D}(A) \subset H$ its domain of definition, and by $\mathsf{R}(A) = \{A\phi \mid \phi \in \mathsf{D}(A)\}$ its range.

Definition 2.2. Let H be a Hilbert space and $E_0 : \Sigma \to \mathcal{L}_+(H)$ a mapping. We call E_0 a *positive operator* (*valued*) *measure* or a *POM* for short, if it weakly σ-additive, i.e. the mapping $X \mapsto \langle \phi | E_0(X)\psi \rangle$ is σ-additive for all $\phi, \psi \in H$. If here $E_0(X)^2 = E_0(X) = E_0(X)^*$ for all $X \in \Sigma$, E_0 is called a projection (valued) measure. A POM $E_0 : \Sigma \to \mathcal{L}(H)$ is called *normalized* if $E_0(\Omega) = I$, the identity operator on H. A normalized POM is also called a *semispectral measure* and a normalized projection measure a *spectral measure*.

Every POM E_0 can be identified with a PSF measure E by setting $E_X(\phi, \psi) := \langle \phi | E_0(X)\psi \rangle$.

Next we exhibit a concrete example of sesquilinear form measures. We show that, for any weighted shift operator, there exists a unique (shift covariant [4,23]) sesquilinear form measure on the circle which is determined by a unique complex matrix. From the structure of the matrix one can easily see when the corresponding sesquilinear form measure is positive and defines a POM. This is the case exactly when the shift operator is contractive. As a byproduct we get the well-known result that the powers of a contractive shift operator are the moment operators of a unique semispectral measure [21, p. 235].

Example 2.3. Consider a Hilbert space H with an orthonormal basis $(e_n)_{n \in \mathbb{Z}}$ and a weighted shift operator $S : e_n \mapsto c_{n-1} e_{n-1}$ where $(c_n)_{n \in \mathbb{Z}} \subset \mathbb{C}$. Let $V := \lim(e_n)_{n \in \mathbb{Z}}$ and define a matrix

$(c_{mn})_{m,n\in\mathbb{Z}}$ by $c_{mm} := 1$, $c_{mn} := \prod_{l=m}^{n-1} c_l$ and $c_{nm} := \overline{c_{mn}}$ for all $m < n$. For any sesquilinear form $\Phi : V \times V \to \mathbb{C}$ we use the formal notation $\sum_{m,n\in\mathbb{Z}} \Phi(e_m, e_n)|e_m\rangle\langle e_n|$ which we understand as $\sum_{m,n\in\mathbb{Z}} \Phi(e_m, e_n)\langle\varphi|e_m\rangle\langle e_n|\psi\rangle = \Phi(\varphi, \psi)$ for all $\varphi, \psi \in V$.

Let $\mathcal{B}(\mathbb{T})$ be the Borel σ-algebra of the unit circle $\mathbb{T}$, and let $\nu : \mathcal{B}(\mathbb{T}) \to [0, 1]$ be the normalized Haar measure of $\mathbb{T}$. Define a sesquilinear form measure $E_S : \mathcal{B}(\mathbb{T}) \to S(V)$ by

$$E_S(X) := \sum_{m,n\in\mathbb{Z}} c_{mn} \int_X \lambda^{m-n}\, d\nu(\lambda)|e_m\rangle\langle e_n|, \quad X \in \mathcal{B}(\mathbb{T}),$$

and, for all $k \in \mathbb{Z}$, the kth moment form by

$$E_S^{(k)} := \sum_{m,n\in\mathbb{Z}} c_{mn} \int_{\mathbb{T}} \lambda^k \lambda^{m-n}\, d\nu(\lambda)|e_m\rangle\langle e_n| \in S(V).$$

Since $\int_{\mathbb{T}} \lambda^k \lambda^{m-n}\, d\nu(\lambda) = 1$ when $k + m - n = 0$ and 0 otherwise, it is easy to see that, for all $\varphi, \psi \in V$,

$$E_S^{(k)}(\varphi, \psi) = \begin{cases} \langle\varphi|S^k\psi\rangle, & k > 0, \\ \langle\varphi|(S^*)^{-k}\psi\rangle, & k < 0, \\ \langle\varphi|\psi\rangle, & k = 0, \end{cases}$$

so that the moment forms of E_S can be (uniquely) extended to the powers of S and S^*. By analyzing the structure of E_S one can derive the following result:

Proposition 2.4. *E_S is positive if and only if S is a contraction, i.e., $|c_n| \leqslant 1$ for all $n \in \mathbb{Z}$. In this case E_S has a (unique) extension to the semispectral measure $\mathcal{B}(\mathbb{T}) \to \mathcal{L}(H)$ which is a spectral measure if and only if $|c_n| = 1$ for all $n \in \mathbb{Z}$ (if and only if S is unitary).*

Proof. It can be shown (see e.g. [4] or [23]) that E_S is positive (and, moreover, has a unique extension to a semispectral measure) if and only if the Hermitian matrix (c_{mn}) is positive semidefinite. Since (c_{mn}) is positive semidefinite if and only if the principal minors of (c_{mn}) are nonnegative, it is easy to show that (c_{mn}) is positive semidefinite if and only if S is a contraction. Indeed, let $s \in \{2, 3, 4, \ldots\}$ and $\{k_1, k_2, \ldots, k_s\} \subset \mathbb{Z}$ where $k_1 < k_2 < \cdots < k_s$. Since $c_{mn} = \prod_{l=m}^{n-1} c_l$, $m < n$, by induction (see [19, Theorem 4.1]), the principal minor can be computed as

$$\begin{vmatrix} 1 & c_{k_1,k_2} & \cdots & c_{k_1,k_s} \\ \overline{c_{k_1,k_2}} & 1 & \cdots & c_{k_2,k_s} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{c_{k_1,k_s}} & \overline{c_{k_2,k_s}} & \cdots & 1 \end{vmatrix} = \prod_{l=1}^{s-1}\bigl[1 - |c_{k_l,k_{l+1}}|^2\bigr].$$

Hence, (c_{mn}) is positive semidefinite if and only if $|c_{mn}| = \prod_{l=m}^{n-1} |c_l| \leqslant 1$ for all $m < n$, and the claim follows. If E_S is positive, then its extension is a spectral measure if and only if $|c_n| = 1$ for all $n \in \mathbb{Z}$ [4, Proposition 3]. $\quad\square$

3. Representing positive sesquilinear form measures

For the rest of the paper, unless otherwise specified, we assume that $(e_n)_{n=0}^{\infty}$ is a (countably infinite) Hamel basis of V, indexed by $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $E : \Sigma \to PS(V)$ is a positive

sesquilinear form measure. We fix a sequence of positive numbers α_n with $\sum_{n=0}^{\infty} \alpha_n < \infty$ and write

$$\mu(X) = \sum_{n=0}^{\infty} \alpha_n E_X(e_n, e_n) \left[1 + E_\Omega(e_n, e_n)\right]^{-1} \tag{3.1}$$

for all $X \in \Sigma$. Then μ is a finite positive measure, and an application of the Cauchy–Schwarz inequality shows that for $X \in \Sigma$, $\mu(X) = 0$ if and only if $E_X(\phi, \psi) = 0$ whenever $\phi, \psi \in V$. The Radon–Nikodým theorem thus yields for any $\phi, \psi \in V$ a unique element $C(\phi, \psi) \in L^1(\mu)$ such that $E_X(\phi, \psi) = \int_X C(\phi, \psi) \, d\mu$ for all $X \in \Sigma$. Clearly the mapping $(\phi, \psi) \mapsto C(\phi, \psi)$ on $V \times V$ is sesquilinear, and $C(\phi, \phi) \geqslant 0$ in the L^1-sense for all $\phi \in V$.

We let $\mathcal{F}$ denote the vector space of Σ-simple V-valued functions on Ω. If $A \subset \Omega$, χ_A is the characteristic function of A, and for any $\phi \in V$, we denote by $\phi \chi_A$ the function $x \mapsto \chi_A(x)\phi$.

Lemma 3.2. *There is a unique sesquilinear form $\theta : \mathcal{F} \times \mathcal{F} \to \mathbb{C}$ satisfying $\theta(\phi \chi_A, \psi \chi_B) = \int_{A \cap B} C(\phi, \psi) \, d\mu$ for all $A, B \in \Sigma$, $\phi, \psi \in V$. For any $f = \sum_{i=1}^{m} \phi_i \chi_{A_i}$ and $g = \sum_{j=1}^{n} \psi_j \chi_{A_j}$, there holds*

$$\theta(f, g) = \sum_{i=1}^{m} \sum_{j=1}^{n} \int_{A_i \cap B_j} C(\phi_i, \psi_j) \, d\mu, \tag{3.3}$$

and θ is a positive sesquilinear form.

Proof. For $f, g \in \mathcal{F}$, choose representations $f = \sum_{i=1}^{m} \phi_i \chi_{A_i}$ and $g = \sum_{j=1}^{n} \psi_j \chi_{B_j}$ with e.g. all the ϕ_i distinct and the A_i disjoint, and define $\theta(f, g)$ by the formula (3.3). Then θ is well defined, and obvious refinement arguments yield the remaining statements. $\square$

We denote $\mathcal{N} = \{f \in \mathcal{F} \mid \theta(f, f) = 0\}$. Then $\mathcal{N}$ is, by the Cauchy–Schwarz inequality, a vector subspace of $\mathcal{F}$, and we get a well-defined inner product of $\mathcal{F}/\mathcal{N}$ via $\langle [f] | [g] \rangle = \theta(f, g)$ where e.g. $[f] = f + \mathcal{N}$. We let K denote the Hilbert space completion of this inner product space and use the notation $\langle \cdot | \cdot \rangle$ for the inner product of K. We refer to K as the *associated Hilbert space* of the PSF measure E (relative to the basis (e_n)) and the sequence (α_k).

Lemma 3.4. *For each $X \in \Sigma$ there is a unique bounded linear operator $F(X) : K \to K$ satisfying $\langle [\phi \chi_A] | F(X)[\psi \chi_B] \rangle = \int_{X \cap A \cap B} C(\phi, \psi) \, d\mu$ for all $A, B \in \Sigma$, $\phi, \psi \in V$. Moreover, $F(X)^2 = F(X)^* = F(X)$ and $F(X)[f] = [\chi_X f]$ for all $X \in \Sigma$, $f \in \mathcal{F}$.*

Proof. If $X \in \Sigma$, $f \in \mathcal{F}$ and $g \in [f]$, then $\theta(\chi_X f - \chi_X g, \chi_X f - \chi_X g) \leqslant \theta(f - g, f - g) = 0$, and so the definition $F_0(X)[f] = [\chi_X f]$ is unambiguous. It is also clear that

$$\langle F_0(X)[f] \mid F_0(X)[f] \rangle \leqslant \langle [f] | [f] \rangle.$$

Thus $F_0(X)$ extends uniquely to a bounded linear map $F(X) : K \to K$. The remaining statements are immediate. $\square$

Lemma 3.5. *The map $X \mapsto F(X)$ on Σ is a spectral measure.*

Proof. Clearly $F(\Omega) = I$, and since $\|F(X)\| \leqslant 1$, for weak σ-additivity it is sufficient to note that the map $X \mapsto \langle [\phi \chi_A] | F(X)[\psi \chi_B] \rangle = \int_{X \cap A \cap B} C(\phi, \psi) \, d\mu$ on Σ is σ-additive for all $A, B \in \Sigma$, $\phi, \psi \in V$. $\square$

We now define $J : V \to K$ by the formula $J\phi = [\phi\chi_\Omega]$. Then J is a linear map. We collect and complement the above arguments in the following theorem.

Theorem 3.6. *Let* $E : \Sigma \to PS(V)$ *be a PSFM.*

(a) *There is a Hilbert space* K *with a spectral measure* $F : \Sigma \to \mathcal{L}(K)$ *and a linear map* $J : V \to K$ *such that* $\langle J\phi | F(X)J\psi\rangle = E_X(\phi, \psi)$ *for all* $X \in \Sigma$ *and* $\phi, \psi \in V$, *and moreover, the linear span of the set* $\{F(X)J\phi \mid X \in \Sigma, \ \phi \in V\}$ *is dense in* K.
(b) *This representation of* E *is essentially unique in the sense that if the triple* (K_1, F_1, J_1) *gives another representation with these properties, there is a unique unitary map* $U : K \to K_1$ *such that* $UF(X)J\phi = F_1(X)J_1\phi$ *for all* $X \in \Sigma$, $\phi \in V$; *in particular,* $UJ\phi = J_1\phi$ *for all* $\phi \in V$. *Moreover,* $UF(X) = F_1(X)U$ *for all* $X \in \Sigma$.
(c) *In the situation of* (a), J *is injective if and only if* E *is strict.*

Proof. (a) In the above construction,

$$\langle J\phi | F(X)J\psi\rangle = \langle [\phi\chi_\Omega] | F(X)[\phi\chi_\Omega]\rangle = \int_X C(\phi, \psi)\, d\mu = E_X(\phi, \psi).$$

The density statement is also an immediate consequence of the construction.

(b) The uniqueness of U is clear since it is determined on a dense subspace of K. On the other hand, if $X_1, \dots, X_n \in \Sigma$ and $\phi_1, \dots, \phi_n \in V$, then

$$\left\| \sum_{i=1}^n F(X_i)J\phi_i \right\|^2 = \sum_{i=1}^n \sum_{j=1}^n \langle F(X_i)J\phi_i | F(X_j)J\phi_j\rangle = \sum_{i=1}^n \sum_{j=1}^n \langle J\phi_i | F(X_i \cap X_j)J\phi_j\rangle$$

$$= \sum_{i=1}^n \sum_{j=1}^n E_{X_i \cap X_j}(\phi_i, \phi_j) = \left\| \sum_{i=1}^n F_1(X_i)J_1\phi_i \right\|^2.$$

Thus there is a well-defined isometric linear map sending each $\sum_{i=1}^n F(X_i)J\phi_i$ to $\sum_{i=1}^n F_1(X_i)J_1\phi_i$, and this map extends by continuity to a unitary $U : K \to K_1$. In particular, $UJ\phi = UF(\Omega)J\phi = F_1(\Omega)J_1\phi = J_1\phi$ for all $\phi \in V$. Moreover, $UF(X)F(Y)J\phi = UF(X \cap Y)J\phi = F_1(X \cap Y)J_1\phi = F_1(X)F_1(Y)J_1\phi = F_1(X)UF(Y)J\phi$ for all $X, Y \in \Sigma$, $\phi \in V$, from which the equation $UF(X) = F_1(X)U$ follows.

(c) Suppose first that the triple (K, F, J) is obtained by the measure theoretic construction preceding the theorem. Then

$$\|J\phi\|^2 = \left\| [\phi\chi_\Omega] \right\|^2 = \theta(\phi\chi_\Omega, \phi\chi_\Omega) = \int_\Omega C(\phi, \phi)\, d\mu = E_\Omega(\phi, \phi), \tag{3.7}$$

so in particular $J\phi$ vanishes if and only if $E_\Omega(\phi, \phi)$ does, and the claim follows. In the case of another triple (K_1, F_1, J_1), let $U : K \to K_1$ be as in (b). Since $UJ\phi = J_1\phi$ and U is bijective, $J\phi = 0$ if and only if $J_1\phi = 0$. $\quad\square$

Remark 3.8. The uniqueness part of the above result shows, in particular, that the choice of the basis (e_n) and the sequence (α_n) does not essentially influence the resulting structure. In fact μ could be replaced by any finite positive measure ν such that every complex measure $X \mapsto E_X(\phi, \psi)$ is absolutely continuous with respect to ν.

Remark 3.9. Let H be a separable Hilbert space, $(e_n)_{n=0}^\infty$ an orthonormal basis of H, and $V := \mathrm{lin}(e_n)_{n=0}^\infty$ its linear span. Suppose that $E_0 : \Sigma \to \mathcal{L}(H)$ is a semispectral measure and let $E : \Sigma \to PS(V)$ be the PSFM defined by $E_X(\phi, \psi) = \langle \phi | E_0(X)\psi \rangle$ for $X \in \Sigma$, $\phi, \psi \in V$. Retaining the notation of the general case, now $J : V \to K$ is isometric, because $\|J\phi\|^2 = E_\Omega(\phi, \phi) = \|\phi\|^2$ by (3.7).

In this case the spectral measure $F : \Sigma \to \mathcal{L}(K)$ is the minimal Naimark dilation of E_0 (see e.g. [22]). It follows form an observation made in [20, p. 171] that the original semispectral measure E_0 is a spectral measure if and only if $J(V)$ is dense in K, or equivalently, $(Je_n)_{n=0}^\infty$ is an orthonormal basis of K.

4. Pointwise representation

In the previous section we obtained the representation $E_X(\phi, \psi) = \int_X C(\phi, \psi) \, d\mu$, where $C : V \times V \to L^1(\mu)$ is sesquilinear and $C(\phi, \phi) \geqslant 0$ in the $L^1(\mu)$ sense. As a step towards the main result of this section, let us now provide a pointwise version of this formula. We use the notion of μ-measurability as in [9] and often call it just measurability. Since μ is a finite measure, for scalar function this simply means measurability with respect to the Lebesgue extension of Σ with respect to μ. We say that $\Omega \ni \omega \mapsto C_\omega \in PS(V)$ is a (μ-)*measurable family of forms* if $\omega \mapsto C_\omega(\phi, \psi)$ is μ-measurable for all $\phi, \psi \in V$.

Lemma 4.1. *For a PSF measure $E : \Sigma \to PS(V)$, there is a measurable family of forms $\Omega \ni \omega \mapsto C_\omega \in PS(V)$, such that for all $\phi, \psi \in V$, the function $\omega \mapsto C_\omega(\phi, \psi)$ is a representative of $C(\phi, \psi) \in L^1(\mu)$.*

Proof. For all $m, n \in \mathbb{N}$, let us pick a function representative $g_{mn} \in C(e_m, e_n)$. For every $\omega \in \Omega$ and any $\phi, \psi \in V$, we define

$$C_\omega(\phi, \psi) := \sum_{m,n=0}^\infty \bar{a}_m g_{mn}(\omega) b_n, \tag{4.2}$$

where $\phi = \sum_{m=0}^\infty a_m e_m$, $\psi = \sum_{n=0}^\infty b_n e_n$ are the unique expansions in the Hamel basis $(e_n)_{n=0}^\infty$, and only finitely many of the coefficients $a_m, b_n \in \mathbb{C}$ are non-zero. Then it is immediate that $C_\omega : V \times V \to \mathbb{C}$ is a sesquilinear form, and the (measurable) function $\omega \mapsto C_\omega(\phi, \psi)$ is a representative of $C(\phi, \psi)$. In particular $C_\omega(\phi, \phi) \geqslant 0$ for μ-a.e. $\omega \in \Omega$. If we only consider the countable set $W := \mathrm{lin}_{\mathbb{Q}+i\mathbb{Q}}(e_n)_{n=0}^\infty$, we can choose a single μ-null set $Z \subset \Omega$ such that $C_\omega(\phi, \phi) \geqslant 0$ for all $\omega \in \Omega \setminus Z$ and $\phi \in W$. But for a general $\phi \in V$, we may approximate the finitely many non-zero coefficient a_m in (4.2) by complex rationals, getting the same result for all $\phi \in V$. Thus $C_\omega \in PS(V)$ for all $\omega \in \Omega \setminus Z$. If we redefine $C_\omega(\phi, \psi) := 0$ for $\omega \in Z$ (which is achieved by changing, if necessary, the functions g_{mn} to have the value zero in Z), we have $C_\omega \in PS(V)$ for all $\omega \in \Omega$, and $\omega \mapsto C_\omega(\phi, \psi)$ is still a representative of $C(\phi, \psi)$ for all $\phi, \psi \in V$. $\quad\square$

We now introduce some notational conventions which depend on the choice of the fixed Hamel basis $(e_n)_{n=0}^\infty$ of V. If $\phi, \psi \in V$ have the basis expansions $\phi = \sum_{n=0}^\infty a_n e_n$ and $\psi = \sum_{n=0}^\infty b_n e_n$ (with only finitely many non-zero terms), we write $\langle \phi | \psi \rangle = \sum_{n=0}^\infty \bar{a}_n b_n$. Then V becomes an inner-product space, and each $\phi \in V$ gives rise to the linear form $\psi \mapsto \langle \phi | \psi \rangle$, denoted by $\langle \phi |$, and to the antilinear form $\psi \mapsto \langle \psi | \phi \rangle$, denoted by $| \phi \rangle$. Clearly, the space of all linear

functionals on V, i.e., the algebraic dual V^* of V, is in a bijective antilinear correspondence with the vector space of all complex sequences (d_n), when $d = (d_n)$ is made to correspond to the functional $\psi = \sum_{n=0}^{\infty} b_n e_n \mapsto \sum_{n=0}^{\infty} \bar{d}_n b_n$, denoted also by $\langle d|$ in the sequel; we may write this as $\langle d|\psi \rangle = \sum_{n=0}^{\infty} \bar{d}_n b_n$. In this kind of situations we also allow the notation $\langle \psi|d \rangle = \sum_{n=0}^{\infty} \bar{b}_n d_n$, so that $\langle \psi|d \rangle = \overline{\langle d|\psi \rangle}$. These notations are consistent when we use the identification of a vector $\phi = \sum_{n=0}^{\infty} a_n e_n \in V$ with the sequence (a_n) of its coefficients. Extending this identification, we sometimes consider V embedded (linearly) in the space of all sequences $(a_n)_{n=0}^{\infty}$ and identify a sequence $(a_n)_{n=0}^{\infty}$ with the formal series $\sum_{n=0}^{\infty} a_n e_n$. In particular note that $\langle e_n|$ is the linear functional on V which maps $\phi = \sum_{k=0}^{\infty} a_k e_k$ into a_n. It is sometimes convenient to denote by the formal series $\sum_{n=0}^{\infty} a_n \langle e_n|$ the element of V^* corresponding in our convention to the sequence $(\bar{a}_n)$. If $\phi = \sum_{n=0}^{\infty} a_n e_n \in V$, i.e., the series is not just formal, then $\langle \phi| = \sum_{n=0}^{\infty} \bar{a}_n \langle e_n|$.

If $\langle d_1|, \langle d_2| \in V^*$, we denote by $|d_1\rangle\langle d_2|$ the sesquilinear form $(\phi, \psi) \mapsto \langle \phi|d_1 \rangle\langle d_2|\psi \rangle$; it can equivalently be viewed as the antilinear map $\phi \mapsto \langle \phi|d_1 \rangle\langle d_2|$ from V to V^*. Note that $|d\rangle\langle d| \in PS(V)$ for any $\langle d| \in V^*$.

In the sequel, we make various measurability assertions concerning vector-valued functions. When saying that a V-valued function $\omega \mapsto F(\omega)$ is measurable, unless otherwise specified, we mean *weak measurability* with respect to the dual pair $\langle V, V^* \rangle$, i.e., that the scalar-valued functions $\omega \mapsto \langle d|F(\omega) \rangle$ are measurable for all $\langle d| \in V^*$.

Lemma 4.3. *A V-valued function $\omega \mapsto F(\omega)$ is measurable if and only if all the coordinate functions $\omega \mapsto \langle e_n|F(\omega) \rangle$, $n \in \mathbb{N}$, are measurable.*

Proof. One direction is clear, since $\langle e_n| \in V^*$. But if the coordinate functions are measurable and $\langle d| = \sum_{k=0}^{\infty} c_k \langle e_k| \in V^*$, then also $\langle d|F(\omega) \rangle = \sum_{k=0}^{\infty} c_k \langle e_k|F(\omega) \rangle$ is measurable as the sum of an everywhere convergent series of measurable functions. $\quad\square$

Lemma 4.4. *For a μ-measurable family of forms $C_\omega \in PS(V)$, $\omega \in \Omega$, there exist μ-measurable mappings $\omega \mapsto n(\omega) \in \mathbb{N} \cup \{\infty\}$ and $\omega \mapsto g_k(\omega) \in V$, $k \in \mathbb{N}$, such that for all $\omega \in \Omega$:*

- *$C_\omega(g_k(\omega), g_\ell(\omega)) = \delta_{k\ell} \chi_{\{\omega'|k < n(\omega')\}}(\omega)$.*
- *$C_\omega(\phi, \psi) = \sum_{k=0}^{n(\omega)-1} C_\omega(\phi, g_k(\omega))C_\omega(g_k(\omega), \psi)$ for all $\phi, \psi \in V$, and only finitely many terms are non-zero for fixed ϕ and ψ even when $n(\omega) = \infty$.*
- *$\omega \mapsto C_\omega(g_k(\omega), \phi)$ is μ-measurable for every $\phi \in V$.*

Proof. For every $\omega \in \Omega$, we denote $\mathcal{N}_\omega := \{\phi \in V \mid C_\omega(\phi, \phi) = 0\}$. In complete analogy with the space $\mathcal{N}$ of the previous section, we see that this is a linear subspace of V, and on $V/\mathcal{N}_\omega$ we get a well-defined inner-product $\langle [\phi]_\omega|[\psi]_\omega \rangle_\omega := C_\omega(\phi, \psi)$, where e.g. $[\phi]_\omega := \phi + \mathcal{N}_\omega$.

We now apply the Gram–Schmidt procedure on $(V/\mathcal{N}_\omega, \langle \cdot|\cdot \rangle_\omega)$, starting from the spanning sequence $([e_n]_\omega)_{n=0}^{\infty}$. Instead of doing this at the completely abstract level, however, we work with concrete representatives in V in order to keep track of the μ-measurability of our operations. We also do not discard possible zero-vectors in the first place.

Denote $\{\phi\}_\omega^0 := C_\omega(\phi, \phi)^{-1/2}\phi$ if $[\phi]_\omega \neq 0$ and $\{\phi\}_\omega^0 := 0$ otherwise. Then let

$$f_0(\omega) := \{e_0\}_\omega^0, \quad f_n(\omega) := \left\{ e_n - \sum_{k=0}^{n-1} C_\omega\big(f_k(\omega), e_n\big) f_k(\omega) \right\}_\omega^0, \quad n = 1, 2, \dots.$$

It is easily seen that $f_n(\omega) \in \mathrm{lin}\{e_0, \ldots, e_n\}$ and

$$\mathrm{lin}\{f_0(\omega), \ldots, f_n(\omega)\} + \mathcal{N}_\omega = \mathrm{lin}\{e_0, \ldots, e_n\} + \mathcal{N}_\omega$$

for all $n \in \mathbb{N}$ and $\omega \in \Omega$. Moreover, the functions $\omega \mapsto f_n(\omega)$ are μ-measurable.

Next, we remove the possible zero-vectors in a measurable way: Let $n_{-1}(\omega) := -1$ and

$$n_k(\omega) := \inf\{n \in \mathbb{N} \mid n > n_{k-1}(\omega),\ f_n(\omega) \neq 0\} \in \mathbb{N} \cup \{\infty\}, \quad k \in \mathbb{N},$$
$$n(\omega) := 1 + \sup\{k \in \mathbb{N} \cup \{-1\} \mid n_k(\omega) < \infty\} \in \mathbb{N} \cup \{\infty\},$$

where $\inf \emptyset := \infty$. Finally, we set $g_k(\omega) := f_{n_k(\omega)}(\omega)$ for $0 \leqslant k < n(\omega)$. These are the non-zero vectors from the Gram–Schmidt procedure, and hence $([g_k(\omega)]_\omega)_{k=0}^{n(\omega)-1}$ is an orthonormal Hamel basis of $(V/\mathcal{N}_\omega, \langle \cdot | \cdot \rangle_\omega)$. This implies the first two claims after setting $g_k(\omega) := 0$ for $k \geqslant n(\omega)$. The last claim follows from the formula

$$C_\omega(g_k(\omega), \phi) = \sum_{j=0}^{n_k(\omega)-1} \bar{f}_{n_k(\omega),j}(\omega) C_\omega(e_j, \phi),$$

where $f_{n_k(\omega),j}(\omega)$ stands for the jth coordinate of $f_{n_k(\omega)}(\omega)$, and from the measurability of sums and products of measurable functions. $\quad\square$

In the above representation, $\phi \mapsto C_\omega(g_k(\omega), \phi)$ is a linear functional on V. There is a unique complex sequence $d_k(\omega)$ such that this functional equals $\langle d_k(\omega)| \in V^*$. By the measurability of V^*-valued functions, we understand the weak* measurability, i.e., the measurability of their pointwise duality pairings with all $\phi \in V$. By an argument similar to, but even easier than, Lemma 4.3, this is seen to be equivalent to the measurability of all the coordinate functions, i.e., it suffices to test the pairings with all $\phi = e_n$, $n \in \mathbb{N}$. For $\omega \mapsto \langle d_k(\omega)|$ this measurability condition is precisely the last claim of the previous lemma. With these remarks, and a combination of Lemmas 4.1 and 4.4, we have the following result, in which $E : \Sigma \to PS(V)$ is a PSFM, and all our earlier choices and notations apply.

Theorem 4.5. *There are μ-measurable mappings $\omega \mapsto n(\omega) \in \mathbb{N} \cup \{\infty\}$, $\omega \mapsto g_k(\omega) \in V$ and $\omega \mapsto \langle d_k(\omega)| \in V^*$, $k \in \mathbb{N}$, such that $\langle d_k(\omega)|g_\ell(\omega)\rangle = \delta_{k\ell} \chi_{\{\omega' \mid k < n(\omega')\}}(\omega)$ for all $\omega \in \Omega$, and the following representation holds for all $\phi, \psi \in V$:*

$$E_X(\phi, \psi) = \int\limits_X \sum_{k=0}^{n(\omega)-1} \langle \phi | d_k(\omega) \rangle \langle d_k(\omega) | \psi \rangle \, d\mu(\omega),$$

where only finitely many terms in the sum are non-zero for each ω, even when $n(\omega) = \infty$. In particular,

$$E_X(\phi, \phi) = \int\limits_X \sum_{k=0}^{n(\omega)-1} \left| \langle d_k(\omega) | \phi \rangle \right|^2 d\mu(\omega). \tag{4.6}$$

5. Relation to direct integrals

In this section we make some comments on the relation of Theorem 4.5 to the direct integral representations which are popular in some of the related literature. We use the following notational conventions. As usual, ℓ^2 is the Hilbert space of square summable sequences $(a_n)_{n=0}^{\infty}$, and

$L^2(\Omega, \mu, \ell^2)$ is the Hilbert space consisting of the (μ-equivalence classes of) μ-measurable ℓ^2-valued functions f for which the function $\omega \mapsto \|f(\omega)\|^2$ is μ-integrable. For any $k \in \mathbb{N} \cup \{\infty\}$ we denote $\ell_k^2 = \{(a_n) \in \ell^2 \mid a_n = 0 \text{ for all } n \geqslant k\}$. In particular, $\ell_0^2 = \{0\}$ and $\ell_\infty^2 = \ell^2$.

Let now $\omega \mapsto n(\omega)$ be a μ-measurable map from Ω into $\mathbb{N} \cup \{\infty\}$. We denote by $L_{n(\cdot)}^2(\Omega, \mu, \ell^2)$ the subset of $L^2(\Omega, \mu, \ell^2)$ consisting of those f for which $f(\omega) \in \ell_{n(\omega)}^2$ μ-almost everywhere. A routine argument based on the fact that a sequence converging in $L^2(\Omega, \mu, \ell^2)$, and hence in μ-measure, contains an almost everywhere convergent subsequence shows that $L_{n(\cdot)}^2(\Omega, \mu, \ell^2)$ is a closed subspace of $L^2(\Omega, \mu, \ell^2)$.

The above construction of the space $L_{n(\cdot)}^2(\Omega, \mu, \ell^2)$ is a relatively simple way of making rigorous the heuristic idea of an L^2 space of functions taking their pointwise values in Hilbert spaces of different dimensions. This type of space is often referred to as the *direct integral* of Hilbert spaces and denoted by $\int_\Omega^\oplus \ell_{n(\omega)}^2 \, d\mu(\omega)$.

We now return to Theorem 4.5 and its notation; in particular, let $\omega \mapsto n(\omega)$ henceforth stand for the fixed map appearing in that theorem. If $\phi \in V$, define $J_1 \phi$ to be the element $\omega \mapsto (\langle d_k(\omega) | \phi \rangle)_{k=0}^\infty$ of $L_{n(\cdot)}^2(\Omega, \mu, \ell^2)$. This produces a linear map $J_1 : V \to L_{n(\cdot)}^2(\Omega, \mu, \ell^2)$. For $X \in \Sigma$, let $F_1(X)$ be the restriction of the multiplication map $f \mapsto \chi_X f$ to the invariant subspace $L_{n(\cdot)}^2(\Omega, \mu, \ell^2)$ of $L^2(\Omega, \mu, \ell^2)$. Then $F_1 : \Sigma \to \mathcal{L}(L_{n(\cdot)}^2(\Omega, \mu, \ell^2))$ is a spectral measure.

Theorem 5.1. *The linear span of the set $\{F_1(X) J_1 \phi \mid X \in \Sigma, \ \phi \in V\}$ is dense in $K_1 := L_{n(\cdot)}^2(\Omega, \mu, \ell^2)$, and $\langle J_1 \phi | F_1(X) J_1 \psi \rangle = E_X(\phi, \psi)$ for any $X \in \Sigma$ and $\phi, \psi \in V$. Thus the triple (K_1, F_1, J_1) is unitarily equivalent to the triple (K, F, J) in the sense of Theorem* 3.6.

Proof. To prove the density statement, let $G = [\omega \mapsto (G_k(\omega))_{k=0}^\infty] \in L_{n(\cdot)}^2(\Omega, \mu, \ell^2)$ be orthogonal to $F_1(X) J_1 e_m$ for all $m \in \mathbb{N}$ and $X \in \Sigma$. This means that

$$\int\limits_X \sum_{k=0}^{n(\omega)-1} \bar{G}_k(\omega) \langle d_k(\omega) | e_m \rangle \, d\mu(\omega) = 0 \quad \text{for all } X \in \Sigma, \ m \in \mathbb{N}.$$

But this implies that $\sum_{k=0}^{n(\omega)-1} \bar{G}_k(\omega) \langle d_k(\omega) | e_m \rangle = 0$ for all $m \in \mathbb{N}$ and μ-a.e. $\omega \in \Omega$, where the exceptional μ-null set, say Z, may be taken independent of $m \in \mathbb{N}$. Recall that $g_\ell(\omega) \in V = \text{lin}(e_m)_{m=0}^\infty$. By linearity, we obtain $\sum_{k=0}^{n(\omega)-1} \bar{G}_k(\omega) \langle d_k(\omega) | g_\ell(\omega) \rangle = 0$ for all $\ell \in \mathbb{N}$ and all $\omega \in \Omega \setminus Z$. But the left-hand side is $\bar{G}_\ell(\omega)$, and hence we have shown that G vanishes as an element of $L_{n(\cdot)}^2(\Omega, \mu, \ell^2)$. This proves our first claim. If $X \in \Sigma$ and $\phi, \psi \in V$, then

$$\langle J_1 \phi | F_1(X) J_1 \psi \rangle = \int\limits_X \sum_{k=0}^{n(\omega)-1} \langle \phi | d_k(\omega) \rangle \langle d_k(\omega) | \psi \rangle \, d\mu(\omega) = E_X(\phi, \psi)$$

by Theorem 4.5. The asserted unitary equivalence now follows from Theorem 3.6. $\square$

As a consequence of Theorem 5.1 and Remark 3.9, we have:

Corollary 5.2. *Let $E_X(\phi, \psi) = \langle \phi | E_0(X) \psi \rangle$ for a semispectral measure E_0. Then E_0 is a spectral measure if and only if*

$$\omega \mapsto \left(\langle d_k(\omega) | e_n \rangle \right)_{k=0}^\infty, \quad n \in \mathbb{N}, \ \text{is an orthonormal basis of } L_{n(\cdot)}^2\big(\Omega, \mu, \ell^2\big). \tag{5.3}$$

6. Extension of the test vector space

Our considerations so far have taken place in the space V, which only has an algebraic vector space structure. We now show that the operations on V that we have been dealing with actually extend to a Hilbert space completion of this initial space that we started with. Let us denote

$$H := \left\{ \psi = \sum_{n=0}^{\infty} b_n e_n \;\middle|\; \|\psi\|_H^2 := \sum_{n=0}^{\infty} |b_n|^2 < \infty \right\}.$$

Then H is a Hilbert space with the orthonormal basis $(e_n)_{n=0}^{\infty}$, whose linear span is V.

Recall the definition of the measure μ from (3.1). We use the same quantities from this definition to introduce the linear operator $\Lambda : H \to H$, given by

$$\Lambda := \sum_{n=0}^{\infty} \beta_n |e_n\rangle\langle e_n|, \qquad \beta_n := \frac{\alpha_n}{1 + E_\Omega(e_n, e_n)}. \tag{6.1}$$

This operator is positive and injective, and its range is dense in H. We shall also use the functional calculus of Λ, i.e., the operators $\eta(\Lambda) := \sum_{n=0}^{\infty} \eta(\beta_n)|e_n\rangle\langle e_n|$ for functions $\eta : \mathbb{R}_+ \to \mathbb{C}$. Note that $\eta(\Lambda)V \subset V$.

Let us now consider the form $C_\omega(\Lambda^{1/2}\cdot, \Lambda^{1/2}\cdot) \in PS(V)$. We define its *trace* by

$$\mathrm{tr}\big(C_\omega\big(\Lambda^{1/2}\cdot, \Lambda^{1/2}\cdot\big)\big) := \sum_{n=0}^{\infty} C_\omega\big(\Lambda^{1/2}e_n, \Lambda^{1/2}e_n\big) = \sum_{n=0}^{\infty} \beta_n C_\omega(e_n, e_n).$$

Integration over an arbitrary $X \in \Sigma$ gives

$$\int_X \mathrm{tr}\big(C_\omega\big(\Lambda^{1/2}\cdot, \Lambda^{1/2}\cdot\big)\big) \, d\mu(\omega) = \sum_{n=0}^{\infty} \beta_n E_X(e_n, e_n) = \mu(X).$$

This means that $\mathrm{tr}(C_\omega(\Lambda^{1/2}\cdot, \Lambda^{1/2}\cdot)) = 1$ for μ-a.e. $\omega \in \Omega$, say for $\omega \in \Omega \setminus Z$, where $\mu(Z) = 0$.

Let us momentarily restrict ourselves to a finite dimensional space $V_N := \mathrm{lin}(e_n)_{n=0}^{N}$, which we make into a Hilbert space such that $(e_n)_{n=0}^{N}$ is an orthonormal basis. The restriction of $C_\omega(\Lambda^{1/2}\cdot, \Lambda^{1/2}\cdot)$ to $V_N \times V_N$ belongs to $PS(V_N)$ and defines a positive operator, say $T_N(\omega)$, on V_N in a natural way. The computation of the previous paragraph implies that the (usual) trace of $T_N(\omega)$, and hence its norm in $\mathcal{L}(V_N)$, is at most 1, for all $\omega \in \Omega \setminus Z$. This uniform estimate, and the density of $V = \bigcup_{n=0}^{\infty} V_N$ in H, imply by a standard limiting argument the existence of an operator $T(\omega) \in \mathcal{L}(H)$ such that $\langle \phi | T(\omega)\psi \rangle = C_\omega(\Lambda^{1/2}\phi, \Lambda^{1/2}\psi)$ for all $\phi, \psi \in V$, and this operator is positive with trace 1. For definiteness, let us define $T(\omega) := 0 \in \mathcal{L}(H)$ for all ω in the exceptional set Z.

We now have the equations

$$C_\omega(\phi, \psi) = \big\langle \Lambda^{-1/2}\phi \big| T(\omega)\Lambda^{-1/2}\psi \big\rangle, \qquad E_X(\phi, \psi) = \int_X \big\langle \Lambda^{-1/2}\phi \big| T(\omega)\Lambda^{-1/2}\psi \big\rangle d\mu(\omega)$$

for all $\phi, \psi \in V$, but we can see that the right-hand sides are meaningful for all $\phi, \psi \in \mathsf{D}(\Lambda^{-1/2}) = \mathsf{R}(\Lambda^{1/2}) =: H_1$, where we define

$$H_\gamma := \left\{ \phi = \sum_{n=0}^{\infty} a_n e_n \;\middle|\; \|\phi\|_{H_\gamma}^2 := \sum_{n=0}^{\infty} \frac{|a_n|^2}{\beta_n^\gamma} < \infty \right\}, \quad \gamma \in \mathbb{R}. \tag{6.2}$$

Since $\sum_{k=0}^{n(\omega)-1} |\langle d_k(\omega)|\phi\rangle|^2 = C_\omega(\phi, \phi) = \langle \Lambda^{-1/2}\phi|T(\omega)\Lambda^{-1/2}\phi\rangle \leqslant \|\Lambda^{-1/2}\phi\|^2$, we also find that the functionals $\langle d_k(\omega)| \in V^*$ in fact extend by continuity to continuous linear functionals on H_1. Observe that under the natural duality of sequence spaces we have $H_1^* \approx H_{-1}$. In particular, the coordinate sequences satisfy

$$\left(\langle d_k(\omega)|e_n\rangle\beta_n^{1/2}\right)_{n=0}^\infty \in \ell^2. \tag{6.3}$$

We collect the results from the above discussion in the following:

Proposition 6.4. *The sesquilinear forms $E_X, C_\omega \in PS(V)$, $X \in \Sigma$, $\omega \in \Omega$, and the functionals $\langle d_k(\omega)| \in V^*$ extend continuously to the Hilbert space completion H_1 of V (defined in (6.2)), which is associated with the PSFM E.*

On the larger Hilbert space $H = H_0$, which is canonically related to V and independent of E, they admit representations as unbounded forms with common dense domain:

$$E_X(\phi, \psi) = \int\limits_X C_\omega(\phi, \psi)\, d\mu(\omega), \quad C_\omega(\phi, \psi) = \langle \Lambda^{-1/2}\phi|T(\omega)\Lambda^{-1/2}\psi\rangle,$$

$$\phi, \psi \in \mathsf{R}\left(\Lambda^{1/2}\right) = H_1,$$

with an injective $\Lambda \in \mathcal{L}_+^1(H)$ (defined in (6.1)) and

$$T(\omega) = \sum_{k=0}^{n(\omega)-1} \Lambda^{1/2}|d_k(\omega)\rangle\langle d_k(\omega)|\Lambda^{1/2} =: \sum_{k=0}^{n(\omega)-1} |h_k(\omega)\rangle\langle h_k(\omega)| \in \mathcal{L}_+^1(H),$$

where $h_k(\omega) \in H$ is defined with the help of the Riesz representation theorem in terms of the linear functional $\langle h_k(\omega)|\phi\rangle := \langle d_k(\omega)|\Lambda^{1/2}\phi\rangle$, $\phi \in H$. Moreover,

$$\operatorname{tr} T(\omega) = \sum_{k=0}^{n(\omega)-1} \left\|h_k(\omega)\right\|_H^2 = 1, \quad \operatorname{rank} T(\omega) = n(\omega) \quad \text{for a.e. } \omega \in \Omega,$$

and the maps $\Omega \ni \omega \mapsto T_\omega \in \mathcal{L}^1(H)$ and $\Omega \ni \omega \mapsto h_k(\omega) \in H$ are Bôchner μ-measurable.

Rest of the proof. What remains to show is in the last two lines of the assertions. Concerning the rank of $T(\omega)$, we know that $\delta_{k\ell}\chi_{\{\omega' \in \Omega | k \leqslant n(\omega')\}}(\omega) = \langle d_k(\omega)|g_\ell(\omega)\rangle = \langle h_k(\omega)|\Lambda^{-1/2}g_\ell(\omega)\rangle$, where $g_\ell(\omega) \in V \subset \mathsf{R}(\Lambda^{1/2})$ so that $\Lambda^{-1/2}g_\ell(\omega) \in H$. This shows that the vectors $h_k(\omega)$, $0 \leqslant k < n(\omega)$, are linearly independent.

As for measurability, we know that $\omega \mapsto \langle h_k(\omega)|\psi\rangle = \langle d_k(\omega)|\Lambda^{-1/2}\psi\rangle$ is measurable for all $\psi \in V$. By the density of V in H, the measurability extends to all $\psi \in H$, and the weak measurability thus established is equivalent to the (strong) Bôchner measurability in the separable Banach space H by the Pettis measurability theorem (see e.g. [7, Theorem II.1.2] or [9, Theorem III.6.11]). This implies the measurability of the finite sums $\sum_{k=0}^{\min(N,n(\omega)-1)} |h_k(\omega)\rangle\langle h_k(\omega)|$ convergent to $T(\omega)$ (pointwise in the norm of $\mathcal{L}^1(H)$), which is hence measurable also. □

7. Positive operator measures

In this section we consider an important special case of the above theory, where the PSFM is defined on the whole Hilbert space H from the beginning. Now $(e_n)_{n=0}^\infty$ is an orthonormal basis of H, but we continue to denote $V = \operatorname{lin}(e_n)_{n=0}^\infty$ as in the previous sections. Every POM

$E_0 : \Sigma \to \mathcal{L}(H)$ can be identified with a PSF measure E by setting $E_X(\phi, \psi) := \langle \phi | E_0(X)\phi \rangle$. Thus the general results give:

Proposition 7.1. *Given a POM $E_0 : \Sigma \to \mathcal{L}(H)$ and defining a measure μ by (3.1), there exists the following representation:*

$$\langle \phi | E_0(X)\psi \rangle = \int_X \langle \Lambda^{-1/2}\phi | T(\omega)\Lambda^{-1/2}\psi \rangle \, d\mu(\omega), \quad \phi, \psi \in \mathsf{R}(\Lambda^{1/2}),$$

where $\Lambda \in \mathcal{L}^1_+(H)$ is injective,

$$T(\omega) = \sum_{k=0}^{n(\omega)-1} |h_k(\omega)\rangle\langle h_k(\omega)| \in \mathcal{L}^1_+(H), \qquad \operatorname{tr} T(\omega) = 1, \qquad \operatorname{rank}\big(T(\omega)\big) = n(\omega),$$

and all functions of ω are Bôchner μ-measurable in their natural range spaces.

Proof. While the result is a specialization of what we showed for general PSF measures, we provide another proof, which is considerably shortened by the use of the well-established Hilbert space operator theory.

Let us define $\Lambda \in \mathcal{L}^1_+(H)$ by (6.1), and consider the positive trace class operator valued measure $F(X) := \Lambda^{1/2}E_0(X)\Lambda^{1/2}$. The total variation of F is

$$\sup \sum_{i=1}^n \big\| F(X_i) \big\|_{\mathcal{L}^1(H)} = \sup \sum_{i=1}^n \operatorname{tr} F(X_i) = \operatorname{tr} F(\Omega) \leqslant \operatorname{tr} \Lambda \cdot \big\| E_0(\Omega) \big\|,$$

where the supremum is over all finite partitions $\Omega = \bigcup_{i=1}^n X_i$, and we made use of the positivity of $F(X_i)$ and Λ, and basic properties of the trace.

Thus F is of bounded total variation. It also has the same null-sets as E_0, which in turn has the same null-sets as the measure μ constructed in (3.1). Since $\mathcal{L}^1(H)$, as a separable (for a reference and an easy direct proof see e.g. [16, p. 794]) dual space has the Radon–Nikodým property (e.g. [7, Theorem III.3.1]), we may apply the vector-valued Radon–Nikodým theorem to deduce the existence of an $\mathcal{L}^1(H)$-valued Bôchner-integrable density $\omega \mapsto T(\omega)$ such that $F(X) = \int_X T(\omega) \, d\mu(\omega)$ for all $X \in \Sigma$. Since $F(X)$ is a positive operator, also $T(\omega)$ must be for a.e. $\omega \in \Omega$; this follows in our separable situation from the corresponding result for scalar-valued measures and densities, since the positivity of an operator $A \in \mathcal{L}(H)$ can be tested in terms of the positivity of $\langle \phi | A\phi \rangle$, where ϕ goes through a countable dense subset of H. Moreover,

$$\int_X \operatorname{tr} T(\omega) \, d\mu(\omega) = \operatorname{tr} F(X) = \sum_{n=1}^\infty \frac{\alpha_n}{1 + \langle e_n | E_0(\Omega)e_n \rangle} \langle e_n | E_0(X)e_n \rangle = \mu(X),$$

and this implies that $\operatorname{tr} T(\omega) = 1$ for a.e. $\omega \in \Omega$.

What remains is the decomposition of $T(\omega)$. For each single operator, this of course is well known from the Hilbert space operator theory, but the point is now to have this in a measurable way. Such a measurable decomposition is proved in [5, Proposition 1.8], based on a classical theorem on *measurable selectors* [17]. The result in [5] also gives the additional properties $\|h_{k-1}(\omega)\| \geqslant \|h_k(\omega)\| > 0$ for $1 \leqslant k < n(\omega)$ and $\langle h_k(\omega)|h_\ell(\omega)\rangle = 0$ for $k \neq \ell$, in addition to those stated in the assertions. $\quad\square$

Remark 7.2. Part of the information in the above proposition was obtained in a different way in [1, Proposition 27, Remark 28 and the proof of Theorem 79].

8. Generalized eigenvectors of normal operators

Here we make a further specialization of the general theory to the case of a spectral measure associated to a normal operator T in a Hilbert space. We are going to show that in this situation the functionals $|d_k(\omega)\rangle$ may be interpreted as generalized eigenvectors of T, in a precise sense to be defined below. Results of this kind have a long history; instead of attempting a comprehensive record, we would only like to mention the general setting for much of the early theory provided by C. Foiaş [12] and the more recent approach due to S.J.L. van Eijndhoven and J. de Graaf [10], from which we borrow an auxiliary result. The technique based on the use of the measure μ and the Radon–Nikodým theorem in our approach bears a certain resemblance to some ideas already present in [12]. Operator densities in connection with positive operator measures are also used in [1], but the generalized eigenvalue problem is not treated there.

Now, let H be a Hilbert space and V a dense subspace of H. Every $\phi \in H$ determines a continuous antilinear functional $\psi \mapsto \langle \psi | \phi \rangle$. We denote this functional, and also its restriction to V, by $|\phi\rangle$. Thus $|\phi\rangle$ belongs to $V^\times$, the linear space of all antilinear functionals on V, and the mapping $\phi \mapsto |\phi\rangle$ is a linear injection from H into $V^\times$. We often write simply $V \subset H \subset V^\times$.

In the following definition we assume that $T : \mathsf{D}(T) \to H$ is a densely defined linear map in H and let $T^* : \mathsf{D}(T^*) \to H$ be its adjoint. We assume that $V \subset \mathsf{D}(T^*)$ and $T^*(V) \subset V$. Let us denote by $\tilde{T} : V^\times \to V^\times$ the linear map defined by

$$(\tilde{T} F)(\psi) := F(T^*\psi), \quad F \in V^\times, \ \psi \in V. \tag{8.1}$$

With $F = |\phi\rangle \in \mathsf{D}(T)$ and $\psi \in V$, this yields $(\tilde{T}|\phi\rangle)(\psi) = \langle T^*\psi | \phi \rangle = \langle \psi | T\phi \rangle$ Thus $\tilde{T}$ may be regarded as an extension of T under the interpretation $V \subset H \subset V^\times$.

Definition 8.2. If $F \in V^\times \setminus \{0\}$ and $\lambda \in \mathbb{C}$ satisfy $\tilde{T} F = \lambda F$, then F is called a *generalized eigenvector* of T belonging to the *generalized eigenvalue* λ of T (relative to V).

The discussion preceding the definition justifies this terminology: any eigenvalue of T is a generalized eigenvalue.

Let then $T : \mathsf{D}(T) \to H$ be a normal operator. According to the well-known *spectral theorem* (see e.g. [24]), associated to T there is a uniquely determined spectral measure $E_0 : \mathcal{B}(\mathbb{C}) \to \mathcal{L}(H)$, supported on the spectrum of T, such that

$$\langle \phi | T\psi \rangle = \int_{\mathbb{C}} \lambda \langle \phi | E_0(d\lambda)\psi \rangle, \quad \phi \in H, \ \psi \in \mathsf{D}(T). \tag{8.3}$$

The following result is based on (8.3) and the application of our diagonalization results to E_0. The existence of invariant subspaces V for T and T^*, as required in the theorem, is always guaranteed by results at the end of the section.

Theorem 8.4. *Let $T : \mathsf{D}(T) \to H$ be a normal operator as in (8.3) with $V \subset \mathsf{D}(T) \cap \mathsf{D}(T^*)$ and $TV \subset V$, $T^*V \subset V$. Then there exist a finite positive Borel measure μ on $\sigma(T)$ having the same null-sets as E_0, and a sequence of μ-measurable mappings $\lambda \mapsto |d_k(\lambda)\rangle \in V^\times$ such that, for μ-almost every $\lambda \in \sigma(T)$ and every $k \in \mathbb{N}$, $|d_k(\lambda)\rangle$ is either zero or a simultaneous*

generalized eigenvector of T and T^* relative to V belonging to their generalized eigenvalues λ and $\bar{\lambda}$, respectively. Moreover,

$$\langle\phi|T\psi\rangle = \int_{\sigma(T)} \lambda \sum_{k=0}^{n(\lambda)-1} \langle\phi|d_k(\lambda)\rangle\langle d_k(\lambda)|\psi\rangle \, d\mu(\lambda), \quad \phi, \psi \in V. \tag{8.5}$$

Proof. From (8.3) we have $\langle\phi|T\psi\rangle = \langle T^*\phi|\psi\rangle = \int_{\mathbb{C}} \lambda \langle\phi|E_0(d\lambda)\psi\rangle$, $\phi, \psi \in V$. On the other hand, we know the existence of μ and $|d_k(\lambda)\rangle$ so that

$$\langle\phi|E_0(d\lambda)\psi\rangle = \sum_{k=0}^{n(\lambda)-1} \langle\phi|d_k(\lambda)\rangle\langle d_k(\lambda)|\psi\rangle \, d\mu(\lambda), \quad \phi, \psi \in V,$$

and hence

$$\langle\phi|T\psi\rangle = \langle T^*\phi|\psi\rangle = \int_{\mathbb{C}} \lambda \sum_{k=0}^{n(\lambda)-1} \langle\phi|d_k(\lambda)\rangle\langle d_k(\lambda)|\psi\rangle \, d\mu(\lambda), \quad \phi, \psi \in V. \tag{8.6}$$

Since $E_0(\mathbb{C}) = I$, we also have

$$\langle\phi|\psi\rangle = \int_{\mathbb{C}} \sum_{k=0}^{n(\lambda)-1} \langle\phi|d_k(\lambda)\rangle\langle d_k(\lambda)|\psi\rangle \, d\mu(\lambda), \quad \phi, \psi \in V. \tag{8.7}$$

Comparing (8.6) and (8.7) with ϕ replaced by $T^*\phi$ (still in V by the assumption that $T^*V \subset V$), and recalling that the functions $\lambda \mapsto (\langle d_k(\lambda)|\psi\rangle)_{k=0}^{\infty}$, $\psi \in V$, are dense in $L^2_{n(\cdot)}(\Omega, \mu, \ell^2)$, we deduce that there must hold, for μ-a.e. $\lambda \in \mathbb{C}$,

$$\langle T^*\phi|d_k(\lambda)\rangle = \lambda\langle\phi|d_k(\lambda)\rangle, \quad \phi \in V. \tag{8.8}$$

By choosing a null-set N such that (8.8) holds in its complement for all $\phi = e_n$, $n \in \mathbb{N}$, it also holds, by linearity, for all $\phi \in V$. Thus, for μ-a.e. $\lambda \in \sigma(T)$, we have the generalized eigenvector equations $\tilde{T}|d_k(\lambda)\rangle = \lambda|d_k(\lambda)\rangle$. Making a similar comparison of (8.6) and (8.7) with ψ replaced by $T\psi$ (where the assumption $TV \subset V$ appears), and proceeding analogously, we deduce the adjoint equation $\tilde{T}^*|d_k(\lambda)\rangle = \bar{\lambda}|d_k(\lambda)\rangle$, which completes the proof. $\quad\square$

Remark 8.9. In the construction of the measure μ in (3.1), any positive sequence $(\alpha_n)_{n=0}^{\infty} \in \ell^1$ could be chosen. For a spectral measure E_0, we have $\langle e_n|E_0(\mathbb{C})e_n\rangle = 1$ for all $n \in \mathbb{N}$, so that the eigenvalues β_n of the operator $\Lambda \in \mathcal{L}_+^1(H)$ in (6.1) are just $\beta_n = \alpha_n/2$. Since we know that the action of the $|d_k(\lambda)\rangle$ can be extended boundedly to $\mathsf{R}(\Lambda^{1/2})$, and since $(\beta_n)_{n=0}^{\infty} \mapsto (\beta_n^{1/2})_{n=0}^{\infty}$ is a bijection between the positive cones of ℓ^1 and ℓ^2, we have the following information concerning the size of the generalized eigenvectors (cf. (6.3)): For any positive $(\varrho_n)_{n=0}^{\infty} \in \ell^2$, for a.e. $\lambda \in \sigma(T)$ (with respect to the spectral measure of the normal operator T), there exist $|d_k(\lambda)\rangle \in V^{\times}$, $0 \leqslant k < n(\lambda)$, which are simultaneous generalized eigenvectors of T and T^*, and satisfy $(\varrho_n\langle e_n|d_k(\lambda)\rangle)_{n=0}^{\infty} \in \ell^2$.

The generalized eigenvector equations can be used, at least in principle, to solve for the $d_k(\lambda)$, from which one may try to construct the spectral decomposition of T. We illustrate this in a toy example below, but also point out in the following section some intrinsic problems related to this approach.

Example 8.10. Consider, as in Example 2.3, a Hilbert space H with an orthonormal basis $(e_n)_{n \in \mathbb{Z}}$ and the simplest shift operator $S : e_n \mapsto e_{n-1}$ (hence $S^* : e_n \mapsto e_{n+1}$). Clearly S is unitary, in particular normal, so that the theory developed in this section applies to it. Writing $|d\rangle = \sum_{j=-\infty}^{\infty} d_j |e_j\rangle$ for $|d_k(\lambda)\rangle$, (8.8) reads for $\phi = e_n$ as $d_{n+1} = \lambda d_n$, giving the unique (up to normalization) solution $d_j = d_0 \lambda^j$, except for the case $\lambda = 0$, where $d_j \equiv 0$ is the only solution. Thus, for every $\lambda \in \mathbb{C} \setminus \{0\}$, there corresponds a one-dimensional generalized eigenspace spanned by $|d(\lambda)\rangle = \sum_{j=-\infty}^{\infty} \lambda^j |e_j\rangle$.

The eigenvector equation for the adjoint S^* gives similarly $d_{n-1} = \bar{\lambda} d_n$, yielding the solution $d_j = \bar{\lambda}^{-j} d_0$. This can only coincide with the generalized eigenvector of S if $\bar{\lambda} = \lambda^{-1}$. Thus the only simultaneous generalized eigenvectors of S and S^* are, up to normalization, $|d(e^{it})\rangle = \sum_{j=-\infty}^{\infty} e^{itj} |e_j\rangle$. Hence the general theory guarantees that, for some finite positive measure μ on the unit circle $\mathbb{T}$, the spectral measure of S is given by $\langle \phi | E_S(X) \psi \rangle = \int_{X \cap \mathbb{T}} \langle \phi | d(\lambda) \rangle \langle d(\lambda) | \psi \rangle \, d\mu(\lambda)$ for all $\phi, \psi \in V$. Testing the equality $\langle \phi | E_S(\mathbb{T}) \psi \rangle = \langle \phi | \psi \rangle$ with $\phi = e_m, \psi = e_n$, we find that $\int_{\mathbb{T}} \lambda^{m-n} \, d\mu(\lambda) = \delta_{m,n}$ for all $m, n \in \mathbb{T}$. This shows that the Fourier coefficients of the measure μ coincide with those of the normalized Haar measure ν of $\mathbb{T}$, and hence $\langle \phi | E_S(X) \psi \rangle = \int_{X \cap \mathbb{T}} \langle \phi | d(\lambda) \rangle \langle d(\lambda) | \psi \rangle \, d\nu(\lambda) = \sum_{m,n \in \mathbb{Z}} \int_{X \cap \mathbb{T}} \lambda^{m-n} \, d\nu(\lambda) \langle \phi | e_m \rangle \langle e_n | \psi \rangle$ for all $\phi, \psi \in V$, gives the spectral measure of S.

We finally address the question of invariant subspaces as required in Theorem 8.4. Following [10], we define the *joint* C^∞-*domain* of linear operators $A_i : D(A_i) \subset H \to H$, $i = 1, \ldots, k$, as $C^\infty(A_1, \ldots, A_k) := \{\psi \in H \mid \text{for all } N \in \mathbb{N} \text{ and } \pi \in \{1, \ldots, k\}^N, \; \psi \in D(A_{\pi(1)} A_{\pi(2)} \cdots A_{\pi(N)})\}$. We quote the following result and deduce an immediate consequence:

Theorem 8.11. *(See [10].) Suppose that $C^\infty(A_1, \ldots, A_k)$ is dense in H. Then there exists an orthonormal basis $(e_n)_{n=0}^\infty$ such that each A_i, $i = 1, \ldots, k$, maps $V = \mathrm{lin}\{e_n \mid n \in \mathbb{N}\}$ into itself.*

Corollary 8.12. *Let $T : D(T) \subset H \to H$ be normal (or bounded). Then there exists an orthonormal basis $(e_n)_{n=0}^\infty$ such that both T and T^* map V into itself.*

Proof. If T is normal, let E_0 be its spectral measure and D_k the disc $\{\zeta \in \mathbb{C} \mid |\zeta| < k\}$, every $\phi \in H$ satisfies $E_0(D_k)\phi \in C^\infty(T, T^*)$ for all $k \in \mathbb{N}$ and $E_0(D_k)\phi \to \phi$ as $k \to \infty$. Hence $C^\infty(T, T^*)$ is dense in H, and we can apply the previous theorem. If T is bounded, then $C^\infty(T, T^*) = H$, and we derive the same conclusion. $\square$

9. A counterexample concerning generalized eigenvectors

In Theorem 8.4 we proved that almost every (with respect to the spectral measure) point λ in the spectrum of a normal operator T is a generalized eigenvalue of T, and moreover Remark 8.9 showed that the associated generalized eigenvectors are in a sense not very far from being vectors in the Hilbert space H. In this section we show that the converse statement fails: even if some $\lambda \in \mathbb{C}$ is a generalized eigenvalue of T with a "nice" associated generalized eigenvector, this λ need not be in the Hilbert space spectrum of T, and this can already happen for a bounded (in fact, Hilbert–Schmidt) self-adjoint operator T. The following technical lemma is the key to the counterexample:

Lemma 9.1. *There exists an infinite matrix $(m_{ij})_{i,j=0}^\infty$ having entries in $[0, 1]$ and with the following properties:*

- $m_{ij} = m_{ji}$ for all $i, j \in \mathbb{N}$;
- for all $i \in \mathbb{N}$, the sequence $(m_{ij})_{j=0}^{\infty}$ has only finitely many non-zero members;
- for all $i \in \mathbb{N}$, there holds $\sum_{j=0}^{\infty} m_{ij} = 1$; and
- $\sum_{i,j=0}^{\infty} m_{ij}^2 < 1$.

Proof. We start with a sequence $(a_i)_{i=0}^{\infty}$ of positive integers, to be chosen later. Denote $\sigma_{-1} := 0$ and $\sigma_i := \sum_{k=0}^{i} a_k$ for $i \in \mathbb{N}$. Let $\phi(j)$, for $j = 1, 2, \ldots$, be the unique $i \in \mathbb{N}$ such that $\sigma_{i-1} < j \leqslant \sigma_i$. Since $\sigma_{i-1} \geqslant i$ (as $a_k \geqslant 1$), we have $\phi(j) < j$. Then we define a sequence $(b_j)_{j=0}^{\infty} \in [0,1]^{\mathbb{N}}$ inductively as follows: $b_0 := 1/a_0$, and $b_j := (1 - b_{\phi(j)})/a_j$ for $j = 1, 2, \ldots$. Finally, the matrix entries m_{ij} are defined: We set $m_{ii} := 0$, and for $j > i$ we let $m_{ij} := b_i$ if $\sigma_{i-1} < j \leqslant \sigma_i$ (i.e., $\phi(j) = i$), and zero otherwise. The entries m_{ij} with $j < i$ are defined so as to satisfy the symmetry requirement.

That $(m_{ij})_{j=0}^{\infty}$ has only finitely many non-zero members is clear from this definition. In fact, for $j \geqslant i$, there are $\sigma_i - \sigma_{i-1} = a_i$ entries equal to $b_i = (1 - b_{\phi(i)})/a_i$, the others being zero. For $j < i$, an entry $m_{ij} = m_{ji}$ can only differ from zero if $j = \phi(i)$, in which case it is equal to $b_j = b_{\phi(i)}$. Thus $\sum_{j=0}^{\infty} m_{ij} = b_{\phi(i)} + a_i \times (1 - b_{\phi(i)})/a_i = 1$, as we wanted. To compute the square sum of the entries m_{ij}, observe that there are exactly $2(\sigma_i - \sigma_{i-1}) = 2a_i$ entries for which we gave the value b_i; hence $\sum_{i,j=0}^{\infty} m_{ij}^2 = \sum_{i=0}^{\infty} 2a_i b_i^2 = \sum_{i=0}^{\infty} 2a_i (1 - b_{\phi(i)})^2/a_i^2 \leqslant \sum_{i=0}^{\infty} 2/a_i$, and this is easily made to satisfy the final requirement by a suitable choice of $(a_i)_{i=0}^{\infty}$. $\square$

We now provide the announced counterexample. Once again, let H be a Hilbert space with an orthonormal basis $(e_n)_{n=0}^{\infty}$ and $V := \lim(e_n)_{n=0}^{\infty}$. We define an operator $T \in \mathcal{L}(H)$ in terms of the matrix $(m_{ij})_{i,j=0}^{\infty}$, i.e., we set

$$T\left(\sum_{i=0}^{\infty} a_i e_i\right) := \sum_{i=0}^{\infty}\left(\sum_{j=0}^{\infty} m_{ij} a_j\right) e_i. \tag{9.2}$$

Since the matrix is real and symmetric, the operator T is self-adjoint, and hence $\sigma(T) \subset \mathbb{R}$. Moreover, T is a Hilbert–Schmidt operator, and $\|T\|_{\mathcal{L}^2(H)}^2 = \sum_{\lambda \in \sigma(T)} \lambda^2 = \sum_{i,j=0}^{\infty} m_{ij}^2 < 1$. In particular, T has a discrete spectrum, and its eigenvalues λ satisfy $-1 < \lambda_- \leqslant \lambda \leqslant \lambda_+ < 1$.

However, let us consider the generalized eigenvalue problem for T. The fact that every column and row of the infinite matrix $(m_{ij})_{i,j=0}^{\infty}$ has only finitely many non-zero entries is equivalent with the properties $TV \subset V$ and $T^*V \subset V$. Thus the generalized eigenvector formalism of the previous section is applicable.

Observe that the same extension $\tilde{T}$ on $V^{\times}$ is obtained by using the original defining formula (9.2), which also makes sense for an arbitrary $\sum_{j=0}^{\infty} a_j |e_j\rangle \in V^{\times}$. We may use this remark to compute $\tilde{T}|e\rangle$, where $|e\rangle := \sum_{j=0}^{\infty} |e_j\rangle$. From self-adjointness and the fact that $\sum_{j=0}^{\infty} m_{ij} = 1$, it follows at once that $\tilde{T}|e\rangle = \tilde{T}^*|e\rangle = |e\rangle$, and hence $|e\rangle$ is a generalized eigenvector of T and T^* corresponding to the generalized eigenvalue 1. By the remarks concerning the spectrum of T above, this generalized eigenvalue is not in $\sigma(T)$. Summarizing, we have shown the following:

Proposition 9.3. *There exists a self-adjoint operator $T \in \mathcal{L}^2(H)$ such that $TV = T^*V \subset V$, and hence extending to $\tilde{T} : V^{\times} \to V^{\times}$ by (8.1), such that*

- $\|T\|_{\mathcal{L}^2(H)} < 1$; *in particular, $\sigma(T) \subset [\lambda_-, \lambda_+] \subset \,]-1, 1[$, but*
- $\tilde{T}|e\rangle = \tilde{T}^*|e\rangle = |e\rangle$ *where $|e\rangle = \sum_{j=0}^{\infty} |e_j\rangle$.*

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