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Trace formulae for the matrix Schrödinger equation with energy-dependent potential

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ABSTRACT

In this paper, we consider the eigenvalue problems for the matrix Schrödinger equation with energy-dependent potential and with separated boundary conditions on the finite interval, and find new trace formulae for the matrix Schrödinger operator.

1. Introduction

In this paper, we will find trace formulae for the following matrix Schrödinger operator $L(P, Q; h, H)$

$$-Y''(x) + [2\lambda P(x) + Q(x)]Y(x) = \lambda^2 Y(x), \quad x \in (0, \pi) \tag{1.1}$$

with Robin boundary conditions

$$Y'(0) - hY(0) = 0 \tag{1.2}$$

and

$$Y'(\pi) + HY(\pi) = 0, \tag{1.3}$$

where λ is a spectral parameter, $Y(x) = [y_k(x)]_{k=\overline{1,d}}$ is a column vector, $P(x) = \mathrm{diag}[p_1(x), \ldots, p_d(x)]$ and $Q(x)$ are $d \times d$ real symmetric matrix-valued functions, h and H are $d \times d$ real symmetric constant matrices, $P \in W_2^2[0, \pi]$ and $Q \in W_2^1[0, \pi]$, where $W_2^k[0, \pi]$ $(k = 1, 2)$ denotes a set whose element is a k-th order continuously differentiable function in $L^2[0, \pi]$.

The study of regularized traces of ordinary differential operators has a long history and there are a large number of papers and books studying this issue. The trace formulae for the scalar differential operators have been found by Gelfand and Levitan [1], Dikii [2], Halberg and Kramer [3] and many other works. The list of the works on this subject is given in [4–6].

A method for calculating trace formulae for general problems involving ordinary differential equations on a finite interval was proposed in [7]. The trace formulae can be used for approximate calculation of the first eigenvalue of an operator [6], and in order to establish necessary and sufficient conditions for a set of complex numbers to be the spectrum of an operator [8].

Afterwards these investigations were continued in many directions, such as Dirac operators, differential operators with abstract operator-valued coefficients, and the case of matrix-valued Sturm–Liouville operators (see, [9–29,5,6,30–33], etc.). Despite the enormous literature on eigenvalue problems for scalar Sturm–Liouville problems, differential operators with operator coefficients and matrix coefficients raise interesting new problems. For differential operators with an operator

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coefficient a similar problem was studied, for example in [10–17,23,27]. For the matrix Sturm–Liouville equation (when $P = 0$ in (1.1)) properties of spectral characteristics were provided in [34–37]. Trace formulae for Sturm–Liouville problems with matrix coefficients were previously considered in [21,28]. Note that there have only a few works on the trace of differential operators with matrix coefficient. In [19,20] the trace of the Sturm–Liouville operator with matrix coefficient has been investigated with the aid of the method of residue computations.

It is pointed out that for $d = 1$ the trace formula for a scalar quadratic operator pencil (1.1) was first studied by Borisov and Freitas in [18]; in the latter papers [32,33] the same was done for other boundary conditions, such as separated boundary conditions, the eigenparameter boundary conditions, and quasiperiodic boundary conditions. However, the trace formula for the matrix Schrödinger operator $L(P, Q; h, H)$ has never been considered before. In this paper, we will discuss the eigenvalue problem for the operator $L(P, Q; h, H)$ and find new trace formulae.

2. Results

New trace formulae for the matrix Schrödinger operator $L(P, Q; h, H)$ are as follows. For simplicity A_{ij} denotes entry of matrix A at the i-th row and the j-th column and $\operatorname{tr} A$ denotes the trace of a matrix A, I_d is a $d \times d$ identity matrix and 0_d is a $d \times d$ zero matrix.

Theorem 2.1. *For the operator $L(P, Q; h, H)$, let $\lambda_n^{(j)}$ $(j = \overline{1, d}, n = \pm 0, \pm 1, \pm 2, \ldots)$ be eigenvalues of the operator $L(P, Q; h, H)$, we have the trace formulae:*

$$\sum_{n=0}^{\infty}\left[\sum_{j=1}^{d}(\lambda_{n,j} + \lambda_{-n,j}) - \frac{2}{\pi}\operatorname{tr}\int_0^{\pi} P(x)\,\mathrm{d}x\right] = \frac{\operatorname{tr}[P(0) + P(\pi)]}{2} - \frac{1}{\pi}\operatorname{tr}\int_0^{\pi} P(x)\,\mathrm{d}x \tag{2.1}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}&\left\{\sum_{j=1}^{d}[\lambda_{n,j}^2 + \lambda_{-n,j}^2 - (\lambda_{n,j}^0)^2 - (\lambda_{-n,j}^0)^2] - \frac{4}{\pi}\operatorname{tr}\left[h + H + \frac{1}{2}\int_0^{\pi}(P^2(x) + Q(x))\mathrm{d}x\right]\right\}\\ &= -\frac{2}{\pi}\operatorname{tr}\left[h + H + \frac{1}{2}\int_0^{\pi}(P^2(x) + Q(x))\mathrm{d}x\right] + \frac{\operatorname{tr}[Q(0) + Q(\pi)]}{2} + \operatorname{tr}[P^2(0) + P^2(\pi)] - \operatorname{tr}[h^2 + H^2],\end{aligned} \tag{2.2}$$

where for $j = \overline{1, d}$,

$$\lambda_{-0,j}^0 = 0, \qquad \lambda_{n,j}^0 = n + \frac{\alpha_j}{\pi} \quad (n = +0, \pm 1, \pm 2, \ldots),\ \alpha_j = \int_0^{\pi} p_j(x)\,\mathrm{d}x.$$

This article is organized as follows. Section 3 is devoted to the representation of the solution to Eq. (1.1). Section 4 contains the analysis of the characteristic determinant. Finally, in Section 5, we present the proof of Theorem 2.1.

3. Representation of the solution to Eq. (1.1)

In this section we will give a representation of the solution to the differential equation (1.1).

Lemma 3.1. *For each $\lambda \in \mathbb{C}$, the solution of the initial value problem*

$$-Y''(x) + [2\lambda P(x) + Q(x)]Y(x) = \lambda^2 Y(x) \tag{3.1}$$

with

$$Y(0, \lambda) - I_d = 0_d = Y'(0, \lambda) - h \tag{3.2}$$

is given, for $x \in [0, \pi]$, by

$$Y(x, \lambda) = \cos(\lambda x - \alpha(x)) + \int_0^x A(x, t)\cos(\lambda t)\,\mathrm{d}t + \int_0^x B(x, t)\sin(\lambda t)\,\mathrm{d}t, \tag{3.3}$$

where the kernels $A(x, t)$, $B(x, t)$ are the solutions of the problem

$$\begin{cases}\dfrac{\partial^2 A(x, t)}{\partial x^2} - 2P(x)\dfrac{\partial B(x, t)}{\partial t} - Q(x)A(x, t) = \dfrac{\partial^2 A(x, t)}{\partial t^2},\\[2mm] \dfrac{\partial^2 B(x, t)}{\partial x^2} + 2P(x)\dfrac{\partial A(x, t)}{\partial t} - Q(x)B(x, t) = \dfrac{\partial^2 B(x, t)}{\partial t^2},\\[2mm] A(0, 0) = h, \qquad B(x, 0) = 0_d, \qquad \left.\dfrac{\partial A(x, t)}{\partial t}\right|_{t=0} = 0_d,\end{cases} \tag{3.4}$$

with $\alpha(x)=\int_0^x P(t)\mathrm{d}t$. *Moreover, there holds*

$$2[\cos\alpha(x)A(x,x)+\sin\alpha(x)B(x,x)]=2h+\int_0^x T_1(t)\,\mathrm{d}t \tag{3.5}$$

and

$$2[\sin\alpha(x)A(x,x)-\cos\alpha(x)B(x,x)]=P(x)-P(0)+\int_0^x T_2(t)\,\mathrm{d}t \tag{3.6}$$

where

$$T_1(x)=P^2(x)+\cos\alpha(x)Q(x)\cos\alpha(x)+\sin\alpha(x)Q(x)\sin\alpha(x)$$

and

$$T_2(x)=\sin\alpha(x)Q(x)\cos\alpha(x)-\cos\alpha(x)Q(x)\sin\alpha(x).$$

Proof. The representation (3.3) can be established using the Paley–Wiener theorem (see [38]) and the kernels $A(x,t)$, $B(x,t)$ have continuous partial derivatives up to order two with respect to x and t and with $\alpha(x)=\int_0^x P(t)\mathrm{d}t$.

From (3.3) we get

$$\begin{aligned}Y'(x,\lambda)\;=\;&-(\lambda-P(x))\sin(\lambda x-\alpha(x))+A(x,x)\cos(\lambda x)+B(x,x)\sin(\lambda x)+\int_0^x A_x(x,t)\cos(\lambda t)\,\mathrm{d}t\\&+\int_0^x B_x(x,t)\sin(\lambda t)\,\mathrm{d}t\end{aligned}$$

and

$$\begin{aligned}Y''(x,\lambda)\;=\;&P'(x)\sin(\lambda x-\alpha(x))-(\lambda-P(x))^2\cos(\lambda x-\alpha(x))+A'(x,x)\cos(\lambda x)+B'(x,x)\sin(\lambda x)\\&-\lambda A(x,x)\sin(\lambda x)+\lambda B(x,x)\cos(\lambda x)+A_x(x,t)|_{t=x}\cos(\lambda x)+B_x(x,t)|_{t=x}\sin(\lambda x)\\&+\int_0^x A_{x^2}(x,t)\cos(\lambda t)\,\mathrm{d}t+\int_0^x B_{x^2}(x,t)\sin(\lambda t)\,\mathrm{d}t.\end{aligned} \tag{3.7}$$

On the other hand, using integration by parts twice, we obtain

$$\begin{aligned}\lambda Y(x,\lambda)\;=\;&\lambda\cos(\lambda x-\alpha(x))+A(x,x)\sin(\lambda x)-B(x,x)\cos(\lambda x)+B(x,0)\\&-\int_0^x A_t(x,t)\sin(\lambda t)\,\mathrm{d}t+\int_0^x B_t(x,t)\cos(\lambda t)\,\mathrm{d}t\\=\;&\lambda\cos(\lambda x-\alpha(x))+A(x,x)\sin(\lambda x)-B(x,x)\cos(\lambda x)+B(x,0)+\frac{1}{\lambda}[A_t(x,t)|_{t=x}\cos(\lambda x)-A_t(x,0)]\\&-\frac{1}{\lambda}\int_0^x A_{t^2}(x,t)\cos(\lambda t)\,\mathrm{d}t+\frac{1}{\lambda}B_t(x,t)|_{t=x}\sin(\lambda x)-\frac{1}{\lambda}\int_0^x B_{t^2}(x,t)\sin(\lambda t)\,\mathrm{d}t.\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8) and using equations

$$\begin{cases}Y''(x)+[\lambda^2-2\lambda P(x)-Q(x)]Y=0,\\Y'(0,\lambda)=h,\end{cases}$$

we obtain

$$A(0,0)=h \tag{3.9}$$

and

$$\begin{aligned}&\lambda B(x,0)-2P(x)B(x,0)-A_t(x,0)+P'(x)\sin(\lambda x-\alpha(x))-(P^2(x)+Q(x))\cos(\lambda x-\alpha(x))\\&+2A'(x,x)\cos(\lambda x)+2B'(x,x)\sin(\lambda x)-2P(x)A(x,x)\sin(\lambda x)+2P(x)B(x,x)\cos(\lambda x)\\&+\int_0^x[A_{x^2}(x,t)-2P(x)B_t(x,t)-Q(x)A(x,t)-A_{t^2}(x,t)]\cos(\lambda t)\,\mathrm{d}t\\&+\int_0^x[B_{x^2}(x,t)+2P(x)A_t(x,t)-Q(x)B(x,t)-B_{t^2}(x,t)]\sin(\lambda t)\,\mathrm{d}t=0.\end{aligned} \tag{3.10}$$

By the Riemann–Lebesgue lemma (3.10) holds for all real λ if and only if

$$B(x,0)=A_t(x,0)=0_d \tag{3.11}$$

and

$$\begin{cases} \dfrac{\partial^2 A(x,t)}{\partial x^2} - 2P(x)\dfrac{\partial B(x,t)}{\partial t} - Q(x)A(x,t) = \dfrac{\partial^2 A(x,t)}{\partial t^2}, \\ \dfrac{\partial^2 B(x,t)}{\partial x^2} + 2P(x)\dfrac{\partial A(x,t)}{\partial t} - Q(x)B(x,t) = \dfrac{\partial^2 B(x,t)}{\partial t^2}, \end{cases} \tag{3.12}$$

with

$$\begin{cases} -P'(x)\sin\alpha(x) - (P^2(x)+Q(x))\cos\alpha(x) + 2A'(x,x) + 2P(x)B(x,x) = 0, \\ P'(x)\cos\alpha(x) - (P^2(x)+Q(x))\sin\alpha(x) + 2B'(x,x) - 2P(x)A(x,x) = 0. \end{cases} \tag{3.13}$$

From (3.13) we have

$$\begin{cases} -\cos\alpha(x)P'(x)\sin\alpha(x) - \cos\alpha(x)(P^2(x)+Q(x))\cos\alpha(x) \\ \qquad + 2\cos\alpha(x)A'(x,x) + 2\cos\alpha(x)P(x)B(x,x) = 0, \\ \sin\alpha(x)P'(x)\cos\alpha(x) - \sin\alpha(x)(P^2(x)+Q(x))\sin\alpha(x) \\ \qquad + 2\sin\alpha(x)B'(x,x) - 2\sin\alpha(x)P(x)A(x,x) = 0, \end{cases}$$

which implies that by adding

$$2\frac{\mathrm{d}}{\mathrm{d}x}[\cos\alpha(x)A(x,x) + \sin\alpha(x)B(x,x)] = T_1(x),$$

where

$$\begin{aligned} T_1(x) &= \cos\alpha(x)(P^2(x)+Q(x))\cos\alpha(x) + \sin\alpha(x)(P^2(x)+Q(x))\sin\alpha(x) \\ &= P^2(x) + \cos\alpha(x)Q(x)\cos\alpha(x) + \sin\alpha(x)Q(x)\sin\alpha(x). \end{aligned} \tag{3.14}$$

Integrating (3.14) and taking into account that $\alpha(0) = 0_d = B(0,0)$ yields

$$2[\cos\alpha(x)A(x,x) + \sin\alpha(x)B(x,x)] = 2h + \int_0^x T_1(t)\,\mathrm{d}t. \tag{3.15}$$

Similarly, we have

$$\begin{cases} -\sin\alpha(x)P'(x)\sin\alpha(x) - \sin\alpha(x)(P^2(x)+Q(x))\cos\alpha(x) \\ \qquad + 2\sin\alpha(x)A'(x,x) + 2\sin\alpha(x)P(x)B(x,x) = 0, \\ \cos\alpha(x)P'(x)\cos\alpha(x) - \cos\alpha(x)(P^2(x)+Q(x))\sin\alpha(x) \\ \qquad + 2\cos\alpha(x)B'(x,x) - 2\cos\alpha(x)P(x)A(x,x) = 0, \end{cases}$$

which implies that by adding

$$2\frac{\mathrm{d}}{\mathrm{d}x}[\sin\alpha(x)A(x,x) - \cos\alpha(x)B(x,x)] = P'(x) + T_2(x),$$

where

$$\begin{aligned} T_2(x) &= \sin\alpha(x)(P^2(x)+Q(x))\cos\alpha(x) - \cos\alpha(x)(P^2(x)+Q(x))\sin\alpha(x) \\ &= \sin\alpha(x)Q(x)\cos\alpha(x) - \cos\alpha(x)Q(x)\sin\alpha(x). \end{aligned}$$

Integrating the above equation yields

$$2[\sin\alpha(x)A(x,x) - \cos\alpha(x)B(x,x)] = P(x) - P(0) + \int_0^x T_2(t)\,\mathrm{d}t. \tag{3.16}$$

Eqs. (3.9), (3.11), (3.12), (3.15) and (3.16) complete the proof of Lemma 3.1. $\square$

4. Analysis of the characteristic determinant

Denote

$$\alpha_j = \int_0^\pi p_j(x)\,\mathrm{d}x, \quad j = \overline{1,d}, \qquad \alpha(\pi) = \mathrm{diag}[\alpha_1, \ldots, \alpha_d].$$

Simple calculations show that the characteristic equation of (1.1)–(1.2) can be reduced to the form $\omega(\lambda) = 0$, where

$$\begin{aligned}\omega(\lambda) &= -[\lambda - P(\pi)]\sin(\lambda\pi - \alpha(\pi)) + \cos(\lambda\pi)A(\pi,\pi) + \sin(\lambda\pi)B(\pi,\pi) + \int_0^\pi A'_x(\pi,t)\cos(\lambda t)\,\mathrm{d}t \\ &\quad + \int_0^\pi B'_x(\pi,t)\sin(\lambda t)\,\mathrm{d}t + H\cos(\lambda\pi - \alpha(\pi)) + H\int_0^\pi A(\pi,t)\cos(\lambda t)\,\mathrm{d}t + H\int_0^\pi B(\pi,t)\sin(\lambda t)\,\mathrm{d}t \\ &= -\lambda\sin(\lambda\pi - \alpha(\pi)) + P(\pi)\sin(\lambda\pi - \alpha(\pi)) + \cos(\lambda\pi)A(\pi,\pi) + \sin(\lambda\pi)B(\pi,\pi) \\ &\quad + H\cos(\lambda\pi - \alpha(\pi)) + O\left(\frac{e^{\tau\pi}}{\lambda}\right), \quad \tau = |\mathrm{Im}\,\lambda|. \end{aligned} \tag{4.1}$$

Using Eq. (3.6)

$$2[\sin\alpha(\pi)A(\pi,\pi) - \cos\alpha(\pi)B(\pi,\pi)] = P(\pi) - P(0) + \int_0^\pi T_2(x)\,\mathrm{d}x$$

and Eq. (3.5)

$$2[\cos\alpha(\pi)A(\pi,\pi) + \sin\alpha(\pi)B(\pi,\pi)] = 2h + \int_0^\pi T_1(x)\,\mathrm{d}x,$$

by calculations we obtain

$$A(\pi,\pi) = \sin\alpha(\pi)\frac{P(\pi) - P(0)}{2} + \frac{\sin\alpha(\pi)}{2}\int_0^\pi T_2(x)\,\mathrm{d}x + \cos\alpha(\pi)\left\{h + \frac{1}{2}\int_0^\pi T_1(x)\,\mathrm{d}x\right\}$$

and

$$B(\pi,\pi) = -\cos\alpha(\pi)\frac{P(\pi) - P(0)}{2} - \frac{\cos\alpha(\pi)}{2}\int_0^\pi T_2(x)\,\mathrm{d}x + \sin\alpha(\pi)\left\{h + \frac{1}{2}\int_0^\pi T_1(x)\,\mathrm{d}x\right\}.$$

Therefore we have

$$\begin{aligned}\cos(\lambda\pi)A(\pi,\pi) + \sin(\lambda\pi)B(\pi,\pi) &= \sin(\lambda\pi - \alpha(\pi))\frac{P(\pi) - P(0)}{2} - \frac{\sin(\lambda\pi - \alpha(\pi))}{2}\int_0^\pi T_2(x)\,\mathrm{d}x \\ &\quad + \cos(\lambda\pi - \alpha(\pi))\left\{h + \frac{1}{2}\int_0^\pi T_1(x)\,\mathrm{d}x\right\}.\end{aligned}$$

Substituting it into Eq. (4.1) yields that

$$\begin{aligned}\omega(\lambda) &= -\lambda\sin(\lambda\pi - \alpha(\pi)) + \frac{P(0) + P(\pi)}{2}\sin(\lambda\pi - \alpha(\pi)) - \frac{\sin(\lambda\pi - \alpha(\pi))}{2}\int_0^\pi T_2(x)\,\mathrm{d}x \\ &\quad + \cos(\lambda\pi - \alpha(\pi))\left\{h + H + \frac{1}{2}\int_0^\pi T_1(x)\,\mathrm{d}x\right\} + O\left(\frac{e^{\tau\pi}}{\lambda}\right).\end{aligned}$$

Denote

$$\omega_{ii} = -\lambda\sin(\lambda\pi - \alpha_i) + \frac{P_{ii}(0) + P_{ii}(\pi)}{2}\sin(\lambda\pi - \alpha_i) + c_i\cos(\lambda\pi - \alpha_i) + O\left(\frac{e^{\tau\pi}}{\lambda}\right),$$

$$c_i = h_{ii} + H_{ii} + \frac{1}{2}\int_0^\pi (P_{ii}^2 + Q_{ii})\,\mathrm{d}x$$

and for $i \neq j$, $\omega_{ij} = O(e^{\tau\pi})$.

Then direct calculation implies that

$$\begin{aligned}\det\omega(\lambda) &= \begin{vmatrix} \omega_{11} & & O(e^{\tau\pi}) & \\ & \omega_{22} & & \\ O(e^{\tau\pi}) & & \ddots & \\ & & & \omega_{dd} \end{vmatrix} \\ &= \prod_{i=1}^{d}\omega_{ii} + O(\lambda^{d-2}e^{\mathrm{d}\tau\pi}) \\ &= \prod_{i=1}^{d}\left[-\lambda\sin(\lambda\pi - \alpha_i) + \frac{P_{ii}(0) + P_{ii}(\pi)}{2}\sin(\lambda\pi - \alpha_i) + c_i\cos(\lambda\pi - \alpha_i) + O\left(\frac{e^{\tau\pi}}{\lambda}\right)\right] + O(\lambda^{d-2}e^{\mathrm{d}\tau\pi})\end{aligned}$$

$$= \prod_{i=1}^{d}(-\lambda \sin(\lambda\pi - \alpha_i)) \left[1 - \frac{P_{ii}(0) + P_{ii}(\pi)}{2\lambda} - \frac{c_i}{\lambda}\cot(\lambda\pi - \alpha_i) + O\left(\frac{1}{\lambda^2}\right)\right] + O(\lambda^{d-2}e^{d\tau\pi})$$

$$= (-\lambda)^d \prod_{i=1}^{d} \sin(\lambda\pi - \alpha_i) \times \left[1 - \frac{\operatorname{tr}(P(0) + P(\pi))}{2\lambda} - \frac{\sum_{i=1}^{d} c_i \cot(\lambda\pi - \alpha_i)}{\lambda}\right] + O(\lambda^{d-2}e^{d\tau\pi}).$$

5. Proof of Theorem 2.1

We only give the proof of Eq. (2.1) in Theorem 2.1. Analogously we can also prove that Eq. (2.2) in Theorem 2.1 holds.

Denote by $C_{n,j}$ the circles of radius $\varepsilon, 0 < \varepsilon < \frac{1}{2}$, centred at the origin $\lambda^0_{n,j}$, $n = \pm 0, \pm 1, \pm 2, \ldots$, where for $j = \overline{1, d}$, $\lambda^0_{-0,j} = 0$ and $\lambda^0_{n,j} = n + \frac{\alpha_j}{\pi}$ $(n = +0, \pm 1, \pm 2, \ldots)$. Denote by Γ_{N_0} the counterclockwise square contour with four vertices

$$A = N_0 + \gamma_1 + N_0 i, \qquad B = -N_0 + \gamma_2 + N_0 i,$$
$$C = -N_0 + \gamma_2 - N_0 i, \qquad D = N_0 + \gamma_1 - N_0 i,$$

where $\gamma_k \neq \frac{\alpha_1}{\pi}, \ldots, \frac{\alpha_d}{\pi}$ $(k = 1, 2)$ and $\gamma_k = \max\{\frac{\alpha_1}{\pi}, \ldots, \frac{\alpha_d}{\pi}\} + (-1)^{k-1}\varepsilon_k$, taking enough small $\varepsilon_k > 0$, and N_0 is a natural number.

Denote

$$\omega_0(\lambda) = (-\lambda)^d \prod_{j=1}^{d} \sin(\lambda\pi - \alpha_j),$$

then its zeros are $\lambda^0_{-0,j} = 0$ with multiplicities d and $\lambda^0_{n,j} = n + \frac{\alpha_j}{\pi}$ (simple) $(n = +0, \pm 1, \pm 2, \ldots)$, $j = \overline{1, d}$.

Obviously, if $\lambda \in C_{n,j}$ or $\lambda \in \Gamma_{N_0}$, then $|\omega_0(\lambda)| \geq M|\lambda|^d e^{d\tau\pi}$ $(M > 0)$ by using a similar method in [39,40]. Thus, on $\lambda \in C_{n,j}$ or $\lambda \in \Gamma_{N_0}$, we have

$$\frac{\det \omega(\lambda)}{\omega_0(\lambda)} = 1 - \frac{\operatorname{tr}(P(0) + P(\pi))}{2\lambda} - \frac{\sum_{j=1}^{d} c_j \cot(\lambda\pi - \alpha_j)}{\lambda} + O\left(\frac{1}{\lambda^2}\right). \tag{5.1}$$

Expanding $\ln \frac{\det \omega(\lambda)}{\omega_0(\lambda)}$ by the Maclaurin formula, we find that

$$\ln \frac{\det \omega(\lambda)}{\omega_0(\lambda)} = -\frac{\operatorname{tr}(P(0) + P(\pi))}{2\lambda} - \frac{\sum_{j=1}^{d} c_j \cot(\lambda\pi - \alpha_j)}{\lambda} + O\left(\frac{1}{\lambda^2}\right). \tag{5.2}$$

Using an identity

$$\lambda_{n,j} - \lambda^0_{n,j} = -\frac{1}{2\pi i} \oint_{C_{n,j}} \ln \frac{\det \omega(\lambda)}{\omega_0(\lambda)} d\lambda,$$

we get

$$\begin{aligned}
\lambda_{n,j} - \left(n + \frac{\alpha_j}{\pi}\right) &= \frac{1}{2\pi i} \oint_{C_{n,j}} \lambda \left[\frac{(\det \omega(\lambda))'}{\det \omega(\lambda)} - \frac{\omega_0'(\lambda)}{\omega_0(\lambda)}\right] d\lambda \\
&= -\frac{1}{2\pi i} \oint_{C_{n,j}} \ln \frac{\det \omega(\lambda)}{\omega_0(\lambda)} d\lambda \\
&= -\frac{1}{2\pi i} \oint_{C_{n,j}} \left[-\frac{\operatorname{tr}(P(0) + P(\pi))}{2\lambda} - \frac{\sum_{j=1}^{d} c_j \cot(\lambda\pi - \alpha_j)}{\lambda} + O\left(\frac{1}{\lambda^2}\right)\right] d\lambda \\
&= \frac{1}{2\pi i} \sum_{j=1}^{d} c_j \oint_{C_{n,j}} \frac{\cot(\lambda\pi - \alpha_j)}{\lambda} d\lambda + O\left(\frac{1}{n^2}\right)
\end{aligned}$$

$$= \frac{c_j}{n\pi + \alpha_j} + O\left(\frac{1}{n^2}\right)$$
$$= \frac{c_j}{n\pi} + O\left(\frac{1}{n^2}\right),$$

that is,

$$\lambda_{n,j} = n + \frac{\alpha_j}{\pi} + \frac{c_j}{n\pi} + O\left(\frac{1}{n^2}\right). \tag{5.3}$$

It is well known that the eigenvalues of (1.1)–(1.2) form a sequence $\lambda_{n,j} = n + \frac{\alpha_j}{\pi} + o(1),\ n = \pm 0, \pm 1, \pm 2, \ldots$. This asymptotic relation for the eigenvalues implies that, for all sufficiently large N_0, the numbers $\lambda_{n,j}$ with $|n| \le N_0$ are inside Γ_{N_0}, and the numbers $\lambda_{n,j}$ with $|n| > N_0$ are outside Γ_{N_0}. It follows that

$$\begin{aligned}
&\sum_{j=1}^{d}(\lambda_{-0,j} - \lambda^0_{-0,j}) + \sum_{j=1}^{d}(\lambda_{+0,j} - \lambda^0_{+0,j}) + \sum_{n=1}^{N_0}\sum_{j=1}^{d}(\lambda_{n,j} + \lambda_{-n,j} - \lambda^0_{n,j} - \lambda^0_{-n,j}) \\
&= \sum_{j=1}^{d}\lambda_{-0,j} + \sum_{j=1}^{d}\lambda_{+0,j} - \frac{1}{\pi}\mathrm{tr}\int_0^{\pi} P(x)\mathrm{d}x + \sum_{n=1}^{N_0}\sum_{j=1}^{d}\left(\lambda_{n,j} + \lambda_{-n,j} - \frac{2\alpha_j}{\pi}\right) \\
&= -\frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\left[-\frac{\mathrm{tr}(P(0)+P(\pi))}{2\lambda} - \frac{\sum_{j=1}^{d} c_j\cot(\lambda\pi - \alpha_j)}{\lambda} + O\left(\frac{1}{\lambda^2}\right)\right]\mathrm{d}\lambda \\
&= \frac{1}{2\pi i}\frac{\mathrm{tr}(P(0)+P(\pi))}{2}\oint_{\Gamma_{N_0}}\frac{1}{\lambda}\,\mathrm{d}\lambda + \frac{1}{2\pi i}\sum_{j=1}^{d} c_j\oint_{\Gamma_{N_0}}\frac{\cot(\lambda\pi - \alpha_j)}{\lambda}\,\mathrm{d}\lambda + O\left(\frac{1}{N_0}\right).
\end{aligned} \tag{5.4}$$

Denote

$$R_{j,N_0} \overset{\text{def}}{=} \frac{1}{2\pi i}\oint_{\Gamma_{N_0}}\frac{\cot(\lambda\pi - \alpha_j)}{\lambda}\,\mathrm{d}\lambda.$$

If $\alpha_i = 0\ (\mathrm{mod}\ \pi)$, then $R_{j,N_0} = 0$. If $\alpha_i \ne 0\ (\mathrm{mod}\ \pi)$, then using the residue calculation we have the following identity:

$$\begin{aligned}
R_{j,N_0} &= \mathrm{Res}\left[\frac{\cot(\lambda\pi - \alpha_j)}{\lambda}, 0\right] + \mathrm{Res}\left[\frac{\cot(\lambda\pi - \alpha_j)}{\lambda}, \frac{\alpha_j}{\pi}\right] + \sum_{n=1}^{N_0}\mathrm{Res}\left[\frac{\cot(\lambda\pi - \alpha_j)}{\lambda}, n + \frac{\alpha_j}{\pi}\right] \\
&\quad + \sum_{n=-1}^{-N_0}\mathrm{Res}\left[\frac{\cot(\lambda\pi - \alpha_j)}{\lambda}, n + \frac{\alpha_j}{\pi}\right] \\
&= -\cot\alpha_j + \frac{1}{\alpha_j} - \frac{2\alpha_j}{\pi^2}\sum_{n=1}^{N_0}\frac{1}{n^2 - \alpha_j^2/\pi^2}.
\end{aligned}$$

Using a well-known formula

$$\sum_{n=1}^{\infty}\frac{1}{n^2 - \alpha^2} = \frac{1}{2\alpha^2} - \frac{\pi}{2\alpha}\cot(\alpha\pi) \quad (\alpha \ne 0),$$

we obtain

$$R_{j,N_0} = \frac{2\alpha_j}{\pi^2}\sum_{n=N_0+1}^{\infty}\frac{1}{n^2 - \alpha_j^2/\pi^2}.$$

Substituting the expression of R_{j,N_0} into (5.4) yields

$$\sum_{n=0}^{N_0}\left[\sum_{j=1}^{d}(\lambda_{n,j} + \lambda_{-n,j}) - \frac{2}{\pi}\mathrm{tr}\int_0^{\pi} P(x)\,\mathrm{d}x\right] = \frac{\mathrm{tr}[P(0)+P(\pi)]}{2} - \frac{1}{\pi}\mathrm{tr}\int_0^{\pi} P(x)\,\mathrm{d}x + \sum_{j=1}^{d} c_j R_{j,N_0} + O\left(\frac{1}{N_0}\right). \tag{5.5}$$

Notice that for $j = \overline{1, d}$,

$$\lim_{N_0\to\infty} R_{j,N_0} = 0.$$

Passing to the limit as $N \to \infty$ in (5.5), we have

$$\sum_{n=0}^{\infty}\left[\sum_{j=1}^{d}(\lambda_{n,j}+\lambda_{-n,j})-\frac{2}{\pi}\mathrm{tr}\int_0^{\pi}P(x)\,\mathrm{d}x\right]=\frac{\mathrm{tr}[P(0)+P(\pi)]}{2}-\frac{1}{\pi}\mathrm{tr}\int_0^{\pi}P(x)\,\mathrm{d}x.$$

This completes the proof of Theorem 2.1.

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References

[1] I.M. Gel'fand, B.M. Levitan, On a simple identity for the characteristic values of a differential operator of the second order, Dokl. Akad. Nauk SSSR 88 (1953) 593–596 (in Russian).
[2] L.A. Dikii, On a formula of Gel'fand–Levitan, Uspekhi Mat. Nauk 8 (1953) 119–123 (in Russian).
[3] C.J. Halberg, V.A. Kramer, A generalization of the trace concept, Duke Math. J. 27 (1960) 607–618.
[4] B.M. Levitan, I.S. Sargsyan, Sturm–Liouville and Dirac Operators, Kluwer, Dordrecht, 1991.
[5] V.A. Sadovnichii, V.E. Podol'skii, Traces of operators, Russian Math. Surveys 61 (2006) 885–953.
[6] V.A. Sadovnichii, V.E. Podol'skii, Traces of differential operators, Differential Equations 45 (2009) 477–493.
[7] V.B. Lidskii, V.A. Sadovnichii, Regularized sums of roots of a class of entire functions, Funct. Anal. i Priložen 1 (1967) 52–59 (in Russian); English transl. Funct. Anal. Appl. 1 (1967) 133–139.
[8] J.-J. Sansuc, V. Tkachenko, Characterization of the periodic and antiperiodic spectra of nonselfadjoint Hill's operators, Oper. Theory Adv. Appl. 98 (1997) 216–224.
[9] E. Abdukadyrov, Computation of the regularized trace for a Dirac system, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 22 (1967) 17–24.
[10] E.E. Adiguzelov, H. Avci, E. Gul, The trace formula for Sturm–Liouville operator with operator coefficient, J. Math. Phys. 42 (2001) 2611–2624.
[11] E.E. Adiguzelov, O. Baykal, A. Bayramov, On the spectrum and regularized trace of the Sturm–Liouville problem with spectral parameter on the boundary condition and with the operator coefficient, Int. J. Differ. Equ. Appl. 2 (2001) 317–333.
[12] I. Albayrak, K. Koklu, A. Bayramov, A regularized trace formula for differential equations with trace class operator coefficients, Rocky Mountain J. Math. 40 (2010) 1095–1110.
[13] N.M. Aslanova, Calculation of the regularized trace of differential operator with operator coefficient, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. 1 (2006) 39–44.
[14] O. Baksi, E. Adiguzelov, The third regularized trace formula for the Sturm–Liouville operator with the bounded operator coefficient, Math. Comput. Modelling 52 (2010) 1846–1857.
[15] M. Bayramoglu, H. Sahinturk, Higher order regularized trace formula for the regular Sturm–Liouville equation contained spectral parameter in the boundary condition, Appl. Math. Comput. 186 (2007) 1591–1599.
[16] M. Bayramogly, On regularized trace of the n-order differential operator with the unbounded operator coefficient. Spectral theory of operators and its applications, Baku: Elm. 8 (1987) 15–40.
[17] A. Bayramov, Z. Oer, O. Baykal, On identity for eigenvalues of second order differential operator equation, Math. Comput. Modelling 49 (2009) 403–412.
[18] D. Borisov, P. Freitas, Eigenvalue asymptotics, inverse problems and a trace formula for the linear damped wave equation, J. Differential Equations 247 (2009) 3028–3039.
[19] R. Carlson, Large eigenvalues and trace formulas for matrix Sturm–Liouville problems, SIAM J. Math. Anal. 30 (1999) 949–962.
[20] R. Carlson, Eigenvalue estimates and trace formulas for the matrix Hill's equation, J. Differential Equations 167 (2000) 211–244.
[21] F. Gesztesy, H. Holden, On trace formulas for Schrodinger-type operators, in: D.G. Truhlar, B. Simon (Eds.), Multi-Particle Quantum Scattering with Applications to Nuclear, Atomic and Molecular Physics, Springer-Verlag, New York, 1997, pp. 121–145.
[22] F. Gesztesy, H. Holden, The damped string problem revisited, J. Differential Equations 251 (2011) 1086–1127.
[23] E. Gul, On the regularized trace of a second order differential operator, Appl. Math. Comput. 198 (2008) 471–480.
[24] N.J. Guliyev, The regularized trace formula for the Sturm–Liouville equation with spectral parameter in the boundary conditions, Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb. 22 (2005) 99–102.
[25] P. Lax, Trace formulas for the Schrodinger operator, Comm. Pure Appl. Math. 47 (1994) 503–512.
[26] A.S. Makin, Regularized trace of the Sturm–Liouville operator with irregular boundary conditions, Electron. J. Differential Equations 2009 (2009) 1–8.
[27] F.G. Maksudov, M. Bairamoglu, E.E. Adiguzelov, On regularized trace of Sturm–Liouville operator on a finite interval with the unbounded operator coefficient, Dokl. Akad. Nauk SSSR 30 (1984) 169–173.
[28] V.G. Papanicolaou, Trace formulas and the behaviour of large eigenvalues, SIAM J. Math. Anal. 26 (1995) 218–237.
[29] V.A. Sadovnichii, On identities for the eigenvalues of a Dirac system and other systems of higher order, Vestnik Moskov. Univ. Ser. I Mat. Mekh. 22 (1967) 37–47.
[30] S. Timischl, A trace formula for one-dimensional Dirac operators, Diploma Thesis, Department of Mathematics, Karl-Franzens-Universitat, Graz, and the Department of Mathematics, University of Missouri, Columbia, 1995.
[31] T.M. Tsai, C.L. Shen, On inverse problems and trace formulae, Doctoral Dissertation for Institute of Mathematics National Tsing Hua University. http://etd.lib.nctu.edu.tw/cgi-bin/gs/hugsweb.cgi?o=dnthucdr?i=sNH840479001.id.
[32] C.F. Yang, New trace formulae for a quadratic pencil of the Schrodinger operator, J. Math. Phys. 51 (2010) 033506. p. 10.
[33] C.F. Yang, Z.Y. Huang, Y.P. Wang, Trace formulae for the Schrodinger equation with energy-dependent potential, J. Phys. A 43 (2010) 415207. p. 15.
[34] N. Bondarenko, Spectral analysis for the matrix Sturm–Liouville operator on a finite interval, arXiv:1008.4339v1.
[35] D. Chelkak, E. Korotyaev, Weyl–Titchmarsh functions of vector-valued Sturm–Liouville operators on the unit interval, J. Funct. Anal. 257 (2009) 1546–1588.
[36] H.H. Chern, On the eigenvalues of some vectorial Sturm–Liouville eigenvalue problems, arXiv:math/9902019v1.
[37] H.I. Dwyer, A. Zettl, Eigenvalue computations for regular matrix Sturm–Liouville problems, Electron. J. Differential Equations 1995 (1995) 1–13.
[38] R.E.A.C. Paley, N. Wiener, Fourier Transforms in Complex Domain, Vol. XIX, Amer. Math. Soc., Colloquium Publications, New York, 1934.
[39] V.A. Yurko, Inverse Spectral Problems for Differential Operators and their Applications, Gordon and Breach, Amsterdam, 2000.
[40] V.A. Yurko, Method of Spectral Mappings in the Inverse Problem Theory, in: Inverse and Ill-Posed Problems Series, VSP, Utrecht, 2002.