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On the error-sum function of tent map base series

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ABSTRACT

We first introduce tent map base series. The tent map base series is special case of generalized Lüroth series which has the tent map as a base map. Then we study some elementary properties of its error-sum function, and show that the function is continuous.

1. Introduction

Over the last years a great deal of attention has been paid to the study of sets defined in terms of arithmetic properties of numbers. This is a renewed interest in problems considered in the half of the 20th century. In the present paper we study a special case of the so-called *generalized Lüroth expansion* (GLS). GLS is a representation of a real number $x \in (0,1) \cap \mathbb{Q}^c$ by an infinite series. In the same spirit as the usual continued fractions, by truncating the series it is possible to define the so-called approximants. For instance, if the generalized expansion of the number x is given by

$$x = [a_1(x), a_2(x), \ldots],$$

then the n-th approximant is defined by $p_n(x)/q_n(x) := [a_1(x) \cdots a_n(x)]$. In the continued fraction case this procedure yields the best possible rational approximations to an irrational number. Ridley and Petruska [2] (in the context of the usual continued fractions) defined the *error-sum function* as

$$P(x) := \sum_{n=1}^{\infty} \left(x - \frac{p_n(x)}{q_n(x)} \right)$$

(actually, the definition was given in the circle, and the summand had the form $xq_n(x) - p_n(x)$). Therefore, this function measures the accumulative error in the approximation of an irrational by its approximants.

In the context of the continued fraction Ridley and Petruska [2] showed that the graph of the error-sum is of fractal nature and they bounded above (by 3/2) the Hausdorff dimension of its graph. Recently, Shen and Wu [3] studied the same problem in the context of the Lüroth expansion and proved that the Hausdorff dimension of the graph of the corresponding error-sum function is equal to one. In all these cases the function is not continuous in a dense set. We consider a particular case of generalized Lüroth expansion which has the tent map as a base map and study the properties of the corresponding error-sum function. The main result is that, in this setting, the error-sum function is continuous, which is completely different from the corresponding properties of other series mentioned.

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2. Tent map base series

To dynamically generate a generalized Lüroth series of x (see [1]), begin with a partition of $[0,1]$ given by a finite or countable collection of intervals, with the sum of the lengths of the intervals equaling one. The digit set is corresponding finite or countably infinite, and the associated map is linear and bijective onto $[0,1]$ on each subinterval. Thus GLS includes n-ary and Lüroth series, but also includes maps such as the tent map.

Let $I_0 = (0, \frac{1}{2}]$, $I_1 = (\frac{1}{2}, 1]$ be two disjoint intervals with length $L_0 = L_1 = \frac{1}{2}$, such that $L_0 + L_1 = 1$. The digit set is $\mathcal{D} = \{0, 1\}$.

Furthermore, let $I_\infty := [0,1] \backslash (I_0 \cup I_1) = \{0\}$, and define the maps $T, S : [0,1] \to [0,1]$ by

$$T(x) := \begin{cases} 2x & \text{if } x \in I_0, \\ 2x-1 & \text{if } x \in I_1, \\ 0 & \text{if } x \in I_\infty, \end{cases}$$

$$S(x) := \begin{cases} 1-2x & \text{if } x \in I_0, \\ 2-2x & \text{if } x \in I_1, \\ 0 & \text{if } x \in I_\infty. \end{cases}$$

Define

$$s(x) := 2, \quad \text{if } x \in I_0 \cup I_1,$$

and

$$h(x) := \begin{cases} 0 & \text{if } x \in I_0, \\ 1 & \text{if } x \in I_1, \end{cases}$$

$$s_n = s_n(x) := \begin{cases} s(T^{n-1}(x)) & \text{if } T^{n-1}(x) \notin I_\infty, \\ \infty & \text{if } T^{n-1}(x) \in I_\infty, \end{cases}$$

and

$$h_n = h_n(x) := \begin{cases} h(T^{n-1}(x)) & \text{if } T^{n-1}(x) \notin I_\infty, \\ 1 & \text{if } T^{n-1}(x) \in I_\infty. \end{cases}$$

For $x \in (0,1)$ such that $T^{n-1}(x) \notin I_\infty$, one has

$$T(x) = s(x)x - h(x) = s_1 x - h_1.$$

Inductively we find

$$\begin{aligned} x &= \frac{h_1}{s_1} + \frac{1}{s_1} T(x) \\ &= \frac{h_1}{s_1} + \frac{1}{s_1}\left(\frac{h_2}{s_2} + \frac{1}{s_2} T^2(x)\right) \\ &= \cdots \\ &= \frac{h_1}{s_1} + \frac{h_2}{s_1 s_2} + \cdots + \frac{h_n}{s_1 \cdots s_n} + \frac{1}{s_1 \cdots s_n} T^n(x). \end{aligned} \tag{2.1}$$

Since $S(x) = 1 - T(x) = 1 + h_1 - s_1 x$ for $x \in I_0 \cup I_1$, one finds

$$\begin{aligned} x &= \frac{h_1 + 1}{s_1} - \frac{S(x)}{s_1} \\ &= \frac{h_1+1}{s_1} - \frac{1}{s_1}\left(\frac{h_2+1}{s_2} - \frac{S^2(x)}{s_2}\right) \\ &= \cdots \\ &= \frac{h_1+1}{s_1} - \frac{h_2+1}{s_1 s_2} + \cdots + (-1)^{n-1}\frac{h_n+1}{s_1 \cdots s_n} + (-1)^n \frac{S^n(x)}{s_1 \cdots s_n}. \end{aligned} \tag{2.2}$$

Now let $\varepsilon = (\varepsilon(0), \varepsilon(1)) = (0, 1)$.

We define the map $T_\varepsilon : [0,1] \to [0,1]$ by $T_\varepsilon(x) := \varepsilon(x)S(x) + (1-\varepsilon(x))T(x)$, $x \in [0,1]$, where

$$\varepsilon(x) = \begin{cases} \varepsilon(0) & \text{if } x \in I_0, \\ \varepsilon(1) & \text{if } x \in I_1, \\ 0 & \text{if } x \in I_\infty. \end{cases}$$

Let $\varepsilon_n := \varepsilon(T_\varepsilon^{n-1}(x))$,

$$s_n = s_n(x) := \begin{cases} s(T_\varepsilon^{n-1}(x)) & \text{if } T_\varepsilon^{n-1}(x) \notin I_\infty, \\ \infty & \text{if } T_\varepsilon^{n-1}(x) \in I_\infty, \end{cases}$$

$$h_n = h_n(x) := \begin{cases} h(T_\varepsilon^{n-1}(x)) & \text{if } T_\varepsilon^{n-1}(x) \notin I_\infty, \\ 1 & \text{if } T_\varepsilon^{n-1}(x) \in I_\infty. \end{cases}$$

By (2.1) and (2.2) one finds that

$$\begin{aligned} x &= \frac{h_1+\varepsilon_1}{s_1} + \frac{(-1)^{\varepsilon_1}}{s_1} T_\varepsilon x \\ &= \frac{h_1+\varepsilon_1}{s_1} + \frac{(-1)^{\varepsilon_1}}{s_1}\left(\frac{h_2+\varepsilon_2}{s_2} + \frac{(-1)^{\varepsilon_2}}{s_2} T_\varepsilon^2 x\right) \\ &= \cdots \\ &= \frac{h_1+\varepsilon_1}{s_1} + (-1)^{\varepsilon_1}\frac{h_2+\varepsilon_2}{s_1 s_2} + (-1)^{\varepsilon_1+\varepsilon_2}\frac{h_3+\varepsilon_3}{s_1 s_2 s_3} + \cdots + (-1)^{\varepsilon_1+\cdots+\varepsilon_{n-1}}\frac{h_n+\varepsilon_n}{s_1\cdots s_n} + \frac{(-1)^{\varepsilon_1+\cdots+\varepsilon_n}}{s_1\cdots s_n} T_\varepsilon^n x. \end{aligned}$$

Let $\varepsilon_0 := 0$, then for each $x \in [0,1]$ one has

$$x = \sum_{n=1}^{\infty} (-1)^{\varepsilon_0+\cdots+\varepsilon_{n-1}} \frac{h_n+\varepsilon_n}{s_1\cdots s_n}. \tag{2.3}$$

The expansion (2.3) is called the *tent map base series* of x (see [1]).

For each $x \in [0,1]$ we define its sequence of digits $a_n = a_n(x)$, $n \geqslant 1$, as follows

$$a_n(x) = k \quad \Leftrightarrow \quad T_\varepsilon^{n-1}(x) \in I_k, \quad \text{for } k \in \{0, 1, \infty\}.$$

Notice that for each $x \in [0,1] \setminus I_\infty$ one finds a unique expansion (2.3), and therefore a unique sequence of digits $a_n(x)$, $n \geqslant 1$. Conversely, each sequence of digits $a_n(x) \in \{0,1,\infty\}$ and $a_1(x) \neq \infty$ defines a unique series expansion (2.3). We denote (2.3) by

$$\begin{bmatrix} \varepsilon_1, & \varepsilon_2, & \varepsilon_3, & \cdots, & \varepsilon_n, & \cdots \\ a_1(x), & a_2(x), & a_3(x), & \cdots, & a_n(x), & \cdots \end{bmatrix}_T. \tag{2.4}$$

Since $\varepsilon_n = \varepsilon(a_n(x))$, $n \geqslant 1$, we might as well replace (2.4) by

$$\left[a_1(x), a_2(x), a_3(x), \ldots, a_n(x), \ldots\right]_T. \tag{2.5}$$

No new information is obtained using (2.5) instead of (2.4).

For each $n \geqslant 1$ and $1 \leqslant i \leqslant n$ one has $s_i \geqslant 2 > 1$, and $|T_\varepsilon^n(x)| \leqslant 1$. Thus,

$$\left|x - \frac{p_n(x)}{q_n(x)}\right| = \frac{|T_\varepsilon^n(x)|}{s_1\cdots s_n} \leqslant \left(\frac{1}{2}\right)^n \to 0 \quad \text{as } n \to \infty,$$

where

$$\frac{p_n(x)}{q_n(x)} = (-1)^{\varepsilon_0}\frac{h_1+\varepsilon_1}{s_1} + (-1)^{\varepsilon_0+\varepsilon_1}\frac{h_2+\varepsilon_2}{s_1 s_2} + (-1)^{\varepsilon_0+\varepsilon_1+\varepsilon_2}\frac{h_3+\varepsilon_3}{s_1 s_2 s_3} + \cdots + (-1)^{\varepsilon_0+\varepsilon_1+\cdots+\varepsilon_{n-1}}\frac{h_n+\varepsilon_n}{s_1\cdots s_n}$$

and $q_n(x) = s_1\cdots s_n$.

We denote the n-th approximant $\frac{p_n(x)}{q_n(x)}$ by

$$\frac{p_n(x)}{q_n(x)} = \left[a_1(x), a_2(x), \ldots, a_n(x)\right]_T.$$

Then

$$x = \frac{p_n(x)}{q_n(x)} + \frac{(-1)^{\varepsilon_0+\varepsilon_1+\cdots+\varepsilon_n}}{s_1\cdots s_n} T_\varepsilon^n(x).$$

For any $x \in [0,1]$, define $P(x) := \sum_{n=1}^{\infty}(x - \frac{p_n(x)}{q_n(x)})$ and we call $P(x)$ the *error-sum function* of tent map base series.

3. Some basic properties of $P(x)$

For any $n \geqslant 1$, let

$$D_n := \{(\sigma_1, \sigma_2, \ldots, \sigma_n) \in \{0,1\}^n\}, \qquad D_\infty := \{(\sigma_1, \sigma_2, \ldots, \sigma_n, \ldots)\colon \sigma_n \in \{0, 1, \infty\}\}.$$

Define $D := \bigcup_{n=0}^{\infty} D_n$ $(D_0 := \phi)$.

For any $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in D_n$, and $\sigma_0 := 0$, write

$$A_\sigma := \sum_{k=1}^{n} (-1)^{\sigma_0 + \cdots + \sigma_{k-1}} \frac{h_{\sigma_k} + \sigma_k}{s_{\sigma_1} \cdots s_{\sigma_k}}, \tag{3.1}$$

$$B_\sigma := A_\sigma + (-1)^{\sigma_0 + \cdots + \sigma_n} \frac{1}{s_{\sigma_1} \cdots s_{\sigma_n}}. \tag{3.2}$$

Finally, define $E := \{A_\sigma, B_\sigma \mid \sigma \in D_n,\ n \geqslant 1\}$.

We use J_σ to denote the following subset of $[0,1]$:

$$J_\sigma := \{x \in (0,1]\colon a_1(x) = \sigma_1,\ a_2(x) = \sigma_2,\ \ldots,\ a_n(x) = \sigma_n\}.$$

We can get the following result:

$J_\sigma = (A_\sigma, B_\sigma]$ when $\sigma_1 + \cdots + \sigma_n$ is even, and $J_\sigma = (B_\sigma, A_\sigma]$ when $\sigma_1 + \cdots + \sigma_n$ is odd.

Lemma 3.1.

(i) *For any* $x \in (0,1]$, $|P(x)| \leqslant 1$.

(ii) *For any* $n \geqslant 1$, $P(x) = \sum_{k=1}^{n}(x - \frac{p_k(x)}{q_k(x)}) + (-1)^{\sigma_1 + \cdots + \sigma_n} \frac{P(T_\varepsilon^n(x))}{s_{\sigma_1} \cdots s_{\sigma_n}}$.

Proof. (i) For any $x \in (0,1]$

$$\begin{aligned}
|P(x)| &= \left| \sum_{n=1}^{\infty} \left(x - \frac{p_n(x)}{q_n(x)} \right) \right| \\
&\leqslant \left| x - \frac{p_1(x)}{q_1(x)} \right| + \left| x - \frac{p_2(x)}{q_2(x)} \right| + \cdots + \left| x - \frac{p_n(x)}{q_n(x)} \right| + \cdots \\
&= \left| (-1)^{\sigma_1} \frac{T_\varepsilon(x)}{s_{\sigma_1}} \right| + \left| (-1)^{\sigma_1 + \sigma_2} \frac{T_\varepsilon^2(x)}{s_{\sigma_1} s_{\sigma_2}} \right| + \cdots + \left| (-1)^{\sigma_1 + \cdots + \sigma_n} \frac{T_\varepsilon^n(x)}{s_{\sigma_1} \cdots s_{\sigma_n}} \right| + \cdots \\
&\leqslant \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} + \cdots \\
&= 1.
\end{aligned}$$

(ii) For any $n \geqslant 1$

$$\begin{aligned}
P(x) &= \sum_{k=1}^{n} \left(x - \frac{p_k(x)}{q_k(x)} \right) + \sum_{k=n+1}^{\infty} \left(x - \frac{p_k(x)}{q_k(x)} \right) \\
&= \sum_{k=1}^{n} \left(x - \frac{p_k(x)}{q_k(x)} \right) + \sum_{k=n+1}^{\infty} \frac{(-1)^{\sigma_1 + \cdots + \sigma_k}}{s_{\sigma_1} \cdots s_{\sigma_k}} T_\varepsilon^k(x) \\
&= \sum_{k=1}^{n} \left(x - \frac{p_k(x)}{q_k(x)} \right) + \frac{(-1)^{\sigma_1 + \cdots + \sigma_n}}{s_{\sigma_1} \cdots s_{\sigma_n}} \sum_{j=1}^{\infty} (-1)^{\sigma_{n+1} + \cdots + \sigma_{n+j}} \frac{T_\varepsilon^{n+j}(x)}{s_1(T_\varepsilon^n(x)) \cdots s_j(T_\varepsilon^n(x))} \\
&= \sum_{k=1}^{n} \left(x - \frac{p_k(x)}{q_k(x)} \right) + \frac{(-1)^{\sigma_1 + \cdots + \sigma_n}}{s_{\sigma_1} \cdots s_{\sigma_n}} \sum_{i=1}^{\infty} \left(T_\varepsilon^n(x) - \frac{p_i(T_\varepsilon^n(x))}{q_i(T_\varepsilon^n(x))} \right) \\
&= \sum_{k=1}^{n} \left(x - \frac{p_k(x)}{q_k(x)} \right) + (-1)^{\sigma_1 + \cdots + \sigma_n} \frac{P(T_\varepsilon^n(x))}{s_{\sigma_1} \cdots s_{\sigma_n}}. \qquad \square
\end{aligned}$$

Corollary 3.2. *$P(x)$ is bounded.*

Theorem 3.3. *Let $E' = E\backslash\{1\}$. $P(x)$ is continuous on E'.*

Proof. For any $n \geqslant 1$ and $\sigma \in D$, write $x_1 = A_\sigma$, $x_2 = B_\sigma$, where A_σ, B_σ are defined by (3.1), (3.2).

We can take $\sigma_n \neq 0$, so

$$\begin{aligned} x_1 &= \big[a_1(x), \ldots, a_{n-1}(x), a_n(x)\big]_T \\ &= [\sigma_1, \ldots, \sigma_{n-1}, 1]_T. \end{aligned}$$

We claim that $x_1 = [\sigma_1, \ldots, \sigma_{n-2}, 0, 1]_T = [\sigma_1, \ldots, \sigma_{n-2}, 1, 1]_T$.

We first prove the claim.

We note that

$$\begin{aligned} [\sigma_1, \ldots, \sigma_{n-2}, 0, 1]_T &= \frac{p_{n-2}(x_1)}{q_{n-2}(x_1)} + (-1)^{\sigma_1+\cdots+\sigma_{n-2}} \frac{h_{\sigma_{n-1}} + \sigma_{n-1}}{s_{\sigma_1} \cdots s_{\sigma_{n-1}}} + (-1)^{\sigma_1+\cdots+\sigma_{n-1}} \frac{h_{\sigma_n} + \sigma_n}{s_{\sigma_1} \cdots s_{\sigma_n}} \\ &= \frac{p_{n-2}(x_1)}{q_{n-2}(x_1)} + (-1)^{\sigma_1+\cdots+\sigma_{n-2}} \frac{0}{2^{n-1}} + (-1)^{\sigma_1+\cdots+\sigma_{n-2}+0} \frac{1+1}{2^n} \\ &= \frac{p_{n-2}(x_1)}{q_{n-2}(x_1)} + (-1)^{\sigma_1+\cdots+\sigma_{n-2}} \frac{1}{2^{n-1}}. \end{aligned}$$

On the other hand

$$\begin{aligned} [\sigma_1, \ldots, \sigma_{n-2}, 1, 1]_T &= \frac{p_{n-2}(x_1)}{q_{n-2}(x_1)} + (-1)^{\sigma_1+\cdots+\sigma_{n-2}} \frac{1+1}{2^{n-1}} + (-1)^{\sigma_1+\cdots+\sigma_{n-2}+1} \frac{1+1}{2^n} \\ &= \frac{p_{n-2}(x_1)}{q_{n-2}(x_1)} + (-1)^{\sigma_1+\cdots+\sigma_{n-2}} \frac{1}{2^{n-1}}. \end{aligned}$$

We distinguish two cases to prove $P(x)$ is continuous on E'.

Case I. $\sigma_1 + \cdots + \sigma_{n-2}$ is even, then let

$$\begin{aligned} x_1 &= [\sigma_1, \ldots, \sigma_{n-2}, 0, 1]_T \\ &= \frac{h_{\sigma_1} + \sigma_1}{2} + (-1)^{\sigma_1} \frac{h_{\sigma_2} + \sigma_2}{2^2} + \cdots + \frac{1}{2^{n-1}}, \\ x_2 &= [\sigma_1, \ldots, \sigma_{n-2}, 0, 1]_T + (-1)^{\sigma_1+\cdots+\sigma_{n-2}+0+1} \frac{1}{2^n} \\ &= \frac{h_{\sigma_1} + \sigma_1}{2} + (-1)^{\sigma_1} \frac{h_{\sigma_2} + \sigma_2}{2^2} + \cdots + \frac{1}{2^{n-1}} - \frac{1}{2^n}, \end{aligned}$$

and $J_\sigma = (B_\sigma, A_\sigma] = (x_2, x_1]$.

Let

$$\begin{aligned} 0 < \alpha &= [\tau_1, \tau_2, \ldots]_T \\ &= [\overbrace{0, \ldots, 0}, \tau_{N_1}, \overbrace{0, \ldots, 0}, \tau_{N_2}, \ldots]_T \\ &= \frac{1}{2^{N_1-1}} - \frac{1}{2^{N_2-1}} + \cdots \\ &= \frac{1}{2^{N_1-1}} - \left(\frac{1}{2^{N_2-1}} - \frac{1}{2^{N_3-1}}\right) - \cdots \\ &\leqslant \frac{1}{2^{N_1-1}} \end{aligned} \tag{3.3}$$

be small enough, where $\overbrace{0, \ldots, 0}$ denotes these elements are all 0.

We assume that $N_1 > n$.

Let

$$\begin{aligned} x_1' &= x_1 - \alpha \\ &= [\sigma_1, \ldots, \sigma_{n-2}, 0, 1, \overbrace{0, \ldots, 0}, \tau_{N_1}, \overbrace{0, \ldots, 0}, \tau_{N_2}, \ldots]_T \\ &= \frac{p_n(x_1)}{q_n(x_1)} + (-1)^{\sigma_1+\cdots+\sigma_{n-2}+1} \frac{1+1}{2^{N_1}} + (-1)^{\sigma_1+\cdots+\sigma_{n-2}+1+1} \frac{1+1}{2^{N_2}} + \cdots \end{aligned}$$

$$= \frac{p_n(x_1)}{q_n(x_1)} - \frac{1}{2^{N_1-1}} + \frac{1}{2^{N_2-1}} - \cdots$$
$$= \frac{p_n(x_1)}{q_n(x_1)} - \alpha.$$

The error-sum function $P(x_1) = \sum_{i=1}^{\infty}(x_1 - \frac{p_i(x_1)}{q_i(x_1)}) = \sum_{i=1}^{n}(x_1 - \frac{p_i(x_1)}{q_i(x_1)})$ and

$$\begin{aligned}
P(x_1') &= \sum_{i=1}^{\infty}\left(x_1' - \frac{p_i(x_1')}{q_i(x_1')}\right) \\
&= \sum_{i=1}^{n}\left(x_1' - \frac{p_i(x_1)}{q_i(x_1)}\right) + \sum_{i=n+1}^{N_1-1}\left(x_1' - \frac{p_n(x_1)}{q_n(x_1)}\right) - \frac{P(T_\varepsilon^{N_1-1}x_1')}{s_{\sigma_1}\cdots s_{\sigma_{N_1-1}}} \\
&= \sum_{i=1}^{n}\left(x_1' - \frac{p_i(x_1)}{q_i(x_1)}\right) + (N_1-n-1)\left(x_1' - \frac{p_n(x_1)}{q_n(x_1)}\right) - \frac{P(T_\varepsilon^{N_1-1}x_1')}{2^{N_1-1}} \\
&= P(x_1) + \sum_{i=1}^{n}\left(x_1' - x_1\right) - (N_1-n-1)\alpha - \frac{P(T_\varepsilon^{N_1-1}x_1')}{2^{N_1-1}}.
\end{aligned}$$

Let $\alpha \to 0^+$, then $N_1 \to +\infty$. From 3.3 and Corollary 3.2, we have

$$\lim_{x_1' \to x_1^-} P(x_1') = P(x_1).$$

Let

$$\begin{aligned}
x_1'' &= x_1 + \alpha \\
&= [\sigma_1, \ldots, \sigma_{n-2}, 1, 1, \overbrace{0, \ldots, 0}, \tau_{N_1}, \overbrace{0, \ldots, 0}, \tau_{N_2}, \ldots]_T \\
&= \frac{p_n(x_1)}{q_n(x_1)} + (-1)^{\sigma_1+\cdots+\sigma_{n-2}+1+1}\frac{1+1}{2^{N_1}} + (-1)^{\sigma_1+\cdots+\sigma_{n-2}+1+1+1}\frac{1+1}{2^{N_2}} + \cdots \\
&= \frac{p_n(x_1)}{q_n(x_1)} + \frac{1}{2^{N_1-1}} - \frac{1}{2^{N_2-1}} + \cdots \\
&= \frac{p_n(x_1)}{q_n(x_1)} + \alpha.
\end{aligned}$$

The error-sum function

$$\begin{aligned}
P(x_1'') &= \sum_{i=1}^{\infty}\left(x_1'' - \frac{p_i(x_1'')}{q_i(x_1'')}\right) \\
&= \sum_{i=1}^{n}\left(x_1'' - \frac{p_i(x_1)}{q_i(x_1)}\right) + \sum_{i=n+1}^{N_1-1}\left(x_1'' - \frac{p_n(x_1)}{q_n(x_1)}\right) + \frac{P(T_\varepsilon^{N_1-1}x_1'')}{s_{\sigma_1}\cdots s_{\sigma_{N_1-1}}} \\
&= P(x_1) + \sum_{i=1}^{n}\left(x_1'' - x_1\right) + (N_1-n-1)\alpha + \frac{P(T_\varepsilon^{N_1-1}x_1'')}{2^{N_1-1}}.
\end{aligned}$$

Let $\alpha \to 0^+$, then $N_1 \to +\infty$. From 3.3 and Corollary 3.2, we have

$$\lim_{x_1'' \to x_1^+} P(x_1'') = P(x_1).$$

Thus

$$\lim_{x \to x_1} P(x) = P(x_1).$$

For $x_2 = \frac{h_{\sigma_1}+\sigma_1}{2} + (-1)^{\sigma_1}\frac{h_{\sigma_2}+\sigma_2}{2^2} + \cdots + \frac{1}{2^{n-1}} - \frac{1}{2^n}$ following the same line as above, we have $\lim_{x\to x_2} P(x) = P(x_2)$.

Case II. $\sigma_1 + \cdots + \sigma_{n-2}$ is odd. Following the same way as Case I, we have

$$\lim_{x \to x_1} P(x) = P(x_1) \quad \text{and} \quad \lim_{x \to x_2} P(x) = P(x_2).$$

Therefore $P(x)$ is continuous on E'. $\square$

Proposition 3.4. *For any $n \geqslant 1$, and $\sigma \in D_n$, write*

$$\alpha_1 = \min\{A_\sigma, B_\sigma\}, \qquad \alpha_2 = \max\{A_\sigma, B_\sigma\}.$$

Then for any $x \in J_\sigma$ we have

$$P(\alpha_1) < P(x) \leqslant P(\alpha_2).$$

Proof. For any $x \in J_\sigma$, let $\alpha_1' = x - \alpha_1 = [\overbrace{0, \ldots, 0}, \tau_{M_1}, \overbrace{0, \ldots, 0}, \tau_{M_2}, \ldots]_T > 0$. Note that $M_1 > n$ and

$$\begin{aligned}
P(x) &= \sum_{i=1}^{\infty} \left(x - \frac{p_i(x)}{q_i(x)} \right) \\
&= \sum_{i=1}^{n-1} \left(x - \frac{p_i(x)}{q_i(x)} \right) + \sum_{i=n}^{\infty} \left(x - \frac{p_i(x)}{q_i(x)} \right) \\
&= \sum_{i=1}^{n-1} \left(\alpha_1 - \frac{p_i(\alpha_1)}{q_i(\alpha_1)} \right) + (n-1)(x - \alpha_1) + (M_1 - n - 1)\alpha_1' + \frac{P(T_\varepsilon^{M_1} x)}{2^{M_1}} \\
&> P(\alpha_1).
\end{aligned}$$

Following the same way as above we have $P(x) \leqslant P(\alpha_2)$. □

From Proposition 3.4, we have for any $\sigma \in D_n$,

$$\sup_{x, y \in J_\sigma} \left| P(x) - P(y) \right| \leqslant P(\alpha_2) - P(\alpha_1).$$

Proposition 3.5. *$P(x)$ is continuous on $(0, 1] \backslash E'$.*

Proof. For any $x \in (0, 1] \backslash E'$ and $x \neq 1$, let

$$x = \left[a_1(x), a_2(x), \ldots, a_n(x), \ldots \right]_T$$

be its tent map base expansion. For any $n \geqslant 1$, write $\sigma^{(n)} = (a_1(x), \ldots, a_n(x))$. By Proposition 3.4, for any $y \in J_{\sigma^{(n)}}$, we have $|P(x) - P(y)| \leqslant |P(A_\sigma) - P(B_\sigma)| \leqslant \frac{n+1}{2^n} \to 0$ as $n \to \infty$. □

Corollary 3.6. *$P(x)$ is continuous on $(0, 1]$.*

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