



# Conjugations between circle maps with a single break point

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## ABSTRACT

Consider two circle homeomorphisms  $f_i \in C^{2+\alpha}(S^1 \setminus \{b_i\})$ ,  $\alpha > 0$ ,  $i = 1, 2$  with a single break point  $b_i$  i.e. a discontinuity in the derivative  $Df_i$ , and identical irrational rotation number  $\rho$ . Suppose the jump ratios  $\sigma_1 = \frac{Df_1(b_1-0)}{Df_1(b_1+0)}$  and  $\sigma_2 = \frac{Df_2(b_2-0)}{Df_2(b_2+0)}$  do not coincide. Then the map  $\psi$  conjugating  $f_1$  and  $f_2$  is a singular function i.e. it is continuous on  $S^1$  and  $D\psi(x) = 0$  a.e. with respect to Lebesgue measure.

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## 1. Introduction

The classification of circle homeomorphisms under changes of variables is one of the important problems in one-dimensional dynamics. It was initiated by Poincaré [14] who was motivated by studies in differential equation more than a century ago and it has been actively studied ever since. Circle maps are also important because of their applications to natural sciences (see [6]). In this paper we study a class of circle homeomorphisms with a single break point i.e. with a jump in the first derivative at exactly one point. We identify the unit circle  $S^1 = \mathbb{R}/\mathbb{Z}^1$  with the half open interval  $[0, 1)$ .

Consider the set of orientation-preserving circle homeomorphisms  $f$  with the following properties:

- (a) there is a point  $b = b(f) \in S^1$  such that the one-sided derivatives  $Df(b \pm 0)$  exist, are positive and  $Df(b-0) \neq Df(b+0)$ ;
- (b)  $f \in C^{2+\alpha}(S^1 \setminus \{b\})$ , for some  $\alpha > 0$  and  $Df(x) > 0$ , for all  $x \in S^1 \setminus \{b\}$ .

The point  $b = b(f)$  is called a **break point** of  $f$ . The ratio  $\sigma_f(b) := \frac{Df(b-0)}{Df(b+0)}$  is called the **jump ratio** of  $f$  in  $b$ . Here and later we'll study circle homeomorphisms with a break point satisfying the conditions (a) and (b). Notice, the parameter  $\sigma_f(b)$  is obviously an invariant under smooth coordinate transformations and characterizes the type of the singularity.

Denote by  $\rho = \rho_f$  the rotation number of the homeomorphism  $f$ . For  $\rho$  irrational the trajectory of an arbitrary point under a sufficiently smooth diffeomorphism  $f$  is dense on the circle, and  $f$  can be conjugated to the pure rotation  $f_\rho x = (x + \rho) \bmod 1$  by the angle  $\rho$  through a change of coordinates.

This result was proved by Denjoy [2]. More precisely, Denjoy proved, that for  $f \in C^1(S^1)$  and  $\text{var}(\log Df) < \infty$ , there exists a circle homeomorphism  $\varphi$  such, that  $\varphi \circ f = f_\rho \circ \varphi$ .

It is also a well-known fact, that a circle homeomorphisms  $f$  with irrational rotation number  $\rho$  is strictly ergodic i.e. admits an unique  $f$ -invariant measure  $\mu_f$  with full support. Indeed, the conjugating map  $\varphi$  and the invariant measure  $\mu_f$

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are related by  $\varphi(x) = \mu_f([0, x])$  (see [5]). This last relation implies, that regularity properties of the conjugating map  $\varphi$  are closely related to the existence of an absolutely continuous invariant measure  $\mu_f$  with regular density.

The problem of smoothness of the conjugacy of smooth diffeomorphisms is now very well understood (see [1,8,10,11,13, 16]). We recall the last two important results in this area.

**Theorem 1.1.** (See Katznelson and Ornstein [10].) Let  $f$  be an orientation preserving  $C^1$ -circle diffeomorphisms. For  $Df$  absolutely continuous,  $D(\log Df) \in L^p$  for some  $p > 1$  and rotation number  $\rho = \rho(f)$  of bounded type the  $f$ -invariant probability measure  $\mu_f$  is absolutely continuous with respect to Lebesgue measure.

**Theorem 1.2.** (See Sinai and Khanin [11].) Let  $f$  be a  $C^{2+\alpha}$ -circle diffeomorphisms for some  $\alpha > 0$ , and let the rotation number  $\rho = \rho_f$  be a Diophantine number with exponent  $\delta \in (0, \alpha)$ , i.e. there is a constant  $c(\rho)$  such that

$$\left| \rho - \frac{p}{q} \right| \geq \frac{c(\rho)}{q^{2+\delta}} \quad \text{for any } \frac{p}{q} \in \mathbb{Q}.$$

Then the conjugation map  $\varphi$  belongs to  $C^{1+\alpha-\delta}$ .

Notice, the condition  $f \in C^{2+\alpha}$  is sharp, because there is a set of full Lebesgue measure in  $[0, 1]$  such that for any rotation number in this set there are  $C^2$ -diffeomorphisms for which the conjugating map  $\varphi$  is singular [9].

The classical Denjoy's theorem can readily be extended to the case of homeomorphisms with a break point. Below we present the exact statement of the corresponding theorem. Khanin and Vul [12] showed, that the renormalization of circle homeomorphism  $f \in C^{2+\alpha}(S^1 \setminus \{b\})$  with irrational rotation number is exponentially close to fractional-linear mappings. The dynamics of homeomorphisms with a single break point exhibits many "critical" properties which are similar to the ones of critical circle mappings (see [7,12,16]).

Dzhalilov and Khanin [4] proved the following result: Let  $f$  be a circle homeomorphisms with a single break point  $b$ . If the rotation number  $\rho$  of  $f$  is irrational and  $f \in C^{2+\alpha}(S^1 \setminus \{b\})$  for some  $\alpha > 0$ , then the  $f$ -invariant probability measure  $\mu_f$  is singular with respect to Lebesgue measure. This implies, that the conjugation between  $f$  and the pure rotation  $f_\rho$  is a singular function.

The homeomorphisms with break points can be considered as natural one-parameter extensions of Herman's rigidity theory [8], where pure rotations are replaced by fractional-linear mappings with break point singularities. Recently, there has been a significant progress in the rigidity theory of circle maps with such singularities (see [3,15]).

Important there is the Teplinskii-Khanin's result [15] on the regularity of the conjugation between two homeomorphisms with one break point: namely, let  $\rho = [k_1, k_2, \dots, k_n, \dots]$  be the continued fraction expansion of the irrational rotation number  $\rho$  and define

$$M_o = \left\{ \rho: (\exists C > 0) (\forall n \in \mathbb{N}) k_{2n-1} \leq C \right\},$$

$$M_e = \left\{ \rho: (\exists C > 0) (\forall n \in \mathbb{N}) k_{2n} \leq C \right\}.$$

Then their result is

**Theorem 1.3.** (See Teplinskii and Khanin [15].) Let  $f_1, f_2$  be circle homeomorphisms with single break points  $b_1 = b(f_1)$  and  $b_2 = b(f_2)$  satisfying conditions (a) and (b). Assume

- (1) the rotation numbers  $\rho(f_i)$  of  $f_i$ ,  $i = 1, 2$  coincide and are irrational, i.e.  $\rho(f_1) = \rho(f_2) = \rho$ ,  $\rho \in \mathbb{R}^1 \setminus \mathbb{Q}$ ;
- (2) the jumps  $\sigma_i = \sigma_i(b)$  of  $f_i$ ,  $i = 1, 2$  coincide i.e.  $\sigma_1 = \sigma_2 = \sigma$ .

Then the map  $\psi$  conjugating the homeomorphisms  $f_1$  and  $f_2$  is a  $C^1$ -diffeomorphism of the circle if either  $\sigma > 1$ ,  $\rho \in M_o$  or  $\sigma < 1$   $\rho \in M_e$ .

In the present paper we consider the case, when the rotation numbers of  $f_i$ ,  $i = 1, 2$  coincide but the jumps  $\sigma_1$  and  $\sigma_2$  are different. In this case the result is quite opposite to the rigidity case: indeed we prove the following

**Theorem 1.4.** Let  $f_i$ ,  $i = 1, 2$  be circle homeomorphisms satisfying conditions (a) and (b). Assume

- (1) the rotation numbers  $\rho_i$  of  $f_i$ ,  $i = 1, 2$  coincide and are irrational, i.e.  $\rho_1 = \rho_2 = \rho$ ,  $\rho \in \mathbb{R}^1 \setminus \mathbb{Q}$ ;
- (2) the jumps  $\sigma_i(b_i)$  of  $f_i$ ,  $i = 1, 2$ , are positive but do not coincide.

Then the homeomorphism  $\psi$  conjugating  $f_1$  and  $f_2$  is a singular function, i.e.  $\psi$  is continuous on  $S^1$  and  $D\psi(x) = 0$  a.e. with respect to Lebesgue measure.

## 2. Preliminaries and notations

Let  $f$  be an orientation preserving homeomorphism of the circle with lift  $\hat{f}$ . The rotation number  $\rho = \rho_f$  of  $f$  is defined by (see [5] for details)

$$\rho(f) := \left( \lim_{n \rightarrow \infty} \frac{\hat{f}^n(x)}{n} \right) \bmod 1,$$

where the limit exists for all  $x \in \mathbb{R}$  and is independent of  $x$ . Here and later on  $g^n$  denotes the  $n$ -th iterate of  $g$ . We denote by  $\{k_n, n \in \mathbb{N}\}$  the sequence of entries in the continued fraction expansion  $\rho = [k_1, k_2, \dots, k_n, \dots] = (1/k_1 + (1/(k_2 + \dots + 1/k_n + \dots)))$ . For  $n \in \mathbb{N}$  denote by  $p_n/q_n = [k_1, k_2, \dots, k_n]$  the convergents of  $\rho$ . Their denominators  $q_n$  satisfy the well-known recursion relation  $q_{n+1} = k_{n+1}q_n + q_{n-1}$ ,  $n \geq 1$ ,  $q_0 = 1$ ,  $q_1 = k_1$ .

Take an arbitrary point  $x_0 \in S^1$ . The  $i$ -th iterate of  $x_0$  under  $f$  is denoted by  $x_i$ . Define  $\Delta_0^{(n)}(x_0)$  as the closed interval on  $S^1$  with endpoints  $x_0$  and  $x_{q_n}$ , such that for  $n$  odd  $x_{q_n}$  is to the left of  $x_0$  and for  $n$  even it is to the right. Denote by  $\Delta_i^{(n)}(x_0) := f^i(\Delta_0^{(n)}(x_0))$ ,  $i \geq 1$  the iterates of the interval  $\Delta_0^{(n)}(x_0)$  under  $f$ .

It is well known from the work of Denjoy that the sets  $\xi_n(x_0)$  of intervals with mutually disjoint interior defined as

$$\xi_n(x_0) = \{ \Delta_i^{(n-1)}(x_0), 0 \leq i < q_n; \Delta_j^{(n)}(x_0), 0 \leq j < q_{n-1} \} \quad (1)$$

determine partitions of the circle. The partition  $\xi_n(x_0)$  is called the  $n$ -th **dynamical partition** of the point  $x_0$  with **generators**  $\Delta_0^{(n-1)}(x_0)$  and  $\Delta_0^{(n)}(x_0)$ . Obviously the partition  $\xi_{n+1}(x_0)$  is a refinement of the partition  $\xi_n(x_0)$ : indeed the intervals of order  $n$  are members of  $\xi_{n+1}(x_0)$  and each interval  $\Delta_i^{(n-1)}(x_0) \in \xi_n(x_0)$ ,  $0 \leq i < q_n$ , is partitioned into  $k_{n+1} + 1$  intervals belonging to  $\xi_{n+1}(x_0)$  such that

$$\Delta_i^{(n-1)}(x_0) = \Delta_i^{(n+1)}(x_0) \cup \bigcup_{s=0}^{k_{n+1}-1} \Delta_{i+q_{n-1}+sq_n}^{(n)}(x_0).$$

The following lemma plays a key role for studying metrical properties of the homeomorphism  $f$ :

**Lemma 2.1.** *Let  $f$  be an orientation preserving homeomorphism of the circle with a single break point  $x = b$  and irrational rotation number  $\rho$ . Assume  $f \in C^1([b, b+1])$  and  $\bar{v} = \text{var}_{[b, b+1]} \log Df < \infty$ . For  $x_0 \in S^1$ ,  $n \geq 1$  and  $b \notin \{f^i(x_0), 0 \leq i < q_n\}$  one has*

$$e^{-v} \leq \prod_{s=0}^{q_n-1} Df(x_s) \leq e^v, \quad (2)$$

with  $v = \bar{v} + |\log \sigma_f(b)|$ .

Inequality (2) is called the **Denjoy inequality**. The proof of Lemma 2.1 is similar to the one for circle diffeomorphisms (see for instance [11]). Using Lemma 2.1 it can be shown, that the length  $l$  of the intervals of the dynamical partition  $\xi_n(x_0)$  in (1) is exponentially small, indeed one has

**Corollary 2.2.** *Let  $\Delta^{(n)}$  be an arbitrary element of the dynamical partition  $\xi_n(x_0)$ . Then*

$$l(\Delta^{(n)}) \leq \text{const } \lambda^n, \quad (3)$$

where  $\lambda = (1 + e^{-v})^{-1/2} < 1$ .

From Corollary 2.2 it follows, that the trajectory of any point  $x \in S^1$  is dense in  $S^1$ . This together with monotonicity of the homeomorphism  $f$  implies the following generalization of the classical Denjoy theorem:

**Theorem 2.3.** *Suppose, the homeomorphism  $f$  satisfies the conditions of Lemma 2.1. Then  $f$  is topologically conjugate to the pure rotation  $f_\rho$ .*

Recall the following definition introduced in [10]:

**Definition 2.4.** An interval  $I = (\tau, t) \subset S^1$  is  $q_n$ -**small** and its endpoints  $\tau, t$  are  $q_n$ -**close** if the intervals  $f^i(I)$ ,  $0 \leq i < q_n$  are disjoint.

It is known that the interval  $(\tau, t)$  is  $q_n$ -small if, depending on the parity of  $n$ , either  $t \leq \tau \leq f^{q_{n-1}}(t)$  or  $f^{q_{n-1}}(\tau) \leq t \leq \tau$ . In the following discussion we have to compare different intervals. For this we use

**Definition 2.5.** Let  $C > 1$ . We call two intervals in  $S^1$  **C-comparable** if the ratio of their lengths is in  $[C^{-1}, C]$ .

Lemma 2.1 then implies (see [10])

**Corollary 2.6.** Suppose the homeomorphism  $f$  satisfies the conditions of Lemma 2.1. Then for any interval  $I \subset S^1$  the intervals  $I$  and  $f^{q_n}(I)$  are  $e^v$ -comparable. If the interval  $I$  is  $q_n$ -small then  $l(f^i(I)) < \text{const } \lambda^n$  for all  $i = 0, 1, \dots, (q_n - 1)$ .

**Lemma 2.7.** Suppose the circle homeomorphism  $f$  satisfies the conditions of Lemma 2.1 and  $x, y \in S^1$  are  $q_n$ -close. Then for any  $0 \leq k < q_n$  the following inequality holds:

$$e^{-v} \leq \frac{Df^k(x)}{Df^k(y)} \leq e^v. \quad (4)$$

**Proof.** Take any two  $q_n$ -close points  $x, y \in S^1$  and  $0 \leq k \leq q_n - 1$ . Denote by  $I$  the open interval with endpoints  $x$  and  $y$ . Because the intervals  $f^i(I)$ ,  $0 \leq i < q_n$  are disjoint, we obtain

$$|\log Df^k(x) - \log Df^k(y)| \leq \sum_{s=0}^{q_n-1} |\log Df(f^s x) - \log Df(f^s y)| \leq v,$$

from which inequality (4) follows immediately.  $\square$

Remember, that homeomorphisms  $f$  of the circle satisfying the conditions of Lemma 2.1 are ergodic with respect to Lebesgue measure, i.e. every  $f$ -invariant set has full or vanishing measure (see [7]).

### 3. Some cross-ratio tools

Let us first recall two definitions:

**Definition 3.1.** The **cross-ratio**  $Cr(z_1, z_2, z_3, z_4)$  of four points  $z_i \in \mathbb{R}$ ,  $i = 1, 2, 3, 4$ ,  $z_1 < z_2 < z_3 < z_4$  is defined as

$$Cr(z_1, z_2, z_3, z_4) = \frac{(z_2 - z_1)(z_4 - z_3)}{(z_3 - z_1)(z_4 - z_2)}.$$

**Definition 3.2.** The **cross-ratio distortion**  $Dist(z_1, z_2, z_3, z_4; g)$  of four points  $z_i \in \mathbb{R}$ ,  $i = 1, 2, 3, 4$ ,  $z_1 < z_2 < z_3 < z_4$  with respect to a strictly increasing function  $g$  on  $\mathbb{R}$  is defined as

$$Dist(z_1, z_2, z_3, z_4; g) = \frac{Cr(g(z_1), g(z_2), g(z_3), g(z_4))}{Cr(z_1, z_2, z_3, z_4)}.$$

Consider then the points  $z_i \in S^1$ ,  $i = 1, \dots, k$ ,  $k \geq 3$  with  $z_1 < z_2 < \dots < z_k$  in the order on the circle. Define  $\hat{z}_1 := z_1$  and for  $i = 2, 3, \dots, k$

$$\hat{z}_i := \begin{cases} z_i, & \text{if } z_1 < z_i < 1, \\ 1 + z_i, & \text{if } 0 \leq z_i < z_1. \end{cases}$$

Then obviously  $\hat{z}_1 < \hat{z}_2 < \dots < \hat{z}_k$ .

The vector  $(\hat{z}_1, \hat{z}_2, \dots, \hat{z}_k)$  is called the **lift** of  $(z_1, z_2, \dots, z_k)$ . Consider now a circle homeomorphism  $f$  with lift  $\hat{f}$ . We define the cross-ratio distortion of  $(z_1, z_2, z_3, z_4)$  with respect to  $f$  by  $Dist(z_1, z_2, z_3, z_4; f) := Dist(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4; \hat{f})$  where  $(\hat{z}_1, \hat{z}_2, \hat{z}_3, \hat{z}_4)$  is the lift of  $(z_1, z_2, z_3, z_4)$ . We need the following

**Lemma 3.3.** (See [4].) Consider a circle homeomorphism  $f$  with  $f \in C^{2+\alpha}([z_1, z_4])$ ,  $\alpha > 0$ ,  $[z_1, z_4] \subset S^1$  and  $Df(x) \geq \text{const} > 0$  for  $x \in [z_1, z_4]$ . Then there is a positive constant  $K$  such that

$$|Dist(z_1, z_2, z_3, z_4; f) - 1| \leq K |\hat{z}_4 - \hat{z}_1|^{1+\alpha}$$

for any quadruple of numbers  $(z_1, z_2, z_3, z_4)$ ,  $z_i \in S^1$ ,  $i = 1, 2, 3, 4$ , with  $z_1 < z_2 < z_3 < z_4$ .

Next we consider the case when the interval  $[z_1, z_4]$  contains the break point  $x = b$  of the homeomorphism  $f$ . We estimate the distortion of the cross-ratio when  $b$  lies outside the middle interval  $[z_2, z_3]$ :

for  $z_i \in S^1$ ,  $i = 1, 2, 3, 4$ , with  $z_1 < z_2 < z_3 < z_4$  and  $b \in [z_1, z_2] \cup [z_3, z_4]$  we set

$$\begin{aligned} \alpha &:= \hat{z}_2 - \hat{z}_1, & \beta &:= \hat{z}_3 - \hat{z}_2, & \gamma &:= \hat{z}_4 - \hat{z}_3, & \tau &:= \hat{z}_2 - \hat{b}, \\ \xi &:= \frac{\beta}{\alpha}, & z &:= \frac{\tau}{\alpha}, & \eta &:= \frac{\beta}{\gamma}, & \vartheta &:= \frac{\hat{b} - \hat{z}_3}{\gamma}. \end{aligned}$$

**Lemma 3.4.** (See [4].) Assume the circle homeomorphism  $f$  satisfies the conditions of Theorem 1.4. Chose  $z_i \in S^1$ ,  $i = 1, 2, 3, 4$ , with  $z_1 < z_2 < z_3 < z_4$  and assume  $f$  has a single break point  $b \in [z_1, z_2] \cup [z_3, z_4]$ . Then

- (i)  $|\text{Dist}(z_1, z_2, z_3, z_4; f) - \frac{[\sigma(b)+(1-\sigma(b))z](1+\xi)}{\sigma(b)+(1-\sigma(b))z+\xi}| \leq K_1 |\hat{z}_4 - \hat{z}_1|$ , if  $b \in [z_1, z_2]$ ,  
 (ii)  $|\text{Dist}(z_1, z_2, z_3, z_4; f) - \frac{[\sigma(b)+(1-\sigma(b))\vartheta](1+\eta)}{\sigma(b)+(1-\sigma(b))\vartheta+\eta}| \leq K_1 |\hat{z}_4 - \hat{z}_1|$ , if  $b \in [z_3, z_4]$

where the constant  $K_1 > 0$  depends only on  $f$ .

#### 4. Proof of Theorem 1.4

For the proof of Theorem 1.4 we need several lemmas which we formulate next. Their proofs will be given later. Consider two copies of the circle  $S^1$  and homeomorphism  $f_i$  with single break point  $b_i$ ,  $i = 1, 2$ , and rotation number  $\rho$  acting on them. Assume that  $f_1$  and  $f_2$  satisfy the conditions of Theorem 1.4.

Let  $\varphi_1$  and  $\varphi_2$  be maps conjugating  $f_1$  and  $f_2$  with the pure rotation  $f_\rho$ , i.e.  $\varphi_1 \circ f_1 = f_\rho \circ \varphi_1$  and  $\varphi_2 \circ f_2 = f_\rho \circ \varphi_2$ . It is easy to check that the map  $\psi = \varphi_2 \circ \varphi_1^{-1}$  conjugates  $f_1$  and  $f_2$ , i.e.

$$\psi(f_1(x)) = f_2(\psi(x)) \quad (5)$$

for all  $x \in S^1$ . Since  $\varphi_i$ ,  $i = 1, 2$ , is unique up to an additive constant we can choose to choose  $\varphi_i$ ,  $i = 1, 2$ , such that  $\varphi_1^{-1}(b_1) = b_2$  and  $\varphi_2(b_2) = b_2$  and hence  $\psi(b_1) = b_2$ . Notice, that  $\psi(x)$  is continuous on  $S^1$ . The derivative  $D\psi(x)$  exists for almost all  $x$  w.r.t. Lebesgue measure because  $\psi(x)$  is monotone. It is enough to show that  $D\psi(x) = 0$  at all points where the derivative is defined. Recall, that the length of an interval  $[a, b] \subset S^1$  is defined by

$$l([a, b]) := \begin{cases} b - a, & \text{if } a < b < 1, \\ 1 + b - a, & \text{if } 0 \leq b < a. \end{cases}$$

**Definition 4.1.** Fix  $R_1 > 1$  and  $\varepsilon > 0$ , four points  $z_i \in S^1$ ,  $i = 1, 2, 3, 4$ ,  $z_1 < z_2 < z_3 < z_4$  satisfy **the conditions**  $(C_{R_1, \varepsilon})$  if:

- (a)  $R_1^{-1} \beta \sqrt{\varepsilon} \leq \alpha \leq R_1 \beta \sqrt{\varepsilon}$ ;  
 (b)  $R_1^{-1} \beta \leq \gamma \leq R_1 \beta$ ;  
 (c)  $\max_{1 \leq i \leq 4} |z_i - x_0| \leq R_1 \beta$ .

Next define  $d(x_0) := \min\{x_0, (1 - x_0)\}$ ,  $x_0 \in S^1$ .

**Lemma 4.2.** Assume that the conjugating map  $\psi$  has a positive derivative  $D\psi(x_0) = \omega$  at the point  $x_0 \in (0, 1)$  and let  $R_1 > 1$  be a constant, then there exists a constant  $C_2 = (C_2(\omega, R_1))$  such that: for any  $\varepsilon > 0$ , there exists  $\delta = \delta(x_0, \varepsilon) \in (0, d(x_0))$  such that for any  $z_i \in (x_0 - \delta, x_0 + \delta)$ ,  $i = 1, 2, 3, 4$  satisfying the conditions  $(C_{R_1, \varepsilon})$  one has:

- (i)  $\frac{\beta}{\alpha} (1 - C_2 \sqrt{\varepsilon}) \leq \frac{l(\psi(z_2), \psi(z_3))}{l(\psi(z_1), \psi(z_2))} \leq \frac{\beta}{\alpha} (1 + C_2 \varepsilon)$ ,  
 (ii)  $\frac{\gamma}{\beta} (1 - C_2 \varepsilon) \leq \frac{l(\psi(z_3), \psi(z_4))}{l(\psi(z_2), \psi(z_3))} \leq \frac{\gamma}{\beta} (1 + C_2 \varepsilon)$ .

**Lemma 4.3.** Assume that the conjugating map  $\psi$  has a positive derivative  $D\psi(x_0) = \omega$  at the point  $x_0 \in (0, 1)$  and let  $R_1 > 1$  be a constant, then there exists a constant  $R_2 = (R_2(\omega, R_1))$  such that: for any  $\varepsilon > 0$ , there exists  $\delta = \delta(x_0, \varepsilon) \in (0, d(x_0))$  such that for any  $z_i \in (x_0 - \delta, x_0 + \delta)$ ,  $i = 1, 2, 3, 4$  satisfying the conditions  $(C_{R_1, \varepsilon})$  one has

$$|\text{Dist}(z_1, z_2, z_3, z_4; \psi) - 1| \leq R_2 \sqrt{\varepsilon}. \quad (6)$$

Suppose now,  $D\psi(x_0) = \omega$  for some point  $x_0 \in (0, 1)$ . Define  $\Delta_i^{(n)} := f_1^i(\Delta_0^{(n)})$ ,  $C_i^{(n)} := f_2^i(C_0^{(n)})$ ,  $i > 0$  where  $\Delta_0^{(n)}$ ,  $C_0^{(n)}$  are the initial intervals of rank  $n$  of the points  $x_0$  and  $y_0 = \psi(x_0)$  determined by  $f_1$  and  $f_2$  respectively. Next, consider the  $n$ -th dynamical partitions

$$\xi_n(x_0) = \{\Delta_i^{(n-1)}, 0 \leq i < q_n; \Delta_j^{(n)}, 0 \leq j < q_{n-1}\},$$

$$\zeta_n(x_0) = \{C_i^{(n-1)}, 0 \leq i < q_n; C_j^{(n)}, 0 \leq j < q_{n-1}\}.$$

Then two cases are possible:  $b_1 \in \Delta_{i_0}^{(n-1)}$ ,  $0 \leq i_0 < q_n$ , or  $b_1 \in \Delta_{j_0}^{(n)}$ ,  $0 \leq j_0 < q_{n-1}$ . Denote by  $\bar{b}_1$  the primage of the break point  $b_1$  which lies inside the interval  $\Delta_0^{(n)}(x_0)$  or  $\Delta_0^{(n-1)}(x_0)$ . It is clear that  $\bar{b}_1 = f_1^{-l}(b_1)$ , for some  $l \in [0, q_n]$ . Notice that the points  $x_0$  and  $\bar{b}_1$  are  $q_n$ -close. Assume  $n$  to be odd. Then  $\Delta_0^{(n-1)} = [x_0, f_1^{q_{n-1}}(x_0)]$  and  $\Delta_0^{(n)} = [f_1^{q_n}(x_0), x_0]$ . The structure

of the  $n$ -th dynamical partition implies that the intervals  $f_1^m([f_1^{-q_{n-1}}\bar{b}_1, f_1^{q_{n-1}}\bar{b}_1])$ ,  $0 \leq m < q_n$  cover the break point  $x = b_1$  exactly once.

Since the rotation number  $\rho$  of  $f_1$  and  $f_2$  is irrational, the order of the points in the orbit  $f^k(x_0)$ ,  $k \in \mathbb{Z}^1$  on the first circle will be precisely the same as for the orbit  $f_2^k(\psi(x_0))$ ,  $k \in \mathbb{Z}^1$  on the second circle. From this and the identities  $\psi(b_1) = b_2$  respectively  $\psi(f_1(x)) = f_2(\psi(x))$ ,  $x \in S^1$  we get

$$\psi(\Delta_i^{(n-1)}) = C_i^{(n-1)}, \quad 0 \leq i < q_n, \quad \psi(\Delta_j^{(n)}) = C_j^{(n)}, \quad 0 \leq j < q_{n-1}.$$

Since  $\bar{b}_1 \in \Delta_0^{(n-1)} \cup \Delta_0^{(n)}$  it follows that  $\psi(\bar{b}_1) \in C_0^{(n-1)} \cup C_0^{(n)}$ . It is clear that

$$f_2^l(\psi(\bar{b}_1)) = f_2^{l-1}(f_2(\psi(\bar{b}_1))) = f_2^{l-1}(\psi(f_1(\bar{b}_1))) = \dots = \psi(f_1^l(\bar{b}_1)) = \psi(b_1) = b_2.$$

Hence the intervals  $f_2^m([f_2^{-q_{n-1}}(\psi(\bar{b}_1)), f_2^{q_{n-1}}(\psi(\bar{b}_1))])$ ,  $0 \leq m < q_n$ , cover the break point  $b_2$  exactly once with  $f_2^l(\psi(\bar{b}_1)) = b_2$ .

Fix  $\varepsilon > 0$ . Next we introduce the following notations:

$$z_2 = \bar{b}_1, \quad z_3 = \frac{\bar{b}_1 + f_1^{q_{n-1}}\bar{b}_1}{2}, \quad z_4 = f_1^{q_{n-1}}\bar{b}_1, \quad z_1 = z_2 - 2\beta\sqrt{\varepsilon} \quad \text{with } \beta = z_3 - z_2. \quad (7)$$

By Corollary 2.6 the intervals  $[f_1^{-q_{n-1}}(\bar{b}_1), \bar{b}_1]$  and  $[\bar{b}_1, f_1^{q_{n-1}}(\bar{b}_1)]$  are  $e^\nu$ -comparable. It is easy to see that for  $\varepsilon \in (0, e^{-2\nu}]$  the point  $z_1$  belongs to the interval  $[f_1^{-q_{n-1}}(\bar{b}_1), \bar{b}_1]$ , where  $\nu = \text{var}_{S^1} \log Df_1$ .

Then one shows

**Lemma 4.4.** Suppose the circle homeomorphism  $f_1$  satisfies the conditions of Lemma 2.1. Let  $\delta$  be the positive constant given by Lemmas 4.2 and 4.3. Then the triple of intervals  $[z_s, z_{s+1}]$ ,  $s = 1, 2, 3$  possesses the following properties:

- (1)  $[z_1, z_4], [f_1^{q_n}(z_1), f_1^{q_n}(z_4)] \subset (x_0 - \delta, x_0 + \delta)$ ;
- (2) the intervals  $[z_s, z_{s+1}], [f_1^{q_n}(z_s), f_1^{q_n}(z_{s+1})]$ ,  $s = 1, 2, 3$ , satisfy the conditions  $(C_{R_1, \varepsilon})$  with some constant  $R_1 > 1$  depending only on the variation of  $\log Df_1$ .

**Lemma 4.5.** Assume the circle homeomorphisms  $f_i$ ,  $i = 1, 2$ , satisfies the conditions of Theorem 1.4 and the conjugacy map  $\psi$  has a positive derivative  $D\psi(x_0) = \omega$  at the point  $x_0 \in (0, 1)$ . Let the points  $z_i \in S^1$ ,  $i = 1, 2, 3, 4$ , be defined as in (7). Then the following inequality holds for sufficiently large  $n$ :

$$\left| \frac{\text{Dist}(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4); f_2^{q_n})}{\text{Dist}(z_1, z_2, z_3, z_4; f_1^{q_n})} - \frac{\sigma_2}{\sigma_1} \right| \leq \text{const} \sqrt{\varepsilon}, \quad (8)$$

where  $\sigma_i$ ,  $i = 1, 2$  are the jump ratios of  $f_i$  and the constant depends only on the functions  $f_i$ ,  $i = 1, 2$ .

After these preparations we can prove Theorem 1.4.

**Proof of Theorem 1.4.** Let  $f_1$  and  $f_2$  be circle homeomorphisms satisfying the conditions of Theorem 1.4. Since their rotation number  $\rho$  is irrational the  $f_i$ -invariant measures  $\mu_i$ ,  $i = 1, 2$ , are non-atomic i.e. every one point subset of the circle has zero  $\mu_i$ -measure. The maps  $\varphi_i$  conjugating  $f_i$  and  $f_\rho$  given by  $\varphi_i(x) = \mu_i([0, x])$ ,  $i = 1, 2$ , are continuous and monotone increasing functions on  $\mathbb{R}^1$ . Hence the map conjugation  $\psi = \varphi_2^{-1} \circ \varphi_1$  conjugating  $f_1$  and  $f_2$  has the same properties. The function  $\psi$  has a finite derivative almost everywhere w.r.t. Lebesgue measure on the circle. We will show  $D\psi(x) = 0$  at all points  $x$  where the derivative exists. Assume that  $D\psi(x_0) = \omega > 0$  at some point  $x_0 \in (0, 1)$ . Fix some  $\varepsilon \in (0, e^{-2\nu})$ . Choose the points  $z_i \in S^1$ ,  $i = 1, 2, 3, 4$  as given in formulas (7). According to Lemma 4.4 the intervals  $[z_s, z_{s+1}]$  and  $[f_1^{q_n}(z_s), f_1^{q_n}(z_{s+1})]$  satisfy the conditions of Lemma 4.2. It follows then from Lemma 4.3 that

$$|\text{Dist}(z_1, z_2, z_3, z_4; \psi) - 1| \leq R_2 \sqrt{\varepsilon} \quad (9)$$

and

$$|\text{Dist}(f_1^{q_n}(z_1), f_1^{q_n}(z_2), f_1^{q_n}(z_3), f_1^{q_n}(z_4); \psi) - 1| \leq R_2 \sqrt{\varepsilon}. \quad (10)$$

Consequently

$$\left| \frac{\text{Dist}(f_1^{q_n}(z_1), f_1^{q_n}(z_2), f_1^{q_n}(z_3), f_1^{q_n}(z_4); \psi)}{\text{Dist}(z_1, z_2, z_3, z_4; \psi)} - 1 \right| \leq R_3 \sqrt{\varepsilon}, \quad (11)$$

where the constant  $R_3 > 0$  does not depend on  $\varepsilon$  and  $n$ .

But by definition

$$\text{Dist}(f_1^{qn}(z_1), f_1^{qn}(z_2), f_1^{qn}(z_3), f_1^{qn}(z_4); \psi) = \frac{\text{Cr}(\psi(f_1^{qn}(z_1)), \psi(f_1^{qn}(z_2)), \psi(f_1^{qn}(z_3)), \psi(f_1^{qn}(z_4)))}{\text{Cr}(f_1^{qn}(z_1), f_1^{qn}(z_2), f_1^{qn}(z_3), f_1^{qn}(z_4))}.$$

Since  $\psi$  is conjugating  $f_1$  and  $f_2$  we can readily see that

$$\text{Cr}(\psi(f_1^{qn}(z_1)), \psi(f_1^{qn}(z_2)), \psi(f_1^{qn}(z_3)), \psi(f_1^{qn}(z_4))) = \text{Cr}(f_2^{qn}(\psi(z_1)), f_2^{qn}(\psi(z_2)), f_2^{qn}(\psi(z_3)), f_2^{qn}(\psi(z_4))).$$

It now follows that

$$\begin{aligned} & \frac{\text{Dist}(f_1^{qn}(z_1), f_1^{qn}(z_2), f_1^{qn}(z_3), f_1^{qn}(z_4); \psi)}{\text{Dist}(z_1, z_2, z_3, z_4; \psi)} \\ &= \frac{\text{Cr}(\psi(f_1^{qn}(z_1)), \psi(f_1^{qn}(z_2)), \psi(f_1^{qn}(z_3)), \psi(f_1^{qn}(z_4)))}{\text{Cr}(f_1^{qn}(z_1), f_1^{qn}(z_2), f_1^{qn}(z_3), f_1^{qn}(z_4))} \times \frac{\text{Cr}(z_1, z_2, z_3, z_4)}{\text{Cr}(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4))} \\ &= \frac{\text{Cr}(f_2^{qn}(\psi(z_1)), f_2^{qn}(\psi(z_2)), f_2^{qn}(\psi(z_3)), f_2^{qn}(\psi(z_4)))}{\text{Cr}(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4))} : \frac{\text{Cr}(f_1^{qn}(z_1), f_1^{qn}(z_2), f_1^{qn}(z_3), f_1^{qn}(z_4))}{\text{Cr}(z_1, z_2, z_3, z_4)} \\ &= \frac{\text{Dist}(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4); f_2^{qn})}{\text{Dist}(z_1, z_2, z_3, z_4; f_1^{qn})}. \end{aligned}$$

Combining this and inequality (11) we get

$$\left| \frac{\text{Dist}(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4); f_2^{qn})}{\text{Dist}(z_1, z_2, z_3, z_4; f_1^{qn})} - 1 \right| \leq R_3 \sqrt{\varepsilon}. \quad (12)$$

But this contradicts Lemma 4.5, according to which

$$\left| \frac{\text{Dist}(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4); f_2^{qn})}{\text{Dist}(z_1, z_2, z_3, z_4; f_1^{qn})} - \frac{\sigma_2}{\sigma_1} \right| \leq \text{const} \sqrt{\varepsilon} \quad (13)$$

for sufficiently large  $n$ . This contradiction proves Theorem 1.4.  $\square$

## 5. Proofs of Lemmas 4.2–4.5

We start with the proof of Lemma 4.2.

**Proof of Lemma 4.2.** Consider a point  $x_0$  where the derivative  $D\psi(x_0)$  exists with  $D\psi(x_0) = \omega > 0$ . It is obvious that  $D\hat{\psi}(x_0) = D\psi(x_0)$ . By the definition of the derivative there exists for any  $\varepsilon > 0$  a number  $\delta_1 = \delta_1(x_0, \varepsilon) > 0$ , such that for all  $x \in (x_0 - \delta_1, x_0 + \delta_1)$

$$\omega - \varepsilon < \frac{\hat{\psi}(x) - \hat{\psi}(x_0)}{x - x_0} < \omega + \varepsilon. \quad (14)$$

Put  $\delta = \min(\delta_1, d(x_0))$ . Take points  $z_i \in (x_0 - \delta, x_0 + \delta)$ ,  $i = 1, 2, 3, 4$  satisfying conditions (a), (b) and (c) of Definition 4.1. W.l.o.g. we can assume that  $[z_1, z_4] \subset (x_0 - \delta, x_0)$ . Relation (6) then implies for  $x = z_i$ ,  $i = 1, 2, 3, 4$ ,

$$(\omega - \varepsilon)(x_0 - z_i) < \hat{\psi}(x_0) - \hat{\psi}(z_i) < (\omega + \varepsilon)(x_0 - z_i).$$

This yields the following inequalities for  $s = 1, 2, 3$ :

$$\begin{aligned} \omega - \varepsilon \frac{(x_0 - z_{s+1}) + (x_0 - z_s)}{z_{s+1} - z_s} &\leq \frac{\hat{\psi}(z_{s+1}) - \hat{\psi}(z_s)}{z_{s+1} - z_s} \\ &\leq \omega + \varepsilon \frac{(x_0 - z_{s+1}) + (x_0 - z_s)}{z_{s+1} - z_s}, \end{aligned} \quad (15)$$

respectively for  $s = 1, 2$

$$\begin{aligned} \omega - \varepsilon \frac{(x_0 - z_{s+2}) + (x_0 - z_s)}{z_{s+2} - z_s} &\leq \frac{\hat{\psi}(z_{s+2}) - \hat{\psi}(z_s)}{z_{s+2} - z_s} \\ &\leq \omega + \varepsilon \frac{(x_0 - z_{s+2}) + (x_0 - z_s)}{z_{s+2} - z_s}. \end{aligned} \quad (16)$$

Conditions (a), (b) and (c) of Definition 4.1 on the other hand imply

$$\max_{1 \leq i \leq 4} \left\{ \frac{x_0 - z_i}{z_2 - z_1} \right\} \leq R_1 \frac{\beta}{\alpha} \leq \frac{R_1^2}{\sqrt{\varepsilon}}, \quad (17)$$

$$\max_{1 \leq i \leq 4} \max \left\{ \frac{x_0 - z_i}{z_3 - z_2}, \frac{x_0 - z_i}{z_4 - z_3} \right\} \leq R_1^2. \quad (18)$$

Combining relations (15), (16), (17) and (18) we get for  $l = 2, 3$

$$\omega - C_4 \sqrt{\varepsilon} \leq \frac{\hat{\psi}(z_2) - \hat{\psi}(z_1)}{z_2 - z_1} \leq \omega + C_4 \sqrt{\varepsilon}, \quad (19)$$

$$\omega - C_4 \varepsilon \leq \frac{\hat{\psi}(z_{l+1}) - \hat{\psi}(z_l)}{z_{l+1} - z_l} \leq \omega + C_4 \varepsilon, \quad (20)$$

and for  $s = 1, 2$

$$\omega - C_4 \varepsilon \leq \frac{\hat{\psi}(z_{s+2}) - \hat{\psi}(z_s)}{z_{s+2} - z_s} \leq \omega + C_4 \varepsilon, \quad (21)$$

where the constant  $C_4 > 0$  depends on  $R_1$ ,  $\omega$  and does not depend on  $\alpha, \beta, \gamma$  and  $\varepsilon$ .

Using the equality

$$\frac{\hat{\psi}(z_{s+1}) - \hat{\psi}(z_s)}{\hat{\psi}(z_s) - \hat{\psi}(z_{s-1})} \cdot \frac{z_{s+1} - z_s}{z_s - z_{s-1}} = \frac{\hat{\psi}(z_{s+1}) - \hat{\psi}(z_s)}{z_{s+1} - z_s} \cdot \frac{z_s - z_{s-1}}{\hat{\psi}(z_s) - \hat{\psi}(z_{s-1})}$$

and relations (19), (20) and (21) we get the assertions of Lemma 4.2.  $\square$

Next we will prove Lemma 4.3.

**Proof of Lemma 4.3.** Since

$$\begin{aligned} \text{Dist}(z_1, z_2, z_3, z_4, \hat{\psi}) &= \frac{\text{Cr}(\hat{\psi}(z_1), \hat{\psi}(z_2), \hat{\psi}(z_3), \hat{\psi}(z_4))}{\text{Cr}(z_1, z_2, z_3, z_4)} \\ &= \frac{\hat{\psi}(z_2) - \hat{\psi}(z_1)}{z_2 - z_1} \cdot \frac{\hat{\psi}(z_4) - \hat{\psi}(z_3)}{z_4 - z_3} \cdot \frac{z_3 - z_1}{\hat{\psi}(z_3) - \hat{\psi}(z_1)} \cdot \frac{z_4 - z_2}{\hat{\psi}(z_4) - \hat{\psi}(z_2)} \end{aligned}$$

inequalities (19)–(21) then prove Lemma 4.3.  $\square$

Next consider the proof of Lemma 4.4.

**Proof of Lemma 4.4.** Let  $D\psi(x_0) = \omega > 0$  for some  $x_0 \in (0, 1)$ . W.l.o.g. we can assume  $n$  to be odd, the case  $n$  even can be deduced from the odd one just by reversing the orientation of the circle. From the definition of the dynamical partition  $\xi_n(x_0)$  it follows, that the primage  $\bar{b}_1$  of the point is in the interval  $[f_1^{q_n}(x_0), f_1^{q_{n-1}}(x_0)]$  with  $\bar{b}_1 = f_1^{-l}(b_1)$  for some  $0 \leq l < q_{n-1}$ . Define the points  $z_i$ ,  $i = 1, 2, 3, 4$  as in formulas [7]. Consider the neighborhood  $[f_1^{-q_{n-1}}(\bar{b}_1), f_1^{q_{n-1}}(\bar{b}_1)]$  of the point  $\bar{b}_1$ .

By Corollary 2.6 the intervals  $[a, b]$ ,  $f_1^{q_n}[a, b]$ ,  $f_1^{-q_n}[a, b]$  are  $e^v$ -comparable for any  $a, b \in S^1$ . Since  $\bar{b}_1 \in [f_1^{q_n}(x_0), f_1^{q_{n-1}}(x_0)]$ , it is easy to check  $[f_1^{-q_{n-1}}(\bar{b}_1), f_1^{q_{n-1}}(\bar{b}_1)] \subset [f_1^{-2q_{n-1}}(x_0), f_1^{2q_{n-1}}(x_0)]$ . But the interval  $[f_1^{-2q_{n-1}}(x_0), f_1^{2q_{n-1}}(x_0)]$  is the union of the four intervals  $[f_1^{-2q_{n-1}}(x_0), f_1^{-q_n}(x_0)]$ ,  $[f_1^{-q_n}(x_0), x_0]$ ,  $[x_0, f_1^{q_n}(x_0)]$ ,  $[f_1^{q_n}(x_0), f_1^{2q_{n-1}}(x_0)]$  and the first three intervals are comparable with  $[x_0, f_1^{q_{n-1}}(x_0)]$ . By Corollary 2.2  $l([x_0, f_1^{q_{n-1}}(x_0)]) < \text{const } \lambda^n$ . Using the last inequality we get  $l([f_1^{-2q_{n-1}}(x_0), f_1^{2q_{n-1}}(x_0)]) < \text{const } \lambda^n$ . If  $n$  is sufficiently large then obviously  $[f_1^{-2q_{n-1}}(x_0), f_1^{2q_{n-1}}(x_0)] \subset [x_0 - \delta, x_0 + \delta]$ . This together with  $[f_1^{-q_{n-1}}(\bar{b}_1), f_1^{q_{n-1}}(\bar{b}_1)] \subset [f_1^{-2q_{n-1}}(\bar{b}_1), f_1^{2q_{n-1}}(\bar{b}_1)]$  imply the first assertion of Definition 4.1.

By Corollary 2.6 the intervals  $[z_s, z_{s+1}]$  and  $[f_1^{q_n}(z_s), f_1^{q_n}(z_{s+1})]$  are  $e^v$ -comparable for all  $s = 1, 2, 3$ . The intervals  $[z_s, z_{s+1}]$ ,  $s = 1, 2, 3$  satisfy the assumptions of Lemma 4.1. This implies that the intervals  $[f_1^{q_n}(z_s), f_1^{q_n}(z_{s+1})]$ ,  $s = 1, 2, 3$  satisfy the assumption (a) and (b) of Definition 4.1.

Next consider assumption (c) of Definition 4.1. It is easy to see, that for  $i = 1, 2, 3, 4$  one has

$$\begin{aligned} l([z_i, x_0]) &\leq l([z_2, x_0]) + l([z_1, z_4]), \\ l([f_1^{q_n}(z_i), x_0]) &\leq l([z_2, x_0]) + l([f_1^{q_n}(z_2), z_2]) + l([f_1^{q_n}(z_1), f_1^{q_n}(z_4)]). \end{aligned}$$



We have  $[z_2, x_0] \subset [f_1^{-q_{n-1}}(x_0), f_1^{q_{n-1}}(x_0)]$  and  $[z_1, z_4] \subset [f_1^{-q_{n-1}}(\bar{b}_1), f_1^{q_{n-1}}(\bar{b}_1)]$ . The interval  $[f_1^{-q_{n-1}}(x_0), f_1^{q_{n-1}}(x_0)]$  is  $e^v$ -comparable with  $[f_1^{-q_{n-1}}(\bar{b}_1), f_1^{q_{n-1}}(\bar{b}_1)]$ . But the length of this last interval is  $4e^v$ -comparable with  $\beta = z_3 - z_2$ . Finally we get that  $l([f_1^{q_n}(z_i), x_0]) \leq K\beta$  for all  $i = 1, 2, 3, 4$ , where the positive constant  $K$  depends on  $R_1$  and the total variation  $v$  of  $\log Df_1$ . Lemma 4.4 is proved.  $\square$

**Proof of Lemma 4.5.** Choose points  $z_i$ ,  $i = 1, 2, 3, 4$  as in formulas (7) and consider the two systems of intervals  $\{[f_1^i(z_s), f_1^i(z_{s+1})], s = 1, 2, 3\}$  and  $\{[f_2^i(\psi(z_s)), f_2^i(\psi(z_{s+1}))], s = 1, 2, 3\}$ . Obviously only the intervals  $(f_1^l[z_1, z_2], f_1^l[z_2, z_3], f_1^l[z_3, z_4])$  respectively  $(f_2^l(\psi(z_1)), f_2^l(\psi(z_2)), f_2^l(\psi(z_3)), f_2^l(\psi(z_4)))$ ,  $0 \leq l < q_n$  cover the break points  $b_1$  respectively  $b_2$ . Next we compare the distortions

$$\text{Dist}(z_1, z_2, z_3, z_4; f_1^{q_n}) \quad \text{and} \quad \text{Dist}(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4); f_2^{q_n}).$$

It is clear that

$$\begin{aligned} \text{Dist}(z_1, z_2, z_3, z_4; f_1^{q_n}) &= \prod_{\substack{i=0 \\ i \neq l}}^{q_n-1} \text{Dist}(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1) \\ &\quad \times \text{Dist}(f_1^l(z_1), f_1^l(z_2), f_1^l(z_3), f_1^l(z_4); f_1), \end{aligned} \quad (22)$$

and

$$\begin{aligned} \text{Dist}(\psi(z_1), \psi(z_2), \psi(z_3), \psi(z_4); f_2^{q_n}) &= \prod_{\substack{i=0 \\ i \neq l}}^{q_n-1} \text{Dist}(f_2^i(\psi(z_1)), f_2^i(\psi(z_2)), f_2^i(\psi(z_3)), f_2^i(\psi(z_4)); f_2) \\ &\quad \times \text{Dist}(f_2^l(\psi(z_1)), f_2^l(\psi(z_2)), f_2^l(\psi(z_3)), f_2^l(\psi(z_4)); f_2). \end{aligned} \quad (23)$$

Applying Lemma 3.3 we obtain

$$\prod_{\substack{i=0 \\ i \neq l}}^{q_n-1} \text{Dist}(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1) = \exp \left\{ \sum_{\substack{i=0 \\ i \neq l}}^{q_n-1} \log(1 + O(|[f_1^i(z_1), f_1^i(z_4)]|^{1+\alpha})) \right\}. \quad (24)$$

It is easy to see, that for  $0 \leq i < q_n$  one has  $[f_1^i(z_1), f_1^i(z_4)] \subset [f_1^{-q_{n-1}}(f_1^i(\bar{b}_1)), f_1^{q_{n-1}}(f_1^i(\bar{b}_1))]$ . The last interval is the union of intervals of rank  $(n-1)$ , namely  $[f_1^{-2q_{n-1}}(f_1^i(\bar{b}_1)), f_1^i(\bar{b}_1)]$  and  $[f_1^i(\bar{b}_1), f_1^{q_{n-1}}(f_1^i(\bar{b}_1))]$ . By Corollary 2.2 the length of the last intervals is bounded by  $\text{const } \lambda^n$ . Thus we get for  $0 \leq i < q_n$ ,

$$|[f_1^i(z_1), f_1^i(z_4)]| \leq \text{const } \lambda^n. \quad (25)$$

Furthermore

$$\sum_{i=0}^{q_n-1} |[f_1^i(z_1), f_1^i(z_4)]| \leq \sum_{i=0}^{q_n-1} |[f_1^{-q_{n-1}}(f_1^i(\bar{b}_1)), f_1^i(\bar{b}_1)]| + \sum_{i=0}^{q_n-1} |[f_1^i(\bar{b}_1), f_1^{q_{n-1}}(f_1^i(\bar{b}_1))]| \leq 2. \quad (26)$$

Combining Eqs. (23), (24) and (26) we get

$$\left| \prod_{\substack{i=0 \\ i \neq l}}^{q_n-1} \text{Dist}(f_1^i(z_1), f_1^i(z_2), f_1^i(z_3), f_1^i(z_4); f_1) - 1 \right| \leq \text{const } \lambda^{n\alpha} \sum_{\substack{i=0 \\ i \neq l}}^{q_n-1} |[f_1^i(z_1), f_1^i(z_4)]| \leq \text{const } \lambda^{n\alpha}. \quad (27)$$

In analogy one shows

$$\left| \prod_{\substack{i=0 \\ i \neq l}}^{q_n-1} \text{Dist}(f_2^i(\psi(z_1)), f_2^i(\psi(z_2)), f_2^i(\psi(z_3)), f_2^i(\psi(z_4)); f_2) - 1 \right| \leq \text{const } \lambda^{n\alpha}. \quad (28)$$

Next we compare

$$\text{Dist}(f_1^l(z_1), f_1^l(z_2), f_1^l(z_3), f_1^l(z_4); f_1) \quad \text{and} \quad \text{Dist}(f_2^l(\psi(z_1)), f_2^l(\psi(z_2)), f_2^l(\psi(z_3)), f_2^l(\psi(z_4)); f_2).$$

Applying Lemma 3.4 we get

$$\left| \text{Dist}(f_1^l(z_1), f_1^l(z_2), f_1^l(z_3), f_1^l(z_4); f_1) - \frac{\sigma_1(1 + \xi_1)}{(\sigma_1 + \xi_1)} \right| \leq K_2 | [f_1^l(z_1), f_1^l(z_4)] |, \quad (29)$$

$$\left| \text{Dist}(f_2^l(\psi(z_1)), f_2^l(\psi(z_2)), f_2^l(\psi(z_3)), f_2^l(\psi(z_4)); f_2) - \frac{\sigma_2(1 + \xi_2)}{(\sigma_2 + \xi_2)} \right| \leq K_2 | [f_2^l(\psi(z_1)), f_2^l(\psi(z_4))] |, \quad (30)$$

where the constant  $K_2 > 0$  depends only on  $f_1, f_2$  and

$$\xi_1 = \frac{f_1^l(z_3) - f_1^l(z_2)}{f_1^l(z_2) - f_1^l(z_1)}, \quad \xi_2 = \frac{f_2^l(\psi(z_3)) - f_2^l(\psi(z_2))}{f_2^l(\psi(z_2)) - f_2^l(\psi(z_1))}.$$

This together with Lemma 2.7 implies

$$\begin{aligned} R_5^{-1} \frac{z_3 - z_2}{z_2 - z_1} &\leq \xi_1 \leq R_5 \frac{z_3 - z_2}{z_2 - z_1}, \\ R_5^{-1} \frac{\psi(z_3) - \psi(z_2)}{\psi(z_2) - \psi(z_1)} &\leq \xi_2 \leq R_5 \frac{\psi(z_3) - \psi(z_2)}{\psi(z_2) - \psi(z_1)}, \end{aligned} \quad (31)$$

where the constant  $R_5 > 1$  does not depend on  $\varepsilon$ . Applying Lemma 4.3 we get

$$(1 - C_2\sqrt{\varepsilon}) \frac{z_3 - z_2}{z_2 - z_1} \leq \frac{\psi(z_3) - \psi(z_2)}{\psi(z_2) - \psi(z_1)} \leq (1 + C_2\sqrt{\varepsilon}) \frac{z_3 - z_2}{z_2 - z_1}.$$

Hence

$$R_5^{-1}(1 - C_2\sqrt{\varepsilon}) \frac{z_3 - z_2}{z_2 - z_1} \leq \xi_2 \leq R_5(1 + C_2\sqrt{\varepsilon}) \frac{z_3 - z_2}{z_2 - z_1}. \quad (32)$$

By construction  $z_3 - z_2 = \beta$ ,  $z_2 - z_1 = 2\beta\sqrt{\varepsilon}$ . This together with relations (31) and (32) yields

$$\frac{R_5^{-1}}{2\sqrt{\varepsilon}} \leq \xi_1 \leq \frac{2R_5}{\sqrt{\varepsilon}}$$

respectively

$$\frac{R_5^{-1}}{2\sqrt{\varepsilon}} \leq \xi_2 \leq \frac{2R_5(1 + C_2\sqrt{\varepsilon})}{\sqrt{\varepsilon}}.$$

Combining the bounds (29), (30) and the last two inequalities we get

$$\left| \frac{\text{Dist}(f_1^l(z_1), f_1^l(z_2), f_1^l(z_3), f_1^l(z_4); f_1)}{\text{Dist}(f_2^l(\psi(z_1)), f_2^l(\psi(z_2)), f_2^l(\psi(z_3)), f_2^l(\psi(z_4)); f_2)} - \frac{\sigma_1}{\sigma_2} \right| \leq \text{const} \sqrt{\varepsilon}.$$

This together with relations (22), (23), (24), (27) and (28) implies the assertion of Lemma 4.5.  $\square$

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## References

- [1] V.I. Arnol'd, Small denominators: I. Mappings from the circle onto itself, *Izv. Akad. Nauk SSSR, Ser. Mat.* 25 (1961) 21–86.
- [2] A. Denjoy, Sur les courbes définies par les équations différentielles à la surface du tore, *J. Math. Pures Appl.* 11 (1932) 333–375.
- [3] E. de Faria, W. de Melo, Rigidity of critical circle mappings. I, *J. Eur. Math. Soc. (JEMS)* 1 (4) (1999) 339–392.
- [4] A.A. Dzhaliilov, K.M. Khanin, On invariant measure for homeomorphisms of a circle with a point of break, *Funct. Anal. Appl.* 32 (3) (1998) 153–161.
- [5] I.P. Cornfeld, S.V. Fomin, Ya.G. Sinai, *Ergodic Theory*, Springer-Verlag, Berlin, 1982.
- [6] P. Cvitanovic (Ed.), *Universality in Chaos*, second ed., Adam Hilger, Bristol, 1989.
- [7] J. Graczyk, G. Świątek, Singular measures in circle dynamics, *Comm. Math. Phys.* 157 (1993) 213–230.
- [8] M. Herman, Sur la conjugaison différentiable des difféomorphismes du cercle à des rotations, *Inst. Hautes Etudes Sci. Publ. Math.* 49 (1979) 225–234.
- [9] J. Hawkins, K. Schmidt, On  $C^2$ -diffeomorphisms of the circle which are of type  $III_1$ , *Invent. Math.* 66 (1982) 511–518.
- [10] Y. Katznelson, D. Ornstein, The absolute continuity of the conjugation of certain diffeomorphisms of the circle, *Ergodic Theory Dynam. Systems* 9 (1989) 681–690.
- [11] K.M. Khanin, Ya.G. Sinai, Smoothness of conjugacies of diffeomorphisms of the circle with rotations, *Russian Math. Surveys* 44 (1989) 69–99; translation of *Uspekhi Mat. Nauk* 44 (1989) 57–82.
- [12] K.M. Khanin, E.B. Vul, Circle homeomorphisms with weak discontinuities, *Adv. Soviet Math.* 3 (1993) 57–98.
- [13] J. Moser, A rapid convergent iteration method and non-linear differential equations. II, *Ann. Sc. Norm. Super. Pisa* 20 (3) (1966) 499–535.
- [14] H. Poincaré, Sur le courbes définies par les équations différentielles, *J. Math. Pures Appl.* 1 (1885) 167–244, reprinted in *Ouvres de Henri Poincaré*, Tome I, Gauthier-Villars, Paris, 1928.
- [15] A.Yu. Teplinskii, K.M. Khanin, Rigidity for circle diffeomorphisms with singularities, *Russian Math. Surveys* 759 (2) (2004) 329–353; translation of *Uspekhi Mat. Nauk* 59 (2) (2004) 137–160.
- [16] J.C. Yoccoz, Il n'y a pas de contre-exemple de Denjoy analytique, *C. R. Acad. Sci. Paris T.* 298 (7) (1984) 141–144.