J. Math. Anal. Appl. 354 (2009) 478–489

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

# The uniqueness of an entire function sharing a small entire function with its derivatives ☆

Jun Wang [a,*], Xiao-Min Li [b]

[a] School of Mathematics Science, Fudan University, Shanghai 200433, People's Republic of China
[b] Department of Mathematics, Ocean University of China, Qingdao, Shandong 266071, People's Republic of China

ARTICLE INFO

*Article history:*
Received 16 August 2007
Available online 13 January 2009
Submitted by M. Putinar

*Keywords:*
Entire functions
Order of growth
Shared values
Uniqueness theorems

ABSTRACT

In this paper, we prove a theorem on the growth of a solution of a linear differential equation. From this we obtain some uniqueness theorems concerning that a nonconstant entire function and its derivatives sharing a small entire function. The results in this paper improve many known results. Some examples are provided to show that the results in this paper are the best possible.



## 1. Introduction and main results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [7,10] and [14]. It will be convenient to let $E$ denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function $h$, we denote by $T(r,h)$ the Nevanlinna characteristic of $h$ and by $S(r,h)$ any quantity satisfying $S(r,h)=o(T(r,h))$ $(r\to\infty,\ r\notin E)$.

Let $f$ and $g$ be two nonconstant meromorphic functions and let $a$ be a complex number. We say that $f$ and $g$ share $a$ CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that $f$ and $g$ share the value $a$ IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. In addition, we say that $f$ and $g$ share $\infty$ CM, if $1/f$ and $1/g$ share 0 CM, and we say that $f$ and $g$ share $\infty$ IM, if $1/f$ and $1/g$ share 0 IM (see [15]). Let $b\not\equiv\infty$ be a nonconstant meromorphic function such that $T(r,b)=S(r,f)$ and $T(r,b)=S(r,g)$. If $f-b$ and $g-b$ share 0 CM, we say that $f$ and $g$ share $b$ CM, and we say that $f$ and $g$ share $b$ IM, if $f-b$ and $g-b$ share 0 IM. In this paper, we also need the following definition.

**Definition 1.1.** For a nonconstant entire function $f$, the order $\sigma(f)$, lower order $\mu(f)$, hyper-order $\sigma_2(f)$ and lower hyper-order $\mu_2(f)$ are defined by

$$\sigma(f)=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r}=\limsup_{r\to\infty}\frac{\log\log M(r,f)}{\log r},$$
$$\mu(f)=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}=\liminf_{r\to\infty}\frac{\log\log M(r,f)}{\log r},$$

☆ Project supported by the NSFC (No. 10871047), the NSFC (No. 10771121), the RFDP (No. 20060422049) and the NSFC (No. A0324617).

* Corresponding author.
*E-mail addresses:* majwang@fudan.edu.cn (J. Wang), xmli01267@gmail.com (X.-M. Li).

0022-247X/$ – see front matter 
doi:10.1016/j.jmaa.2008.12.067

$$\sigma_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r} = \limsup_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r}$$

and

$$\mu_2(f) = \liminf_{r\to\infty} \frac{\log\log T(r,f)}{\log r} = \liminf_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r}$$

respectively, where and in what follows, $M(r,f) = \max_{|z|=r} |f(z)|$.

In 1977, L.A. Rubel and C.C. Yang [12] proved that if an entire function $f$ shares two distinct complex numbers CM with its derivative $f'$, then $f = f'$. How is the relation between $f$ and $f'$, if an entire function $f$ shares one complex number $a$ CM with its derivative $f'$? In 1996, R. Brück [1] made a conjecture that if $f$ is a nonconstant entire function satisfying $\sigma_2(f) < \infty$, where $\sigma_2(f)$ is not a positive integer, and if $f$ and $f'$ share one complex number $a$ CM, then $f - a = c(f' - a)$ for some constant $c \neq 0$. For the case that $a = 0$, the above conjecture had been proved by R. Brück [1]. In the same paper, R. Brück proved the above conjecture is true, provided that $a \neq 0$ and $N(r, 1/f') = S(r,f)$. In 1998, G.G. Gundersen and L.Z. Yang proved that the conjecture is true for $a \neq 0$, provided that $f$ satisfies the additional assumption $\sigma(f) < \infty$ (see [6]). In 1999, L.Z. Yang proved that if a nonconstant entire function $f$ and one of its derivatives $f^{(k)}$ $(k \geqslant 1)$ share one complex number $a$ $(\neq 0)$ CM, where $f$ satisfies $\sigma(f) < \infty$ and $k$ $(\geqslant 1)$ is a positive integer, then $f - a = c(f^{(k)} - a)$ for some complex number $c \neq 0$ (see [16]). In 2004, J.P. Wang proved the following theorem.

**Theorem A.** *(See [13, Theorem 1].) Let $f$ be a nonconstant entire function of finite order, let $P$ be a polynomial with degree $p \geqslant 1$, and let $k$ be a positive integer. If $f - P$ and $f^{(k)} - P$ share 0 CM, then $f^{(k)} - P = c(f - P)$ for some complex number $c \neq 0$.*

Regarding Theorem A, it is natural to ask the following question.

**Question 1.1.** What can be said if a nonconstant entire function $f$ and one of its derivatives $f^{(k)}$ $(k \geqslant 1)$ share a small entire function $a$ related to $f$?

For dealing with Question 1.1, we will prove the following result which improves Theorem A in this paper.

**Theorem 1.1.** *If $f$ is a nonconstant solution of the differential equation*

$$f^{(k)} - a_1 = (f - a_2)\cdot e^{Q}, \tag{1.1}$$

*where $a_1$ and $a_2$ are two entire functions such that $\sigma(a_j) < 1$ $(j = 1,2)$, $k$ $(\geqslant 1)$ is a positive integer, and $Q$ is a polynomial, then $\mu_2(f) = \sigma_2(f) = \deg(Q)$, where and in what follows, $\deg(Q)$ denotes the degree of $Q$.*

From Theorem 1.1 we get the following corollary that improves Theorem 1 in [6].

**Corollary 1.1.** *If $f$ is a nonconstant solution of the differential equation* (1.1)*, where $a_1$ and $a_2$ are two entire functions such that $\sigma(a_j) < 1$ $(j = 1,2)$, $k$ $(\geqslant 1)$ is a positive integer, and $Q$ is a nonconstant polynomial, then $\mu_2(f) = \sigma_2(f) = \deg(Q) \geqslant 1$, and $f$ is an entire function of infinite order.*

From Theorem 1.1 we also get the following two corollaries which improve Theorem A and deal with Question 1.1.

**Corollary 1.2.** *Let $f$ be a nonconstant solution of the differential equation*

$$f^{(k)} - a = (f - a)\cdot e^{Q}, \tag{1.2}$$

*where $k$ $(\geqslant 1)$ is a positive integer, $a$ $(\not\equiv 0,\infty)$ is an entire function such that $\sigma(a) < 1$, and $Q$ is a polynomial. If $\mu_2(f) < \infty$ and $\mu_2(f)$ is not a positive integer, then $f - a = c(f^{(k)} - a)$ for some complex number $c \neq 0$.*

**Corollary 1.3.** *Let $f$ be a nonconstant solution of the differential equation* (1.2)*, where $k$ $(\geqslant 1)$ is a positive integer, $a$ $(\not\equiv 0,\infty)$ is an entire function such that $\sigma(a) < 1$, and $Q$ is a polynomial. If $\mu(f) < \infty$, then $f - a = c(f^{(k)} - a)$ for some complex number $c \neq 0$.*

**Proof.** First, from (1.2) and Lemma 2.1 in Section 2 of this paper, we get

$$T\left(r, e^{Q}\right) \leqslant T(r,f) + 2T(r,a) + O\left(\log T(r,f) + \log r\right) \quad (r \notin E). \tag{1.3}$$

From (1.3), Lemma 2.2 and the condition $\mu(f) < \infty$, we get

$$\sigma\left(e^{Q}\right) = \mu\left(e^{Q}\right) \leqslant \mu(f) < \infty. \tag{1.4}$$

From (1.4) we see that $Q$ is a polynomial. From (1.2), (1.4) and Theorem 1.1 we get

$$\mu_2(f) = \sigma_2(f) = \deg(Q) = 0. \tag{1.5}$$

From (1.5) we see that $Q$, and so $e^Q$ is a constant. From this and (1.2) we get the conclusion of Corollary 1.3. □

**Example 1.1.** Let $f(z)$ be a solution of the differential equation

$$f'(z) - z = \big(f(z) - z\big) \cdot e^z.$$

Then $\sigma(z) = 0$, and from Lemma 2.11 in Section 2 of this paper we see that $f$ is a nonconstant entire function. Moreover, it immediately follows from Theorem 1.1 that $\mu_2(f) = \sigma_2(f) = \sigma(e^z) = 1$. This example shows that the conclusions of Theorem 1.1 and Corollary 1.1 occur. This example also shows that the condition "$\mu_2(f) < \infty$ and $\mu_2(f)$ is not a positive integer" in Corollary 1.2 is the best possible.

From Corollary 1.2 we get the following corollary.

**Corollary 1.4.** *Let $f$ be a nonconstant solution of the differential equation* (1.2)*, where $k$ ($\geqslant 1$) is a positive integer, $a$ ($\not\equiv 0, \infty$) is an entire function such that $\sigma(a) < 1$, and $Q$ is a polynomial. If $\mu_2(f) < \infty$ and $\mu_2(f)$ is not a positive integer, and if $f$ and $f^{(k)}$ share $b$ IM, where $b$ ($\not\equiv a, \infty$) is a small function related to $f$, then $f = f^{(k)}$.*

**Proof.** First, from Corollary 1.2 we get

$$f - a = c\big(f^{(k)} - a\big), \tag{1.6}$$

where $c$ ($\neq 0$) is a complex number. If $c = 1$, from (1.6) we get the conclusion of Corollary 1.4. Next we suppose that $c \neq 1$. Then it follows from (1.6) that $f \not\equiv f^{(k)}$. If $\overline{N}(r, 1/(f - b)) \neq S(r, f)$, from $a \not\equiv b$ we see that there exists one point $z_0$ such that $f(z_0) = f^{(k)}(z_0) = b(z_0) \neq a(z_0)$. From this and (1.6) we get the conclusion of Corollary 1.4. Next we suppose that

$$\overline{N}\left(r, \frac{1}{f - b}\right) = S(r, f). \tag{1.7}$$

From (1.7) and Nevanlinna's three small functions theorem (see [15, Theorem 1.36]) we get

$$T(r, f) = T\big(r, f^{(k)}\big) + O(1) = N_{1)}\left(r, \frac{1}{f - a}\right) + S(r, f), \tag{1.8}$$

where and in what follows, $N_{1)}(r, 1/(f - a))$ denotes the counting function of simple zeros of $f - a$. Let

$$F = \frac{f - b}{a - b}, \qquad G = \frac{f^{(k)} - b}{a - b}. \tag{1.9}$$

Then from (1.7)–(1.9) we get

$$\overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) = S(r, f), \qquad \overline{N}(r, G) + \overline{N}\left(r, \frac{1}{G}\right) = S(r, f) \tag{1.10}$$

and

$$\limsup_{\substack{r \to \infty \\ r \notin E}} \frac{\overline{N}_0(r, 1)}{T(r, F) + T(r, G)} = \frac{1}{2}, \tag{1.11}$$

where and in what follows, $\overline{N}_0(r, 1)$ denotes the reduced counting function of the common 1-points of $F$ and $G$. Thus from (1.10), (1.11) and Lemma 2.9 in Section 2 of this paper we have $F = G$ or $FG = 1$. If $F = G$, from (1.9) we get $f = f^{(k)}$, which is impossible. If $FG = 1$, from (1.9) we get

$$(f - b)\big(f^{(k)} - b\big) = (a - b)^2. \tag{1.12}$$

From (1.6), (1.12) and $c \neq 1$ we have

$$f^2 + (ca - cb - a - b)f + \big(ab - a^2 c + abc\big) = 0. \tag{1.13}$$

From (1.13) we get $T(r, f) = S(r, f)$, which is impossible.

Corollary 1.4 is thus completely proved. □

In 1995, H.X. Yi and C.C. Yang posed the following question.

**Question 1.2.** (See [15, p. 398].) Let $f$ be a nonconstant meromorphic function, and let $a$ be a finite nonzero complex constant. If $f$, $f^{(n)}$ and $f^{(m)}$ share the value $a$ CM, where $n$ and $m$ $(n<m)$ are distinct positive integers not all even or odd, then can we get the result $f=f^{(n)}$?

Regarding Question 1.2, G.G. Gundersen and L.Z. Yang proved the following result in 1998.

**Theorem B.** *(See [6, Theorem 2].) Let $f$ be a nonconstant entire function of finite order, let $a$ $(\neq 0)$ be a complex number, and let $n$ be a positive integer. If the value $a$ is shared by $f$, $f^{(n)}$ and $f^{(n+1)}$ IM, and shared by $f^{(n)}$ and $f^{(n+1)}$ CM, then $f=f'$.*

In this paper, we will prove the following two theorems that improve Theorem B.

**Theorem 1.2.** *Let $f$ be a nonconstant solution of the differential equation*

$$f^{(n+1)}(z)-P(z)=\big(f^{(n)}(z)-P(z)\big)\cdot e^{Q(z)}, \tag{1.14}$$

*where $n$ $(\geqslant 1)$ is a positive integer, $P$ $(\not\equiv 0)$ is a nonconstant polynomial, and $Q$ is a polynomial. If $f-P$ and $f^{(n)}-P$ share 0 IM, and if $\mu_2(f)<\infty$ and $\mu_2(f)$ is not a positive integer, then $e^Q$ is a constant, and $f$ is given as one of the following three expressions:*

(i) *$f=P+\frac{f^{(l+n)}}{(l+n)!}\cdot(z-z_1)^{j_1}(z-z_2)^{j_2}\cdots(z-z_k)^{j_k}$, where $l=\deg(P)=\deg(f^{(n)})$ is the degree of $f^{(n)}$ and $P$, $k$ $(\geqslant 1)$ is a positive integer, $z_1,z_2,\ldots,z_{k-1},z_k$ are $k$ distinct elements in the set $\{z\colon -\frac{1}{c}\cdot P+(c-1)\sum_{j=1}^{l}\frac{P^{(j)}}{c^{j+1}}=0\}=\{z_1,z_2,\ldots,z_{k-1},z_k\}$, in which $c\neq 0,1$ is a complex number, $j_1,j_2,\ldots,j_{k-1},j_k$ are positive integers such that $j_1+j_2+\cdots+j_k=l+n$.*

(ii) *$f=P+\gamma_k(z-z_1)^{j_1}(z-z_2)^{j_2}\cdots(z-z_k)^{j_k}$, where $\gamma_k$ $(\neq 0)$ is a certain complex number, $z_1,z_2,\ldots,z_{k-1}$ and $z_k$ are $k$ distinct elements in the set $\{z\colon P(z)=0\}=\{z_1,z_2,\ldots,z_{k-1},z_k\}$, $k$ $(\geqslant 1)$ is a positive integer, and $1\leqslant\max\{\deg(P),j_1+j_2+\cdots+j_k\}\leqslant n-1$.*

(iii) *$f=\gamma e^z$, where $\gamma$ is a certain nonzero complex number.*

**Theorem 1.3.** *Let $f$ be a nonconstant entire function such that $\mu(f)<\infty$, and let $a$ $(\not\equiv 0,\infty)$ be an entire function such that $\sigma(a)<\mu(f)$, and let $n$ $(\geqslant 1)$ be a positive integer. If $a$ is shared by $f$, $f^{(n)}$ and $f^{(n+1)}$ IM, and shared by $f^{(n)}$ and $f^{(n+1)}$ CM, then $f=\gamma e^z$, where $\gamma$ is a certain nonzero complex number.*

**Example 1.2.** (See [4].) Let $f(z)=e^{e^z}+e^z$, and $a(z)=e^z$. Then

$$f'(z)-a(z)=\big(f(z)-a(z)\big)\cdot e^z.$$

Moreover, we verify that $\sigma(a)<\mu(f)$ and $\mu(f)=\infty$. This example shows that the condition "$\mu(f)<\infty$" in Theorem 1.3 is the best possible.

From Theorem 1.2 we get the following corollary.

**Corollary 1.5.** *Let $f$ be a nonconstant solution of (1.14), where $P$ $(\not\equiv 0)$ is a nonconstant polynomial and $n=1$. If $f-P$ and $f'-P$ share 0 IM, and if $\mu_2(f)<\infty$ and $\mu_2(f)$ is not a positive integer, then $e^Q$ is a constant and $f=f'$ or $f$ is expressed as $f=P+\frac{f^{(l+1)}}{(l+1)!}\cdot(z-z_1)^{j_1}(z-z_2)^{j_2}\cdots(z-z_k)^{j_k}$, where $l=\deg(P)=\deg(f')$ is the degree of $f'$ and $P$, $k$ $(\geqslant 1)$ is a positive integer, $z_1,z_2,\ldots,z_{k-1},z_k$ are $k$ distinct elements in the set $\{z\colon -\frac{1}{c}\cdot P+(c-1)\sum_{j=1}^{l}\frac{P^{(j)}}{c^{j+1}}=0\}=\{z_1,z_2,\ldots,z_{k-1},z_k\}$, in which $c\neq 0,1$ is a complex number, $j_1,j_2,\ldots,j_{k-1},j_k$ are positive integers such that $j_1+j_2+\cdots+j_k=l+1$.*

From Theorem 1.3 we get the following corollary.

**Corollary 1.6.** *Let $f$ be a nonconstant entire function such that $\mu(f)<\infty$, and let $a$ $(\not\equiv 0,\infty)$ be an entire function such that $\sigma(a)<\mu(f)$. If $a$ is shared by $f$, $f'$ and $f''$ IM, and shared by $f'$ and $f''$ CM, then $f=f'$.*

## 2. Some lemmas

Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be an entire function. Next we define by $\mu(r)=\max\{|a_n|r^n\colon n=0,1,2,\ldots\}$ the maximum term of $f$, and define by $\nu(r,f)=\max\{m\colon \mu(r)=|a_m|r^m\}$ the central index of $f$ (see [10, p. 50]).

**Lemma 2.1.** *(See [10, Corollary 2.3.4] or [15, Lemma 1.4].) Let $f$ be a transcendental meromorphic function and $k\geqslant 1$ be an integer. Then $m(r,f^{(k)}/f)=O(\log(rT(r,f)))$, outside of a possible exceptional set $E$ of finite linear measure, and if $f$ is of finite order of growth, then $m(r,f^{(k)}/f)=O(\log r)$.*

**Lemma 2.2.** *(See [10, Lemma 1.1.1].) Let $g:(0,+\infty)\to\mathbb{R}$, $h:(0,+\infty)\to\mathbb{R}$ be monotone increasing functions such that $g(r)\leqslant h(r)$ outside of an exceptional set $E$ of finite linear measure. Then, for any $\alpha>1$, there exists $r_0>0$ such that $g(r)\leqslant h(\alpha r)$ for all $r>r_0$.*

**Lemma 2.3.** *(See [9, pp. 36–37] or [10, Theorem 3.1].) If $f$ is an entire function of order $\sigma(f)$, then*

$$\sigma(f)=\limsup_{r\to\infty}\frac{\log\nu(r,f)}{\log r}.$$

**Lemma 2.4.** *(See [2, Lemma 2] or [3, Lemma 4].) If $f$ is a transcendental entire function of hyper-order $\sigma_2(f)$, then $\sigma_2(f)=\limsup_{r\to\infty}(\log\log\nu(r,f))/\log r$.*

**Lemma 2.5.** *Let $f$ be an entire function of infinite order, with the lower hyper-order $\mu_2(f)$. Then*

$$\mu_2(f)=\liminf_{r\to\infty}\frac{\log\log\nu(r,f)}{\log r}.$$

**Proof.** Set $f(z)=\sum_{n=0}^{\infty}a_nz^n$. Without loss of generality, we assume $|a_0|\neq 0$. By Theorem 1.9 in [8], we see that the maximum term $\mu(r)$ of $f$ satisfies

$$\log\mu(2r)=\log|a_0|+\int_0^{2r}\frac{\nu(t,f)}{t}dt\geqslant\log|a_0|+\nu(r,f)\log 2. \tag{2.1}$$

By Cauchy's inequality, we have

$$\mu(2r)\leqslant M(2r,f). \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\nu(r,f)\log 2\leqslant\log M(2r,f)+C, \tag{2.3}$$

where $C$ $(>0)$ is a suitable constant. By definition of $\mu_2(f)$, we have

$$\mu_2(f)=\liminf_{r\to\infty}\frac{\log\log\log M(r,f)}{\log r}=\liminf_{r\to\infty}\frac{\log\log T(r,f)}{\log r}. \tag{2.4}$$

From (2.3) and (2.4) we get

$$\liminf_{r\to\infty}\frac{\log\log\nu(r,f)}{\log r}\leqslant\liminf_{r\to\infty}\frac{\log\log\log M(r,f)}{\log r}=\mu_2(f). \tag{2.5}$$

On the other hand, by Theorem 1.10 in [8] we have

$$M(r,f)<\mu(r)\big\{\nu(2r,f)+2\big\}=|a_{\nu(r,f)}|r^{\nu(r,f)}\cdot\big\{\nu(2r,f)+2\big\}. \tag{2.6}$$

Since $\{|a_n|\}$ is bounded, from (2.6) we get

$$\begin{aligned}\log\log M(r,f)&\leqslant\log\nu(r,f)+\log\log\nu(2r,f)+\log\log r+C_1\\&\leqslant\log\nu(2r,f)\cdot\left(1+\frac{\log\log\nu(2r,f)}{\log\nu(r,f)}\right)+\log\log r+C_2,\end{aligned} \tag{2.7}$$

where $C_j$ $(>0)$ $(j=1,2)$ are suitable constants. By (2.4) and (2.7) we get

$$\mu_2(f)=\liminf_{r\to\infty}\frac{\log\log\log M(r,f)}{\log r}\leqslant\liminf_{r\to\infty}\frac{\log\log\nu(2r,f)}{\log 2r}=\liminf_{r\to\infty}\frac{\log\log\nu(r,f)}{\log r}. \tag{2.8}$$

By (2.5) and (2.8), Lemma 2.5 follows. □

**Lemma 2.6.** *(See [10, Lemma 1.1.2].) Let $g,h:(0,+\infty)\to\mathbb{R}$ be monotonically increasing functions such that $g(r)\leqslant h(r)$ outside of an exceptional set $F$ of finite logarithmic measure. Then, for any $\alpha>1$, there exists $r_0>0$ such that $g(r)\leqslant h(r^{\alpha})$ for all $r>r_0$.*

**Lemma 2.7.** *(See [5] or [15, corollary of Theorem 1.20].) Suppose that $f$ is meromorphic in the complex plane. Then $T(r,f)\leqslant O(T(2r,f')+\log r)$, as $r\to\infty$.*

**Lemma 2.8.** *(See [4, Lemma 8].) Let $f$ be a nonconstant entire function, let $a$ $(\not\equiv 0,\infty)$ be a small function related to $f$, and let $k$ $(\geqslant 1)$ be a positive integer. If $f^{(k)}-a=(f-a)\cdot e^{\alpha}$ and $f^{(k+1)}-a=(f-a)\cdot e^{\beta}$, where $\alpha$ and $\beta$ are two entire functions such that $T(r,e^{\alpha})+T(r,e^{\beta})=S(r,f)$, then $f=f'$.*

**Lemma 2.9.** *(See [15, Theorem 3.30].) Let $f$ and $g$ be two nonconstant meromorphic functions such that $\overline{N}(r,f)+\overline{N}(r,1/f)=S(r,f)$ and $\overline{N}(r,g)+\overline{N}(r,1/g)=S(r,g)$. If $\limsup_{r\underset{r\notin E}{\longrightarrow}\infty}\overline{N}_0(r,1)/(T(r,f)+T(r,g))>1/3$, where $\overline{N}_0(r,1)$ denotes the reduced counting function of the common 1-points of $f$ and $g$, then $f=g$ or $fg=1$.*

**Lemma 2.10.** *(See [11, proof of Corollary 2].) Let $f$ and $a$ be two nonconstant polynomials. If there exists a complex number $c\neq 0,1$ such that $f'-a=c(f-a)$, then $f$ is expressed as $f=(c-1)\sum_{j=0}^{p}\frac{a^{(j)}}{c^{j+1}}$, where $p=\deg(f)=\deg(a)$ is the degree of $f$ and $a$.*

**Lemma 2.11.** *(See [10, Proposition 8.1].) Suppose that all the coefficients $a_0$ $(\not\equiv 0)$, $a_1,a_2,\ldots,a_{n-1}$ and $g$ $(\not\equiv 0)$ of the non-homogeneous linear differential equation*

$$f^{(n)}(z)+a_{n-1}(z)f^{(n-1)}(z)+\cdots+a_1(z)f'(z)+a_0(z)f(z)=g(z) \tag{2.9}$$

*are entire functions. Then all solutions of (2.9) are entire functions.*

## 3. Proof of theorems

**Proof of Theorem 1.1.** Suppose that $f(z)$ is a nonconstant polynomial. Then from (1.1) and $\sigma(a_j)<1$ $(j=1,2)$, we see that for sufficiently large positive number $r_0$ we have $T(r,e^Q)\leqslant T(r,a_1)+T(r,a_2)+O(\log r)$ $(r\geqslant r_0)$, from which we get $\sigma(e^Q)=\mu(e^Q)\leqslant\max\{\sigma(a_1),\sigma(a_2)\}<1$. Combining the condition that $Q(z)$ is a polynomial we get $\deg(Q)=\sigma(e^Q)=\mu(e^Q)<1$, which implies that $Q$ is a constant. Thus $\mu_2(f)=\sigma_2(f)=\deg(Q)=0$, this reveals the conclusion of Theorem 1.1. Next we suppose that $f$ is a transcendental entire function. We discuss the following two cases.

**Case 1.** Suppose that $e^Q$ is a constant. Let $e^Q=c$, where and in what follows, $c$ $(\neq 0)$ is a complex number. Then (1.1) can be rewritten by

$$f^{(k)}-a_1=c(f-a_2). \tag{3.1}$$

If $\sigma(f)<\infty$, then $\mu_2(f)=\sigma_2(f)=\deg(Q)=0$, this reveals the conclusion of Theorem 1.1. Next we suppose that

$$\sigma(f)=\infty. \tag{3.2}$$

From the condition that $f$ is a nonconstant entire function we have

$$M(r,f)\to\infty, \tag{3.3}$$

as $r\to\infty$. Let

$$M(r,f)=\left|f(z_r)\right|, \tag{3.4}$$

where $z_r=re^{i\theta(r)}$, $\theta(r)\in[0,2\pi)$ is some nonnegative real number. From (3.4) and the Wiman–Valiron theory (see [10, Theorem 3.2]) we see that there exists a subset $F\subset(1,\infty)$ with finite logarithmic measure, i.e., $\int_F\frac{dt}{t}<\infty$, such that for some point $z_r=re^{i\theta(r)}$ $(\theta(r)\in[0,2\pi))$ satisfying $|z_r|=r\notin F$ and $M(r,f)=|f(z_r)|$, we have

$$\frac{f^{(k)}(z_r)}{f(z_r)}=\left(\frac{\nu(r,f)}{z_r}\right)^k\bigl(1+o(1)\bigr), \tag{3.5}$$

as $r\to\infty$, $r\notin F$. From (3.2), the condition $\sigma(a_j)<1$ $(j=1,2)$ and Definition 1.1 we see that there exists an infinite sequence of points $z_{r_n}$ such that

$$\lim_{r_n\to\infty}\frac{\log\log M(r_n,f)}{\log r_n}=\limsup_{r\to\infty}\frac{\log\log M(r,f)}{\log r}=\infty \tag{3.6}$$

and

$$\lim_{r_n\to\infty}\frac{|a_j(z_{r_n})|}{|f(z_{r_n})|}=\lim_{r_n\to\infty}\frac{|a_j(z_{r_n})|}{M(r_n,f)}=0\quad(j=1,2). \tag{3.7}$$

Since (3.1) can be rewritten by

$$c=\frac{\frac{f^{(k)}}{f}-\frac{a_1}{f}}{1-\frac{a_2}{f}}, \tag{3.8}$$

from (3.3)–(3.5), (3.7) and (3.8) we have

$$c = \left(\frac{\nu(r_n, f)}{z_{r_n}}\right)^k \bigl(1 + o(1)\bigr), \tag{3.9}$$

as $r_n \to \infty$. Proceeding as in the proof of Lemma 2.5 and applying (3.6), we get

$$\lim_{r_n\to\infty} \frac{\log\log M(r_n, f)}{\log r_n} = \lim_{r_n\to\infty} \frac{\log \nu(r_n, f)}{\log r_n} = \infty,$$

which contradicts (3.9).

**Case 2.** Suppose that $e^Q$ is a nonconstant entire function. Then

$$\sigma\bigl(e^Q\bigr) = \deg(Q) \geqslant 1. \tag{3.10}$$

From (1.1), Lemma 2.1 and the assumptions of Theorem 1.1 we get

$$T\bigl(r, e^Q\bigr) + O\bigl(r^{\max\{\sigma(a_1),\sigma(a_2)\}+\varepsilon}\bigr) \leqslant 2T(r, f) + O\bigl(\log T(r, f) + \log r\bigr) \quad (r \notin E), \tag{3.11}$$

where $\varepsilon$ is an arbitrary positive number. From (3.10), (3.11), Lemma 2.2 and the condition $\max\{\sigma(a_1), \sigma(a_2)\} < 1$, we get

$$1 \leqslant \deg(Q) = \sigma\bigl(e^Q\bigr) = \mu\bigl(e^Q\bigr) \leqslant \mu(f). \tag{3.12}$$

From (3.3), (3.4), (3.12), Definition 1.1 and the condition $\sigma(a_j) < 1$ $(j = 1, 2)$ we get

$$\lim_{r\to\infty} \frac{|a_j(z_r)|}{|f(z_r)|} = \lim_{r\to\infty} \frac{|a_j(z_r)|}{M(r, f)} = 0 \quad (j = 1, 2). \tag{3.13}$$

On the other hand, from (1.1) we get

$$\bigl|Q(z)\bigr| = \bigl|\log e^{Q(z)}\bigr| = \left|\log \frac{\frac{f^{(k)}(z)}{f(z)} - \frac{a_1(z)}{f(z)}}{1 - \frac{a_2(z)}{f(z)}|}\right|. \tag{3.14}$$

Substituting (3.3)–(3.5) and (3.13) into (3.14) we get

$$e^{Q(z_r)} = \left(\frac{\nu(r, f)}{z_r}\right)^k \bigl(1 + o(1)\bigr), \tag{3.15}$$

as $|z_r| = r \to \infty$, $r \notin F$. From (3.15) we get

$$\bigl|Q(z_r)\bigr| = k\bigl|\log \nu(r, f) - \log r - i\theta(r)\bigr|\bigl(1 + o(1)\bigr), \tag{3.16}$$

as $|z_r| = r \to \infty$, $r \notin F$. We discuss the following two subcases.

**Subcase 2.1.** Suppose that

$$\sigma(f) < \infty. \tag{3.17}$$

From (3.17) and Lemma 2.3 we see that there exists a sufficiently large positive number $r_0$, such that

$$\log \nu(r, f) = O(\log r) \quad (r \geqslant r_0). \tag{3.18}$$

Noting that $\theta(r) \in [0, 2\pi)$, from (3.16) and (3.18) we get

$$\bigl|Q(z_r)\bigr| = O\bigl(\log|z_r|\bigr) \quad \bigl(|z_r| \geqslant r_0,\ r \notin F\bigr). \tag{3.19}$$

From (3.19) and the condition that $Q$ is a polynomial we see that $Q$ is a constant, thus $\deg(Q) = 0$. From this and (3.17) we get $\mu_2(f) = \sigma_2(f) = \deg(Q) = 0$, this reveals the conclusion of Theorem 1.1.

**Subcase 2.2.** Suppose that

$$\sigma(f) = \infty. \tag{3.20}$$

From (3.20) and Lemma 2.3 we see that

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log \nu(r, f)}{\log r} = \infty. \tag{3.21}$$

Let

$$Q = q_n z^n + q_{n-1} z^{n-1} + \cdots + q_1 z + q_0, \tag{3.22}$$

where $q_n$ $(\neq 0)$, $q_{n-1}, \ldots, q_1$ and $q_0$ are complex numbers. From (3.22) we get $\lim_{|z|\to\infty} |Q|/|q_n z^n| = 1$. From this we see that there exists some sufficiently large positive number $r_0$, such that $|Q|/|q_n z^n| > 1/e$ $(|z| > r_0)$. Combining (1.1) we get

$$n \log r + \log |q_n| - 1 = \log \left|\log e^{Q(z)}\right| \leqslant \left|\log \log e^{Q(z)}\right| = \left|\log \log \frac{f^{(k)} - a_1}{f - a_2}\right| \quad \big(|z| > r_0\big). \tag{3.23}$$

Since

$$\frac{f^{(k)} - a_1}{f - a_2} = \frac{\frac{f^{(k)}}{f} - \frac{a_1}{f}}{1 - \frac{a_2}{f}}, \tag{3.24}$$

by substituting (3.3)–(3.5) and (3.13) into (3.24) we get

$$\frac{f^{(k)}(z_r) - a_1(z_r)}{f(z_r) - a_2(z_r)} = \left(\frac{\nu(r, f)}{z_r}\right)^k \big(1 + o(1)\big), \tag{3.25}$$

as $|z_r| \to \infty$, $r \notin F$. From (3.23) and (3.25) we have

$$n \log |z_r| + \log |q_n| - 1 \leqslant \left|\log \log \left(\frac{\nu(r, f)}{z_r}\right)^k \big(1 + o(1)\big)\right|, \tag{3.26}$$

as $|z_r| = r \to \infty$, $r \notin F$. Since

$$\log\left(\left(\frac{\nu(r, f)}{z_r}\right)^k \big(1 + o(1)\big)\right) = k\left(1 - \frac{\log r}{\log \nu(r, f)} - \frac{i\theta(r)}{\log \nu(r, f)}\right) \log \nu(r, f) + o(1), \tag{3.27}$$

as $r \to \infty$, $r \notin F$, from (3.21), (3.26), (3.27), Lemma 2.4 and the conditions $\theta(r) \in [0, 2\pi)$ and $|z_r| = r$, we get

$$\begin{aligned} n &\leqslant \limsup_{r\to\infty} \frac{|\log \log ((\frac{\nu(r,f)}{z_r})^k (1 + o(1)))|}{\log r} \leqslant \limsup_{r\to\infty} \frac{\log \log \nu(r, f)}{\log r} \\ &\quad + \limsup_{r\to\infty} \frac{|\log (1 - \frac{\log r}{\log \nu(r,f)} - \frac{i\theta(r)}{\log \nu(r,f)})|}{\log r} + \lim_{r\to\infty} \frac{\log 2k}{\log r} + \limsup_{r\to\infty} \frac{2k_1 \pi}{\log r} \\ &= \limsup_{r\to\infty} \frac{\log \log \nu(r, f)}{\log r} = \sigma_2(f), \end{aligned} \tag{3.28}$$

where $k_1$ is some nonnegative integer. Namely

$$n \leqslant \sigma_2(f). \tag{3.29}$$

From (3.22) we have

$$\sigma\big(e^Q\big) = \deg(Q) = n. \tag{3.30}$$

From (3.29) and (3.30) we get

$$n = \sigma\big(e^Q\big) \leqslant \sigma_2(f). \tag{3.31}$$

On the other hand, from (3.15) we have

$$\limsup_{r\to\infty} \frac{\log \log (\frac{(\nu(r,f))^k}{|z_r|^k} \cdot |1 + o(1)|)}{\log r} \leqslant \limsup_{r\to\infty} \frac{\log \log M(r, e^Q)}{\log r}. \tag{3.32}$$

Since

$$\limsup_{r\to\infty} \frac{\log \log \nu(r, f)}{\log r} = \limsup_{r\to\infty} \frac{\log \log \frac{(\nu(r,f))^k}{2r^k}}{\log r} \tag{3.33}$$

and

$$\limsup_{r\to\infty} \frac{\log \log \frac{(\nu(r,f))^k}{2r^k}}{\log r} \leqslant \limsup_{r\to\infty} \frac{\log \log (\frac{\nu(r,f))^k}{|z_r|^k} \cdot |1 + o(1)|)}{\log r}, \tag{3.34}$$

from (3.32)–(3.34), Lemma 2.4 and Definition 1.1 we get

$$\sigma_2(f) \leqslant \sigma\big(e^Q\big). \tag{3.35}$$

From (3.30), (3.31) and (3.35) we get

$$\sigma_2(f) = \deg(Q) = n. \tag{3.36}$$

On the other hand, from (3.23), (3.15) and the conditions $z_r = re^{i\theta(r)}$, $\theta(r) \in [0, 2\pi)$ and $|z_r| = r$, we get

$$n \log|z_r| + \log|q_n| - 1 \leqslant \log\big|Q(z_r)\big| \leqslant \big|\log\log e^{Q(z_r)}\big| \quad \big(|z_r| > r_0,\ |z_r| \notin F\big) \tag{3.37}$$

and

$$\log e^{Q(z_r)} = k\big(\log\nu(r, f) - \log r - i\theta(r) + o(1)\big) = k\big(\log\nu(r, f) - \log r\big)\big(1 + o(1)\big), \tag{3.38}$$

as $r \to \infty$, $r \notin F$. From (3.37), (3.38), Lemmas 2.5 and 2.6 we get

$$n \leqslant \liminf_{r\to\infty} \frac{\log\log\nu(r, f)}{\log r} = \mu_2(f). \tag{3.39}$$

Noting that $\mu_2(f) \leqslant \sigma_2(f)$ and $\deg(Q) = n$, from (3.36) and (3.39) we get the conclusion of Theorem 1.1.

Theorem 1.1 is thus completely proved. $\square$

**Proof of Theorem 1.2.** First, we will prove

$$\mu_2(f) = \mu_2\big(f^{(n)}\big). \tag{3.40}$$

If $f(z)$ is a nonconstant polynomial, then $\mu_2(f) = \mu_2(f^{(n)}) = 0$, and so (3.40) holds. Next we suppose that $f$, and so $f^{(k)}$ is a transcendental entire function, where $k$ is an arbitrary positive integer. Next we will verify that (3.40) holds. In fact, from Lemma 2.7 we have

$$T(r, f) \leqslant O\big(T(2r, f') + \log r\big), \tag{3.41}$$

as $r \to \infty$. Noting that $f$ and $f'$ are transcendental entire functions, from (3.41) and the definition of the lower hyper-order of a nonconstant entire function we get

$$\mu_2(f) \leqslant \mu_2(f'). \tag{3.42}$$

On the other hand, since

$$T(r, f') \leqslant 2T(r, f) + O\big(\log rT(r, f)\big) \quad (r \notin E), \tag{3.43}$$

from (3.43) and Lemma 2.2 we get

$$\mu_2(f') \leqslant \mu_2(f). \tag{3.44}$$

From (3.42) and (3.44) we get

$$\mu_2(f) = \mu_2(f'). \tag{3.45}$$

Similarly

$$\mu_2\big(f^{(j)}\big) = \mu_2\big(f^{(j+1)}\big) \quad (1 \leqslant j \leqslant n-1). \tag{3.46}$$

From (3.45) and (3.46) we get (3.40). From (3.40) and the condition $\mu_2(f) < \infty$ we have $\mu_2(f^{(n)}) < \infty$, where $\mu_2(f^{(n)})$ is not a positive integer. Combining (1.14) and Theorem 1.1 we have

$$f^{(n+1)}(z) - P(z) = c\big(f^{(n)}(z) - P(z)\big), \tag{3.47}$$

where and in what follows, $c$ $(\neq 0)$ is a complex number. We discuss the following three cases.

**Case 1.** Suppose that $f$ is a nonconstant polynomial and $c \neq 1$. Then from (3.47) and Lemma 2.10 we get

$$f^{(n)} = (c-1)\sum_{j=0}^{l} \frac{P^{(j)}}{c^{j+1}}, \tag{3.48}$$

where $l = \deg(f^{(n)}) = \deg(P)$ is the degree of $f^{(n)}$ and $P$. From (3.48) we see that $f$ is a polynomial with its degree $\deg(f) = l + n$. Combining (3.48) and the condition that $f - P$ and $f^{(n)} - P$ share 0 IM we get (i) of Theorem 1.2.

**Case 2.** Suppose that $f$ is a nonconstant polynomial and $c=1$. Then from (3.47) we have

$$f^{(n)}=f^{(n+1)}=0. \tag{3.49}$$

From (3.49) we see that $f$ is a nonconstant polynomial with degree $\deg(f)\leqslant n-1$. Combining (3.49) and the condition that $f-P$ and $f^{(n)}-P$ share 0 IM we get the conclusion (ii) of Theorem 1.2.

**Case 3.** Suppose that $f$, and so $f^{(n)}$ is a transcendental entire function. First, from the condition that $P$ is a nonconstant polynomial we have

$$P'-P\not\equiv 0. \tag{3.50}$$

Since (3.47) can be rewritten by

$$\frac{f^{(n+1)}(z)-P'}{f^{(n)}-P}+\frac{P'-P}{f^{(n)}-P}=c, \tag{3.51}$$

from (3.50) and Lemma 2.1 we get

$$m\left(r,\frac{1}{f^{(n)}-P}\right)=O\big(\log T(r,f)+\log r\big)\quad (r\notin E),$$

and so

$$N\left(r,\frac{1}{f^{(n)}-P}\right)=T\big(r,f^{(n)}\big)+O\big(\log T(r,f)+\log r\big)\quad (r\notin E). \tag{3.52}$$

From (3.47) and (3.50) we get

$$\begin{aligned} N_{(2}\left(r,\frac{1}{f^{(n)}-P}\right)&\leqslant 2N\left(r,\frac{1}{(f^{(n+1)}-P')-(f^{(n+1)}-P)}\right)\\ &=2N\left(r,\frac{1}{P'-P}\right)=O(\log r), \end{aligned} \tag{3.53}$$

where $N_{(2}(r,1/(f^{(n)}-P))$ denotes the counting function of those zeros of $f^{(n)}-P$ with multiplicity $\geqslant 2$ (see [15]). From (3.52) and (3.53) we get

$$\begin{aligned} N_{1)}\left(r,\frac{1}{f^{(n)}-P}\right)&=N\left(r,\frac{1}{f^{(n)}-P}\right)+O\big(\log T(r,f)+\log r\big)\\ &=T\big(r,f^{(n)}\big)+O\big(\log T(r,f)+\log r\big)\quad (r\notin E). \end{aligned} \tag{3.54}$$

Again from (3.47) we get

$$f^{(n+1-j)}(z)-P^{(-j)}(z)=c\big(f^{(n-j)}(z)-P^{(-j)}(z)+P_{j-1}(z)\big), \tag{3.55}$$

where $j$ is a positive integer satisfying $1\leqslant j\leqslant n$, $P^{(-j)}$ denotes a polynomial such that $(P^{(-j)})^{(j)}=P$, $P_{j-1}=0$ or $P_{j-1}$ is a polynomial such that its degree $\deg(P_{j-1})\leqslant j-1$. From (3.55) we get

$$T(r,f)=T\big(r,f^{(n)}\big)+O(\log r)=T\big(r,f^{(n+1)}\big)+O(\log r). \tag{3.56}$$

From the condition that $f-P$, $f^{(n)}-P$ and $f^{(n+1)}-P$ share 0 IM we get

$$\overline{N}_{1)}\left(r,\frac{1}{f^{(n)}-P}\right)\leqslant\overline{N}\left(r,\frac{1}{f^{(n+1)}-P}\right)\leqslant N\left(r,\frac{1}{f^{(n+1)}-P}\right)\leqslant T\big(r,f^{(n+1)}\big)+O(\log r) \tag{3.57}$$

and

$$\overline{N}_{1)}\left(r,\frac{1}{f^{(n)}-P}\right)\leqslant\overline{N}\left(r,\frac{1}{f-P}\right)\leqslant N\left(r,\frac{1}{f-P}\right)\leqslant T(r,f)+O(\log r). \tag{3.58}$$

From (3.54) and (3.56)–(3.58) we get

$$N_{(2}\left(r,\frac{1}{f-P}\right)+N_{(2}\left(r,\frac{1}{f^{(n)}-P}\right)+N_{(2}\left(r,\frac{1}{f^{(n+1)}-P}\right)=O\big(\log T(r,f)+\log r\big)\quad (r\notin E) \tag{3.59}$$

and

$$m\left(r,\frac{1}{f-P}\right)=O\big(\log T(r,f)+\log r\big)\quad (r\notin E). \tag{3.60}$$

By (3.47) we let

$$\frac{f^{(n)}-P}{f-P}=\frac{f^{(n)}-P^{(n)}}{f-P}+\frac{P^{(n)}-P}{f-P}=h \tag{3.61}$$

and

$$\frac{f^{(n+1)}-P}{f-P}=\frac{f^{(n+1)}-P^{(n+1)}}{f-P}+\frac{P^{(n+1)}-P}{f-P}=ch. \tag{3.62}$$

From (3.59)–(3.62) and the condition that $f-P$, $f^{(n)}-P$ and $f^{(n+1)}-P$ share 0 IM we get

$$T(r,h)=O\big(\log T(r,f)+\log r\big)\quad (r\notin E). \tag{3.63}$$

Again from (3.47) and Lemma 2.11 we see that $f^{(n+l+1)}$, and so $f$ is a transcendental entire function such that $\sigma(f)<\infty$, where $l=\deg(P)$ is the degree of $P$. From this and (3.63) we get $T(r,h)=O(\log r)$, which implies that $h$ is a rational function. From (3.55) we get

$$f'-P^{(-n)}=c\big(f-P^{(-n)}+P_{n-1}\big). \tag{3.64}$$

From (3.61) and (3.62) we get

$$(ch-h')f=hf'+(ch-1)P-(Ph)'. \tag{3.65}$$

By (3.64) and (3.65) and eliminating $f'$ we get

$$(ch-h')f=chf+h\big(cP_{n-1}-cP^{(-n)}+P^{(-n)}\big)+(ch-1)P-(Ph)'. \tag{3.66}$$

From (3.66), the supposition that $f$ is a transcendental entire function and the fact $T(r,h)+T(r,P)=O(\log r)$ we get $ch-h'=ch$, and so $h'=0$, which implies that $h$ is a constant. From this and (3.61) we see that $h$ is a nonzero complex number. Again from (3.61), (3.62) and Lemma 2.8 we get (iii) of Theorem 1.2.

Theorem 1.2 is thus completely proved. □

**Proof of Theorem 1.3.** First, from the assumptions of Theorem 1.3 we have

$$f^{(n+1)}-a=\big(f^{(n)}-a\big)\cdot e^{Q}, \tag{3.67}$$

where $Q$ is an entire function. From (3.67), Lemma 2.1 and the condition $\sigma(a)<\mu(f)$ we see that for a sufficiently large positive number $r_0$ and a sufficiently small positive number $\varepsilon$, we have

$$T\big(r,e^{Q}\big)\leqslant 2T(r,f)+2r^{\mu(f)-\varepsilon}+O\big(\log T(r,f)+\log r\big)\quad (r\notin E,\ r\geqslant r_0). \tag{3.68}$$

From (3.68), Lemma 2.2 and the condition $\mu(f)<\infty$, we get

$$\sigma\big(e^{Q}\big)=\mu\big(e^{Q}\big)\leqslant\mu(f)<\infty, \tag{3.69}$$

which implies that $Q$ is a polynomial. Combining (3.67), (3.69) and proceeding as in the proof of Theorem 1.1, we have $\mu_2(f)=\sigma_2(f)=\deg(Q)=0$. Thus $e^{Q}$ is a nonzero complex number $c$, so (3.67) can be rewritten by

$$f^{(n+1)}-a=c\big(f^{(n)}-a\big). \tag{3.70}$$

From (3.70), Lemma 2.11 and the condition $\sigma(a)<\mu(f)$ we see that $f$ is a transcendental entire function. Next from (3.70) and by proceeding as in Case 3 in the proof of Theorem 1.2 we get the conclusion of Theorem 1.3. □

**Acknowledgment**

The authors wish to express their thanks to the referee for his valuable suggestions and comments.

## References

[1] R. Brück, On entire functions which share one value CM with their first derivative, Results Math. 30 (1996) 21–24.
[2] Z.X. Chen, C.C. Yang, Some further results on the zeros and growths of entire solutions of second order linear differential equation, Kodai Math. J. 22 (1999) 273–285.
[3] Z.X. Chen, The growth of solutions of $f'' + e^{-z} f' + Q(z) f = 0$, Sci. China Ser. A 31 (2001) 775–784.
[4] J.M. Chang, M.L. Fang, Entire functions that share a small function with their derivatives, Complex Var. Theory Appl. 49 (2004) 871–895.
[5] C.T. Chuang, Sur la comparaison de la croissanxe d'une fonction méromorphe et celle de sa dérivée, Bull. Sci. Math. 75 (1951) 171–190.
[6] G.G. Gundersen, L.Z. Yang, Entire functions that share one value with one or two of their derivatives, J. Math. Anal. Appl. 223 (1998) 88–95.
[7] W.K. Hayman, Meromorphic Functions, Clarendon Press, Oxford, 1964.
[8] Y.Z. He, X.Z. Xiao, Algebroid Functions and Ordinary Differential Equations, Science Press, Beijing, 1988 (in Chinese).
[9] G. Jank, L. Volkmann, Einführung in die Theorie der ganzen und meromorphen Funktionen mit Anwendungen auf Differentialgleichungen, Birkhäuser, Basel/Boston, 1985.
[10] I. Laine, Nevanlinna Theory and Complex Differential Equations, Walter de Gruyter, Berlin, 1993.
[11] P. Li, W.J. Wang, Entire functions that share a small function with its derivative, J. Math. Anal. Appl. 328 (2007) 743–751.
[12] L. Rubel, C.C. Yang, Values shared by an entire function and its derivative, in: Complex Analysis, Kentucky, 1976, in: Lecture Notes in Math., vol. 599, Springer-Verlag, Berlin, 1977, pp. 101–103.
[13] J.P. Wang, Entire functions that share a polynomial with one of their derivatives, Kodai Math. J. 27 (2004) 144–151.
[14] L. Yang, Value Distribution Theory, Springer-Verlag, Berlin, 1993.
[15] C.C. Yang, H.X. Yi, Uniqueness Theory of Meromorphic Functions, Kluwer Academic Publishers, Dordrecht/Boston/London, 2003.
[16] L.Z. Yang, Solution of a differential equation and its applications, Kodai Math. J. 22 (1999) 458–464.