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Biseparating maps between Lipschitz function spaces ☆

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ABSTRACT

For complete metric spaces X and Y, a description of linear biseparating maps between spaces of vector-valued Lipschitz functions defined on X and Y is provided. In particular it is proved that X and Y are bi-Lipschitz homeomorphic, and the automatic continuity of such maps is derived in some cases. Besides, these results are used to characterize the separating bijections between scalar-valued Lipschitz function spaces when Y is compact.

1. Introduction

Separating maps, also called disjointness preserving maps, between spaces of scalar-valued continuous functions defined on compact or locally compact spaces have drawn the attention of researchers in last years (see for instance [10,15,17,21]). Roughly speaking, a (bijective) linear operator T between two spaces of functions is said to be separating if $(Tf) \cdot (Tg) = 0$ whenever $f \cdot g = 0$ (see Definition 2.1).

Recently, separating maps and related operators have been studied in the context of Lipschitz function spaces. For instance, Jiménez-Vargas obtained the representation of separating maps defined between *little* Lipschitz algebras on *compact* metric spaces (see [18]). Unfortunately proofs rely heavily on the properties of these algebras and on the compactness of spaces, so that they cannot carry over to the general case. More general results concerning spaces of vector-valued little Lipschitz functions on compact and locally compact metric spaces have been given later in [19] and [20]. On the other hand, in the recent paper [12], Garrido and Jaramillo studied a related problem: find those metric spaces X for which the algebra of bounded Lipschitz functions on X determines the Lipschitz structure of X. But even if separating maps are related with algebra isomorphisms, their techniques cannot be used here either. As for the spaces of scalar-valued bounded Lipschitz functions, biseparating maps (i.e., separating bijections whose inverse is also separating) have been studied in [26] in the case when the underlying spaces are compact, where a first description of them is included (as pointed out to us by the referee).

The aim of this paper is to study biseparating maps when they are defined between spaces of bounded Lipschitz functions and obtain their general representation in a much more general context. In this way, we do not restrict ourselves to the scalar setting and we deal with the vector-valued case as well. As usual, when spaces of functions taking values in arbitrary normed spaces are involved, the condition for an operator of being separating is not enough to ensure a good representation, and we must require the inverse map to be separating too (see for instance [1–4,7,13,14,16]; see also [5, Theorem 5.4]

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and [9] for special cases where this may not be true). We also drop any requirement of compactness on the metric spaces where functions are defined, and completeness is assumed instead.

Other papers where related operators have been recently studied in similar contexts are [8,11] and [22] (see also [23] and [24]).

The paper is organized as follows. In Section 2 we give some definitions and notation that we use throughout the paper. In Section 3 we state the main results. In Section 4 we give some properties of spaces of Lipschitz functions that we use later. Section 5 is devoted to prove the main results concerning biseparating maps between spaces of vector-valued Lipschitz functions. In particular, apart from obtaining their general form, we show that the underlying spaces are bi-Lipschitz homeomorphic and, when E and F are complete, we obtain the automatic continuity of some related maps. Finally, in Section 6 we prove that every bijective separating map between spaces of scalar-valued Lipschitz functions defined on compact metric spaces is indeed biseparating.

2. Preliminaries and notation

Let (X,d_1) and (Y,d_2) be metric spaces. Recall that a map $f : X \to Y$ is said to be *Lipschitz* if there exists a constant $k \geqslant 0$ such that

$$d_2\big(f(x), f(y)\big) \leqslant k d_1(x, y)$$

for each $x, y \in X$. The least such k is called the *Lipschitz number* of f and will be denoted by $L(f)$. Equivalently, $L(f)$ can be defined as

$$L(f) := \sup\left\{\frac{d_2(f(x), f(y))}{d_1(x, y)}:\ x, y \in X,\ x \neq y\right\}.$$

When f is bijective and both f and f^{-1} are Lipschitz, we will say that f is *bi-Lipschitz*.

If E is a $\mathbb{K}$-normed space, where $\mathbb{K}$ stands for the field of real or complex numbers, then $\operatorname{Lip}(X, E)$ will denote the space of all *bounded* E-valued Lipschitz functions defined on X. If $E = \mathbb{K}$, then we put $\operatorname{Lip}(X) := \operatorname{Lip}(X, E)$.

It is well known that $\operatorname{Lip}(X, E)$ is a normed space endowed with the norm

$$\|f\|_L = \max\big\{\|f\|_\infty, L(f)\big\}$$

for each $f \in \operatorname{Lip}(X, E)$ (where $\|\cdot\|_\infty$ denotes the usual supremum norm), which is complete when E is a Banach space.

From now on, unless otherwise stated, we will suppose that X and Y are *bounded* complete metric spaces (see Remark 3.6). In general, we will use d to denote the metric in both spaces.

For $x_0 \in X$ and $r > 0$, $B(x_0, r)$ will denote the open ball $\{x \in X:\ d(x, x_0) < r\}$. Finally, if A is a subset of a topological space Z, $\operatorname{cl}_Z A$ stands for the closure of A in Z.

We will suppose that E and F are $\mathbb{K}$-normed spaces. Given a function f defined on X and taking values on E, we define the *cozero set* of f as $\operatorname{coz}(f) := \{x \in X:\ f(x) \neq 0\}$. Also, for each $\mathbf{e} \in E$, $\widehat{\mathbf{e}} : X \to E$ will be the constant function taking the value $\mathbf{e}$. On the other hand, if (f_n) is a sequence of functions, then $\sum_{n=1}^{\infty} f_n$ denotes its (pointwise) sum.

Finally, we will denote by $L'(E, F)$ the set of linear and bijective maps from E to F, and by $L(E, F)$ the subset of all *continuous* operators of $L'(E, F)$.

We now give the definition of separating and biseparating maps in the context of Lipschitz function spaces.

Definition 2.1. A linear map $T : \operatorname{Lip}(X, E) \to \operatorname{Lip}(Y, F)$ is said to be *separating* if $\operatorname{coz}(Tf) \cap \operatorname{coz}(Tg) = \emptyset$ whenever $f, g \in \operatorname{Lip}(X, E)$ satisfy $\operatorname{coz}(f) \cap \operatorname{coz}(g) = \emptyset$. Moreover, T is said to be *biseparating* if it is bijective and both T and T^{-1} are separating.

Equivalently, a map $T : \operatorname{Lip}(X, E) \to \operatorname{Lip}(Y, F)$ is *separating* if it is linear and $\|Tf(y)\|\|Tg(y)\| = 0$ for all $y \in Y$, whenever $f, g \in \operatorname{Lip}(X, E)$ satisfy $\|f(x)\|\|g(x)\| = 0$ for all $x \in X$.

3. Main results

Our first result gives a general description of biseparating maps.

Theorem 3.1. *Let $T : \operatorname{Lip}(X, E) \to \operatorname{Lip}(Y, F)$ be a biseparating map. Then there exist a bi-Lipschitz homeomorphism $h : Y \to X$ and a map $J : Y \to L'(E, F)$ such that*

$$Tf(y) = (Jy)\big(f\big(h(y)\big)\big)$$

for all $f \in \operatorname{Lip}(X, E)$ and $y \in Y$.

Due to the representation given above, we see that when T is continuous, then Jy belongs to $L(E,F)$ for every $y \in Y$. In particular we also have that, for $y, y' \in Y$ and $\mathbf{e} \in E$, the map $\|T\widehat{\mathbf{e}}(y) - T\widehat{\mathbf{e}}(y')\| \leqslant \|T\|\|\mathbf{e}\|d(y,y')$. Consequently, the map $y \in Y \mapsto Jy \in L(E,F)$ is continuous when $L(E,F)$ is endowed with the usual norm.

Of course Theorem 3.1 does not give an answer to whether or not a biseparating map is necessarily continuous. In fact, automatic continuity cannot be derived in general. Nevertheless, in some cases an associated continuous operator can be defined. This is done in Theorem 3.4. We first give a result concerning continuity of maps Jy.

Given a biseparating map $T : \mathrm{Lip}(X,E) \to \mathrm{Lip}(Y,F)$, we denote

$$Y_d := \{y \in Y:\ Jy \text{ is discontinuous}\}.$$

Proposition 3.2. *Let $T : \mathrm{Lip}(X,E) \to \mathrm{Lip}(Y,F)$ be a biseparating map. Then the set $\{\|Jy\|:\ y \in Y \setminus Y_d\}$ is bounded. Moreover, Y_d is finite and each point of Y_d is isolated in Y.*

An immediate consequence is the following.

Corollary 3.3. *Let $T : \mathrm{Lip}(X,E) \to \mathrm{Lip}(Y,F)$ be a biseparating map. If X is infinite, then E and F are isomorphic.*

Another immediate consequence of Proposition 3.2 and Theorem 3.1 is that $Y \setminus Y_d$ is complete, and that the restriction of h to this set is a homeomorphism onto $X \setminus h(Y_d)$. This allows us to introduce in a natural way a new biseparating map defined in a related domain.

Theorem 3.4. *Suppose that E and F are complete. Let $T : \mathrm{Lip}(X,E) \to \mathrm{Lip}(Y,F)$ be a biseparating map, and let J and h be as in Theorem* 3.1. *Then $T_d : \mathrm{Lip}(X \setminus h(Y_d), E) \to \mathrm{Lip}(Y \setminus Y_d, F)$, defined as*

$$T_d f(y) := (Jy)\big(f\big(h(y)\big)\big)$$

for all $f \in \mathrm{Lip}(X \setminus h(Y_d), E)$ and $y \in Y \setminus Y_d$, is biseparating and continuous.

In the case when Y is compact and we deal with spaces of scalar-valued functions, the assumption on T of being just separating and bijective is enough to obtain both its automatic continuity and the fact that it is biseparating.

Theorem 3.5. *Let $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a bijective and separating map. If Y is compact, then T is biseparating and continuous.*

Remark 3.6. Recall that we are assuming that the metrics in X and Y are bounded. Nevertheless results can be translated to the case of unbounded metric spaces. Let d_1 be an unbounded metric in X such that (X, d_1) is complete. Then $d_1' := \min\{2, d_1\}$ is a *bounded* complete metric in X and the topology induced by both metrics is the same. Following the same ideas as in [25, Proposition 1.7.1], we can also see that the identity map of the space $\mathrm{Lip}(X,E)$ (with respect to d_1) onto itself (with respect to d_1') is an isometric isomorphism. It is easy to see now that if d_2 is a (bounded or unbounded) complete metric in Y, then a map $f : (Y, d_2) \to (X, d_1')$ is Lipschitz if and only if $f : (Y, d_2) \to (X, d_1)$ is what is called *Lipschitz in the small*, that is, there exist $r, k > 0$ such that $d_1(f(y), f(y')) \leqslant k d_2(y, y')$ whenever $d_2(y, y') < r$.

4. Lipschitz function spaces

Notice that since every complete metric space X is completely regular, it admits a Stone–Čech compactification, which will be denoted by βX. Recall that this implies that every continuous map $f : X \to \mathbb{K}$ can be extended to a continuous map $f^{\beta X}$ from βX into $\mathbb{K} \cup \{\infty\}$ (which takes all values in $\mathbb{K}$ if f is bounded). In particular, given a continuous map $f : X \to E$, we will denote by $\|f\|^{\beta X}$ the extension of $\|.\| \circ f : X \to \mathbb{K} \cup \{\infty\}$ to βX.

Now, we suppose that $A(X)$ is a subring of the space of continuous functions $C(X)$ which separates each point of X from each point of βX. We introduce in βX the equivalence relation

$$x \sim y \quad \Leftrightarrow \quad f^{\beta X}(x) = f^{\beta X}(y)$$

for all $f \in A(X)$. In this way, we obtain the quotient space $\gamma X := \beta X / \sim$, which is a new compactification of X. Besides, each $f \in A(X)$ is continuously extendable to a map $f^{\gamma X}$ from γX into $\mathbb{K} \cup \{\infty\}$. In this context, $A(X)$ is said to be *strongly regular* if given $x_0 \in \gamma X$ and a nonempty closed subset K of γX that does not contain x_0, there exists $f \in A(X)$ such that $f^{\gamma X} \equiv 1$ on a neighborhood of x_0 and $f^{\gamma X}(K) \equiv 0$.

Finally, assume that $A(X,E) \subset C(X,E)$ is an $A(X)$-module. We will say that $A(X,E)$ is *compatible* with $A(X)$ if, for every $x \in X$, there exists $f \in A(X,E)$ with $f(x) \neq 0$, and if, given any points $x, y \in \beta X$ such that $x \sim y$, we have $\|f\|^{\beta X}(x) = \|f\|^{\beta X}(y)$ for every $f \in A(X,E)$. In this case, it is easy to see that $\|.\| \circ f : X \to \mathbb{K} \cup \{\infty\}$ can be continuously extended to $\|f\|^{\gamma X}$ from γX into $\mathbb{K} \cup \{\infty\}$.

It is straightforward to check that, if $f \in \mathrm{Lip}(X)$ and $g \in \mathrm{Lip}(X,E)$, then $f \cdot g \in \mathrm{Lip}(X,E)$, that is,

Lemma 4.1. $\mathrm{Lip}(X,E)$ *is a* $\mathrm{Lip}(X)$*-module.*

Remark 4.2. We introduce two families of Lipschitz functions that will be used later. Given $x_0 \in X$ and $r > 0$, the function $\psi_{x_0,r} : X \to \mathbb{K}$ defined as

$$\psi_{x_0,r}(x) := \max\left\{0, 1 - \frac{d(x,x_0)}{r}\right\}$$

for all $x \in X$, belongs to $\mathrm{Lip}(X)$ and satisfies $\psi_{x_0,r}(x_0) = 1$, $\mathrm{coz}(\psi_{x_0,r}) = B(x_0,r)$, $\|\psi_{x_0,r}\|_\infty = 1$, and $L(\psi_{x_0,r}) = 1/r$. On the other hand, another Lipschitz function we will use is

$$\varphi_{x_0,r}(x) := \max\left\{0, 1 - \frac{d(x,B(x_0,r))}{r}\right\}$$

for all $x \in X$, which satisfies $\varphi_{x_0,r}(B(x_0,r)) \equiv 1$, $\mathrm{coz}(\varphi_{x_0,r}) = B(x_0,2r)$, $\|\varphi_{x_0,r}\|_\infty = 1$, and $L(\varphi_{x_0,r}) = 1/r$.

Clearly, given $f \in \mathrm{Lip}(X,E)$, $\|.\| \circ f \in \mathrm{Lip}(X)$. Then, by the definition of the equivalence relation $\sim$ in βX given above and the function $\psi_{x_0,r} \in \mathrm{Lip}(X)$ for each $x_0 \in X$ (see Remark 4.2), we obtain the next lemma.

Lemma 4.3. $\mathrm{Lip}(X,E)$ *is compatible with* $\mathrm{Lip}(X)$.

Lemma 4.4. $\mathrm{Lip}(X)$ *is strongly regular.*

Proof. Let K and L be two disjoint closed subsets of γX. Since γX is compact, there exists $f_0 \in C(\gamma X)$, $0 \leqslant f_0 \leqslant 1$, satisfying $f_0(K) \equiv 0$ and $f_0(L) \equiv 1$. Obviously $K_0 := \{x \in \gamma X\colon\ f_0(x) \leqslant 1/3\}$ and $L_0 := \{x \in \gamma X\colon\ f_0(x) \geqslant 2/3\}$ are disjoint compact neighborhoods of K and L, respectively. Consider now $K_1 := K_0 \cap X$ and $L_1 := L_0 \cap X$. We claim that $d(K_1, L_1) > 0$.

Suppose this is not true, so for each $n \in \mathbb{N}$ there exist $x_n \in K_1$ and $z_n \in L_1$ such that $d(x_n, z_n) < 1/n$. Since K_0 is compact, $\{x_n\colon\ n \in \mathbb{N}\}$ has a limit point x_0 in K_0. Consequently, there exists a net $(x_\alpha)_{\alpha\in\Omega}$ in $\{x_n\colon\ n \in \mathbb{N}\}$ which converges to x_0. Clearly, by using the Axiom of Choice if necessary, we can define a map sending each $\alpha \in \Omega$ to $n_\alpha \in \mathbb{N}$ with the property that $x_\alpha = x_{n_\alpha}$. Next, we consider the net $(z_\alpha)_{\alpha\in\Omega}$ in $\{z_n\colon\ n \in \mathbb{N}\}$ defined, for each $\alpha \in \Omega$, as $z_\alpha := z_{n_\alpha}$. By the compactness of L_0, we know that there exists a subnet $(z_\lambda)_{\lambda\in\Lambda}$ of $(z_\alpha)_{\alpha\in\Omega}$ converging to a point z_0 in L_0.

We are going to prove that $x_0 = z_0$, which is absurd because $K_0 \cap L_0 = \emptyset$. Obviously if x_0 or z_0 belongs to X, then we would have $x_0 = z_0$, so we assume that this is not the case. Let U and V be open neighborhoods of x_0 and z_0, respectively, and let $n_0 \in \mathbb{N}$. We are going to see that there exists $n \geqslant n_0$, $n \in \mathbb{N}$, such that $x_n \in U$ and $z_n \in V$. Without loss of generality we assume that $x_1, \ldots, x_{n_0} \notin U$ and $z_1, \ldots, z_{n_0} \notin V$. Since $(x_\lambda)_{\lambda\in\Lambda}$ and $(z_\lambda)_{\lambda\in\Lambda}$ converge to x_0 and z_0, respectively, there exist $\lambda_1^{x_0} \in \Lambda$ and $\lambda_1^{z_0} \in \Lambda$ such that $x_\lambda \in U$ for all $\lambda \geqslant \lambda_1^{x_0}$ and $z_\lambda \in V$ for all $\lambda \geqslant \lambda_1^{z_0}$. Taking $\lambda \in \Lambda$ such that $\lambda \geqslant \lambda_1^{x_0}, \lambda_1^{z_0}$, it is clear that $x_\lambda \in U$ and $z_\lambda \in V$. Now, there exists $n_\lambda \in \mathbb{N}$ such that $x_\lambda = x_{n_\lambda}$ and $z_\lambda = z_{n_\lambda}$, as we wanted to show.

Thus, if we take any $g \in \mathrm{Lip}(X)$ with associated constant k, and n as above, we have that

$$\left|g^{\gamma X}(x_n) - g^{\gamma X}(z_n)\right| \leqslant k d(x_n, z_n).$$

Clearly this implies that $g^{\gamma X}(x_0) = g^{\gamma X}(z_0)$. By the definition of γX, we have $x_0 = z_0$, and we are done.

Therefore we conclude that $d(K_1, L_1) > 0$. This lets us consider the function

$$f(x) := \max\left\{0, 1 - \frac{d(x, L_1)}{d(K_1, L_1)}\right\}$$

for all $x \in X$, defined in a similar way as in Remark 4.2, which belongs to $\mathrm{Lip}(X)$ and satisfies $0 \leqslant f \leqslant 1$, $f(K_1) \equiv 0$, and $f(L_1) \equiv 1$. This proves the lemma. □

The next lemma is a Lipschitz version (with a similar proof) of the result given in [6, Lemma 3.4] in the context of uniformly continuous functions.

Lemma 4.5. *Let X be a complete metric space and let $x \in \gamma X$. Then, x is a G_δ-set in γX if and only if $x \in X$.*

We close this section with a result concerning sums of Lipschitz functions that will be used in next sections.

Lemma 4.6. *Let (f_n) be a sequence of functions in* $\mathrm{Lip}(X,E)$ *with pairwise disjoint cozero sets and suppose that there exists a constant $M > 0$ such that $L(f_n) \leqslant M$ for all $n \in \mathbb{N}$. If $f := \sum_{n=1}^\infty f_n$ belongs to $C(X,E)$, then f is a Lipschitz function.*

Proof. Let $x, y \in X$. Suppose first that $f(x) = f_{n_0}(x)$ and $f(y) = f_{n_0}(y)$ for some $n_0 \in \mathbb{N}$. Then $\|f(x) - f(y)\| = \|f_{n_0}(x) - f_{n_0}(y)\| \leqslant M d(x,y)$. Next assume that $f(x) = f_n(x) \neq 0$ and $f(y) = f_m(y) \neq 0$ with $n \neq m$. Then $\|f(x) - f(y)\| = \|f_n(x) - f_m(y)\| \leqslant \|f_n(x)\| + \|f_m(y)\| = \|f_n(x) - f_n(y)\| + \|f_m(y) - f_m(x)\| \leqslant 2M d(x,y)$. Consequently $L(f) \leqslant 2M$ and f is a Lipschitz function. □

5. Biseparating maps. Proofs

In this section we give the proofs of Theorems 3.1 and 3.4 and that of Proposition 3.2, and some corollaries as well. We start with the notions of support point and support map.

Definition 5.1. Let $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ be a biseparating map. A point $x \in \gamma X$ is said to be a *support point* of $y \in Y$ if, for every neighborhood U of x in γX, there exists $f \in \mathrm{Lip}(X, E)$ with $\mathrm{coz}(f) \subset U$ such that $Tf(y) \neq 0$.

Remark 5.2. For each $y \in Y$, the support point $x \in \gamma X$ of $y \in Y$ exists and is unique (see [4, Lemma 4.3]). This fact lets us define a map $h_T : Y \to \gamma X$ sending each $y \in Y$ to its support point $h_T(y) \in \gamma X$. This map is usually called the *support map* of T. If there is no chance of confusion, we will denote it just by h (instead of h_T).

Proposition 5.3. *Let $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ be a biseparating map. Then $h(Y) \subset X$ and $h : Y \to X$ is a homeomorphism.*

Proof. In view of [4, Lemma 4.7], we can define the extension $\widetilde{h} : \gamma Y \to \gamma X$ of h. Besides, taking into account Lemmas 4.1, 4.3, and 4.4, we deduce that $\widetilde{h}$ is a homeomorphism by applying [4, Theorem 3.1]. On the other hand, we have characterizated the points in X as being the only G_δ-points in γX (see Lemma 4.5). Then, for each $y \in Y$, $h(y)$ clearly belongs to X and $h : Y \to X$ is a homeomorphism. □

Lemma 5.4. *If $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ is a biseparating map and $f \in \mathrm{Lip}(X, E)$ satisfies $f \equiv 0$ on a neighborhood of $h(y)$, then $Tf \equiv 0$ on a neighborhood of y.*

Proof. See [4, Lemma 4.4]. □

Lemma 5.5. *Let $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ be a biseparating map. Let $f \in \mathrm{Lip}(X, E)$ and $y_0 \in Y$ be such that $f(h(y_0)) = 0$. Then $Tf(y_0) = 0$.*

Proof. Let (r_n) be a sequence in $\mathbb{R}^+$ which converges to 0 and satisfies $2r_{n+1} < r_n$ for every $n \in \mathbb{N}$. We set $B_n := B(h(y_0), r_n)$, $B_n^2 := B(h(y_0), 2r_n)$, and $\varphi_n := \varphi_{h(y_0), r_n}$ for each $n \in \mathbb{N}$, where $\varphi_{h(y_0), r_n}$ is given as in Remark 4.2.

Claim 1. *Let $n, m \in \mathbb{N}$, $n \neq m$. Then*

$$\left(B_{2n}^2 \setminus B_{2n+1}\right) \cap \left(B_{2m}^2 \setminus B_{2m+1}\right) = \emptyset = \left(B_{2n-1}^2 \setminus B_{2n}\right) \cap \left(B_{2m-1}^2 \setminus B_{2m}\right).$$

The proof of Claim 1 follows directly from the fact that, for all $k \in \mathbb{N}$, $2r_{k+1} < r_k$, and consequently $B_{k+1}^2 \subset B_k$.

Claim 2. *$L(f\varphi_n) \leqslant 3L(f)$ for all $n \in \mathbb{N}$.*

It is clear that $f\varphi_n \in \mathrm{Lip}(X, E)$ for all $n \in \mathbb{N}$. Now, by definition of φ_n, $\mathrm{coz}(f\varphi_n) \subset B_n^2$, and if $x \in B_n^2$, then $\|f(x)\| = \|f(x) - f(h(y_0))\| \leqslant L(f)d(x, h(y_0)) < 2r_n L(f)$. Consequently, if $x, y \in B_n^2$,

$$\begin{aligned}\left\|(f\varphi_n)(x) - (f\varphi_n)(y)\right\| &\leqslant \left\|f(x)\right\|\left|\varphi_n(x) - \varphi_n(y)\right| + \left|\varphi_n(y)\right|\left\|f(x) - f(y)\right\| \\ &\leqslant 2r_n L(f)(1/r_n)d(x, y) + L(f)d(x, y) \\ &= 3L(f)d(x, y).\end{aligned}$$

Besides, if $x \in B_n^2$ and $y \notin B_n^2$,

$$\left\|(f\varphi_n)(x) - (f\varphi_n)(y)\right\| \leqslant 2r_n L(f)(1/r_n)d(x, y) = 2L(f)d(x, y).$$

Thus Claim 2 is proved.

Next we consider the function $g := f\varphi_1$, and define $g_1 := \sum_{n=1}^{\infty} f(\varphi_{2n} - \varphi_{2n+1})$ and $g_2 := \sum_{n=1}^{\infty} f(\varphi_{2n-1} - \varphi_{2n})$. It is obvious that $g = g_1 + g_2$, and since $f(h(y_0)) = 0$, we see that $g_1(h(y_0)) = 0$ and $g_2(h(y_0)) = 0$. This implies that both g_1 and g_2 are continuous. Taking into account of Claim 2, $L(f(\varphi_n - \varphi_{n+1})) \leqslant L(f\varphi_n) + L(f\varphi_{n+1}) \leqslant 6L(f)$ for all $n \in \mathbb{N}$. Besides, since $\mathrm{coz}(\varphi_{2n} - \varphi_{2n+1}) \subset B_{2n}^2 \setminus B_{2n+1}$, we deduce from Claim 1 that

$$\mathrm{coz}(\varphi_{2n} - \varphi_{2n+1}) \cap \mathrm{coz}(\varphi_{2m} - \varphi_{2m+1}) = \emptyset$$

whenever $n \neq m$. Applying Lemma 4.6, we conclude that g_1 (and similarly g_2) belongs to $\mathrm{Lip}(X, E)$. Besides, $g \equiv f$ on B_1, and by Lemma 5.4, $Tg(y_0) = Tf(y_0)$. Therefore, to see that $Tf(y_0) = 0$, it is enough to prove that $Tg_1(y_0) = 0$ and $Tg_2(y_0) = 0$.

Claim 3. *Given $n_0 \in \mathbb{N}$,*

$$\mathrm{cl}_X\big(\mathrm{coz}(g_1)\big) \subset \mathrm{cl}_X\big(B^2_{2n_0}\big) \cup \bigcup_{n=1}^{n_0-1} \mathrm{cl}_X\big(B^2_{2n} \setminus B_{2n+1}\big).$$

To see this, notice that

$$\mathrm{coz}\left(\sum_{n=n_0}^{\infty} \varphi_{2n} - \varphi_{2n+1}\right) \subset \bigcup_{n=n_0}^{\infty} \mathrm{coz}(\varphi_{2n} - \varphi_{2n+1}) \subset B^2_{2n_0},$$

and that $\mathrm{coz}(\varphi_{2n} - \varphi_{2n+1}) \subset B^2_{2n} \setminus B_{2n+1}$ for $n < n_0$.

If we consider, for each $n \in \mathbb{N}$, a point $y_n \in h^{-1}(B_{2n-1}) \setminus \mathrm{cl}_Y\, h^{-1}(B^2_{2n})$, then the sequence (y_n) converges to y_0 because $\bigcap_{n=1}^{\infty} B_n = \{h(y_0)\}$ and h is a homeomorphism.

Claim 4. $h(y_n) \notin \mathrm{cl}_X(\mathrm{coz}(g_1))$ *for all $n \in \mathbb{N}$.*

Let us prove the claim. Fix $n_0 \in \mathbb{N}$. It is clear by construction that $h(y_{n_0}) \notin \mathrm{cl}_X(B^2_{2n_0})$ and that, if $n < n_0$, then $h(y_{n_0}) \in B_{2n_0-1} \subseteq B_{2n+1}$, that is, $h(y_{n_0}) \notin \mathrm{cl}_X(B^2_{2n} \setminus B_{2n+1})$. Therefore Claim 4 follows from Claim 3.

Finally, since $h(y_n) \notin \mathrm{cl}_X(\mathrm{coz}(g_1))$ for all $n \in \mathbb{N}$, then $g_1 \equiv 0$ on a neighborhood of $h(y_n)$. Applying Lemma 5.4, $Tg_1(y_n) = 0$ for all $n \in \mathbb{N}$, and by continuity, we conclude that $Tg_1(y_0) = 0$. In the same way it can be proved that $Tg_2(y_0) = 0$. □

Proposition 5.6. *Let $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ be a biseparating map. For each $y \in Y$, there exists a linear and bijective map $Jy : E \to F$ such that*

$$Tf(y) = (Jy)\big(f\big(h(y)\big)\big)$$

for all $f \in \mathrm{Lip}(X, E)$ and $y \in Y$.

Proof. For $y \in Y$ and $f \in \mathrm{Lip}(X, E)$ fixed, consider the function $g := f - \widehat{f(h(y))} \in \mathrm{Lip}(X, E)$. Clearly $g(h(y)) = 0$, and by Lemma 5.5, $Tg(y) = 0$. Consequently $Tf(y) = T\,\widehat{f(h(y))}(y)$ for all $f \in \mathrm{Lip}(X, E)$ and $y \in Y$. Next, we define $Jy : E \to F$ as $(Jy)(\mathbf{e}) := T\,\widehat{\mathbf{e}}(y)$ for all $\mathbf{e} \in E$, which is linear and bijective (see [3, Theorem 3.5]). We easily see that T has the desired representation. □

Remark 5.7. Notice that, if $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ is a biseparating map, $T^{-1} : \mathrm{Lip}(Y, F) \to \mathrm{Lip}(X, E)$ is also biseparating, so there exist a homeomorphism $h_{T^{-1}} : X \to Y$ and a map $Kx : F \to E$ for all $x \in X$ such that

$$T^{-1}g(x) = (Kx)\big(g\big(h_{T^{-1}}(x)\big)\big)$$

for all $g \in \mathrm{Lip}(Y, F)$ and $x \in X$. Besides, it is not difficult to check that $h_{T^{-1}} \equiv h_T^{-1}$ (see Claim 1 in the proof of Theorem 3.1 in [4]).

Lemma 5.8. *Let $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ be a biseparating map. Then $\inf\{\|(Jy)(\mathbf{e})\|\colon y \in Y\} > 0$ for each non-zero $\mathbf{e} \in E$.*

Proof. Suppose this is not true. Then there exist (y_n) in Y and $\mathbf{e} \in E$ with $\|\mathbf{e}\| = 1$ such that $\|(Jy_n)(\mathbf{e})\| < 1/n^3$ for each $n \in \mathbb{N}$.

If we assume first that there exists a limit point $y_0 \in Y$ of $\{y_n\colon n \in \mathbb{N}\}$, then we can consider a subsequence (y_{n_k}) of (y_n) converging to y_0, so that $\|(Jy_0)(\mathbf{e})\| = 0$, which is absurd since Jy_0 is injective.

Therefore, there exists $r > 0$ such that $d(y_n, y_m) > r$ whenever $n \neq m$. Also, on the one hand, $[T^{-1}(T\,\widehat{\mathbf{e}})](h(y_n)) = \widehat{\mathbf{e}}(h(y_n)) = \mathbf{e}$ for all $n \in \mathbb{N}$, and on the other hand, by Remark 5.7, $[T^{-1}(T\,\widehat{\mathbf{e}})](h(y_n)) = (Kh(y_n))(T\,\widehat{\mathbf{e}}(y_n))$. Consequently $\|(Kh(y_n))(T\,\widehat{\mathbf{e}}(y_n))\| = \|\mathbf{e}\| = 1$ for each $n \in \mathbb{N}$. If we take $\mathbf{f}_n \in F$ defined as $\mathbf{f}_n := T\,\widehat{\mathbf{e}}(y_n)/\|T\,\widehat{\mathbf{e}}(y_n)\|$ for each $n \in \mathbb{N}$, it is clear that $\|\mathbf{f}_n\| = 1$ and

$$\big\|\big(Kh(y_n)\big)(\mathbf{f}_n)\big\| = \big(1/\big\|T\,\widehat{\mathbf{e}}(y_n)\big\|\big)\big\|\big(Kh(y_n)\big)\big(T\,\widehat{\mathbf{e}}(y_n)\big)\big\| > n^3.$$

Next, we define, in a similar way as in Remark 4.2,

$$\psi_{y_n, r/3}(y) := \max\left\{0, 1 - \frac{3d(y, y_n)}{r}\right\}$$

for all $y \in Y$ and $n \in \mathbb{N}$ (denoted for short ψ_n) which belongs to $\mathrm{Lip}(Y)$, and finally, we consider the function

$$g := \sum_{n=1}^{\infty} \frac{\psi_n \mathbf{f}_n}{n^2}.$$

It is immediate to see that $\|\psi_n \mathbf{f}_n/n^2\|_\infty \leqslant 1/n^2$ and $L(\psi_n \mathbf{f}_n/n^2) = (\|\mathbf{f}_n\|/n^2)L(\psi_n) = 3/(rn^2)$ for all $n \in \mathbb{N}$, which lets us conclude by Lemma 4.6 that g belongs to $\mathrm{Lip}(Y, F)$.

It is apparent that $g(y_n) = \mathbf{f}_n/n^2$, and applying Lemma 5.5 for the biseparating map T^{-1}, we deduce that $T^{-1}g(h(y_n)) = (1/n^2)T^{-1}\widehat{\mathbf{f}_n}(h(y_n))$. Consequently, $\|T^{-1}g(h(y_n))\| = (1/n^2)\|(Kh(y_n))(\mathbf{f}_n)\| > n$ for all $n \in \mathbb{N}$, which contradicts the fact that $T^{-1}g$ is bounded. $\square$

Proof of Proposition 3.2. Suppose on the contrary that there exist sequences (y_n) in Y and $(\mathbf{e}_n)$ in E with $\|\mathbf{e}_n\| = 1$ and $\|T\,\widehat{\mathbf{e}_n}(y_n)\| > n^2$ for every $n \in \mathbb{N}$. Take $\mathbf{f} \in F$ with $\|\mathbf{f}\| = 1$. By Lemma 5.8 there exists $M > 0$ such that $\|T^{-1}\widehat{\mathbf{f}}(h(y_n))\| > M$ for every n. Consider a sequence (r_n) in $(0, 1)$ such that $B(y_n, r_n) \cap B(y_m, r_m) = \emptyset$ whenever $n \neq m$ (this can be done by taking a subsequence of (y_n) if necessary). Without loss of generality we may also assume that (r_n) is decreasing and converging to 0.

We define, for each $n \in \mathbb{N}$,

$$\xi_n(y) := \max\{0, r_n - d(y, y_n)\}$$

for all $y \in Y$, which belongs to $\mathrm{Lip}(Y)$ and satisfies $\xi_n(y_n) = r_n$, $\mathrm{coz}(\xi_n) = B(y_n, r_n)$, $\|\xi_n\|_\infty = r_n$, and $L(\xi_n) = 1$. Finally, we consider the function

$$g := \sum_{n=1}^{\infty} \xi_n \mathbf{f}.$$

The fact that g belongs to $\mathrm{Lip}(Y, F)$ follows from Lemma 4.6. Now let $f := T^{-1}g$. It is clear from the description of T^{-1} given in Remark 5.7 that $f = \sum_{n=1}^{\infty} T^{-1}(\xi_n \mathbf{f})$. Consequently, if for each $n \in \mathbb{N}$, we define $f_n(x) := \|T^{-1}(\xi_n \mathbf{f})(x)\|$ $(x \in X)$, then $f_0 := \|f\| = \sum_{n=1}^{\infty} f_n$ belongs to $\mathrm{Lip}(X)$ and $f_0(h(y_n)) \geqslant Mr_n$ for every $n \in \mathbb{N}$. Therefore $f_0' := \sum_{n=1}^{\infty} f_n \mathbf{e}_n$ belongs to $\mathrm{Lip}(X, E)$. Finally $\|Tf_0'(y_n)\| \geqslant Mr_n n^2$, and it is easily seen that $L(Tf_n \mathbf{e}_n) \geqslant Mn^2$, for every $n \in \mathbb{N}$. We conclude that Tf_0' does not belong to $\mathrm{Lip}(Y, F)$, which is absurd.

Now, the fact that each $y \in Y_d$ is isolated follows easily. $\square$

Remark 5.9. We will use later the fact that, since Y_d is a finite set of isolated points and h is a homeomorphism, then $d(X \setminus h(Y_d), h(Y_d)) > 0$.

The restriction to $X \setminus h(Y_d)$ (respectively, $Y \setminus Y_d$) of a function $f \in \mathrm{Lip}(X, E)$ (respectively, $f \in \mathrm{Lip}(Y, F)$), is obviously a bounded Lipschitz function, which will be denoted by f_d. The converse is also true, that is, we can obtain a Lipschitz function as an extension of an element of $\mathrm{Lip}(X \setminus h(Y_d), E)$, as it is done in the next lemma.

Lemma 5.10. *Let $f \in \mathrm{Lip}(X \setminus h(Y_d), E)$. Then the function*

$$f^d(x) := \begin{cases} f(x) & \text{if } x \in X \setminus h(Y_d), \\ 0 & \text{if } x \in h(Y_d) \end{cases}$$

belongs to $\mathrm{Lip}(X, E)$.

Proof. Since $h(Y_d)$ is a finite set of isolated points, f^d is clearly a continuous function. Besides, if we consider $x_1 \in X \setminus h(Y_d)$ and $x_2 \in h(Y_d)$,

$$\frac{\|f^d(x_1) - f^d(x_2)\|}{d(x_1, x_2)} \leqslant \frac{\|f(x_1)\|}{d(X \setminus h(Y_d), h(Y_d))} \leqslant \frac{\|f\|_\infty}{d(X \setminus h(Y_d), h(Y_d))}.$$

Therefore

$$L(f^d) \leqslant \max\left\{L(f), \frac{\|f\|_\infty}{d(X \setminus h(Y_d), h(Y_d))}\right\} < \infty,$$

which implies that $f^d \in \mathrm{Lip}(X, E)$. $\square$

Proof of Theorem 3.4. By definition of T_d and Lemma 5.10 (see also the comment before it), we clearly see that

$$T_d(f) = (Tf^d)_d$$

for all $f \in \mathrm{Lip}(X \setminus h(Y_d), E)$, so T_d is well defined and it is biseparating. To prove that T_d is continuous, we will see that given a sequence (f_n) in $\mathrm{Lip}(X \setminus h(Y_d), E)$ converging to 0 and such that $(T_d f_n)$ converges to $g \in \mathrm{Lip}(Y \setminus Y_d, F)$, we have $g \equiv 0$.

If we consider, for each $n \in \mathbb{N}$, the extension f_n^d of f_n given in Lemma 5.10, we can show that

$$\|f_n^d\|_L \leqslant \max\left\{\|f_n\|_\infty, \max\left\{L(f_n), \frac{\|f_n\|_\infty}{d(X \setminus h(Y_d), h(Y_d))}\right\}\right\} \leqslant \|f_n\|_L \max\bigl\{1, 1/d\bigl(X \setminus h(Y_d), h(Y_d)\bigr)\bigr\},$$

which allows us to deduce that (f_n^d) converges to 0. By continuity, if we fix $y \in Y \setminus Y_d$, the sequence $((Jy)(f_n^d(h(y))))$ converges to 0. Besides, since $Tf_n^d(y) = T_d f_n(y)$, we conclude that $(T_d f_n(y))$ converges to 0.

On the other hand, $\|T_d f_n(y) - g(y)\| \leqslant \|T_d f_n - g\|_L$ for each $n \in \mathbb{N}$, and as $(T_d f_n)$ converges to g, we deduce that $(T_d f_n(y))$ converges to $g(y)$. Combined with the above, $g(y) = 0$ for all $y \in Y \setminus Y_d$. □

The proof of the two following results is now immediate.

Corollary 5.11. *Suppose that E and F are complete and let $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ be a biseparating map. If Y has no isolated points, then T is continuous.*

Corollary 5.12. *Let $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ be a biseparating map. If E has finite dimension, then F has the same dimension as E and T is continuous.*

Proposition 5.13. *Let $T : \mathrm{Lip}(X, E) \to \mathrm{Lip}(Y, F)$ be a biseparating map. Then $h : Y \to X$ is a bi-Lipschitz map.*

Proof. Associated to T, we define a linear map $S : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$. For $f \in \mathrm{Lip}(X)$, define

$$Sf(y) := f\bigl(h(y)\bigr)$$

for every $y \in Y$. It is obvious that Sf is a continuous bounded function on Y. Next we are going to see that it is also Lipschitz. It is clear that it is enough to prove it in the case when $f \geqslant 0$.

Fix any $\mathbf{e} \neq 0$ in E. By Lemma 5.8, we know that there exists $M > 0$ such that $\|T\widehat{\mathbf{e}}(y)\| \geqslant M$ for every $y \in Y$, so the map $y \mapsto 1/\|T\widehat{\mathbf{e}}(y)\|$ belongs to $\mathrm{Lip}(Y)$. On the other hand, taking into account that $f \geqslant 0$, we have that for $y, y' \in Y$,

$$\begin{aligned}
\bigl|Sf(y)\bigl\|T\widehat{\mathbf{e}}(y)\bigr\| - Sf(y')\bigl\|T\widehat{\mathbf{e}}(y')\bigr\|\bigr| &= \bigl|\bigl\|(Jy)\bigl(f\bigl(h(y)\bigr)\mathbf{e}\bigr)\bigr\| - \bigl\|(Jy')\bigl(f\bigl(h(y')\bigr)\mathbf{e}\bigr)\bigr\|\bigr| \\
&\leqslant \bigl\|(Jy)\bigl(f\bigl(h(y)\bigr)\mathbf{e}\bigr) - (Jy')\bigl(f\bigl(h(y')\bigr)\mathbf{e}\bigr)\bigr\| \\
&= \bigl\|T(f\mathbf{e})(y) - T(f\mathbf{e})(y')\bigr\| \\
&\leqslant L\bigl(T(f\mathbf{e})\bigr)d(y, y').
\end{aligned}$$

We deduce that Sf is Lipschitz. A similar process can be done with the map T^{-1}, and we conclude that $S : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ is bijective and biseparating.

Next we prove that h is Lipschitz. Let $K_0 := \max\{1, \mathrm{diam}(X)\}$. We take $y, y' \in Y$ and define $f_1(x) := d(h(y), x)$ for all $x \in X$. Clearly f_1 belongs to $\mathrm{Lip}(X)$. Notice also that S is a biseparating map between scalar-valued spaces of Lipschitz functions, so by Corollary 5.12 it is continuous. Thus it is not difficult to see that

$$\frac{\|Sf_1(y) - Sf_1(y')\|}{d(y, y')} \leqslant \|Sf_1\|_L \leqslant \|S\|\|f_1\|_L \leqslant K_0\|S\|.$$

On the other hand, $Sf_1(y) = 0$ and $Sf_1(y') = d(h(y), h(y'))$. Then, replacing in the above inequality,

$$d\bigl(h(y), h(y')\bigr) \leqslant K_0\|S\|d(y, y'),$$

and we are done.

Moreover, h^{-1} is also Lipschitz because $h^{-1} = h_{T^{-1}}$ (see Remark 5.7). □

Proof of Theorem 3.1. It follows immediately from Propositions 5.3, 5.6 and 5.13. □

Taking into account Theorem 3.1, Lemma 5.8, and Corollary 5.12, we can give the general form of biseparating maps in the scalar-valued case (see also Theorem 3.5 and Corollary 6.1). Of course it also applies to algebra isomorphisms.

Corollary 5.14. *Let $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a biseparating map. Then T is continuous and there exist a bi-Lipschitz homeomorphism $h : Y \to X$ and a nonvanishing function $\tau \in \mathrm{Lip}(Y)$ such that*

$$Tf(y) = \tau(y)f\bigl(h(y)\bigr)$$

for every $f \in \mathrm{Lip}(X)$ and $y \in Y$.

Corollary 5.15. *Let $I : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be an algebra isomorphism. Then I is continuous and there exists a bi-Lipschitz homeomorphism $h : Y \to X$ such that*

$$If(y) = f\big(h(y)\big)$$

for every $f \in \mathrm{Lip}(X)$ and $y \in Y$.

6. Separating maps. Proof of Theorem 3.5

In this section we give the proof of Theorem 3.5 and the representation of bijective separating maps in the scalar setting when Y is compact.

Proof of Theorem 3.5. Let $f, g \in \mathrm{Lip}(X)$ be such that $\mathrm{coz}(f) \cap \mathrm{coz}(g) \neq \emptyset$, that is, there exists $x_0 \in X$ satisfying $f(x_0) \neq 0$ and $g(x_0) \neq 0$. Since T is onto, $Tk \equiv 1$ for some $k \in \mathrm{Lip}(X)$, and we can take $\alpha, \beta \in \mathbb{K}$ such that $(\alpha f + k)(x_0) = 0$ and $(\beta g + k)(x_0) = 0$. We denote $l := \alpha f + k$.

Let (r_n), B_n, B_n^2, and φ_n be as in the proof of Lemma 5.5 (where $h(y_0)$ is replaced by x_0); indeed, we closely follow that proof. Now, we take $y_n \in \mathrm{coz}(T(\varphi_n - \varphi_{n+1}))$ for each $n \in \mathbb{N}$. By the compactness of Y, $\{y_n\colon n \in \mathbb{N}\}$ has a limit point y_0 in Y. Then, we can consider a subsequence (y_{n_i}) of (y_n) converging to y_0 whose indexes satisfy $|n_i - n_j| \geqslant 3$ whenever $i \neq j$.

We claim that $Tl(y_0) = 0$. To prove it, we define

$$l_1 := \sum_{k=1}^{\infty} l(\varphi_{n_{2k}-1} - \varphi_{n_{2k}+2})$$

and $l_2 := l - l_1$, and we will see that $Tl_1(y_0) = 0$ and $Tl_2(y_0) = 0$ (in the rest of the proof we will set $\xi_k := \varphi_{n_{2k}-1} - \varphi_{n_{2k}+2}$ for every $k \in \mathbb{N}$).

First, we will check that l_1 and l_2 are both Lipschitz functions. As in Claim 2 in the proof of Lemma 5.5, we know that $L(l\varphi_n) \leqslant 3L(l)$ for all $n \in \mathbb{N}$. Consequently $L(l\xi_k) \leqslant L(l\varphi_{n_{2k}-1}) + L(l\varphi_{n_{2k}+2}) \leqslant 6L(l)$ for all $k \in \mathbb{N}$. Since $\mathrm{coz}(\xi_k) \cap \mathrm{coz}(\xi_j) = \emptyset$ if $k \neq j$, by Lemma 4.6 we conclude that $l_1 \in \mathrm{Lip}(X)$, and then l_2 also belongs to $\mathrm{Lip}(X)$.

Now, we will see that $Tl_1(y_{n_{2k-1}}) = 0$ for all $k \in \mathbb{N}$. Fix $k_0 \in \mathbb{N}$ and consider $y_{n_{2k_0-1}}$. It is not difficult to see that $\mathrm{coz}(\varphi_{n_{2k_0-1}} - \varphi_{n_{2k_0-1}+1}) \subset B^2_{n_{2k_0-1}} \setminus B_{n_{2k_0-1}+1}$ and that, for every $k \in \mathbb{N}$, $\mathrm{coz}(\xi_k) \subset B^2_{n_{2k}-1} \setminus B_{n_{2k}+2}$, so

$$\mathrm{coz}(\varphi_{n_{2k_0-1}} - \varphi_{n_{2k_0-1}+1}) \cap \mathrm{coz}(\xi_k) = \emptyset,$$

which allows us to deduce that

$$\mathrm{coz}(\varphi_{n_{2k_0-1}} - \varphi_{n_{2k_0-1}+1}) \cap \mathrm{coz}\left(\sum_{k=1}^{\infty} l\xi_k\right) = \emptyset.$$

Next, since T is a separating map,

$$\mathrm{coz}\big(T(\varphi_{n_{2k_0-1}} - \varphi_{n_{2k_0-1}+1})\big) \cap \mathrm{coz}\left(T\left(\sum_{k=1}^{\infty} l\xi_k\right)\right) = \emptyset,$$

and we conclude that $Tl_1(y_{n_{2k_0-1}}) = 0$ because $y_{n_{2k_0-1}} \in \mathrm{coz}(T(\varphi_{n_{2k_0-1}} - \varphi_{n_{2k_0-1}+1}))$. By continuity, it is clear that $Tl_1(y_0) = 0$.

On the other hand, if $x \in \mathrm{coz}(\varphi_{n_{2k}} - \varphi_{n_{2k}+1}) = B^2_{n_{2k}} \setminus B_{n_{2k}+1} \subset B_{n_{2k}-1} \setminus B^2_{n_{2k}+2}$, then $\xi_k(x) = 1$. This fact allows us to deduce that $\mathrm{coz}(\varphi_{n_{2k}} - \varphi_{n_{2k}+1}) \cap \mathrm{coz}(l_2) = \emptyset$, and consequently $\mathrm{coz}(T(\varphi_{n_{2k}} - \varphi_{n_{2k}+1})) \cap \mathrm{coz}(Tl_2) = \emptyset$. For this reason $Tl_2(y_{n_{2k}}) = 0$ for all $k \in \mathbb{N}$, and as above we conclude that $Tl_2(y_0) = 0$.

Therefore $0 = Tl(y_0) = T(\alpha f + k)(y_0) = \alpha Tf(y_0) + 1$, which implies that $Tf(y_0) \neq 0$. The same reasoning can be applied to the function $\beta g + k$ and we obtain that $Tg(y_0) \neq 0$. Then, we deduce that $\mathrm{coz}(Tf) \cap \mathrm{coz}(Tg) \neq \emptyset$, and T^{-1} is separating.

The fact that T is continuous follows from Corollary 5.12. □

Corollary 6.1. *Let $T : \mathrm{Lip}(X) \to \mathrm{Lip}(Y)$ be a bijective and separating map. If Y is compact, then there exist a bi-Lipschitz homeomorphism $h : Y \to X$ and a nonvanishing function $\tau \in \mathrm{Lip}(Y)$ such that*

$$Tf(y) = \tau(y) f\big(h(y)\big)$$

for every $f \in \mathrm{Lip}(X)$ and $y \in Y$.

Proof. Immediate by Theorem 3.5 and Corollary 5.14. □

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