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A regularizing effect of radiation in one-dimensional compressible MHD equations ☆

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ABSTRACT

This paper is concerned with the global existence of classical solutions to an initial–boundary value problem of the one-dimensional (1D) equations of compressible radiative magnetohydrodynamics (MHD). The key point here is that there is no growth restriction imposed on the heat conductivity, in particular, the heat-conducting coefficient is allowed to be a positive constant. This in particular implies that the radiation is indeed a mathematically "regularizing" effect on the fluid dynamics, since the global existence of the classical solution to the one-dimensional full perfect MHD equations without radiation is still unknown when all the viscosity, magnetic diffusivity and heat conductivity coefficients are constant.

1. Introduction

The motion of a conducting fluid (plasma) in an electromagnetic field is governed by the compressible MHD equations in Eulerian coordinates (see, for example, [1,2]):

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + \left(\rho u^2 + p + \frac{1}{2}|\boldsymbol{b}|^2\right)_x = (\lambda u_x)_x, \\ (\rho \boldsymbol{w})_t + (\rho u \boldsymbol{w} - \boldsymbol{b})_x = (\mu \boldsymbol{w}_x)_x, \\ \boldsymbol{b}_t + (u\boldsymbol{b} - \boldsymbol{w})_x = (\nu \boldsymbol{b}_x)_x, \\ \mathcal{E}_t + \left(u\left(\mathcal{E} + p + \frac{1}{2}|\boldsymbol{b}|^2\right) - \boldsymbol{w}\cdot\boldsymbol{b}\right)_x = (\kappa\theta_x + \lambda u u_x + \mu \boldsymbol{w}\cdot\boldsymbol{w}_x + \nu \boldsymbol{b}\cdot\boldsymbol{b}_x)_x, \end{cases} \tag{1.1}$$

where the unknown functions $\rho, u, \boldsymbol{w} = (w_1, w_2), \boldsymbol{b} = (b_1, b_2)$ and θ are the density, longitudinal velocity, transverse velocity, transverse magnetic field, and absolute temperature, respectively. The pressure p and the internal energy e are related with the density and the temperature of the flow through the equations of state:

$$p \triangleq p(\rho, \theta), \qquad e \triangleq e(\rho, \theta), \tag{1.2}$$

and the total energy $\mathcal{E}$ is given by

$$\mathcal{E} \triangleq \rho\left(e + \frac{1}{2}(u^2 + |\boldsymbol{w}|^2)\right) + \frac{1}{2}|\boldsymbol{b}|^2, \tag{1.3}$$

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where $\rho(u^2 + |\boldsymbol{w}|^2)/2$ and $|\boldsymbol{b}|^2/2$ are the kinetic energy and the magnetic energy, respectively. The physical positive constants λ, μ are the viscosity coefficients of the flow, ν is the resistivity coefficient acting as the magnetic diffusion coefficient of the magnetic field, and κ is the heat-conducting coefficient.

The MHD equations are derived from fluid mechanics with appropriate modifications to account for the electrical forces. In particular, if $\boldsymbol{b} = 0$, then (1.1) turns into the compressible planar Navier–Stokes equations describing the motion of a shear flow (cf. [3]). If in addition $\boldsymbol{w} = 0$, then it becomes the standard one-dimensional Navier–Stokes equations for compressible heat-conducting viscous gases:

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = (\lambda u_x)_x, \\ \tilde{\mathcal{E}}_t + (u(\tilde{\mathcal{E}} + p))_x = (\kappa\theta_x + \lambda u u_x)_x \end{cases} \tag{1.4}$$

with $\tilde{\mathcal{E}} \triangleq \rho(e + u^2/2)$ being the total energy.

The one-dimensional Navier–Stokes equations (1.4) have been extensively studied by many people; see, for example, [4–9], and the references therein. Comparing (1.4) with (1.1), the additional presence of the magnetic field and its interaction with the hydrodynamic motion in MHD flows will cause some serious difficulties, and hence, the extension of known results for the Navier–Stokes equations to the MHD equations does not always appear to be a simple matter. For example, consider an initial–boundary value problem of (1.4) with the following initial and boundary conditions:

$$(\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0)(x), \qquad (u, \theta_x)|_{\partial\Omega} = 0, \tag{1.5}$$

where $\Omega \subset \mathrm{R}$ is a bounded spatial domain with smooth boundary $\partial\Omega$. Assume that the initial data ρ_0, u_0, θ_0 are appropriately smooth and satisfy

$$0 < \inf \rho_0(x) \le \sup \rho_0(x) < \infty, \qquad 0 < \inf \theta_0(x) \le \sup \theta_0(x) < \infty. \tag{1.6}$$

Then, for a perfect polytropic gas obeying the equations of state (cf. [17]):

$$p \triangleq R\rho\theta, \qquad e \triangleq c_V\theta, \tag{1.7}$$

where $R > 0$ is the gas constant, $c_V = R/(\gamma - 1)$ is the heat capacity of the gas at constant volume, and $\gamma > 1$ is the adiabatic exponent, it has been known for a long time (see [7]) that there exists a unique global solution of (1.4)–(1.7) with fixed positive constants λ, κ. However, as it was pointed out in [2,10] that such a global existence result of the classical solution to the 1D MHD equations (1.1), (1.3), (1.7) with large initial data is still unknown when all the viscosity, diffusivity and heat conductivity coefficients are constant.

In this paper we study the MHD equations subject to thermally radiative effects at high temperature, which are of particular interest in astrophysical models since stars may be viewed as gaseous objects (cf. [11,12]) whose dynamics are often shaped and controlled by intense magnetic fields and high temperature radiation effects (cf. [13–15]). In view of the classical theory in (see [16]), the total pressure p and the internal energy e in radiation hydrodynamics are decomposed into two parts: a thermal part (for perfect gas) and a radiative part,

$$p(\rho, \theta) = p_G(\rho, \theta) + p_R(\rho, \theta), \qquad e(\rho, \theta) = e_G(\rho, \theta) + e_R(\rho, \theta).$$

In agreement with the Boyle law for gas dynamics and the Stefan–Boltzmann law for radiation hydrodynamics, it holds for radiative gases in quasi Local Thermodynamical Equilibrium (LTE) that (see [17,16])

$$\begin{cases} p_G(\rho, \theta) = R\rho\theta, & e_G(\rho, \theta) = c_V\theta, \\ p_R(\rho, \theta) = \dfrac{a}{3}\theta^4, & e_R(\rho, \theta) = \dfrac{a}{\rho}\theta^4, \end{cases}$$

where $a > 0$ is the Stefan–Boltzmann constant. Thus,

$$p(\rho, \theta) = R\rho\theta + \frac{a}{3}\theta^4, \qquad e(\rho, \theta) = c_V\theta + \frac{a}{\rho}\theta^4. \tag{1.8}$$

This, together with (1.1) and (1.3), forms a complete system for $\rho, u, \boldsymbol{w}, \boldsymbol{b}$ and θ.

Without loss of generality, let $\Omega \triangleq (0, 1)$. In this paper, we study an initial–boundary value problem of (1.1), (1.3) and (1.8) with the following initial and boundary conditions:

$$\begin{cases} (\rho, u, \boldsymbol{w}, \boldsymbol{b}, \theta)|_{t=0} = (\rho_0, u_0, \boldsymbol{w}_0, \boldsymbol{b}_0, \theta_0)(x), \\ (u, \boldsymbol{w}, \boldsymbol{b}, \theta_x)|_{x=0} = (u, \boldsymbol{w}, \boldsymbol{b}, \theta_x)|_{x=1} = 0. \end{cases} \tag{1.9}$$

The boundary conditions in $(1.9)_2$ particularly imply that the boundary is non-slip, impermeable, and thermally insulated.

Before stating our main results, we first recall some recent studies on 1D compressible MHD equations. For the 1D perfect MHD flows satisfying the equations of state (1.7), the global existence of smooth solutions with small initial data was obtained in [18,19], while the global existence and uniqueness of strong solutions with large initial data was studied in [20,21] provided the heat conductivity is in a particular form:

$$\kappa(\rho, \theta) = k\theta \quad \text{or} \quad \kappa(\rho, \theta) = k/\rho, \tag{1.10}$$

where $k > 0$ is a positive constant. The existence and stability of weak solutions with Lebesgue initial data to the 1D perfect full MHD equations was proved in [22], and the vanishing shear viscosity limit (i.e. $\mu \to 0$) of weak solutions was considered in [23]. It is worth mentioning that the condition (1.10) is very crucial for the analysis in the works [20–23].

For real gases, let $v \triangleq 1/\rho$ be the specific volume in Lagrangian coordinates. Assume that the pressure $p(v, \theta)$, internal energy $e(v, \theta)$ and heat conductivity $\kappa(v, \theta)$ satisfy the following growth conditions with $r \in [0, 1]$ and $q \geq 2(1 + r)$:

$$0 \leq vp(v, \theta) \leq p_0(1 + \theta^{1+r}), \qquad e_\theta(v, \theta) \geq e_0(1 + \theta^r), \quad \text{and} \quad \kappa(v, \theta) \geq \kappa_0(1 + \theta^q) \tag{1.11}$$

for some positive constants p_0, e_0 and κ_0, Chen–Wang [1] and Wang [2] proved the global existence of strong/classical solutions to the MHD equations (1.1), (1.3) and (1.11) with appropriately smooth initial data. The continuous dependence of large solutions was also studied in [1,24]. Note that, the growth conditions (1.11) include the case of perfect flows (1.7) (i.e. $r = 0$), but exclude the radiation case (1.8).

For the 1D compressible MHD flows subject to radiation effects at high temperature with the equations of state (1.8), the global existence of a unique classical solution with large initial data was studied in [25,26] under the following growth condition on the heat conductivity:

$$\kappa_1(1 + \theta^q) \leq \kappa(\rho, \theta), \qquad \kappa_\rho(\rho, \theta) \leq \kappa_2(1 + \theta^q), \tag{1.12}$$

where $q > 5/2$ and $q > (2 + \sqrt{211})/9$ were assumed respectively in [25,26]. We also refer the reader to [27–32], . . . for the studies of other 1D models for radiative gases, among all of which the growth condition (1.12) with different exponent $q > 0$ was required; see, for instance, $q \geq 4$ in [27,28,31], $q \geq 2$ in [30,32], and $q \geq 1$ in [29].

For radiative gas-dynamics, it was pointed out in [27,28] that the growth condition (1.12) with suitably large $q > 0$ plays a key role in the proof of the global a priori estimates. Indeed, compared with the perfect flows satisfying the equations of state (1.7), the nonlinear radiative terms (i.e. θ^4) in (1.8) will cause some new difficulties in the study of radiative gas-dynamics, and hence, the growth condition (1.12) with suitably large $q > 0$ is technically needed in the previous works to obtain some better estimates of the temperature (see, e.g. [25–32]). Motivated by the results achieved, it is therefore mathematically interesting to improve or remove such growth restriction on the heat conductivity, although (1.12) with $q \geq 3$ is physically valid for radiative gases in LTE (see [16]).

So, placing emphasis on the case of constant heat conductivity, we aim to prove the global existence of classical solutions for the 1D compressible radiative MHD system (1.1), (1.3), (1.8), (1.9) without any growth condition on the heat conductivity. More precisely, our main result in this paper reads as follows.

Theorem 1.1. *Assume that all the viscosity, magnetic diffusivity and heat-conductivity coefficients of the flows (i.e. λ, μ, ν and κ) are positive constants. Assume also that the initial data $(\rho_0, u_0, \boldsymbol{w}_0, \boldsymbol{b}_0, \theta_0)$ satisfies*

$$0 < \inf_{0\leq x\leq 1} \rho_0 \leq \sup_{0\leq x\leq 1} \rho_0 < \infty, \qquad 0 < \inf_{0\leq x\leq 1} \theta_0 \leq \sup_{0\leq x\leq 1} \theta_0 < \infty,$$

$$\rho_0 \in C^{1+\alpha}(\Omega), \qquad (u_0, \boldsymbol{w}_0, \boldsymbol{b}_0, \theta_0) \in \left(C^{2+\alpha}(\Omega)\right)^6 \quad \textit{for some } \alpha \in (0, 1).$$

Then there exists a unique classical solution $(\rho, u, \boldsymbol{w}, \boldsymbol{b}, \theta)$ of (1.1), (1.3), (1.8) and (1.9) such that for any $T > 0$,

$$0 < \rho(x, t), \qquad \theta(x, t) < \infty \quad \textit{for all } (x, t) \in \Omega_T \triangleq \Omega \times (0, T),$$

$$(\rho, \rho_x, \rho_t) \in \left(C^{\alpha,\alpha/2}_{x,t}(\Omega_T)\right)^3, \qquad (u, \boldsymbol{w}, \boldsymbol{b}, \theta) \in \left(C^{2+\alpha,1+\alpha/2}_{x,t}(\Omega_T)\right)^6.$$

Remark 1.1. Theorem 1.1 especially implies that the thermal radiation is indeed a mathematically "regularizing" effect for compressible MHD flows, since the global existence of the classical solution to the 1D perfect full MHD equations (1.1), (1.3), (1.7) is still unknown when all the viscosity, diffusivity and heat conductivity coefficients are constant (cf. [2,10]).

Remark 1.2. A regularizing effect of radiation in the multi-dimensional equations of fluid dynamics was also observed in [33]. However, the regularizing mechanisms between [33] and the one observed in the present paper are different from the mathematical point of view. Roughly speaking, the regularizing effect in [33] mainly comes from the specific form of the heat conductivity which contains an extra "radiative" term ($\sim \theta^3$). However, the mechanism of the regularizing effect in this paper is mainly due to the additional radiative parts of the pressure and the internal energy in the equations of state (1.8).

Remark 1.3. The conclusion of Theorem 1.1 is also valid for other models of radiative gases satisfying the equations of state (1.8). We hope that the ideas of proof can be adopted to remove/improve the growth conditions (1.11) for real gases considered in [1,2,5] and to study the 1D perfect MHD equations with constant physical coefficients λ, μ, ν and κ.

We now comment on the proof of Theorem 1.1. As usual, the global existence will be proved by continuing the local solutions with respect to the time based on the global a priori estimates. The local existence can be proved by using the fixed point theorem in a standard way, whose proof is therefore omitted and referred to [4] for simplicity. So, to extend the local solution to be a global classical one, it suffices to establish some global a priori estimates on the solution and its derivatives. It turns out that the key steps are to bound the L^2-norm of the gradient of the density and to estimate the upper bound of the temperature. To achieve these, the growth condition (1.12) with suitably large $q > 0$ in the previous works [25–28,30–32] is technically needed for their analysis. So, in the case that the heat conductivity is only a positive constant, the situation becomes somewhat different and some new ideas have to be developed. A key observation here is that we can utilize the radiative parts in the equations of state (1.8) to derive some preliminary estimates of the higher integrability of the temperature (see Lemma 2.4), which in turn give a preliminary estimate of $\|\nabla\rho\|_{L^2(\Omega)}$ in terms of $\|u_{xx}\|^{\alpha}_{L^2(\Omega_T)}$ with $0 < \alpha < 1$ (see Lemma 2.5). Thus, the gradient of the longitudinal velocity can be bounded only by the upper bound of the temperature with appropriate order. With the help of these preliminary estimates, we then utilize the specific form of the additional radiative internal energy again to obtain the upper bound of the temperature (see Lemma 2.7), which particularly close the proofs of Lemmas 2.4–2.6. With these estimates at hand, the global estimates of the second order derivatives can then be proved in a standard way.

The method used for the proof of Theorem 1.1 can be adopted to deal with the case of non-constant heat conductivity. More precisely, we have the following.

Theorem 1.2. *Assume that $(\rho_0, u_0, \boldsymbol{w}_0, \boldsymbol{b}_0, \theta_0)$ satisfy the conditions of Theorem 1.1. Assume also that the heat conductivity $\kappa = \kappa(\rho, \theta)$ is strictly positive, continuously differentiable on $\mathrm{R}^+ \times \mathrm{R}^+$ and there exist two positive constants κ_1 and κ_2 such that for any $q > 0$,*

$$\kappa_1(1+\theta)^q \le \kappa(\rho,\theta) \le \kappa_2(1+\theta)^q, \tag{1.13}$$

$$|\kappa_\rho(\rho,\theta)| \le \kappa_2(1+\theta)^q, \qquad |\kappa_\theta(\rho,\theta)| \le \kappa_2(1+\theta)^{q-1} \tag{1.14}$$

for all $\rho, \theta \in (0,\infty)$. Then for any $T > 0$, the problem (1.1), (1.3), (1.8), (1.9) with fixed positive $\mu > 0$ has a unique classical solution $(\rho, u, \boldsymbol{w}, \boldsymbol{b}, \theta)$ on Ω_T as the one in Theorem 1.1.

2. Proof of Theorem 1.1

This section is devoted to the derivation of the global estimates which are needed for the proof of Theorem 1.1. We begin with the following lemma which is mainly concerned with the standard energy–entropy estimates and the upper and lower bounds of the density. The proof can be found in [25, Lemmas 2.1–2.3] and is therefore omitted here for simplicity.

Lemma 2.1. *There exists a positive constant C, which may depend on T, such that*

$$m(t) \triangleq \int_0^1 \rho(x,t)dx = \int_0^1 \rho_0(x)dx \triangleq m_0, \tag{2.1}$$

$$\begin{aligned} E(t) &\triangleq \int_0^1 \left[\rho\left(e + \frac{1}{2}(u^2 + |\boldsymbol{w}|^2)\right) + \frac{1}{2}|\boldsymbol{b}|^2\right](x,t)dx \\ &= \int_0^1 \left[\rho\left(e + \frac{1}{2}(u^2 + |\boldsymbol{w}|^2)\right) + \frac{1}{2}|\boldsymbol{b}|^2\right](x,0)dx \triangleq E(0), \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\sup_{0\le t\le T}\int_0^1 \left[c_V\rho(\theta - \ln\theta - 1) + R(\rho\ln\rho - \rho + 1) + \frac{1}{2}\left(\rho u^2 + \rho|\boldsymbol{w}|^2 + |\boldsymbol{b}|^2\right)\right]dx \\ &\quad + \int_0^T\int_0^1 \left(\frac{\kappa\theta_x^2}{\theta^2} + \frac{\lambda u_x^2 + \mu|\boldsymbol{w}_x|^2 + \nu|\boldsymbol{b}_x|^2}{\theta}\right)dxdt \le C, \end{aligned} \tag{2.3}$$

$$\int_0^T \left(\|\theta\|^4_{L^\infty} + \|\boldsymbol{b}\|^2_{L^\infty} + \|\theta\|^8_{L^8}\right)dt \le C, \tag{2.4}$$

$$C^{-1} \le \rho(x,t) \le C, \quad (x,t) \in \overline{\Omega}_T \triangleq [0,1]\times[0,T], \tag{2.5}$$

and

$$\int_0^T \left(\|u_x\|^2_{L^2} + \|\boldsymbol{w}_x\|^2_{L^2} + \|\boldsymbol{b}_x\|^2_{L^2}\right)dt \le C. \tag{2.6}$$

As a result of (2.5) and (2.6), one has the following.

Lemma 2.2. *For any given $T > 0$, one has*

$$\sup_{0\le t\le T}\left(\|\boldsymbol{w}_x\|^2_{L^2} + \|\boldsymbol{b}_x\|^2_{L^2}\right) + \int_0^T \left(\|\boldsymbol{w}_t\|^2_{L^2} + \|\boldsymbol{b}_t\|^2_{L^2} + \|\boldsymbol{w}_{xx}\|^2_{L^2} + \|\boldsymbol{b}_{xx}\|^2_{L^2}\right)dt \le C. \tag{2.7}$$

Proof. It follows from (1.1)$_3$ and (1.1)$_4$ that

$$\rho|\boldsymbol{w}_t|^2+\mu^2\rho^{-1}|\boldsymbol{w}_{xx}|^2-2\mu\boldsymbol{w}_t\cdot\boldsymbol{w}_{xx}=\rho^{-1}\left(\rho u\boldsymbol{w}_x-\boldsymbol{b}_x\right)^2,$$
$$|\boldsymbol{b}_t|^2+\nu^2|\boldsymbol{b}_{xx}|^2-2\nu\boldsymbol{b}_t\cdot\boldsymbol{b}_{xx}=\left(u\boldsymbol{b}_x+u_x\boldsymbol{b}-\boldsymbol{w}_x\right)^2.$$

Thus, adding them together and integrating by parts, we infer from (2.5) that

$$\begin{aligned}&\frac{d}{dt}\int_0^1\left(\mu|\boldsymbol{w}_x|^2+\nu|\boldsymbol{b}_x|^2\right)dx+\int_0^1\left(\rho|\boldsymbol{w}_t|^2+\mu^2\rho^{-1}|\boldsymbol{w}_{xx}|^2+|\boldsymbol{b}_t|^2+\nu^2|\boldsymbol{b}_{xx}|^2\right)dx\\&\quad\le C\left(\|u\|_{L^\infty}^2\|\boldsymbol{w}_x\|_{L^2}^2+\|\boldsymbol{b}_x\|_{L^2}^2+\|u\|_{L^\infty}^2\|\boldsymbol{b}_x\|_{L^2}^2+\|\boldsymbol{b}\|_{L^\infty}^2\|u_x\|_{L^2}^2+\|\boldsymbol{w}_x\|_{L^2}^2\right)\\&\quad\le C\left(1+\|u_x\|_{L^2}^2\right)\left(1+\|\boldsymbol{w}_x\|_{L^2}^2+\|\boldsymbol{b}_x\|_{L^2}^2\right),\end{aligned}\tag{2.8}$$

where we have used (2.2), (2.5) and the Sobolev type inequality:

$$\|u\|_{L^\infty}^2\le C\|u\|_{L^2}\|u_x\|_{L^2}\le C\|u_x\|_{L^2},\qquad\|\boldsymbol{b}\|_{L^\infty}^2\le C\|\boldsymbol{b}\|_{L^2}\|\boldsymbol{b}_x\|_{L^2}\le C\|\boldsymbol{b}_x\|_{L^2}.\tag{2.9}$$

Combining (2.6), (2.8) with the Gronwall inequality immediately leads to (2.7). □

Next, we present a preliminary estimate of $\|uu_x\|_{L^2(0,T;L^2)}$ which will be used later.

Lemma 2.3. *For any given $T>0$, there exists a positive constant $C>0$ such that*

$$\sup_{0\le t\le T}\|u\|_{L^4}^4+\int_0^T\|uu_x\|_{L^2}^2dt\le C+C\sup_{0\le t\le T}\|u_x\|_{L^2}.\tag{2.10}$$

Proof. Multiplying (1.1)$_2$ by $4u^3$ and integrating the resulting equation by parts over Ω give

$$\begin{aligned}\frac{d}{dt}\int_0^1\rho u^4(x,t)dx+12\lambda\int_0^1u^2u_x^2dx&=12\int_0^1\left(p+\frac{1}{2}|\boldsymbol{b}|^2\right)u^2u_xdx\\&\le\lambda\int_0^1u^2u_x^2dxds+C\|u\|_{L^\infty}^2\int_0^1\left(1+\theta^8+|\boldsymbol{b}|^4\right)dxds,\end{aligned}$$

which, integrated over $(0,T)$ and combined with (2.4), (2.7), (2.9), yields

$$\sup_{0\le t\le T}\|u\|_{L^4}^4+\int_0^T\|uu_x\|_{L^2}^2dt\le C+C\sup_{0\le t\le T}\|u\|_{L^\infty}^2\le C+C\sup_{0\le t\le T}\|u_x\|_{L^2}.$$

This finishes the proof of (2.10). Note that, here we have also used the fact that $\|\boldsymbol{b}\|_{L^\infty}\le C$ due to (2.2), (2.7) and (2.9). □

By making use of the specific form of the radiative internal energy in (1.8), we can obtain some preliminary estimates of the higher integrability of the temperature which play a key role in the entire analysis.

Lemma 2.4. *Let κ be the heat-conducting coefficient of the flows. Then for any $T>0$,*

$$\sup_{0\le t\le T}\int_0^1\theta^8(x,t)dx+\int_0^T\int_0^1\kappa\theta^3\theta_x^2dxdt\le C+C\left(\int_0^T\|u_{xx}\|_{L^2}^2dt\right)^{1/2}\tag{2.11}$$

and

$$\sup_{0\le t\le T}\int_0^1\theta^{11}(x,t)dx+\int_0^T\int_0^1\kappa\theta^6\theta_x^2dxdt\le C+C\left(\int_0^T\|u_{xx}\|_{L^2}^2dt\right)^{7/8}.\tag{2.12}$$

Proof. In view of (1.1)$_1$–(1.1)$_4$ and (1.8), we can write (1.1)$_5$ in the form:

$$\left(c_V\rho\theta+a\theta^4\right)_t+\left(c_V\rho u\theta+au\theta^4\right)_x+\left(R\rho\theta+\frac{a}{3}\theta^4\right)u_x=(\kappa\theta_x)_x+\lambda u_x^2+\mu|\boldsymbol{w}_x|^2+\nu|\boldsymbol{b}_x|^2,\tag{2.13}$$

which, multiplied by θ^4 and integrated over $\Omega\times(0,T)$, gives

$$\begin{aligned}&\sup_{0\le t\le T}\int_0^1\theta^8(x,t)dx+\int_0^T\int_0^1\kappa\theta^3\theta_x^2dxdt\\&\le C+C\int_0^T\int_0^1\left(\theta^8|u_x|+\theta^4\left(u_x^2+|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2\right)\right)dxdt,\end{aligned}\tag{2.14}$$

where we have also used (2.5) and integration by parts to get that

$$\int_0^T\int_0^1 (u\theta^4)_x\theta^4 dxdt = -4\int_0^T\int_0^1 u\theta^7\theta_x dxdt = \frac{1}{2}\int_0^T\int_0^1 u_x\theta^8 dxdt.$$

Note that (2.2), together with (1.8), implies $\|\theta\|_{L^4} \le C$. So, using (2.4), (2.6) and (2.7), we can bound the second term on the right-hand side of (2.14) as follows:

$$\begin{aligned}
&\int_0^T\int_0^1 \left(|u_x|\theta^8 + \theta^4 u_x^2 + \theta^4\left(|\boldsymbol{w}_x|^2 + |\boldsymbol{b}_x|^2\right)\right) dxdt \\
&\le C\int_0^T \|\theta\|_{L^4}^4\left(\|\theta\|_{L^\infty}^4\|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^2\right) dt + C\int_0^T \|\theta\|_{L^\infty}^4\left(\|\boldsymbol{b}_x\|_{L^2}^2 + \|\boldsymbol{w}_x\|_{L^2}^2\right) dt \\
&\le C + C\int_0^T \left(\|\theta\|_{L^\infty}^8 + \|u_x\|_{L^\infty}^2\right) dt. \qquad (2.15)
\end{aligned}$$

Using the boundedness of $\|\theta\|_{L^4}$ again, we deduce from the Hölder inequality that

$$\|\theta\|_{L^\infty}^{9/2} \le C + C\int_0^1 \theta^2\theta^{3/2}|\theta_x| dx \le C + C\left(\int_0^1 \kappa\theta^3\theta_x^2 dx\right)^{1/2},$$

and consequently,

$$\int_0^T \|\theta\|_{L^\infty}^9 dt \le C + C\int_0^T\int_0^1 \kappa\theta^3\theta_x^2 dxdt,$$

which, combined with the Young inequality, yields

$$\int_0^T \|\theta\|_{L^\infty}^8 dt \le \varepsilon\int_0^T \|\theta\|_{L^\infty}^9 dt + C\varepsilon^{-1} \le C\varepsilon\int_0^T\int_0^1 \kappa\theta^3\theta_x^2 dxdt + C\varepsilon^{-1}. \qquad (2.16)$$

Thus, putting (2.15), (2.16) into (2.14) and choosing $\varepsilon > 0$ sufficiently small, we find

$$\begin{aligned}
\sup_{0\le t\le T}\int_0^1 \theta^8(x,t)dx + \int_0^T\int_0^1 \kappa\theta^3\theta_x^2 dxdt &\le C + C\int_0^T \left(\|u_x\|_{L^2}^2 + \|u_x\|_{L^2}\|u_{xx}\|_{L^2}\right) dt \\
&\le C + C\left(\int_0^T \|u_{xx}\|_{L^2}^2 dt\right)^{1/2}, \qquad (2.17)
\end{aligned}$$

which ends the proof of (2.11). Here, we have also used (2.6), the Hölder inequality and the following Sobolev type inequality:

$$\|v_x\|_{L^\infty}^2 \le C\left(\|v_x\|_{L^2}^2 + \|v_x\|_{L^2}\|v_{xx}\|_{L^2}\right). \qquad (2.18)$$

Similar to the derivation of (2.14), we multiply (2.13) by θ^7, integrate the resulting equation by parts over $\Omega \times (0,T)$ and use the Cauchy–Schwarz inequality to get that

$$\begin{aligned}
&\sup_{0\le t\le T}\int_0^1 \theta^{11}(x,t)dx + \int_0^T\int_0^1 \kappa\theta^6\theta_x^2 dxdt \\
&\le C + C\int_0^T\int_0^1 \left(\theta^{11}|u_x| + \theta^7\left(u_x^2 + |\boldsymbol{b}_x|^2 + |\boldsymbol{w}_x|^2\right)\right) dxdt \\
&\le C + C\int_0^T\int_0^1 \theta^{15} dxdt + C\int_0^T\int_0^1 \theta^7\left(u_x^2 + |\boldsymbol{b}_x|^2 + |\boldsymbol{w}_x|^2\right) dxdt. \qquad (2.19)
\end{aligned}$$

Note that the boundedness of $\|\theta\|_{L^4}$ implies

$$\|\theta\|_{L^\infty}^6 \le C + C\int_0^1 \theta^2\theta^3|\theta_x| dx \le C + C\left(\int_0^1 \kappa\theta^6\theta_x^2 dx\right)^{1/2},$$

and thus,

$$\begin{aligned}
\int_0^T\int_0^1 \theta^{15} dxdt &\le \int_0^T \|\theta\|_{L^\infty}^{11}\|\theta\|_{L^4}^4 dt \le C\left(\int_0^T \|\theta\|_{L^\infty}^{12} dt\right)^{11/12} \\
&\le C + C\left(\int_0^T\int_0^1 \kappa\theta^6\theta_x^2 dxdt\right)^{11/12}. \qquad (2.20)
\end{aligned}$$

On the other hand, using (2.6), (2.7), (2.11), (2.18) and the Hölder inequality, one gets

$$\begin{aligned}\int_0^T\int_0^1 \theta^7\left(u_x^2+|\boldsymbol{b}_x|^2+|\boldsymbol{w}_x|^2\right)dxdt &\le \int_0^T \|\theta\|_{L^7}^7\left(\|u_x\|_{L^\infty}^2+\|\boldsymbol{b}_x\|_{L^\infty}^2+\|\boldsymbol{w}_x\|_{L^\infty}^2\right)dt\\ &\le C\sup_{0\le t\le T}\|\theta\|_{L^7}^7+C\sup_{0\le t\le T}\|\theta\|_{L^7}^7\left(\int_0^T\|u_{xx}\|_{L^2}^2dt\right)^{1/2}\\ &\le C+C\left(\int_0^T\|u_{xx}\|_{L^2}^2dt\right)^{7/8},\end{aligned}\tag{2.21}$$

since it follows from the boundedness of $\|\theta\|_{L^4}$ and (2.11) that

$$\sup_{0\le t\le T}\|\theta\|_{L^7}^7\le\sup_{0\le t\le T}\left(\|\theta\|_{L^4}\|\theta\|_{L^8}^6\right)\le C+C\left(\int_0^T\|u_{xx}\|_{L^2}^2ds\right)^{3/8}.$$

Thus, putting (2.20), (2.21) into (2.19) and using the Young inequality, we arrive at the desired estimate in (2.12). The proof of Lemma 2.4 is thus complete. □

By virtue of Lemma 2.4, we now can estimate the gradient of the density, which is one of the most important step in the proofs of the boundedness of the temperature and the higher order estimates of the longitudinal velocity.

Lemma 2.5. *For any given $T>0$, one has*

$$\sup_{0\le t\le T}\int_0^1\rho_x^2(x,t)dx+\int_0^T\int_0^1\theta\rho_x^2dxdt\le C+C\sup_{0\le t\le T}\|\theta\|_{L^\infty}+C\left(\int_0^T\|u_{xx}\|_{L^2}^2dt\right)^{7/8}.\tag{2.22}$$

Proof. It follows from $(1.1)_1$ and (2.5) that $(1.1)_2$ can be written as follows:

$$\left[\rho\left(\lambda\rho^{-2}\rho_x+u\right)\right]_t+\left[\rho u\left(\lambda\rho^{-2}\rho_x+u\right)\right]_x=-\left(p+\frac{1}{2}|\boldsymbol{b}|^2\right)_x,$$

which, multiplied by $(\lambda\rho^{-2}\rho_x+u)$ and integrated over $\Omega\times(0,T)$, gives

$$\begin{aligned}&\frac{1}{2}\int_0^1\rho\left(\lambda\rho^{-2}\rho_x+u\right)^2dx\Big|_0^T+\lambda R\int_0^T\int_0^1\rho^{-2}\theta\rho_x^2dxdt\\ &=-\int_0^T\int_0^1\left[Ru\theta\rho_x+\left(R\rho\theta_x+\frac{4a}{3}\theta^3\theta_x+\boldsymbol{b}\cdot\boldsymbol{b}_x\right)\left(\lambda\rho^{-2}\rho_x+u\right)\right]dxdt.\end{aligned}\tag{2.23}$$

The right-hand side of (2.23) can be estimated term by term as follows. First, by (2.2), (2.4) and (2.5), we find

$$\begin{aligned}\left|R\int_0^T\int_0^1 u\theta\rho_xdxdt\right| &\le \varepsilon\int_0^T\int_0^1\theta\rho_x^2dxdt+C\varepsilon^{-1}\sup_{0\le t\le T}\|u\|_{L^2}^2\int_0^T\|\theta\|_{L^\infty}dt\\ &\le\varepsilon\int_0^t\int_0^1\theta\rho_x^2dxdt+C\varepsilon^{-1},\quad\forall\,\varepsilon\in(0,1).\end{aligned}\tag{2.24}$$

Second, using (2.2)–(2.5) and the Cauchy–Schwarz inequality, we have

$$\begin{aligned}&\left|R\int_0^T\int_0^1\rho\theta_x\left(\lambda\rho^{-2}\rho_x+u\right)dxdt\right|\\ &\le\varepsilon\int_0^T\int_0^1\theta\rho_x^2dxdt+C\varepsilon^{-1}\int_0^T\int_0^1\left(\frac{\kappa\theta_x^2}{\theta}+\theta u^2\right)dxdt\\ &\le\varepsilon\int_0^T\int_0^1\theta\rho_x^2dxdt+C\varepsilon^{-1}+C\varepsilon^{-1}\sup_{0\le t\le T}\|\theta\|_{L^\infty}\int_0^T\int_0^1\frac{\kappa\theta_x^2}{\theta^2}dxdt\\ &\le\varepsilon\int_0^T\int_0^1\theta\rho_x^2dxds+C\varepsilon^{-1}+C\varepsilon^{-1}\sup_{0\le t\le T}\|\theta\|_{L^\infty}.\end{aligned}\tag{2.25}$$

Next, it follows from (2.2), (2.4), (2.5) and Lemma 2.4 that

$$\begin{aligned}
&\left|\frac{4a}{3}\int_0^T\int_0^1 \theta^3\theta_x\left(\lambda\rho^{-2}\rho_x+u\right)dxdt\right| \\
&\quad\le \varepsilon\int_0^T\int_0^1 \theta\rho_x^2 dxdt + C\varepsilon^{-1}\int_0^T\int_0^1\left(\theta^5\theta_x^2+\theta u^2\right)dxdt \\
&\quad\le \varepsilon\int_0^T\int_0^1 \theta\rho_x^2 dxdt + C\varepsilon^{-1} + C\varepsilon^{-1}\int_0^T\int_0^1 (\kappa\theta^3\theta_x^2+\kappa\theta^6\theta_x^2)dxdt \\
&\quad\le \varepsilon\int_0^T\int_0^1 \theta\rho_x^2 dxdt + C\varepsilon^{-1} + C\varepsilon^{-1}\left(\int_0^T \|u_{xx}\|_{L^2}^2 dt\right)^{7/8}. \qquad (2.26)
\end{aligned}$$

Finally, it is easily seen from (2.2), (2.3), (2.5) and (2.7) that

$$\begin{aligned}
&\left|\int_0^T\int_0^1 (\boldsymbol{b}\cdot\boldsymbol{b}_x)\left(\lambda\rho^{-2}\rho_x+u\right)dxdt\right| \\
&\quad\le \varepsilon\int_0^T\int_0^1 \theta\rho_x^2 dxdt + C\varepsilon^{-1}\int_0^T\int_0^1\left(|\boldsymbol{b}|^2\frac{|\boldsymbol{b}_x|^2}{\theta}+\theta u^2\right)dxdt \\
&\quad\le \varepsilon\int_0^T\int_0^1 \theta\rho_x^2 dxdt + C\varepsilon^{-1} + C\varepsilon^{-1}\sup_{0\le t\le T}\|\boldsymbol{b}\|_{L^\infty}^2\int_0^T\int_0^1\frac{|\boldsymbol{b}_x|^2}{\theta}dxdt \\
&\quad\le \varepsilon\int_0^t\int_0^1 \theta\rho_x^2 dxds + C\varepsilon^{-1}. \qquad (2.27)
\end{aligned}$$

Now, plugging (2.24)–(2.27) into (2.23) and taking (2.5) into account, we immediately obtain (2.22) by choosing $\varepsilon > 0$ sufficiently small. □

By Lemma 2.5, we have the following temporary estimates on the gradient of the longitudinal velocity.

Lemma 2.6. *For any given $T > 0$, one has*

$$\sup_{0\le t\le T}\|u_x\|_{L^2}^2 + \int_0^T\left(\|u_{xx}\|_{L^2}^2+\|u_t\|_{L^2}^2\right)dt \le C + C\sup_{0\le t\le T}\|\theta\|_{L^\infty}^2, \qquad (2.28)$$

$$\int_0^T \|u_x\|_{L^\infty}^2 dt \le C + C\sup_{0\le t\le T}\|\theta\|_{L^\infty}, \qquad (2.29)$$

and

$$\int_0^T \|u_x\|_{L^4}^4 dt \le C + C\sup_{0\le t\le T}\|\theta\|_{L^\infty}^3. \qquad (2.30)$$

Proof. Multiplying $(1.1)_2$ by u_t and integrating it by parts over Ω_t, we deduce from (2.5) and the Cauchy–Schwarz inequality that

$$\sup_{0\le t\le T}\int_0^1 u_x^2(x,t)dx + \int_0^T\|u_t\|_{L^2}^2 dt \le C + C\int_0^T\int_0^1\left(u^2u_x^2+\theta_x^2+\theta^2\rho_x^2+\theta^6\theta_x^2+|\boldsymbol{b}|^2|\boldsymbol{b}_x|^2\right)dxdt. \qquad (2.31)$$

The terms on the right-hand side of (2.31) can be estimated as follows. By (2.7) and (2.10), we easily see that

$$\int_0^T\int_0^1\left(u^2u_x^2+|\boldsymbol{b}|^2|\boldsymbol{b}_x|^2\right)dxdt \le C + C\sup_{0\le t\le T}\|u_x\|_{L^2}, \qquad (2.32)$$

while, by (2.4) and (2.22), we find

$$\begin{aligned}
\int_0^T\int_0^1 \theta^2\rho_x^2 dxdt &\le \sup_{0\le t\le T}\|\rho_x\|_{L^2}^2\int_0^T\|\theta\|_{L^\infty}^2 dt \\
&\le C + C\sup_{0\le t\le T}\|\theta\|_{L^\infty} + C\left(\int_0^T\|u_{xx}\|_{L^2}^2 dt\right)^{7/8}. \qquad (2.33)
\end{aligned}$$

Moreover, due to (2.3) and (2.12), it holds that

$$\begin{aligned}\int_0^T\int_0^1\left(\theta_x^2+\theta^6\theta_x^2\right)dxdt &\le C+C\sup_{0\le t\le T}\|\theta\|_{L^\infty}^2\int_0^T\int_0^1\frac{\kappa\theta_x^2}{\theta^2}dxdt+C\left(\int_0^T\|u_{xx}\|_{L^2}^2dt\right)^{7/8}\\ &\le C+C\sup_{0\le t\le T}\|\theta\|_{L^\infty}^2+C\left(\int_0^T\|u_{xx}\|_{L^2}^2dt\right)^{7/8}.\end{aligned}\tag{2.34}$$

So, putting (2.32)–(2.34) into (2.31) and using the Cauchy–Schwarz inequality give

$$\sup_{0\le t\le T}\|u_x\|_{L^2}^2+\int_0^T\|u_t\|_{L^2}^2dt\le C+C\sup_{0\le t\le T}\|\theta\|_{L^\infty}^2+C\left(\int_0^T\|u_{xx}\|_{L^2}^2dt\right)^{7/8},\tag{2.35}$$

which, together with $(1.1)_2$, (2.5) and (2.32)–(2.34), also yields

$$\begin{aligned}\int_0^T\|u_{xx}\|_{L^2}^2dt &\le C\int_0^T\int_0^1\left(u^2u_x^2+\theta^6\theta_x^2+\theta^2\rho_x^2+\theta_x^2+|\boldsymbol{b}|^2|\boldsymbol{b}_x|^2+u_t^2\right)dxdt\\ &\le C+C\sup_{0\le t\le T}\|\theta\|_{L^\infty}^2+C\left(\int_0^T\|u_{xx}\|_{L^2}^2dt\right)^{7/8}.\end{aligned}\tag{2.36}$$

Thus, combining (2.35), (2.36) with the Young inequality leads to (2.28). As an immediate result, (2.29) follows from (2.6), (2.18), (2.28) and the Hölder inequality, and consequently, (2.30) holds due to (2.28) and (2.29). The proof of Lemma 2.6 is therefore complete. □

With the help of Lemmas 2.4–2.6, we now can prove the upper bound of the temperature, which in turn concludes the proofs of the above preliminary estimates.

Lemma 2.7. *Let $\kappa>0$ be a positive constant. Then for any given $T>0$,*

$$\sup_{0\le t\le T}\int_0^1\theta_x^2(x,t)dx+\int_0^T\int_0^1(1+\theta^3)\theta_t^2dxdt\le C,\tag{2.37}$$

which particularly implies

$$\sup_{0\le t\le T}\|\theta\|_{L^\infty}\le C.\tag{2.38}$$

Proof. Taking the L^2-inner product of (2.13) with θ_t and integrating the resulting equation by parts, we have from (2.5), (2.7), (2.30) and the Cauchy–Schwarz inequality that

$$\begin{aligned}&\sup_{0\le t\le T}\int_0^1\theta_x^2(x,t)dx+\int_0^T\int_0^1(1+\theta^3)\theta_t^2dxdt\\ &\le C+C\int_0^T\int_0^1\left(u^2\theta_x^2+u^2\theta^3\theta_x^2+\theta^2u_x^2+\theta^5u_x^2+u_x^4+|\boldsymbol{w}_x|^4+|\boldsymbol{b}_x|^4\right)dxdt\\ &\le C+C\sup_{0\le t\le T}\|\theta(t)\|_{L^\infty}^3+C\int_0^T\int_0^1\left(u^2\theta_x^2+u^2\theta^3\theta_x^2+\theta^2u_x^2+\theta^5u_x^2\right)dxdt,\end{aligned}\tag{2.39}$$

since it follows from (2.7) and (2.18) that

$$\begin{aligned}\int_0^T\left(\|\boldsymbol{w}_x\|_{L^4}^4+\|\boldsymbol{b}_x\|_{L^4}^4\right)dt &\le \sup_{0\le t\le T}\left(\|\boldsymbol{w}_x\|_{L^2}^2+\|\boldsymbol{b}_x\|_{L^2}^2\right)\int_0^T\left(\|\boldsymbol{w}_x\|_{L^\infty}^2+\|\boldsymbol{b}_x\|_{L^\infty}^2\right)dt\\ &\le C+C\int_0^T\left(\|\boldsymbol{w}_{xx}\|_{L^2}^2+\|\boldsymbol{b}_{xx}\|_{L^2}^2\right)dt\le C.\end{aligned}$$

Using (2.2), (2.3), (2.5), (2.9), (2.11) and (2.28), we find

$$\begin{aligned}\int_0^T\int_0^1\left(u^2\theta_x^2+u^2\theta^3\theta_x^2\right)dxdt &\le \sup_{0\le t\le T}\|u\|_{L^\infty}^2\left(\sup_{0\le t\le T}\|\theta\|_{L^\infty}^2\int_0^T\int_0^1\frac{\kappa\theta_x^2}{\theta^2}dxdt+\int_0^T\int_0^1\theta^3\theta_x^2dxdt\right)\\ &\le C\sup_{0\le t\le T}\|u_x\|_{L^2}\left(1+\sup_{0\le t\le T}\|\theta\|_{L^\infty}^2+\left(\int_0^T\|u_{xx}\|_{L^2}^2dt\right)^{1/2}\right)\\ &\le C+C\sup_{0\le t\le T}\|\theta\|_{L^\infty}^3,\end{aligned}$$

while, it is easily deduced from (2.6) that

$$\int_0^T\int_0^1 \left(\theta^2 u_x^2+\theta^5 u_x^2\right)dxdt \le \sup_{0\le t\le T}\left(\|\theta\|_{L^\infty}^2+\|\theta\|_{L^\infty}^5\right)\int_0^T \|u_x\|_{L^2}^2dt \le C+C\sup_{0\le t\le T}\|\theta\|_{L^\infty}^5,$$

which, inserted into (2.39), gives

$$\sup_{0\le t\le T}\|\theta_x\|_{L^2}^2+\int_0^T\int_0^1(1+\theta^3)\theta_t^2dxdt \le C+C\sup_{0\le t\le T}\|\theta\|_{L^\infty}^5. \tag{2.40}$$

By virtue of the boundedness of $\|\theta\|_{L^4}$ due to (1.8) and (2.2), we have

$$\|\theta\|_{L^\infty}^3 \le C+C\int_0^1\theta^2|\theta_x|dx \le C\left(1+\|\theta\|_{L^4}^2\|\theta_x\|_{L^2}\right)\le C\left(1+\|\theta_x\|_{L^2}\right),$$

which, together with (2.40) and the Young inequality, immediately yields (2.37) and (2.38). □

As a result of (2.37) and (2.38), we deduce the following from Lemmas 2.5 and 2.6.

Lemma 2.8. *For any given $T>0$, there exists a positive constant $C>0$ such that*

$$\sup_{0\le t\le T}\left(\|\rho_x\|_{L^2}+\|\rho_t\|_{L^2}+\|u\|_{L^\infty}+\|u_x\|_{L^2}\right)+\int_0^T\left(\|u_{xx}\|_{L^2}^2+\|u_x\|_{L^\infty}^2+\|u_x\|_{L^4}^4+\|u_t\|_{L^2}^2\right)dt\le C. \tag{2.41}$$

By Lemmas 2.1, 2.2, 2.7 and 2.8, the lower positive bound of the temperature and the higher order estimates of the solution can be easily obtained.

Lemma 2.9. *For any given $T>0$, one has*

$$\sup_{0\le t\le T}\left(\|(u_t,\boldsymbol{w}_t,\boldsymbol{b}_t,u_{xx},\boldsymbol{w}_{xx},\boldsymbol{b}_{xx})\|_{L^2}^2\right)+\int_0^T\|(u_{xt},\boldsymbol{w}_{xt},\boldsymbol{b}_{xt})\|_{L^2}^2dt\le C, \tag{2.42}$$

$$\inf_{(x,t)\in\overline{\Omega}_T}\theta(x,t)\ge C^{-1}, \tag{2.43}$$

$$\sup_{0\le t\le T}\left(\|\theta_t\|_{L^2}^2+\|\theta_{xx}\|_{L^2}^2\right)+\int_0^T\|\theta_{xt}\|_{L^2}^2dt\le C. \tag{2.44}$$

Proof. The proofs are sketched here for completeness. Differentiating $(1.1)_2$ with respect to t, multiplying the resulting equation by u_t in L^2, and integrating by parts, we have from Lemmas 2.1, 2.2, 2.7 and 2.8 and the Cauchy–Schwarz inequality that

$$\begin{aligned}
&\frac12\int_0^1\rho u_t^2(x,t)dx-\frac12\int_0^1\rho u_t^2(x,0)dx+\nu\int_0^t\|u_{xt}\|_{L^2}^2ds\\
&=\int_0^t\int_0^1\left[\left(R\rho\theta+\frac a3\theta^4+\frac12|\boldsymbol{b}|^2\right)_t u_{xt}-(\rho u)\left(u_t^2+uu_xu_t\right)_x-\rho u_xu_t^2\right]dxds\\
&\le\frac{\nu}{2}\int_0^t\|u_{xt}\|_{L^2}^2ds+C+C\int_0^t\left(1+\|u_x\|_{L^\infty}\right)\|u_t\|_{L^2}^2ds.
\end{aligned} \tag{2.45}$$

So, combining (2.45) with (2.5) and the Gronwall inequality gives

$$\sup_{0\le t\le T}\|u_t\|_{L^2}^2+\int_0^T\|u_{xt}\|_{L^2}^2dt\le C$$

which, together with $(1.1)_2$, implies that $\|u_{xx}\|_{L^2}\le C$. Analogously, one can also obtain the same estimates of $(\boldsymbol{w},\boldsymbol{b})$. The proof of (2.42) is thus complete.

Let $\mathcal{L}$ be a parabolic operator defined as follows:

$$\mathcal{L}(f)\triangleq\frac{\partial f}{\partial t}+u\frac{\partial f}{\partial x}+\frac{u_xp_\theta}{\rho e_\theta}f-\frac{1}{\rho e_\theta}\frac{\partial}{\partial x}\left(\kappa\frac{\partial f}{\partial x}\right).$$

Then, it follows from (1.8) and (2.13) that

$$\mathcal{L}(\theta)=\theta_t+u\theta_x+\frac{u_xp_\theta}{\rho e_\theta}\theta-\frac{(\kappa\theta_x)_x}{\rho e_\theta}=\frac{\lambda u_x^2+\mu|\boldsymbol{w}_x|^2+\nu|\boldsymbol{b}_x|^2}{\rho e_\theta}\geq 0.$$

Set

$$\tilde\theta(t)\triangleq e^{-Kt}\inf_{x\in[0,1]}\theta_0(x)\quad\text{with } K\triangleq\left\|\frac{u_xp_\theta}{\rho e_\theta}\right\|_{L^\infty(\Omega_T)}.$$

It is easy to check that

$$\mathcal{L}(\tilde\theta)\leq 0\leq\mathcal{L}(\theta),\qquad \theta|_{t=0}\geq\tilde\theta|_{t=0}\quad\text{and}\quad \theta_x|_{x=0,1}=\tilde\theta_x|_{x=0,1}=0.$$

Thus, the standard comparison argument yields

$$\theta(x,t)\geq\tilde\theta(t)\quad\text{for all } (x,t)\in\overline{\Omega}_T,$$

which finishes the proof of (2.43).

Finally, multiplying $(2.13)_t$ by θ_t in L^2, integrating by parts, and using Lemmas 2.1, 2.2, 2.7 and 2.8 and (2.42), one can prove (2.44) in a similar manner as that used for (2.42). The proof of Lemma 2.9 is therefore complete. □

With all the global a priori estimates established in Lemmas 2.1, 2.2 and 2.7–2.9, we can prove Theorem 1.1 in a standard way. Indeed, similar to the proofs in [31], one easily deduces from Lemmas 2.2 and 2.7–2.9 that

$$(u,\boldsymbol{w},\boldsymbol{b},\theta)\in\left(C^{1,1/2}_{x,t}(\Omega_T)\right)^6,\qquad (u_x,\boldsymbol{w}_x,\boldsymbol{b}_x,\theta_x)\in\left(C^{1/2,1/4}_{x,t}(\Omega_T)\right)^6\quad\text{and}\quad \rho\in C^{1/2,1/4}_{x,t}(\Omega_T).$$

Furthermore, by means of the classical theory in Ref. [34] of parabolic equations and the results in Ref. [35], we can obtain the desired Hölder continuity of the solutions stated in Theorem 1.1. The uniqueness of the solution can also be proved in a very standard way, basing on the energy method. Therefore, the proof of Theorem 1.1 is complete.

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. To this end, we first notice that the global estimates established in Lemmas 2.1–2.6 still hold under the conditions (1.13), (1.14) of Theorem 1.2. So, to complete the proof of Theorem 1.2, it suffices to prove the following.

Lemma 3.1. *Let the heat-conducting coefficient $\kappa=\kappa(\rho,\theta)$ be a function of (ρ,θ), satisfying (1.13) and (1.14) with $q>0$. Then, besides the estimates in Lemmas 2.1–2.6, one has*

$$\sup_{0\leq t\leq T}\left(\|\theta\|_{L^\infty}+\int_0^1(1+\theta)^{2q}\theta_x^2dx\right)+\int_0^T\int_0^1(1+\theta)^{q+3}\theta_t^2dxdt\leq C.\tag{3.1}$$

Proof. By the boundedness of $\|\theta\|_{L^4}$ due to (1.8) and (2.2), we infer from (1.13) that

$$\|\theta\|_{L^\infty}^{(q+4)/2}\leq C+C\int_0^1\theta^2\theta^{(q-2)/2}|\theta_x|dx\leq C+C\left(\int_0^1\frac{\kappa\theta_x^2}{\theta^2}dx\right)^{1/2},$$

which, combined with (2.3) and the boundedness of $\|\theta\|_{L^4}$, gives

$$\int_0^T\left(\|\theta\|_{L^\infty}^{q+4}+\|\theta\|_{L^{q+8}}^{q+8}\right)dt\leq C,\quad\forall\, q\geq 0.\tag{3.2}$$

Moreover, using (2.3), (2.11) and (2.28), we have

$$\int_0^T\int_0^1(1+\theta)^3\theta_x^2dxdt\leq C\int_0^T\int_0^1\left(\theta^2\frac{\kappa\theta_x^2}{\theta^2}+\kappa\theta^3\theta_x^2\right)dxdt\leq C\left(1+\Theta^2\right),\tag{3.3}$$

where and whereafter we denote $\Theta\triangleq\sup_{0\leq t\leq T}\|\theta\|_{L^\infty}$ for simplicity.

To prove (3.1), we first observe from $(1.1)_1$ that

$$\begin{aligned}(\kappa\theta_x)(\kappa\theta_t)_x&=(\kappa\theta_x)(\kappa\theta_x)_t+(\kappa\theta_x)\left(\kappa_\rho\rho_x\theta_t-\kappa_\rho\rho_t\theta_x\right)\\&=\frac{1}{2}\frac{d}{dt}(\kappa\theta_x)^2+\kappa\kappa_\rho\theta_x\left(u\rho_x\theta_x+\rho u_x\theta_x+\rho_x\theta_t\right).\end{aligned}$$

Now, similar to the derivations of (2.39) and (2.40), multiplying (2.13) by $\kappa\theta_t$ in L^2, and integrating by parts, we deduce from (1.13), (1.14) and the Cauchy–Schwarz inequality that

$$\begin{aligned}
&\sup_{0\le t\le T}\int_0^1(\kappa\theta_x)^2(x,t)dx+\int_0^T\int_0^1(1+\theta)^{q+3}\theta_t^2dxdt\\
&\le C\int_0^T\int_0^1(1+\theta)^q\left(u^2\theta_x^2+u^2\theta^3\theta_x^2+\theta^5u_x^2+\theta^2u_x^2+u_x^4+|\boldsymbol{w}_x|^4+|\boldsymbol{b}_x|^4\right)dxdt\\
&\quad+C\int_0^T\int_0^1\left(\left|\kappa\kappa_\rho\theta_x^2(u\rho_x+\rho u_x)\right|+\left|\kappa\kappa_\rho\theta_x\rho_x\theta_t\right|\right)dxdt+C\\
&\le C\left(1+\Theta^{q+5}\right)+C\int_0^T\int_0^1\left(\left|\kappa\kappa_\rho\theta_x^2(u\rho_x+\rho u_x)\right|+\left|\kappa\kappa_\rho\theta_x\rho_x\theta_t\right|\right)dxdt. \qquad (3.4)
\end{aligned}$$

Using (1.13) and (1.14), we see that

$$\begin{aligned}
&\int_0^T\int_0^1\left(\left|\kappa\kappa_\rho\theta_x^2(u\rho_x+\rho u_x)\right|+\left|\kappa\kappa_\rho\theta_x\rho_x\theta_t\right|\right)dxdt\\
&\le C\int_0^T\int_0^1(|u\rho_x|+|u_x|)\,|\kappa\theta_x|^2dxdt+\varepsilon\int_0^T\int_0^1(1+\theta)^{q+3}\theta_t^2dxdt\\
&\quad+C\varepsilon^{-1}\int_0^T\int_0^1(1+\theta)^{q-3}\,\rho_x^2|\kappa\theta_x|^2dxdt\\
&\le\varepsilon\int_0^T\int_0^1(1+\theta)^{q+3}\theta_t^2dxdt+C\varepsilon^{-1}\left(1+\Theta^{2+(q-3)_+}\right)\int_0^T\|\kappa\theta_x\|_{L^\infty}^2dt, \qquad (3.5)
\end{aligned}$$

since it follows from (2.2), (2.5), (2.22) and (2.28) and the Hölder inequality that

$$\|u\rho_x\|_{L^1}+\|u_x\|_{L^1}+\|\rho_x\|_{L^2}^2\le C\left(1+\|u_x\|_{L^2}^2+\|\rho_x\|_{L^2}^2\right)\le C\left(1+\Theta^2\right).$$

Thus, putting (3.5) into (3.4) and choosing $\varepsilon>0$ small enough give

$$\sup_{0\le t\le T}\int_0^1(\kappa\theta_x)^2dx+\int_0^T\int_0^1(1+\theta)^{q+3}\theta_t^2dxdt\le C\left(1+\Theta^{q+5}\right)+C\left(1+\Theta^{2+(q-3)_+}\right)\int_0^T\|\kappa\theta_x\|_{L^\infty}^2dt. \qquad (3.6)$$

To deal with the second term on the right-hand side of (3.6), we first utilize (2.5) and (2.13) to get that

$$|(\kappa\theta_x)_x|\le C(1+\theta)^3\left(|\theta_t|+|u_x\theta|+|u\theta_x|\right)+u_x^2+|\boldsymbol{w}_x|^2+|\boldsymbol{b}_x|^2,$$

and consequently,

$$\int_0^T\int_0^1(1+\theta)^{q-3}|(\kappa\theta_x)_x|^2dxdt\le C\int_0^T\int_0^1(1+\theta)^{q+3}\theta_t^2dxdt+C\left(1+\Theta^{3+(q-3)_+}\right). \qquad (3.7)$$

Here, we have used (2.2), (2.5)–(2.7), (2.28), (2.30), (3.2) and (3.3) to get that

$$\int_0^T\int_0^1(1+\theta)^{q+3}\theta^2u_x^2dxdt\le C\Theta\sup_{0\le t\le T}\|u_x\|_{L^2}^2\le C\left(1+\Theta^3\right),$$

$$\int_0^T\int_0^1(1+\theta)^{q+3}u^2\theta_x^2dxdt\le C\left(1+\Theta^2\right)\sup_{0\le t\le T}\|u\|_{L^\infty}^2\le C\left(1+\Theta^3\right),$$

and

$$\int_0^T\int_0^1(1+\theta)^{q-3}\left(u_x^4+|\boldsymbol{w}_x|^4+|\boldsymbol{b}_x|^4\right)dxdt\le C\left(1+\Theta^{3+(q-3)_+}\right).$$

As a result of (3.3) and (3.7), we find

$$\begin{aligned}
\left(1+\Theta^{2+(q-3)_+}\right)\int_0^T\|\kappa\theta_x\|_{L^\infty}^2dt&\le C\left(1+\Theta^{2+(q-3)_+}\right)\int_0^T\int_0^1|\kappa\theta_x|\,|(\kappa\theta_x)_x|dxdt\\
&\le C\left(1+\Theta^{3+(q-3)_+}\right)\left(\int_0^T\int_0^1(1+\theta)^{q-3}|(\kappa\theta_x)_x|^2dxdt\right)^{1/2}
\end{aligned}$$

$$\leq C\left(1+\Theta^{3+(q-3)_+}\right)\left(\int_0^T\int_0^1(1+\theta)^{q+3}\theta_t^2dxdt\right)^{1/2}+C\left(1+\Theta^{(9+3(q-3)_+)/2}\right)$$

$$\leq \varepsilon\int_0^T\int_0^1(1+\theta)^{q+3}\theta_t^2dxdt+C\varepsilon^{-1}\left(1+\Theta^{6+2(q-3)_+}\right). \tag{3.8}$$

Therefore, putting (3.8) into (3.6) and choosing $\varepsilon > 0$ sufficiently small, we arrive at

$$\sup_{0\leq t\leq T}\int_0^1(1+\theta)^{2q}\theta_x^2dx+\int_0^T\int_0^1(1+\theta)^{q+3}\theta_t^2dxdt\leq C\left(1+\Theta^{6+2(q-3)_+}\right). \tag{3.9}$$

Thanks to (2.12) and (2.28), we have

$$\|\theta\|_{L^\infty}^{q+\frac{13}{2}}\leq C+C\int_0^1\theta^{\frac{11}{2}}\theta^q\theta_xdx\leq C+C(1+\Theta)\left(\int_0^1\theta^{2q}\theta_x^2dx\right)^{1/2},$$

from which and (3.9) it follows that

$$\Theta^{2q+13}\leq C+C\left(1+\Theta^2\right)\sup_{0\leq t\leq T}\int_0^1\theta^{2q}\theta_x^2dx\leq C\left(1+\Theta^{8+2(q-3)_+}\right). \tag{3.10}$$

Combining (3.10) with the Young inequality yields $\Theta \leq C$. This, together with (3.9), finishes the proof of (3.1) immediately. □

With the help of (3.1), we can obtain all the estimates as those in Lemmas 2.8 and 2.9 in the exactly same way, and thus, the proof of Theorem 1.2 is complete.

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