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Robustness of nonuniform exponential trichotomies in Banach spaces ☆

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ABSTRACT

We establish the robustness under sufficiently small linear perturbations of *nonuniform* exponential trichotomies defined by linear equations $x' = A(t)x$ in Banach spaces. We also establish the continuous dependence on the perturbation of the constants in the notion of trichotomy. We consider both trichotomies in semi-infinite intervals and trichotomies in $\mathbb{R}$.

1. Introduction

1.1. Robustness of nonuniform exponential trichotomies

The purpose of this note is to show that a *nonuniform* exponential trichotomy defined by a nonautonomous linear equation

$$x' = A(t)x \tag{1}$$

in a Banach space, persists under sufficiently small linear perturbations in the equation

$$x' = \big[A(t) + B(t)\big]x. \tag{2}$$

This is the so-called robustness problem.

In the special case of *uniform* exponential trichotomies, we recover the following result, where $\mathcal{B}(X)$ denotes the space of bounded linear operators in the Banach space X.

Theorem 1. *Let $A, B: \mathbb{R} \to \mathcal{B}(X)$ be continuous functions such that Eq. (1) admits a uniform exponential trichotomy in $\mathbb{R}$. If $\sup_{t\in\mathbb{R}} \|B(t)\|$ is sufficiently small, then Eq. (2) admits a uniform exponential trichotomy in $\mathbb{R}$.*

Theorem 1 should be considered classical although we are not able to indicate an appropriate reference. We note that the notion of (uniform) exponential trichotomy plays a central role in the study of center manifolds, which are powerful tools in the analysis of the asymptotic behavior of dynamical systems. Namely, when a linear dynamics possesses no unstable directions the stability of the system is completely determined by the behavior on any center manifold. The study of center manifolds can be traced back to the works of Pliss [21] and Kelley [14]. A very detailed exposition in the case of autonomous

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equations is given in [24], adapting results in [26]. See also [17,25] for the case of infinite-dimensional systems. We refer the reader to [7,8,10,11,22,24] for more details and further references.

Our main objective is to consider the much more general notion of *nonuniform* exponential trichotomy and obtain an appropriate version of Theorem 1 (see Theorem 5), building on our work in [5] for nonuniform exponential dichotomies. We refer the reader to [1,6] for related discussions on the ubiquity of the nonuniform exponential behavior, particularly in the context of ergodic theory. Due to the central role played by the notion of exponential trichotomy, most importantly in the theory of center manifolds which are crucial in the study of the asymptotic behavior of trajectories, it is important to understand how exponential trichotomies vary under perturbations.

We note that the study of robustness has a long history. In particular, the problem was discussed by Massera and Schäffer [15] (building on earlier work of Perron [20]; see also [16]), Coppel [12], and in the case of Banach spaces by Dalec'kiĭ and Kreĭn [13], with different approaches and successive generalizations. The continuous dependence of the projections on the perturbation was obtained by Palmer [19]. For more recent works we refer to [9,18,22,23] and the references therein. We emphasize that all these works consider only the case of *uniform* exponential behavior.

1.2. Applications of the robustness

Now we discuss several applications of our results. These include the persistence of center manifolds under sufficiently small linear and nonlinear perturbations, the partial linearization of a nonlinear dynamics, and the approximation of invariant sets obtaining from varying parameters:

Persistence of center manifolds. Center manifold theorems are powerful tools in the analysis of the asymptotic behavior of a dynamical system. Namely, when a linear equation has some elliptic directions and no unstable directions, all solutions converge exponentially to the center manifold, and thus the stability of the zero solution under sufficiently small perturbations is completely determined by the behavior on any center manifold. Accordingly, one often considers a reduction of the dynamics to a center manifold, and one determines the quantitative behavior on it. This has also the advantage of reducing the dimension of the system. We refer the reader to [7] for details and references, and to [4,6] for corresponding results in the case of *nonuniform* exponential trichotomies.

In particular, our results imply that even in the nonuniform setting the center manifolds persist not only under sufficiently small nonlinear perturbations but also under sufficiently small *linear* perturbations. This is of importance particularly in numerical applications when the information about the linear part of the dynamics is perhaps not sufficiently precise, although our results guarantee that any *nonuniform* exponential trichotomy persists under sufficiently small perturbations.

Partial linearization. A fundamental problem in the study of the local behavior of a dynamical system is whether the linearization of the system along a given solution approximates well the solution itself in some open neighborhood. This means that we look for an appropriate local change of variables, called a conjugacy, that can transform the system into a linear one. The problem goes back to the pioneering work of Poincaré.

When a given dynamics admits an exponential dichotomy, the Grobman–Hartman theorem shows that locally the original dynamics is topologically conjugated to its linearization. There is also a version of the Grobman–Hartman theorem in the case of *nonuniform* exponential dichotomies (see [2,6] for details). When the linearization has some elliptic directions, that is, when it admits an exponential trichotomy one can still obtain a conjugacy that linearizes the stable and unstable parts of the dynamics. While in the case of exponential dichotomies it is well know that one can obtain conjugacies both under linear and nonlinear perturbations (see [3] for the case of nonuniform exponential dichotomies), our results show that a similar statement holds for exponential trichotomies. Namely, the conjugacy obtained in the partial linearization of the stable and unstable directions persists under sufficiently small linear and nonlinear perturbations.

Approximation of invariant sets. When we work with equations depending on parameters it is crucial to understand whether the invariant sets of the dynamics vary in a sufficiently regular manner with the parameters. For example, in the study of the stability of a given solution using invariant center manifolds and normal forms one needs to approximate the center manifolds to sufficiently high order. First of all this requires knowing that the center manifolds are sufficiently regular. We can make not only nonlinear perturbations but also linear ones, for example given by changes of parameters in the linear part. In the last case, it is important to know that the linear structure persists. This is precisely given by our robustness result which shows that the initial nonuniform exponential trichotomy persists under sufficiently small linear perturbations.

2. Basic notions

Let $\mathcal{B}(X)$ be the space of bounded linear operators in the Banach space X. We consider the linear equation (1), where $A: I \to \mathcal{B}(X)$ is a continuous function in an interval I. We note that each solution of (1) is defined on the whole I. We denote by $T(t,s)$ the associated evolution operator, that is, the linear operator such that $T(t,s)x(s) = x(t)$ for every $t, s \in I$, where $x(t)$ is any solution of (1). Clearly, $T(t,t) = \mathrm{Id}$, and

$$T(t,\tau)T(\tau,s) = T(t,s), \quad t, \tau, s \in I.$$

We say that Eq. (1) admits a *nonuniform exponential trichotomy in I* if there exist projections $P(t), Q(t), R(t): X \to X$ for each $t \in I$ such that

$$T(t,s)P(s) = P(t)T(t,s), \qquad T(t,s)Q(s) = Q(t)T(t,s), \qquad T(t,s)R(s) = R(t)T(t,s),$$

and

$$P(t) + Q(t) + R(t) = \mathrm{Id}$$

for every $t, s \in I$, and there exist constants

$$0 \leqslant a < b, \qquad 0 \leqslant c < d, \qquad \varepsilon \geqslant 0 \quad \text{and} \quad D \geqslant 1 \tag{3}$$

such that for every $t, s \in I$ with $t \geqslant s$ we have

$$\begin{aligned} \|T(t,s)P(s)\| &\leqslant De^{-d(t-s)+\varepsilon|s|}, \\ \|T(t,s)R(s)\| &\leqslant De^{a(t-s)+\varepsilon|s|}, \end{aligned} \tag{4}$$

and

$$\begin{aligned} \|T(t,s)^{-1}Q(t)\| &\leqslant De^{-b(t-s)+\varepsilon|t|}, \\ \|T(t,s)^{-1}R(t)\| &\leqslant De^{c(t-s)+\varepsilon|t|}. \end{aligned} \tag{5}$$

We notice that setting $t = s$ in (4) and (5) we obtain

$$\|P(t)\| \leqslant De^{\varepsilon|t|}, \qquad \|Q(t)\| \leqslant De^{\varepsilon|t|} \quad \text{and} \quad \|R(t)\| \leqslant De^{\varepsilon|t|} \tag{6}$$

for every $t \in I$. We say that Eq. (1) admits a *uniform exponential trichotomy* if it admits a nonuniform exponential trichotomy with $\varepsilon = 0$.

3. Robustness in semi-infinite intervals

3.1. Nonuniform exponential trichotomies in $I = [0, +\infty)$

The following is our robustness result for nonuniform exponential trichotomies in intervals of the form $[\varrho, +\infty)$ with $\varrho \leqslant 0$.

Theorem 2. *Let $A, B: I \to \mathcal{B}(X)$ be continuous functions in an interval $I = [\varrho, +\infty)$ with $\varrho \leqslant 0$ such that Eq.* (1) *admits a nonuniform exponential trichotomy in I satisfying*

$$\varepsilon < \min\{(d-c)/2, (b-a)/2\}, \tag{7}$$

and assume that $\|B(t)\| \leqslant \delta e^{-2\varepsilon|t|}$ for every $t \in I$. If δ is sufficiently small, then there exist projections $\hat{P}(t)$, $\hat{Q}(t)$ and $\hat{R}(t)$ for $t \in I$ such that:

1. *Eq.* (2) *admits a nonuniform exponential trichotomy in* $[0, +\infty)$ *with respect to these projections*;
2. *for the new trichotomy, the corresponding estimates to the ones in* (4) *and* (6) *are valid for all $t \geqslant s$ with $t, s \in I$.*

Proof. We start by recalling a result in [5] concerning the robustness of nonuniform exponential dichotomies. We also recall that Eq. (1) admits a *nonuniform exponential dichotomy in I* if there exist projections $\bar{P}(t): X \to X$ for $t \in I$ such that

$$T(t,s)\bar{P}(s) = \bar{P}(t)T(t,s)$$

for every $t, s \in I$, and there exist constants α, $C > 0$ and $\varepsilon \geqslant 0$ such that

$$\|T(t,s)\bar{P}(s)\| \leqslant Ce^{-\alpha(t-s)+\varepsilon|s|}, \quad t \geqslant s,$$

and

$$\|T(t,s)\bar{Q}(s)\| \leqslant Ce^{-\alpha(s-t)+\varepsilon|s|}, \quad s \geqslant t,$$

where $\bar{Q}(t) = \mathrm{Id} - \bar{P}(t)$. Set

$$\tilde{a} = \alpha\sqrt{1 - 2\delta C/\alpha} \quad \text{and} \quad \tilde{D} = \frac{C}{1 - \delta C/(\tilde{a} + \alpha)},$$

and denote by $\hat{T}(t, \tau)$ the evolution operator for Eq. (2).

Lemma 1. *(See [5].) Let $A, B : I \to \mathcal{B}(X)$ be continuous functions in an interval $I = [\varrho, +\infty)$ with $\varrho \leqslant 0$ such that Eq. (1) admits a nonuniform exponential dichotomy in I with $\varepsilon < a$, and assume that $\|B(t)\| \leqslant \delta e^{-2\varepsilon|t|}$ for every $t \in I$. If δ is sufficiently small, then there exist projections $\hat{P}(t)$ and $\hat{Q}(t) = \mathrm{Id} - \hat{P}(t)$ for $t \in I$ such that for every $t, s \in I$:*

1. (*see* (30), (41), (42) *and* (58) *in* [5])

$$\hat{P}(t) = \hat{T}(t,0)\hat{P}(0)\hat{T}(0,t), \qquad \hat{Q}(t) = \hat{T}(t,0)\hat{Q}(0)\hat{T}(0,t),$$

and

$$\hat{P}(0)\bar{P}(0) = \hat{P}(0), \qquad \bar{P}(0)\hat{P}(0) = \bar{P}(0),$$
$$\hat{Q}(0)\bar{Q}(0) = \bar{Q}(0), \qquad \bar{Q}(0)\hat{Q}(0) = \hat{Q}(0);$$

2. (*see* (21) *and* (22) *in* [5])

$$\|\hat{T}(t,s)|\operatorname{Im}\hat{P}(s)\| \leqslant \tilde{D}e^{-\tilde{a}(t-s)+\varepsilon|s|}, \quad t \geqslant s,$$

and

$$\|\hat{T}(t,s)|\operatorname{Im}\hat{Q}(s)\| \leqslant \tilde{D}e^{-\tilde{a}(s-t)+\varepsilon|s|}, \quad s \geqslant t \geqslant 0;$$

3. (*see* (48) *in* [5])

$$\|\hat{P}(t)\| \leqslant 4De^{\varepsilon|t|} \quad \text{and} \quad \|\hat{Q}(t)\| \leqslant 4De^{\varepsilon|t|}. \tag{8}$$

Let $x(t) = T(t,s)x(s)$ be a solution of Eq. (1). We consider the change of variables $y(t) = x(t)e^{\kappa t}$, where $\kappa = (c+d)/2$. Then $y(t)$ satisfies the linear equation

$$y' = \big(A(t) + \kappa\big)y, \tag{9}$$

and denoting by $T_\kappa(t,s)$ its evolution operator we have

$$T_\kappa(t,s) = T(t,s)e^{\kappa(t-s)}.$$

Since Eq. (1) admits a nonuniform exponential trichotomy in I, we conclude that Eq. (9) admits a nonuniform exponential dichotomy in I with $\alpha = (d-c)/2$, and projections

$$P_1(t) = P(t) \quad \text{and} \quad Q_1(t) = Q(t) + R(t)$$

for each $t \in I$. It follows from Lemma 1 that the equation

$$y' = \big[A(t) + \kappa + B(t)\big]y \tag{10}$$

admits a nonuniform exponential dichotomy, say with projections $\hat{P}_1(t)$ and $\hat{Q}_1(t)$. In particular, the linear subspaces $\hat{E}_1(t) = \hat{P}_1(t)(X)$ and $\hat{F}_1(t) = \hat{Q}_1(t)(X)$ satisfy

$$\hat{E}_1(t) \oplus \hat{F}_1(t) = X. \tag{11}$$

Now we consider a second change of variables $z(t) = x(t)e^{\kappa' t}$, where $\kappa' = -(a+b)/2$. Then $z(t)$ satisfies the linear equation

$$z' = \big(A(t) + \kappa'\big)z, \tag{12}$$

and denoting by $T_{\kappa'}(t,s)$ its evolution operator we have

$$T_{\kappa'}(t,s) = T(t,s)e^{\kappa'(t-s)}.$$

Since (1) admits a nonuniform exponential trichotomy in I, we conclude that Eq. (12) admits a nonuniform exponential dichotomy in I with α replaced by $\alpha' = (b-a)/2$, and projections

$$P_2(t) = P(t) + R(t) \quad \text{and} \quad Q_2(t) = Q(t)$$

for each $t \in I$. It follows from Lemma 1 that the equation

$$z' = \big[A(t) + \kappa' + B(t)\big]z \tag{13}$$

admits a nonuniform exponential dichotomy, say with projections $\hat{P}_2(t)$ and $\hat{Q}_2(t)$. In particular, the linear subspaces $\hat{E}_2(t) = \hat{P}_2(t)(X)$ and $\hat{F}_2(t) = \hat{Q}_2(t)(X)$ satisfying

$$\hat{E}_2(t) \oplus \hat{F}_2(t) = X. \tag{14}$$

We also consider the evolution operators of Eqs. (10) and (13), namely

$$\hat{T}_\kappa(t,s) = e^{\kappa(t-s)}\hat{T}(t,s) \quad \text{and} \quad \hat{T}_{\kappa'}(t,s) = e^{\kappa'(t-s)}\hat{T}(t,s). \tag{15}$$

Lemma 2. *For every $s \in I$ we have*

$$\hat{E}_1(s) \subset \hat{E}_2(s) \quad \textit{and} \quad \hat{F}_2(s) \subset \hat{F}_1(s).$$

Proof. Set

$$\mu(x) = \limsup_{t\to+\infty} \frac{1}{t} \log \left\| \hat{T}_\kappa(t,s)x \right\|.$$

If there exists $x \in \hat{E}_1(s) \setminus \hat{E}_2(s)$, then we write $x = y + z$ with $y \in \hat{E}_2(s)$ and $z \in \hat{F}_2(s)$. Since $x \in \hat{E}_1(s)$, by Lemma 1 we have

$$\left\| \hat{T}_\kappa(t,s)x \right\| \leqslant \tilde{D} e^{-\Gamma(\alpha)(t-s)+\varepsilon|s|} \|x\|,$$

and hence $\mu(x) \leqslant -\Gamma(\alpha)$, where

$$\Gamma(x) = x\sqrt{1 - 2\delta D/x}. \tag{16}$$

Moreover, we have $z \neq 0$ (otherwise $x = y \in \hat{E}_2(s)$ which is impossible), and hence

$$\mu(x) = \max\{\mu(y), \mu(z)\} = \mu(z) = \limsup_{t\to+\infty} \frac{1}{t} \log \left\| T_\kappa(t,s)z \right\|.$$

Since $z \in \hat{F}_2(s)$, for $t \geqslant s$ we have

$$\left\| \hat{T}_\kappa(t,s)z \right\| = e^{(\kappa-\kappa')(t-s)} \left\| \hat{T}_{\kappa'}(t,s)z \right\| \geqslant \frac{1}{\tilde{D}} \|z\| e^{(\kappa-\kappa'+\Gamma(\alpha'))(t-s)} e^{-\varepsilon|t|},$$

and hence

$$\mu(x) = \kappa - \kappa' + \Gamma(\alpha') - \varepsilon,$$

but this contradicts to the inequality $\mu(x) \leqslant -\Gamma(\alpha)$ provided that $\varepsilon < b + d$ (which follows from (7)) and δ is sufficiently small. Therefore, $\hat{E}_1(s) \subset \hat{E}_2(s)$. We show in a similar manner that $\hat{F}_1(s) \subset \hat{F}_2(s)$ for each $s \in \mathbb{R}$. □

Lemma 3. *For every $s \in I$ we have*

$$\left(\hat{E}_2(s) \cap \hat{F}_1(s)\right) \oplus \hat{E}_1(s) \oplus \hat{F}_2(s) = X. \tag{17}$$

Proof. It follows from (11) that

$$\left(\hat{E}_2(s) \cap \hat{E}_1(s)\right) \oplus \left(\hat{E}_2(s) \cap \hat{F}_1(s)\right) = \hat{E}_2(s).$$

But in view of Lemma 2 we have $\hat{E}_2(s) \cap \hat{E}_1(s) = \hat{E}_1(s)$, and hence

$$\hat{E}_1(s) \oplus \left(\hat{E}_2(s) \cap \hat{F}_1(s)\right) = \hat{E}_2(s).$$

The desired statement follows now immediately from (14). □

Lemma 4. *For each $t \in I$ we have*

$$\hat{P}_1(t)\hat{Q}_2(t) = \hat{Q}_2(t)\hat{P}_1(t) = 0.$$

Proof. By Lemma 2, for each $x \in X$ we have

$$\hat{Q}_2(t)x \in \hat{F}_2(t) \subset \hat{F}_1(t),$$

and hence,

$$\hat{P}_1(t)\hat{Q}_2(t)x \in \hat{P}_1(t)\hat{F}_1(t) = \hat{P}_1(t)\operatorname{Im}\hat{Q}_1(t) = \{0\}.$$

Similarly, again by Lemma 2, for each $x \in X$ we have

$$\hat{P}_1(t)x \in \hat{E}_1(t) \subset \hat{E}_2(t)$$

and hence,

$$\hat{Q}_2(t)\hat{P}_1(t)x \in \hat{Q}_2(t)\hat{E}_2(t) = \hat{Q}_2(t)\operatorname{Im}\hat{P}_2(t) = \{0\}.$$

This completes the proof of the lemma. $\square$

We proceed with the proof of Theorem 2. We set

$$\hat{P}(t) = \hat{P}_1(t), \qquad \hat{Q}(t) = \hat{Q}_2(t) \quad \text{and} \quad \hat{R}(t) = \mathrm{Id} - \hat{P}_1(t) - \hat{Q}_2(t).$$

In view of the first property in Lemma 1, we have

$$\hat{T}_\kappa(t,s)\hat{P}(s) = \hat{P}(t)\hat{T}_\kappa(t,s) \quad \text{and} \quad \hat{T}_{\kappa'}(t,s)\hat{Q}(s) = \hat{Q}(t)\hat{T}_{\kappa'}(t,s),$$

which by (15) yields that

$$\hat{T}(t,s)\hat{P}(s) = \hat{P}(t)\hat{T}(t,s) \quad \text{and} \quad \hat{T}(t,s)\hat{Q}(s) = \hat{Q}(t)\hat{T}(t,s).$$

This readily implies that

$$\hat{T}(t,s)\hat{R}(s) = \hat{R}(t)\hat{T}(s,t).$$

Furthermore, the operators $\hat{P}(t)$ and $\hat{Q}(t)$ are projections, and by Lemma 4 we also have

$$\begin{aligned}\hat{R}(t)^2 &= \big(\mathrm{Id} - \hat{P}_1(t) - \hat{Q}_2(t)\big)^2 = \mathrm{Id} - 2\hat{P}_1(t) - 2\hat{Q}_2(t) + \hat{P}_1(t)^2 + \hat{Q}_2(t)^2 + \hat{P}_1(t)\hat{Q}_2(t) + \hat{Q}_2(t)\hat{P}_1(t)\\ &= \mathrm{Id} - \hat{P}_1(t) - \hat{Q}_2(t) = \hat{R}(t).\end{aligned}$$

Now we consider the subspaces

$$\hat{E}(t) = \hat{P}(t)(X), \qquad \hat{F}(t) = \hat{Q}(t)(X) \quad \text{and} \quad \hat{G}(t) = \hat{R}(t)(X). \tag{18}$$

We note that

$$\hat{E}(t) = \hat{E}_1(t), \qquad \hat{F}(t) = \hat{F}_2(t) \quad \text{and} \quad \hat{G}(t) = \hat{E}_2(t) \cap \hat{F}_1(t). \tag{19}$$

The first two identities are immediate from (18). Moreover, since $\hat{P}(t)$ and $\hat{Q}(t)$ are projections respectively onto $\hat{E}_1(t)$ and $\hat{F}_2(t)$, it follows from Lemma 3 (see (17)) that the image of (the projection) $\mathrm{Id} - \hat{P}(t) - \hat{Q}(t) = \hat{R}(t)$ is $\hat{E}_2(t) \cap \hat{F}_1(t)$. This yields the third identity in (18).

It follows from (8) that

$$\|\hat{R}(t)\| = \|\mathrm{Id} - \hat{P}_1(t) - \hat{Q}_2(t)\| \leqslant 1 + 8De^{\varepsilon|t|} \leqslant (1+8D)e^{\varepsilon|t|}. \tag{20}$$

By Lemma 1, since $\hat{P}(t) = \hat{P}_1(t)$, for every $t \geqslant s$ we have

$$\begin{aligned}\|\hat{T}(t,s)|\hat{E}(s)\| &= \|\hat{T}_\kappa(t,s)e^{-\kappa(t-s)}|\hat{E}_1(s)\|\\ &= K_1 e^{-\kappa(t-s)}e^{-\Gamma(\alpha)(t-s)+2\varepsilon|s|},\end{aligned}$$

for some constant $K_1 > 0$. Similarly, since $\hat{Q}(t) = \hat{Q}_2(t)$, for every $t \geqslant s \geqslant 0$ we have

$$\begin{aligned}\|\hat{T}(t,s)^{-1}|\hat{F}(t)\| &= \|\hat{T}_{\kappa'}(t,s)^{-1}e^{\kappa'(t-s)}|\hat{F}_2(t)\|\\ &\leqslant K_2 e^{\kappa'(t-s)}e^{-\Gamma(\alpha')(t-s)+2\varepsilon|t|},\end{aligned}$$

for some constant $K_2 > 0$. Furthermore, by (19), for every $t \geqslant s$ we have

$$\begin{aligned}\|\hat{T}(t,s)\hat{R}(s)\| &\leqslant \|\hat{T}(t,s)|\hat{G}(s)\| \cdot \|\hat{R}(s)\| = \|\hat{T}(t,s)|\big(\hat{E}_2(s) \cap \hat{F}_1(s)\big)\| \cdot \|\hat{R}(s)\|\\ &\leqslant \|\hat{T}(t,s)|\hat{E}_2(s)\| \cdot \|\hat{R}(s)\| = e^{-\kappa'(t-s)}\|\hat{T}_{\kappa'}(t,s)|\hat{E}_2(s)\| \cdot \|\hat{R}(s)\|.\end{aligned} \tag{21}$$

Analogously, again by (19), for every $t \geqslant s \geqslant 0$,

$$\begin{aligned}\|\hat{T}(t,s)^{-1}\hat{R}(t)\| &\leqslant \|\hat{T}(t,s)^{-1}|\hat{G}(t)\| \cdot \|\hat{R}(t)\| \leqslant \|\hat{T}(t,s)^{-1}|\hat{F}_1(t)\| \cdot \|\hat{R}(t)\|\\ &= e^{\kappa(t-s)}\|\hat{T}_\kappa(t,s)^{-1}|\hat{F}_1(t)\| \cdot \|\hat{R}(t)\|.\end{aligned} \tag{22}$$

By Lemma 1 and (20), it follows from (21) that for every $t \geqslant s$,

$$\left\|\hat{T}(t,s)\hat{R}(s)\right\| \leqslant (1+8D)K_2 e^{-\kappa'(t-s)} e^{-\Gamma(\alpha')(t-s)+2\varepsilon|s|}$$

and it follows from (22) that for every $t \geqslant s \geqslant 0$,

$$\left\|\hat{T}(t,s)^{-1}\hat{R}(s)\right\| \leqslant (1+8D)K_1 e^{\kappa(t-s)} e^{-\Gamma(\alpha)(t-s)+2\varepsilon|s|}.$$

This shows that Eq. (2) admits a nonuniform exponential trichotomy in $[0,+\infty)$. The second property follows also immediately from the above lemmas. This completes the proof of the theorem. □

It follows from the proof of Theorem 2 that Eq. (2) admits a nonuniform exponential trichotomy with the constants in (3) replaced respectively by

$$\hat{a} = \frac{a+b}{2} - \Gamma\left(\frac{b-a}{2}\right), \qquad \hat{b} = \frac{a+b}{2} + \Gamma\left(\frac{b-a}{2}\right),$$
$$\hat{c} = \frac{c+d}{2} - \Gamma\left(\frac{d-c}{2}\right), \qquad \hat{d} = \frac{c+d}{2} + \Gamma\left(\frac{d-c}{2}\right),$$
$$3\varepsilon \quad \text{and} \quad \max\left\{\frac{D}{1-\delta D/(\hat{d}-c)}, \frac{D}{1-\delta D/(\hat{b}-a)}\right\},$$

with Γ as in (16).

3.2. Robustness of strong exponential trichotomies

We can also consider a stronger version of exponential trichotomy and establish a corresponding robustness result. Namely, we say that Eq. (1) admits a *strong nonuniform exponential trichotomy in I* if it admits a nonuniform exponential trichotomy in I and there exist constants $\beta \geqslant d$ and $\gamma \geqslant b$ such that for every $t, s \in I$ with $t \geqslant s$ we have

$$\left\|T(t,s)^{-1}P(t)\right\| \leqslant De^{\beta(t-s)+\varepsilon|t|},$$

and

$$\left\|T(t,s)Q(s)\right\| \leqslant De^{\gamma(t-s)+\varepsilon|s|}.$$

The following is a robustness result for strong trichotomies in the interval $\mathbb{R}_0^+ = [0,+\infty)$.

Theorem 3. *Let $A, B : \mathbb{R}_0^+ \to \mathcal{B}(X)$ be continuous functions such that Eq.* (1) *admits a nonuniform exponential trichotomy in $\mathbb{R}_0^+$ satisfying* (7), *and assume that $\|B(t)\| \leqslant \delta e^{-2\varepsilon|t|}$ for every $t \in \mathbb{R}_0^+$. If δ is sufficiently small, then Eq.* (2) *admits a strong nonuniform exponential trichotomy in $\mathbb{R}_0^+$.*

Proof. We recall that Eq. (1) is said to admit a *strong nonuniform exponential dichotomy* if it admits a nonuniform exponential dichotomy and there exists $\beta \geqslant \alpha$ such that for every $t \geqslant s$ we have

$$\left\|T(t,s)^{-1}P(t)\right\| \leqslant Ce^{\beta(t-s)+\varepsilon|t|},$$

and

$$\left\|T(t,s)Q(s)\right\| \leqslant Ce^{\beta(t-s)+\varepsilon|s|}.$$

Lemma 5. *(See [5].) Let $A, B : \mathbb{R}_0^+ \to \mathcal{B}(X)$ be continuous functions such that Eq.* (1) *admits a nonuniform exponential dichotomy in $\mathbb{R}_0^+$ with $\varepsilon < a$, and assume that $\|B(t)\| \leqslant \delta e^{-2\varepsilon|t|}$ for every $t \in \mathbb{R}_0^+$. If δ is sufficiently small, then Eq.* (2) *admits a strong nonuniform exponential dichotomy in $\mathbb{R}_0^+$.*

Proceeding as in the proof of Theorem 2 we consider the projections $\hat{P}(t) = \hat{P}_1(t)$ and $\hat{Q}(t) = \hat{Q}_2(t)$. It then follows from Lemma 5 that for every $t \geqslant s$ we have

$$\left\|\hat{T}(t,s)^{-1}\hat{P}(t)\right\| = e^{\kappa(t-s)}\left\|\hat{T}_\kappa(t,s)^{-1}\hat{P}(t)\right\| \leqslant 8D^2\tilde{D}e^{\kappa(t-s)}e^{\tilde{\beta}(t-s)+3\varepsilon|t|},$$

and

$$\left\|\hat{T}(t,s)\hat{Q}(s)\right\| = e^{-\kappa'(t-s)}\left\|\hat{T}_{\kappa'}(t,s)\hat{Q}(s)\right\| \leqslant \tilde{D}e^{-\kappa'(t-s)}e^{\tilde{\beta}(t-s)+3\varepsilon|s|}.$$

This completes the proof of the theorem. □

3.3. Nonuniform exponential trichotomies in $I = (-\infty, 0]$

Now we consider intervals of the form $(-\infty, \varrho]$ with $\varrho \geqslant 0$. Our robustness result is entirely analogous to Theorem 2, and can be readily obtained simply by reversing the time.

Theorem 4. *Let $A, B : I \to \mathcal{B}(X)$ be continuous functions in a interval $I = (-\infty, \varrho]$ with $\varrho \geqslant 0$ such that Eq. (1) admits a nonuniform exponential trichotomy in I satisfying (7), and assume that $\|B(t)\| \leqslant \delta e^{-2\varepsilon|t|}$ for every $t \in I$. If δ is sufficiently small, then there exist projections $\hat{P}^-(t)$, $\hat{Q}^-(t)$ and $\hat{R}^-(t)$ for $t \in I$ such that:*

1. *Eq. (2) admits a nonuniform exponential trichotomy in $(-\infty, 0]$ with respect to these projections;*
2. *for the new trichotomy, the corresponding estimates to the ones in (4) and (6) are valid for all $t \geqslant s$ with $t, s \in I$.*

Proof. The following is a version of Lemma 1 for intervals of the form $(-\infty, \varrho]$, and is obtained by reversing the time.

Lemma 6. *Let $A, B : I \to \mathcal{B}(X)$ be continuous functions in an interval $(-\infty, \varrho]$ with $\varrho \geqslant 0$ such that Eq. (1) admits a nonuniform exponential dichotomy in I with $\varepsilon < a$, and assume that $\|B(t)\| \leqslant \delta e^{-2\varepsilon|t|}$ for every $t \in I$. If δ is sufficiently small, then there exist projections $\tilde{P}(t)$ and $\tilde{Q}(t) = \mathrm{Id} - \tilde{P}(t)$ for $t \in I$ such that for every $t, s \in I$:*

1.
$$\tilde{P}(t) = \hat{T}(t,0)\tilde{P}(0)\hat{T}(0,t), \qquad \tilde{Q}(t) = \hat{T}(t,0)\tilde{Q}(0)\hat{T}(0,t),$$
and
$$\tilde{P}(0)\bar{P}(0) = \bar{P}(0), \qquad \bar{P}(0)\tilde{P}(0) = \tilde{P}(0),$$
$$\tilde{Q}(0)\bar{Q}(0) = \tilde{Q}(0), \qquad \bar{Q}(0)\tilde{Q}(0) = \bar{Q}(0);$$

2.
$$\left\|\hat{T}(t,s)|\operatorname{Im}\tilde{P}(s)\right\| \leqslant \tilde{D}e^{-\tilde{a}(t-s)+\varepsilon|s|}, \quad 0 \geqslant t \geqslant s,$$
$$\left\|\hat{T}(t,s)|\operatorname{Im}\tilde{Q}(s)\right\| \leqslant \tilde{D}e^{-\tilde{a}(s-t)+\varepsilon|s|}, \quad s \geqslant t;$$

3.
$$\left\|\tilde{P}(t)\right\| \leqslant 4De^{\varepsilon|t|} \quad \textit{and} \quad \left\|\tilde{Q}(t)\right\| \leqslant 4De^{\varepsilon|t|}.$$

The desired statement follows readily from Lemmas 3 and 6. □

We can obtain in a similar manner a version of Theorem 3 for intervals of the form $(-\infty, \varrho]$.

4. Exponential trichotomies in $\mathbb{R}$

Theorem 5. *Let $A, B : \mathbb{R} \to \mathcal{B}(X)$ be continuous functions such that Eq. (1) admits a nonuniform exponential trichotomy in $\mathbb{R}$ satisfying (7), and assume that $\|B(t)\| \leqslant \delta e^{-2\varepsilon|t|}$ for every $t \in \mathbb{R}$. If δ is sufficiently small, then Eq. (2) admits a nonuniform exponential trichotomy in $\mathbb{R}$.*

Proof. We need the following result.

Lemma 7. *(See [5].) Let $A, B : \mathbb{R} \to \mathcal{B}(X)$ be continuous functions such that Eq. (1) admits a nonuniform exponential dichotomy in $\mathbb{R}$ with $\varepsilon < a$, and assume that $\|B(t)\| \leqslant \delta e^{-2\varepsilon|t|}$ for every $t \in \mathbb{R}$. If δ is sufficiently small, then Eq. (2) admits a nonuniform exponential dichotomy in $\mathbb{R}$, with the constants a, D and ε replaced respectively by $\tilde{a}$, $4D\tilde{D}$ and 2ε. Moreover, for every $t, s \in \mathbb{R}$ the associated projections $\bar{P}(t)$ and $\bar{Q}(t) = \mathrm{Id} - \bar{P}(t)$ satisfy:*

1.
$$\bar{P}(t) = \hat{T}(t,0)S\bar{P}(0)S^{-1}\hat{T}(0,t)$$
and
$$\bar{Q}(t) = \hat{T}(t,0)S\bar{Q}(0)S^{-1}\hat{T}(0,t)$$
with $S = \hat{P}(0) + \tilde{Q}(0)$ invertible, where $\hat{P}(0)$ is the projection given by Lemma 1 and $\tilde{Q}(0)$ is the projection given by Lemma 6;

2.

$$S\bar{P}(0) = \hat{P}(0), \qquad S\bar{Q}(0) = \tilde{Q}(0),$$
$$\bar{P}(t)P(t) = \bar{P}(t), \qquad P(t)\bar{P}(t) = P(t),$$
$$\bar{Q}(t)Q(t) = \bar{Q}(t), \qquad Q(t)\bar{Q}(t) = Q(t);$$

3.

$$\|\hat{T}(t,s)|\operatorname{Im}\bar{P}(s)\| \leqslant \tilde{D}e^{-\tilde{a}(t-s)+\varepsilon|s|}, \quad t \geqslant s,$$

and

$$\|\hat{T}(t,s)|\operatorname{Im}\bar{Q}(s)\| \leqslant \tilde{D}e^{-\tilde{a}(s-t)+\varepsilon|s|}, \quad s \geqslant t;$$

4.

$$\|\bar{P}(t)\| \leqslant 4De^{\varepsilon|t|} \quad \textit{and} \quad \|\bar{Q}(t)\| \leqslant 4De^{\varepsilon|t|}.$$

By Lemma 7, Eq. (10) admits a nonuniform exponential dichotomy, say with projections $\bar{P}_1$ and $\bar{Q}_1$, and Eq. (13) admits a nonuniform exponential dichotomy, say with projections $\bar{P}_2$ and $\bar{Q}_2$. We consider the subspaces

$$\bar{E}_1(t) = \bar{P}_1(t)(X) \quad \text{and} \quad \bar{F}_1(t) = \bar{Q}_1(t)(X),$$

and

$$\bar{E}_2(t) = \bar{P}_2(t)(X) \quad \text{and} \quad \bar{F}_2(t) = \bar{Q}_2(t)(X).$$

Since the map S in Lemma 7 is invertible, it follows from Lemmas 2 and 3 that

$$\bar{E}_1(t) \subset \bar{E}_2(t), \qquad \bar{F}_2(t) \subset \bar{F}_1(t),$$

and

$$\left(\bar{E}_2(t) \cap \bar{F}_1(t)\right) \oplus \bar{E}_1(t) \oplus \bar{F}_2(t) = X$$

for every $t \in \mathbb{R}$. This allows us to proceed exactly as in the proof of Theorem 2 to show that Eq. (2) admits a nonuniform exponential trichotomy in $\mathbb{R}$ with projections $\bar{P}_1(t)$, $\bar{Q}_2(t)$ and $\mathrm{Id} - \bar{P}_1(t) - \bar{Q}_2(t)$. □

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