

Toeplitz and Hankel operators with $L^{\infty,1}$ symbols on Dirichlet spaceYong Chen^{a,b,*}, Nguyen Quang Dieu^{c,d}^a College of Mathematics, Physics and Information Engineering, Zhejiang Normal University, Jinhua 321004, PR China^b School of Mathematical Sciences, Fudan University, Shanghai 200433, PR China^c Department of Mathematics, Hanoi University of Education (Dai Hoc Pham Hanoi), Hanoi, Viet Nam^d Department of Mathematics, Chonnam National University, Gwangju 700-757, South Korea

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ABSTRACT

We show that Toeplitz or small Hankel operators with symbols in $L^{\infty,1}$ is a generalization of the case with the harmonic symbol in $\mathbb{C} \oplus \mathcal{D} \oplus \overline{\mathcal{D}}$, where \mathcal{D} is the Dirichlet space on \mathbb{D} . We also give a decomposition theorem for the Sobolev space of first order on \mathbb{D} . Using this result, some characterizations for algebraic properties of Toeplitz or small Hankel operators with symbols in $L^{\infty,1}$ are given.

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1. Introduction

Let \mathbb{D} be the open unit disk in the complex plane \mathbb{C} and dA be the normalized area measure on \mathbb{D} . The Sobolev space $L^{2,1} = L^{2,1}(\mathbb{D})$ is the completion of the space of smooth functions u such that

$$\|u\|_{\frac{1}{2}} = \left(\left| \int_{\mathbb{D}} u dA \right|^2 + \int_{\mathbb{D}} \left(\left| \frac{\partial u}{\partial z} \right|^2 + \left| \frac{\partial u}{\partial \bar{z}} \right|^2 \right) dA \right)^{\frac{1}{2}} < \infty.$$

$L^{2,1}$ is a Hilbert space with the inner product

$$\langle u, v \rangle_{\frac{1}{2}} = \int_{\mathbb{D}} u dA \int_{\mathbb{D}} \bar{v} dA + \left\langle \frac{\partial u}{\partial z}, \frac{\partial v}{\partial z} \right\rangle_{L^2} + \left\langle \frac{\partial u}{\partial \bar{z}}, \frac{\partial v}{\partial \bar{z}} \right\rangle_{L^2},$$

where the symbol $\langle \cdot, \cdot \rangle_{L^2}$ means the inner product in the Hilbert space $L^2(\mathbb{D}, dA)$. The Dirichlet space \mathcal{D} is the closed subspace of all holomorphic functions $f \in L^{2,1}$ with $f(0) = 0$. Let P be the orthogonal projection from $L^{2,1}$ onto \mathcal{D} . Then

$$P(u)(w) = \langle u, K_w \rangle_{\frac{1}{2}}, \quad u \in L^{2,1}$$

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where $K_w(z) = -\log(1 - z\bar{w}) = \sum_{k=1}^{\infty} \frac{z^k \bar{w}^k}{k}$ is a reproducing kernel for \mathcal{D} . More precisely, P is an integral operator represented by

$$P(u)(w) = \int_{\mathbb{D}} \frac{\partial u}{\partial z} \frac{\overline{\partial K_w(z)}}{\partial z} dA(z), \quad u \in L^{2,1}. \quad (1.1)$$

Given a function φ in $L^{2,1}$, the Toeplitz operator $T_\varphi : \mathcal{D} \rightarrow \mathcal{D}$, the (big) Hankel operator $H_\varphi : \mathcal{D} \rightarrow \mathcal{D}^\perp$ and the small Hankel operator $\Gamma_\varphi : \mathcal{D} \rightarrow \mathcal{D}$, with symbol φ are densely defined on \mathcal{D} respectively by

$$T_\varphi f = P(\varphi f), \quad H_\varphi f = (I - P)(\varphi f), \quad \Gamma_\varphi f = P(J(\varphi f)),$$

where J is the unitary $L^{2,1} \rightarrow L^{2,1}$ defined by $Jh(z) = h(\bar{z})$ for $h \in L^{2,1}$, and \mathcal{D}^\perp is the orthogonal complement of \mathcal{D} in $L^{2,1}$.

Let $L^\infty(\mathbb{D})$ denote the algebra of all essentially bounded measurable functions on \mathbb{D} and H^∞ denote the space of bounded analytic function on \mathbb{D} . Define

$$L^{\infty,1} = \left\{ \varphi \in L^{2,1} : \varphi, \frac{\partial \varphi}{\partial z}, \frac{\partial \varphi}{\partial \bar{z}} \in L^\infty(\mathbb{D}) \right\}.$$

It is well known that Toeplitz operator or (small) Hankel operator with symbol $\varphi \in L^{\infty,1}$ is bounded on \mathcal{D} (see [2]).

In the last few decades, much attention have been paid to Toeplitz operators and Hankel operators on the classical Hardy space H^2 and Bergman space L_a^2 . Recently, Toeplitz operators on the Dirichlet space have been studied intensively (see [10, 11, 2, 7, 8, 3]). G.F. Cao considered Fredholm properties of Toeplitz operators with $C^1(\overline{\mathbb{D}})$ symbols in [2]. In case of bounded harmonic symbols in $L^{\infty,1}$, Y.J. Lee in [7] studied the commutativity of two Toeplitz operators while in [8] he studied the zero or compactness of finite sum Toeplitz products. Moreover, L.K. Zhao investigated properties of Hankel operator in [12]. The question that arises naturally is whether the results in [2, 7, 8, 12] holds for symbols in $L^{\infty,1}$, or $L^{2,1}$? In this note we show that Toeplitz operators or small Hankel operators with symbol in $L^{\infty,1}$, is just a generalization of the case with the harmonic symbol in $L^{\infty,1}$. More precisely, we show that if $f \in L^{\infty,1}$, then there is a harmonic function $F \in \mathbb{C} \oplus \mathcal{D} \oplus \overline{\mathcal{D}}$ such that $T_F = T_f$ and $\Gamma_F = \Gamma_f$ on \mathcal{D} . Finally, as a byproduct of our results, based on the above-mentioned work of Lee and Zhao, we obtain algebraic properties of Toeplitz and small Hankel operator with symbols in $L^{\infty,1}$.

It should be pointed out that in this note, the derivative is taken in the *distribution sense* whereas in [3] the derivative is defined in the classical mode, which in turn provides different properties than the properties presented here. We point out that there is a flaw in [3] but the result is correct if the Sobolev space $L^{2,1}$ is removed and the Toeplitz operator is directly defined by the integral operator (1.1).

2. Preliminaries

Throughout this note we will write f' for $\frac{\partial f}{\partial z}$, \bar{f}' for $\frac{\partial \bar{f}}{\partial \bar{z}}$ when $f \in L^{2,1}$ and $\|\cdot\|_{\mathcal{H}}$ for the norm in a Hilbert space \mathcal{H} . We first need the following result which is probably well known.

Proposition 1. Let $E := [0, 1) \times [0, 1)$. Assume that $u \in L^{2,1}(E)$. Then the following assertions hold.

(1) For almost all $x \in [0, 1)$, $u(x, \cdot)$ is absolutely continuous on $[0, 1)$ and

$$\lim_{y \rightarrow 1} u(x, y) \in L^1[0, 1).$$

(2) For almost all $y \in [0, 1)$, $u(\cdot, y)$ is absolutely continuous on $[0, 1)$ and

$$\lim_{x \rightarrow 1} u(x, y) \in L^1[0, 1).$$

Proof. Due to the lack of an explicit reference, we give a detailed proof. It suffices to prove (1) since (2) can be treated analogously. Since $u \in L^{2,1}(E)$, by Fubini's theorem for almost all $x \in [0, 1)$ the function $\frac{\partial u}{\partial y}(x, \cdot) \in L^2[0, 1)$. Thus for almost all $x \in [0, 1)$, the function

$$\hat{u}(x, y) := \int_0^y \frac{\partial u}{\partial y}(x, t) dt$$

is well defined. Moreover,

$$\int_0^1 \int_0^1 |\hat{u}(x, y)| dx dy \leq \int_0^1 \int_0^1 \left| \frac{\partial u}{\partial y}(x, t) \right| dx dt < \infty.$$

This implies that $\tilde{u} := u - \hat{u} \in L^1(E)$. Next we claim that $\frac{\partial \hat{u}}{\partial y} = \frac{\partial u}{\partial y}$ in the sense of distributions. To see this, we let $\{p_k\}_{k \geq 1}$ be a sequence of polynomials on \mathbb{R}^2 such that $p_k \rightarrow \frac{\partial u}{\partial y}$ in $L^2(E)$ as $k \rightarrow \infty$. Define

$$u_k(x, y) = \int_0^y p_k(x, t) dt.$$

Then $\frac{\partial u_k}{\partial y}(x, y) = p_k(x, y)$ in the classical sense for every $k \in \mathbb{Z}_+$. Since $p_k \rightarrow \frac{\partial u}{\partial y}$ in $L^2(E)$, using Fubini's theorem, after passing to a subsequence we may assume that

$$\lim_{k \rightarrow \infty} \int_0^1 \left| p_k(x, t) - \frac{\partial u}{\partial y}(x, t) \right| dt = 0 \quad \text{for almost all } x \in [0, 1).$$

It follows that $u_k \rightarrow \hat{u}$ in $L^1(E)$ and the claim follows. Therefore $\frac{\partial \hat{u}}{\partial y} = 0$ in the sense of distributions. Thus, by Theorem 3.1.4' in [9], we conclude that \hat{u} is a function of x . Hence $u(x, \cdot)$ is absolutely continuous on $[0, 1)$ for almost all x . Moreover, since for almost all $x \in [0, 1)$, $\frac{\partial u}{\partial y}(x, \cdot) \in L^2[0, 1)$ we have

$$\lim_{y \rightarrow 1} u(x, y) = \hat{u}(x) + \int_0^1 \frac{\partial u}{\partial y}(x, t) dt \quad \text{for a.e. } x.$$

Since $\frac{\partial u}{\partial y} \in L^2(E)$ and since $\hat{u} \in L^1[0, 1)$, using Fubini's theorem again we infer that the limit function $\lim_{y \rightarrow 1} u(x, y) \in L^1[0, 1)$. The proof is complete. \square

Given a function $f \in L^{2,1}$. In the polar coordinates $z = re^{i\theta}$, we have

$$\frac{\partial f}{\partial z} = \frac{1}{2} e^{-i\theta} \left(\frac{\partial f}{\partial r} - \frac{i}{r} \frac{\partial f}{\partial \theta} \right), \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} e^{i\theta} \left(\frac{\partial f}{\partial r} + \frac{i}{r} \frac{\partial f}{\partial \theta} \right).$$

It follows that $f(re^{i\theta}) \in L^{2,1}(E)$, where $E = [0, 1) \times [0, 2\pi)$. By Proposition 1, we see that $f(re^{i\theta})$ is absolutely continuous in r for almost all θ and absolutely continuous in θ for almost all r . In particular, the radial limit $f|_{\partial \mathbb{D}} := \lim_{r \rightarrow 1} f(re^{i\theta})$ exists for almost all θ . Moreover, from the final conclusions of Proposition 1 we also have $f|_{\partial \mathbb{D}} \in L^1(\partial \mathbb{D})$. Thus we can define for $k \in \mathbb{Z}$,

$$f_k(1) = \frac{1}{2\pi} \int_0^{2\pi} f|_{\partial \mathbb{D}}(e^{i\theta}) e^{-ik\theta} d\theta.$$

The result below roughly says that Toeplitz operators on \mathcal{D} depends only on boundary values of the symbols.

Proposition 2. Let $f \in L^{2,1}$. Then for each $n \in \mathbb{Z}_+$,

$$T_f(z^n) = \sum_{k \in \mathbb{Z}_+} f_{k-n}(1) z^k.$$

Proof. For $n \in \mathbb{Z}_+$ we have

$$T_f(z^n)(w) = P(fz^n)(w) = \langle fz^n, K_w \rangle_{\frac{1}{2}} = \left\langle \frac{\partial(fz^n)}{\partial z}, \frac{\partial K_w}{\partial z} \right\rangle_{L^2} = \sum_{k \in \mathbb{Z}_+} \frac{1}{k} \left\langle \frac{\partial(fz^n)}{\partial z}, \frac{\partial z^k}{\partial z} \right\rangle_{L^2} w^k.$$

Hence

$$T_f(z^n)(w) = \sum_{k \in \mathbb{Z}_+} \frac{1}{\pi} \left[\int_0^1 \int_0^{2\pi} \frac{\partial f}{\partial z}(re^{i\theta}) r^{n+k} e^{i(n-k+1)\theta} dr d\theta + n \int_0^1 \int_0^{2\pi} f(re^{i\theta}) r^{n+k-1} e^{i(n-k)\theta} dr d\theta \right] w^k.$$

Notice $\frac{\partial f}{\partial z} = \frac{1}{2} e^{-i\theta} \left(\frac{\partial f}{\partial r} - \frac{i}{r} \frac{\partial f}{\partial \theta} \right)$, then by Fubini's theorem we obtain

$$\frac{1}{\pi} \int_0^1 \int_0^{2\pi} \frac{\partial f}{\partial z}(re^{i\theta}) r^{n+k} e^{i(n-k+1)\theta} dr d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-k)\theta} d\theta \int_0^1 \frac{\partial f}{\partial r} r^{n+k} dr - \frac{i}{2\pi} \int_0^{2\pi} r^{n+k-1} dr \int_0^{2\pi} \frac{\partial f}{\partial \theta} e^{i(n-k)\theta} d\theta.$$

Using the absolute continuity of f on r and θ , and integration by parts, we get the desired result. The proof is complete. \square

The following proposition is a characterization for functions in $L^{\infty,1}$. We claim no originality for this result.

Proposition 3. Let f be a measurable function on \mathbb{D} . Then $f \in L^{\infty,1}$ if and only if there exists a continuous function \tilde{f} on \mathbb{D} such that $\tilde{f} = f$ a.e. on \mathbb{D} and that

$$|\tilde{f}(z) - \tilde{f}(w)| \leq M|z - w| \quad \forall z, w \in \mathbb{D},$$

where $M > 0$ is a constant.

Proof. First assume that $f \in L^{\infty,1}$. Fix a smooth radial nonnegative function ρ on \mathbb{C} with compact support in \mathbb{D} such that $\int \rho(z) dA(z) = 1$. For $\delta > 0$, we define $\rho_\delta(z) := \delta^{-2} \rho(|z|/\delta)$. The convolution of a locally integrable function f and ρ_δ is defined on $\mathbb{D}_\delta := \{z \in \mathbb{D} : d(z, \partial\mathbb{D}) > \delta\}$ as

$$f_\delta(z) = (f * \rho_\delta)(z) := \int_{|w| < \delta} f(z - w) \rho_\delta(w) dA(w) \quad \forall z \in \mathbb{D}_\delta.$$

It is known that f_δ is C^∞ smooth and that f_δ converges to f in $L^p(\mathbb{D}, dA)$ as $\delta \rightarrow 0$ for $1 \leq p < \infty$ if $f \in L^p(\mathbb{D}, dA)$ (cf. Chapter 4 in [9]).

Since $f \in L^{\infty,1}$ and since

$$\frac{\partial f_\delta}{\partial z} = \frac{\partial f}{\partial z} * \rho_\delta, \quad \frac{\partial f_\delta}{\partial \bar{z}} = \frac{\partial f}{\partial \bar{z}} * \rho_\delta,$$

we deduce that for every $\delta > 0$, all the partial derivatives of f_δ are bounded on \mathbb{D}_δ . Moreover, this bound does not depend on δ . It follows that, there is a constant $M > 0$ satisfying

$$|f_\delta(z) - f_\delta(w)| \leq M|z - w| \quad \forall z, w \in \mathbb{D}.$$

We then choose a sequence $\delta_k \rightarrow 0$ such that $f_{\delta_k} \rightarrow f$ outside a set $E \subset \mathbb{D}$ of measure 0. Clearly

$$|f(z) - f(w)| \leq M|z - w| \quad \forall z, w \in \mathbb{D} \setminus E.$$

Since E has empty interior, we may extend f to a continuous function \tilde{f} on \mathbb{D} satisfying

$$|\tilde{f}(z) - \tilde{f}(w)| \leq M|z - w| \quad \forall z, w \in \mathbb{D}.$$

Conversely, suppose that f is a continuous function on \mathbb{D} such that

$$|f(z) - f(w)| \leq M|z - w| \quad \forall z, w \in \mathbb{D},$$

where $M > 0$ is a constant. We have to show $f \in L^{\infty,1}$. For this, we note that by the Rademacher theorem (cf. [5, p. 80]), $g := \frac{\partial f}{\partial \bar{z}}$ (in the classical sense) exists almost everywhere on \mathbb{D} . Moreover $g \in L^\infty(\mathbb{D})$. We define f_δ as in the first part. By the Lebesgue dominated convergence theorem, we verify that

$$\frac{\partial f_\delta}{\partial z} = g * \rho_\delta.$$

Since the right-hand side converges to g in $L^1(\mathbb{D}, dA)$ as $\delta \rightarrow 0$, we infer that g is actually the distributional derivative of f . Thus $f \in L^{\infty,1}$. The proof is complete. \square

Remark 1. From this proposition, we can regard every $f \in L^{\infty,1}$ as a Lipschitz continuous function on the closed unit disk $\overline{\mathbb{D}}$. In addition, if $f|_{\partial\mathbb{D}}$ is the radial limit value of f , then $f|_{\partial\mathbb{D}}$ is also Lipschitz continuous on the unit circle $\partial\mathbb{D}$.

Proposition 4. Let $f \in L^{\infty,1}$ and F be the Poisson extension of $f|_{\partial\mathbb{D}}$. Then $F', \bar{F}' \in H^2$ and $F \in \mathbb{C} \oplus \mathcal{D} \oplus \overline{\mathcal{D}}$.

Proof. By Remark 1, there exists a constant $M > 0$ such that $f|_{\partial\mathbb{D}}$ satisfying

$$|f|_{\partial\mathbb{D}}(e^{i\theta_1}) - f|_{\partial\mathbb{D}}(e^{i\theta_2})| \leq M|e^{i\theta_1} - e^{i\theta_2}|$$

for every $\theta_1, \theta_2 \in [0, 2\pi]$. Let $P(r, \theta) = \frac{1-r^2}{1-2r\cos\theta+r^2}$ be the Poisson kernel. Then

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - t) f|_{\partial\mathbb{D}}(e^{it}) dt \quad (z = re^{i\theta}) \quad (2.1)$$

is harmonic in \mathbb{D} , continuous on the closed unit disk and $F|_{\partial\mathbb{D}} = f|_{\partial\mathbb{D}}$.

Differentiating with respect to θ in both sides of (2.1), we obtain

$$izF'(z) + i\overline{z\overline{F'(z)}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{\partial P}{\partial \theta}(r, \theta - t) f|_{\partial \mathbb{D}}(e^{it}) dt.$$

Using integration by parts and the absolute continuity of $f|_{\partial \mathbb{D}}(e^{it})$ we get

$$izF'(z) + i\overline{z\overline{F'(z)}} = \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - t) ie^{it} (f|_{\partial \mathbb{D}})'(e^{it}) dt.$$

Let f'_+ denote the analytic part of $ie^{it}(f|_{\partial \mathbb{D}})'$ and $f'_- = ie^{it}(f|_{\partial \mathbb{D}})' - f'_+$. Since $(f|_{\partial \mathbb{D}})'$ is bounded on $\partial \mathbb{D}$, we infer that f'_+ and f'_- are in $L^2(\partial \mathbb{D})$ [6]. Moreover, since F is harmonic we deduce that

$$F' = \frac{\partial F}{\partial z}, \quad \overline{F'} = \frac{\partial \overline{F}}{\partial \overline{z}}$$

are analytic functions on \mathbb{D} . It follows that $izF'(z)$ and $iz\overline{F'(z)}$ are analytic on \mathbb{D} . Therefore, by comparing analytic and anti-analytic parts in both sides of the last identity we obtain

$$izF'(z) = \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - t) f'_+(e^{it}) dt, \quad \overline{iz\overline{F'(z)}} = \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - t) f'_-(e^{it}) dt.$$

Thus $izF'(z)$ and $iz\overline{F'(z)}$ in H^2 . Since $F', \overline{F'}$ are already analytic functions on \mathbb{D} , we conclude that $F'(z)$ and $\overline{F'(z)}$ are in H^2 as well. Hence $F \in \mathbb{C} \oplus \mathcal{D} \oplus \overline{\mathcal{D}}$. The proof is complete. \square

Put

$$H_1^\infty = \{\varphi \in \mathcal{D}: \varphi' \in H^\infty\}.$$

Clearly $\mathbb{C} \oplus H_1^\infty \oplus \overline{H_1^\infty} \subset L^{\infty,1}$. Moreover, $f \in L^{\infty,1}$ is harmonic if and only if $f \in \mathbb{C} \oplus H_1^\infty \oplus \overline{H_1^\infty}$. Now the following natural question arises: Given $f \in L^{\infty,1}$, let F be the Poisson extension of $f|_{\partial \mathbb{D}}$. Does F belong to $L^{\infty,1}$? The example given at the end of the section provides a negative answer to the above question. Before describing the example, we remark without proof that, by following the same arguments as in Proposition 4, we can obtain the following characterization of the boundary function $f|_{\partial \mathbb{D}}$ in order that its harmonic extension F belongs to $L^{\infty,1}$.

Proposition 5. Let $f \in L^{2,1}$ and F be the Poisson extension of $f|_{\partial \mathbb{D}}$ on \mathbb{D} . Then $F \in L^{\infty,1}$ if and only if $f|_{\partial \mathbb{D}}$ is Lipschitz continuous and both f'_+ and f'_- are in $L^\infty(\partial \mathbb{D})$. Here f'_+ denotes the analytic part of $(f|_{\partial \mathbb{D}})'$ and $f'_- = (f|_{\partial \mathbb{D}})' - f'_+$.

Now we formulate the promised example.

Example 1. Let

$$g(\theta) = \sum_{k \in \mathbb{Z}^*} \frac{1}{k^2} e^{ik\theta},$$

where \mathbb{Z}^* denotes the set of nonzero integers. Then it is easy to see that $g \in C(\partial \mathbb{D})$. Note that the series

$$\sum_{k \in \mathbb{Z}^*} \frac{e^{ik\theta}}{k} = 2i \sum_{k \in \mathbb{Z}_+} \frac{\sin(k\theta)}{k}$$

is the Fourier series of the function $i(\pi - \theta)$ [6]. Thus g' is uniformly bounded on $(0, 2\pi)$. Hence g is Lipschitz on $[0, 2\pi]$, i.e., there exists a constant $M > 0$ such that

$$|g(\theta_1) - g(\theta_2)| \leq M|\theta_1 - \theta_2| \quad \forall \theta_1, \theta_2 \in [0, 2\pi].$$

Now we apply a result of McShane (cf. Proposition 1 in [1] or p. 80 in [5]) to find a function f on $\overline{\mathbb{D}}$ such that $f|_{\partial \mathbb{D}} = g$ and that f is Lipschitz with the same constant M . Using Proposition 3, we conclude $f \in L^{\infty,1}$.

Next we prove that

$$F(z) = \sum_{k \in \mathbb{Z}_+} \frac{z^k}{k^2} + \sum_{k \in \mathbb{Z}_+} \frac{\overline{z}^k}{k^2},$$

the Poisson extension of g , is not in $L^{\infty,1}$. Indeed,

$$zF'(z) = \sum_{k \in \mathbb{Z}_+} \frac{z^k}{k} = -\log(1-z).$$

Therefore F' is not bounded on \mathbb{D} . It follows that $F \notin L^{\infty,1}$ by Proposition 3 (this also follows from Proposition 5, since neither $g'_+ := i \sum_{k \in \mathbb{Z}_+} \frac{e^{ik\theta}}{k}$ nor $g'_- := g' - g'_+$ belongs to $L^\infty(\partial\mathbb{D})$).

3. Main results

Given a function f in $L^2(\mathbb{D}, dA)$, we have the following polar decomposition (cf. [4])

$$f(re^{i\theta}) = \sum_{k \in \mathbb{Z}} e^{ik\theta} f_k(r)$$

for almost all $r \in [0, 1)$, where $f_k(r) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-ik\theta} d\theta$, and

$$\|f\|_{L^2}^2 = 2 \sum_{k \in \mathbb{Z}} \int_0^1 |f_k(r)|^2 r dr < \infty.$$

Here $\|\cdot\|_{L^2}$ denotes the $L^2(\mathbb{D}, dA)$ -norm. Moreover, if $f \in L^{2,1}$, then by the same argument as in [3, Lemma 2.3], using Proposition 1, we can check that $\sum_{|k| \leq N} e^{ik\theta} f_k(r)$ converges to f in $L^{2,1}$ as N tends to infinity.

We first give a decomposition of the Sobolev space $L^{2,1}$. Let $\Omega = \Omega_0 + \mathbb{C}$, where

$$\Omega_0 = \left\{ \sum_{k \in \mathbb{Z}} [f_k(r) - f_k(1)r^{|k|}] e^{ik\theta} : f(re^{i\theta}) = \sum_{k \in \mathbb{Z}} e^{ik\theta} f_k(r) \in L^{\infty,1} \right\}.$$

Notice that the quantities $f_k(1)$ are well defined for all $f \in L^{2,1}$ in view of Proposition 1 (see the argument before Proposition 2).

Theorem 1. Let Δ_0 denote the closure of Ω_0 in the space $L^{2,1}$ and $\Delta = \Delta_0 + \mathbb{C}$. Then $L^{2,1} = \Delta \oplus \mathcal{D} \oplus \overline{\mathcal{D}}$. Moreover,

$$\Delta_0 = \left\{ \sum_{k \in \mathbb{Z}} [f_k(r) - f_k(1)r^{|k|}] e^{ik\theta} : f(re^{i\theta}) = \sum_{k \in \mathbb{Z}} e^{ik\theta} f_k(r) \in L^{2,1} \right\}. \quad (3.1)$$

Proof. First we show that $\Omega_0 \perp \mathcal{D}$ and $\Omega_0 \perp \overline{\mathcal{D}}$. For $n \in \mathbb{Z}_+$, we have

$$\left\langle \sum_{k \in \mathbb{Z}} f_k(1)r^{|k|} e^{ik\theta}, z^n \right\rangle_{\frac{1}{2}} = nf_n(1).$$

Since $\frac{\partial}{\partial z} = \frac{1}{2} e^{-i\theta} \left(\frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \theta} \right)$, we get

$$\frac{\partial}{\partial z} [f_k(r) e^{ik\theta}] = \frac{1}{2} e^{i(k-1)\theta} \left[f'_k(r) + \frac{k}{r} f_k(r) \right].$$

Observe that f_k is absolutely continuous for every k by Proposition 3, so we have

$$\begin{aligned} \left\langle \sum_{k \in \mathbb{Z}} f_k(r) e^{ik\theta}, z^n \right\rangle_{\frac{1}{2}} &= \left\langle \sum_{k \in \mathbb{Z}} \frac{\partial}{\partial z} (f_k e^{ik\theta}), n z^{n-1} \right\rangle_{L^2} = \sum_{k \in \mathbb{Z}} \left\langle \frac{1}{2} e^{i(k-1)\theta} \left[f'_k + \frac{k}{r} f_k \right], n r^{n-1} e^{i(n-1)\theta} \right\rangle_{L^2} \\ &= \left\langle \frac{1}{2} e^{i(n-1)\theta} \left[f'_n + \frac{n}{r} f_n \right], n r^{n-1} e^{i(n-1)\theta} \right\rangle_{L^2} = \int_0^1 \left(f'_n + \frac{n}{r} f_n \right) n r^n dr = n f_n r^n \Big|_0^1 = n f_n(1). \end{aligned}$$

It follows that

$$\left\langle \sum_{k \in \mathbb{Z}} [f_k(r) - f_k(1)r^{|k|}] e^{ik\theta}, z^n \right\rangle_{\frac{1}{2}} = 0.$$

Since $\frac{\partial}{\partial \bar{z}} = \frac{1}{2} e^{i\theta} \left(\frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \theta} \right)$, by an analogous argument we can prove $\Omega_0 \perp \overline{\mathcal{D}}$. The details are omitted.

By combining the last result with Proposition 4, we infer $L^{\infty,1} \subset \Omega \oplus \mathcal{D} \oplus \overline{\mathcal{D}}$. Since the set of smooth functions with compact support is dense in $L^{2,1}$ (cf. [9]), we get the required decomposition for $L^{2,1}$.

Finally, by Proposition 2, $T_{f-F}(z^n) = 0$ for every $n \in \mathbb{Z}_+$ if $f \in L^{\infty,1}$ and F is the Poisson extension of $f|_{\partial\mathbb{D}}$. Clearly, if $f \in \Delta_0$, then for every $n \in \mathbb{Z}_+$, $T_f(z^n) = 0$. It follows that for $f \in L^{2,1}$, there exists a harmonic function $F \in \mathbb{C} \oplus \mathcal{D} \oplus \overline{\mathcal{D}}$ such that $T_f(z^n) = T_F(z^n)$, $n \in \mathbb{Z}_+$. Moreover, Proposition 2 shows that F is the Poisson extension of $f|_{\partial\mathbb{D}}$. Hence the set in the right side of (3.1) equals to Δ_0 . The proof is complete. \square

Remark 2. From the above theorem, we see easily that if $f \in L^{2,1}$, then $f \in \Delta_0$ if and only if $f|_{\partial\mathbb{D}} = 0$ and $f \in \Delta$ if and only if $f|_{\partial\mathbb{D}}$ is a constant.

The following theorem asserts that Toeplitz operator or small Hankel operator with symbol in $L^{\infty,1}$ is just a generalization with the harmonic symbol in $\mathbb{C} \oplus \mathcal{D} \oplus \overline{\mathcal{D}}$.

Theorem 2. Let $f \in L^{\infty,1}$ and F be the Poisson extension of $f|_{\partial\mathbb{D}}$. Then

- (1) T_F is bounded on \mathcal{D} and $T_F = T_f$;
- (2) Γ_F is bounded on \mathcal{D} and $\Gamma_F = \Gamma_f$.

Proof. (1) By Proposition 2, it suffice to show T_F is bounded on \mathcal{D} . First we note that $F \in L^\infty(\mathbb{D})$ and $F' \in H^2$ by Proposition 4.

For $g, h \in \mathcal{D}$, we have

$$\langle T_F g, h \rangle_{\frac{1}{2}} = \langle Fg, h \rangle_{\frac{1}{2}} = \left\langle \frac{\partial(Fg)}{\partial z}, \frac{\partial h}{\partial z} \right\rangle_{L^2} = \langle F'g, h' \rangle_{L^2} + \langle Fg', h' \rangle_{L^2}.$$

This implies

$$\begin{aligned} |\langle T_F g, h \rangle_{\frac{1}{2}}| &\leq \|F'g\|_{L^2} \|h'\|_{L^2} + \|Fg'\|_{L^2} \|h'\|_{L^2} \\ &\leq \|F'g\|_{L^2} \|h\|_{\mathcal{D}} + \|F\|_{\infty} \|g\|_{\mathcal{D}} \|h\|_{\mathcal{D}}. \end{aligned}$$

Let $g(z) = \sum_{n \in \mathbb{Z}_+} a_n z^n$ and $F'(z) = \sum_{n \geq 0} b_n z^n$. Then $\|g\|_{\mathcal{D}}^2 = \sum_{n \in \mathbb{Z}_+} n|a_n|^2 < \infty$ and $\|F'\|_{H^2}^2 = \sum_{n \geq 0} |b_n|^2 < \infty$. Observe that

$$\begin{aligned} \|F'g\|_{L^2}^2 &= \int_{\mathbb{D}} |g(z)F'(z)|^2 dA(z) = \int_{\mathbb{D}} \left| \sum_{n \in \mathbb{Z}_+} a_n z^n \sum_{n \geq 0} b_n z^n \right|^2 dA(z) \\ &= \int_{\mathbb{D}} \left| \sum_{n \in \mathbb{Z}_+} \left(\sum_{k=0}^{n-1} a_{n-k} b_k \right) z^n \right|^2 dA(z) = \sum_{n \in \mathbb{Z}_+} \frac{1}{n+1} \left| \sum_{k=0}^{n-1} a_{n-k} b_k \right|^2. \end{aligned}$$

Thus using the Cauchy–Schwartz inequality, we obtain

$$\begin{aligned} \|F'g\|_{L^2}^2 &\leq \sum_{n \in \mathbb{Z}_+} \frac{1}{n+1} \left(n \sum_{k=0}^{n-1} |a_{n-k} b_k|^2 \right) \\ &= \sum_{n \in \mathbb{Z}_+} \frac{n}{n+1} \sum_{k=0}^{n-1} |a_{n-k}|^2 |b_k|^2 \leq \sum_{n \in \mathbb{Z}_+} \sum_{k=0}^{n-1} |a_{n-k}|^2 |b_k|^2 \\ &= \sum_{n \in \mathbb{Z}_+} |a_n|^2 \sum_{n \geq 0} |b_n|^2 \leq \|g\|_{\mathcal{D}}^2 \|F'\|_{H^2}^2. \end{aligned}$$

It follows that

$$|\langle T_F g, h \rangle_{\frac{1}{2}}| \leq (\|F'\|_{H^2} + \|F\|_{\infty}) \|g\|_{\mathcal{D}} \|h\|_{\mathcal{D}}.$$

Therefore T_F is bounded on \mathcal{D} .

- (2) Let $g(z) = \sum_{n \in \mathbb{Z}_+} a_n z^n \in \mathcal{D}$ and $F(z) = \sum_{n \geq 0} b_n z^n + \sum_{n \in \mathbb{Z}_+} b_{-n} \bar{z}^n$. Since

$$P(\bar{z}^n z^m) = T_{\bar{z}^n} z^m = z^{m-n}$$

when $m > n > 0$ and

$$P(\bar{z}^n z^m) = 0$$

when $0 < m \leq n$, thus

$$\begin{aligned}\Gamma_F g &= P(J(Fg)) = P(F(\bar{z})g(\bar{z})) = P\left(\sum_{n \in \mathbb{Z}_+} a_n \bar{z}^n \sum_{n \in \mathbb{Z}_+} b_{-n} z^n\right) \\ &= \sum_{m, n \in \mathbb{Z}_+} a_n b_{-m} P(\bar{z}^n z^m) = \sum_{m > n > 0} a_n b_{-m} z^{m-n} \\ &= \sum_{k \in \mathbb{Z}_+} \left(\sum_{n \in \mathbb{Z}_+} a_n b_{-(n+k)} \right) z^k.\end{aligned}$$

Again using the Cauchy–Schwartz inequality, we have

$$\begin{aligned}\|\Gamma_F g\|_{\mathcal{D}}^2 &= \left\| \sum_{k \in \mathbb{Z}_+} \left(\sum_{n \in \mathbb{Z}_+} a_n b_{-(n+k)} \right) z^k \right\|_{\mathcal{D}}^2 = \sum_{k \in \mathbb{Z}_+} k \left| \sum_{n \in \mathbb{Z}_+} a_n b_{-(n+k)} \right|^2 \\ &\leq \sum_{k \in \mathbb{Z}_+} k \sum_{n \in \mathbb{Z}_+} n |a_n|^2 \sum_{n \in \mathbb{Z}_+} \frac{|b_{-(n+k)}|^2}{n} \\ &= \|g\|_{\mathcal{D}}^2 \sum_{n \in \mathbb{Z}_+} \frac{1}{n} \sum_{k \in \mathbb{Z}_+} (n+k)^2 |b_{-(n+k)}|^2 \frac{k}{(n+k)^2} \\ &\leq \|g\|_{\mathcal{D}}^2 \sum_{n \in \mathbb{Z}_+} \frac{1}{n^2} \sum_{k \in \mathbb{Z}_+} (n+k)^2 |b_{-(n+k)}|^2.\end{aligned}$$

Because $\bar{F}' \in H^2$ and $\|\bar{F}'\|_{H^2}^2 = \sum_{n \geq 0} n^2 |b_{-n}|^2 < \infty$, we have

$$\|\Gamma_F g\|_{\mathcal{D}}^2 \leq \|g\|_{\mathcal{D}}^2 \sum_{n \in \mathbb{Z}_+} \frac{1}{n^2} \|\bar{F}'\|_{H^2}^2 < \infty.$$

So Γ_F is bounded on \mathcal{D} .

It remains to show $\Gamma_f(z^n) = \Gamma_F(z^n)$ for every $n \in \mathbb{Z}_+$. By Remark 2, it is easy to see $J\Omega_0 \subset \Omega_0$, and $z^n \Omega_0 \subset \Omega_0$, $\bar{z}^n \Omega_0 \subset \Omega_0$ for $n \in \mathbb{Z}_+ \cup \{0\}$. On the other hand, since $f - F \in \Omega_0$ and $P|_{\Omega_0} = 0$, we have

$$\Gamma_f(z^n) = P(J(fz^n)) = P(J((f - F)z^n)) + \Gamma_F(z^n) = \Gamma_F(z^n) \quad \forall n \in \mathbb{Z}_+.$$

The proof is complete. \square

Remark 3. In [2], it was claimed that the Toeplitz operator T_F with the symbol $F(z) = \sum_{k \in \mathbb{Z}_+} \frac{z^k}{k^\alpha}$ is an unbounded operator. This is false, since for $F(z) = \sum_{k \in \mathbb{Z}_+} \frac{z^k}{k^\alpha}$ or $F(z) = \sum_{k \in \mathbb{Z}^*} \frac{r^{|k|} e^{ik\theta}}{k^\alpha}$, by the proof of Theorem 2 (1), T_F is bounded on \mathcal{D} when $\alpha > \frac{3}{2}$.

By combining Theorem 2 with [7, Theorem 5] and [12, Theorem 2.9], we have the following two theorems on commutativity of Toeplitz and Hankel operators with symbols in $L^{\infty,1}$.

Theorem 3. Let $f, g \in L^{\infty,1}$. Then the following assertions hold.

- (a) $T_f T_g = T_g T_f$ if and only if $f, g \in \Omega \oplus \mathcal{D}$ or $f, g \in \Omega \oplus \bar{\mathcal{D}}$ or a nontrivial linear combination of f, g belongs to Ω .
- (b) $T_f T_g = T_{fg}$ on \mathcal{D} if and only if $f \in \Omega \oplus \bar{\mathcal{D}}$ or $g \in \Omega \oplus \mathcal{D}$.

Theorem 4. Let $f, g \in L^{\infty,1}$. Then $\Gamma_f \Gamma_g = \Gamma_g \Gamma_f$ if and only if there exists a constant c such that $f - cg \in \Omega \oplus \mathcal{D} \oplus \mathbb{C} \cdot \bar{z}$.

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