

A note on the γ topology in dual Banach spaces

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ARTICLE INFO

Article history:

Received 6 March 2012

Available online 7 June 2013

Submitted by Bernardo Cascales

Keywords:

Dual spaces

 w^* -compact subsets

Convex sets

 γ topology

Separation theorems

ABSTRACT

If X is a Banach space, the γ topology of the dual X^* is the topology of the uniform convergence on bounded separable subsets of X . We prove, among other things, that $\overline{\text{co}}(K) = \overline{\text{co}}^{\gamma}(K)$ for every w^* -compact subset K of X^* . As an application we obtain some new separation results.

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1. Introduction

$|A|$ will be the cardinality of a set A . If $(X, \|\cdot\|)$ is a Banach space, let $B(X)$ and $S(X)$ be the closed unit ball and unit sphere of X , respectively, and X^* its topological dual. The weak* topology of X^* is denoted by w^* and the weak topology of X by w . If A is a subset of X , then $[A]$ and $\overline{[A]}$ denote the linear hull and the closed linear hull of A , respectively. $\text{co}(A)$ denotes the convex hull of A , $\overline{\text{co}}(A)$ is the $\|\cdot\|$ -closure of $\text{co}(A)$ and, if $A \subset X^*$, we put $\overline{\text{co}}^{w^*}(A)$ for the w^* -closure of $\text{co}(A)$. If κ is a cardinal and $A \subset X^{**}$ a subset of the bidual X^{**} , A_κ will denote the following subset of X^{**} :

$$A_\kappa := \bigcup \{\overline{D}^{w^*} : D \subset A, |D| \leq \kappa\}.$$

Clearly, X_κ is a norm-closed subspace of X^{**} . For every cardinal $\kappa \geq 1$ let γ_κ and \mathcal{T}_κ be the following topologies of X^* : (i) γ_κ will be the topology on X^* of the uniform convergence on bounded subsets $D \subset X$ with $|D| \leq \kappa$; (ii) $\mathcal{T}_\kappa := \sigma(X^*, X_\kappa)$, that is, \mathcal{T}_κ will be the topology on X^* of the pointwise convergence on X_κ . Let us denote the topology γ_{\aleph_0} by γ . Cascales, Muñoz, Namioka, Orihuela, etc., used this topology γ very profitably to demonstrate numerous and profound results (see [3,4,2,1,7]).

In this paper we are concerned with the topologies γ_κ and \mathcal{T}_κ , and mainly with the topologies $\gamma = \gamma_{\aleph_0}$ and \mathcal{T}_{\aleph_0} . Actually, the topology \mathcal{T}_κ is the weak topology corresponding to the space (X^*, γ_κ) and so $(X^*, \gamma_\kappa)^* = (X^*, \mathcal{T}_\kappa)^* = X_\kappa$. Thus the topologies γ_κ and \mathcal{T}_κ have the same closed convex subsets. The main goal of this paper is to show that, when $K \subset X^*$ is a w^* -compact subset, then $\overline{\text{co}}(K) = \overline{\text{co}}^{\gamma_\kappa}(K) = \overline{\text{co}}^{\mathcal{T}_\kappa}(K)$ for every $\kappa \geq \aleph_0$. From this equality we obtain, for instance, that if $K_1, K_2 \subset X^*$ are two w^* -compact subsets such that $\inf\{\|a - b\| : a \in \overline{\text{co}}(K_1), b \in \overline{\text{co}}(K_2)\} = d > 0$, the subspace X_{\aleph_0} completely separates the convex subsets $\overline{\text{co}}(K_1)$ and $\overline{\text{co}}(K_2)$, that is, there exists $\psi \in S(X_{\aleph_0})$ such that

$$\inf\langle \psi, \overline{\text{co}}(K_1) \rangle = \sup\langle \psi, \overline{\text{co}}(K_2) \rangle + d.$$

Haydon proved in [6] that a Banach space X does not contain ℓ_1 if and only if for every w^* -compact subset $K \subset X^*$ we have that $\overline{\text{co}}^{w^*}(K) = \overline{\text{co}}(K)$. Hence, since always $\overline{\text{co}}(K) \subset \overline{\text{co}}^{\gamma_\kappa}(K) = \overline{\text{co}}^{\mathcal{T}_\kappa}(K) \subset \overline{\text{co}}^{w^*}(K)$, $\forall \kappa \geq \aleph_0$ (because τ_κ is stronger

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than w^*), our results are new only for non-separable Banach spaces containing ℓ_1 . If $\ell_1 \subset X$, in general $\overline{\text{co}}^{w^*}(K) \neq \overline{\text{co}}(K)$ and, moreover, if $z \in X_{\aleph_0}$, then z is a w^* -cluster point (but in general not a limit point) of a sequence in X , and these facts are the reasons for the difficulties in proving the equality $\overline{\text{co}}(K) = \overline{\text{co}}^{\gamma_\kappa}(K) = \overline{\text{co}}^{\mathcal{T}_\kappa}(K)$ for every $\kappa \geq \aleph_0$.

The organization of the paper is the following. In Section 2 we see some auxiliary facts. We prove in Section 3 that $\overline{\text{co}}(K) = \overline{\text{co}}^{\gamma_\kappa}(K)$ for every w^* -compact subset $K \subset X^*$ and every cardinal $\kappa \geq \aleph_0$. Finally, in Section 4 we give some separation theorems as an application of the main result.

Let us introduce some notation. If X is a Banach space, A and B are two subsets of X and $x \in X$, we define the following distances:

$$\text{dist}(x, A) := \inf\{\|x - a\| : a \in A\} \quad \text{and} \quad \text{dist}(A, B) := \inf\{\|a - b\| : a \in A, b \in B\}.$$

If (Y, τ) is a topological space, A a subset of Y and $x \in \bar{A}$, we define the “tightness” of x with respect to A (for short, $t_\tau(x, A)$) as follows:

$$t_\tau(x, A) = \min\{|D| : D \subset A, x \in \bar{D}\}.$$

2. Preliminaries and basic facts

If X is a Banach space, the following facts are elementary:

(1) If $A \subset X^{**}$, then $(A_\kappa)_\rho = A_{\kappa \vee \rho}$ for every pair of cardinals κ, ρ . Clearly, $X = X_\kappa$ if $1 \leq \kappa < \aleph_0$, and $X_\kappa = X^{**}$ whenever $\kappa \geq \text{Dens}(X)$. Moreover, if $\kappa \leq \rho$, then $X_\kappa \subset X_\rho$.

(2) If X is reflexive, then $X^{**} = X_\kappa$, $\forall \kappa \geq 1$. If X is separable, then $X^{**} = X_{\aleph_0}$, but this equality also holds for some non-reflexive non-separable Banach spaces, for instance, for the Banach spaces X such that X^* has the property (C) of Corson (by Pol [9, p. 147]).

(3) Obviously, if $\kappa \geq \aleph_0$, then $X_\kappa = \bigcup\{\bar{E}^{w^*} : E \subset X \text{ is a subspace such that } \text{Dens}(E) \leq \kappa\}$. Thus $B(X_\kappa) = (B(X))_\kappa$ and $X_\kappa = \bigcup\{\bar{A}^{w^*} : A \subset X \text{ is a bounded subset such that } |A| \leq \kappa\}$ for all $\kappa \geq 1$.

(4) Clearly, $w^* \leq \mathcal{T}_\kappa \leq w$ and $\mathcal{T}_\kappa \leq \gamma_\kappa \leq \mathcal{T}_{\|\cdot\|}$ (= the norm topology). Moreover it is easy to see that if $\kappa \geq \aleph_0$, $B_0 := X_\kappa \cap B(X^{**}) = B(X_\kappa)$ is a (James) boundary of $B(X^{**})$, which means that every $x \in X$ attains on B_0 its maximum on $B(X^*)$. So by Pfitzner [8] a subset $D \subset X^*$ is w -compact if and only if D is \mathcal{T}_{\aleph_0} -compact. Actually all the linear spaces (X^*, \mathcal{T}) where either $\mathcal{T} = w$ or $\mathcal{T} = \mathcal{T}_\kappa$, $\aleph_0 \leq \kappa$ have the same compact subsets and obey the Krein–Smulian theorem, that is, if $K \subset X^*$ is a \mathcal{T} -compact subset, then $\overline{\text{co}}^{\mathcal{T}}(K)$ is also a \mathcal{T} -compact subset.

(5) It is easy to see that for every cardinal κ , the following statements are equivalent:

(5-1) $X^{**} = X_\kappa$, that is, $\mathcal{T}_\kappa = w$.

(5-2) X_κ separates points and norm-closed convex subsets of X^* .

(5-3) $\overline{\text{co}}(D) = \overline{\text{co}}^{\mathcal{T}_\kappa}(D)$ for every subset $D \subset X^*$.

Proposition 2.1. *Let X be a Banach space. For every cardinal $\kappa \geq 1$ we have $(X^*, \gamma_\kappa)^* = (X^*, \mathcal{T}_\kappa)^* = X_\kappa$. So, the spaces (X^*, γ_κ) and $(X^*, \mathcal{T}_\kappa)$ have the same convex closed subsets.*

Proof. Obviously $(X^*, \mathcal{T}_\kappa)^* = X_\kappa$ because $\mathcal{T}_\kappa := \sigma(X^*, X_\kappa)$. Since the identity $\text{id} : (X^*, \gamma_\kappa) \rightarrow (X^*, \mathcal{T}_\kappa)$ is continuous, clearly $X_\kappa \subset (X^*, \gamma_\kappa)^*$. Let us see that $(X^*, \gamma_\kappa)^* \subset X_\kappa$. Let $u \in (X^*, \gamma_\kappa)^*$. First, $u \in X^{**}$ because $\gamma_\kappa \leq \tau_{\|\cdot\|}$. Since u is continuous for the γ_κ topology, there exists a bounded subset $A \subset X$ with $|A| \leq \kappa$ such that for some $\epsilon > 0$ we have

$$\{x^* \in X^* : \sup |\langle x^*, A \rangle| \leq \epsilon\} \subset \{x^* \in X^* : |\langle u, x^* \rangle| \leq 1\}. \quad (2.1)$$

Claim. $u \in \overline{[A]}^{w^*}$.

Indeed, if $u \notin \overline{[A]}^{w^*}$, there exists $x_0^* \in X^*$ such that $x_0^* \perp \overline{[A]}^{w^*}$, but $\langle u, x_0^* \rangle \neq 0$. The fact that $x_0^* \perp \overline{[A]}^{w^*}$ implies from (2.1) that $|\langle u, \lambda x_0^* \rangle| \leq 1$, $\forall \lambda \in \mathbb{R}$, and this contradicts the fact that $\langle u, x_0^* \rangle \neq 0$ and proves the claim.

Finally, from the claim it follows that $u \in X_\kappa$ because $\overline{[A]}^{w^*} \subset X_\kappa$. \square

Lemma 2.2. *Let X be a Banach space, $A \subset X$ an infinite subset, $C \subset X$ a convex subset and $u \in X^{**} \setminus X$ a vector such that $u \in \overline{A}^{w^*} \cap \overline{C}^{w^*}$. Then*

$$t_{w^*}(u, C) \leq t_{w^*}(u, A) \leq |A|.$$

Proof. Without loss of generality we assume that $|A| = t_{w^*}(u, A)$. Let $\text{co}_{\mathbb{Q}}(A)$ be the family of convex combinations of elements of A with rational coefficients. Clearly, $|\text{co}_{\mathbb{Q}}(A)| = |A|$ because A is infinite. For each $a \in \text{co}_{\mathbb{Q}}(A)$ choose a vector $c_a \in \bar{C}$ such that $\|c_a - a\| \leq 2\text{dist}(a, C)$. Clearly $\{c_a : a \in \text{co}_{\mathbb{Q}}(A)\} \leq |A|$. Thus taking into account that $t_{w^*}(u, C) = t_{w^*}(u, \bar{C})$, it is enough to prove the following claim.

Claim. $u \in \overline{\{c_a : a \in \text{co}_{\mathbb{Q}}(A)\}}^{w^*}$.

Indeed, if $\epsilon > 0$ and $x_1^*, \dots, x_p^* \in S(X^*)$, consider the following convex w^* -neighborhood of u :

$$W(u; x_1^*, \dots, x_p^*; \epsilon) := \{z \in X^{**} : |\langle z - u, x_i^* \rangle| \leq \epsilon : i = 1, \dots, p\}.$$

Let $A_0 := A \cap W(u; x_1^*, \dots, x_p^*; \epsilon/2)$. Obviously $\text{co}(A_0) \subset W(u; x_1^*, \dots, x_p^*; \epsilon/2)$.

Subclaim. $\inf\{\|c - a\| : c \in \bar{C}, a \in \overline{\text{co}}(A_0)\} = \inf\{\|c - a\| : c \in \bar{C}, a \in \overline{\text{co}}_{\mathbb{Q}}(A_0)\} = 0$.

Indeed, clearly $\overline{\text{co}}(A_0) = \overline{\text{co}}_{\mathbb{Q}}(A_0)$ and so

$$\inf\{\|c - a\| : c \in \bar{C}, a \in \overline{\text{co}}(A_0)\} = \inf\{\|c - a\| : c \in \bar{C}, a \in \overline{\text{co}}_{\mathbb{Q}}(A_0)\}.$$

Suppose that $\inf\{\|c - a\| : c \in \bar{C}, a \in \overline{\text{co}}_{\mathbb{Q}}(A_0)\} > 0$. By the Hahn–Banach separation theorem there exists $x^* \in X^*$ such that

$$\inf\langle x^*, C \rangle > \sup\langle x^*, \text{co}_{\mathbb{Q}}(A_0) \rangle. \quad (2.2)$$

As $\langle u, x^* \rangle \geq \inf\langle x^*, C \rangle$, because $u \in \bar{C}^{w^*}$, and also $\sup\langle x^*, \text{co}_{\mathbb{Q}}(A_0) \rangle \geq \langle u, x^* \rangle$, since $u \in \bar{A}_0^{w^*} \subset \overline{\text{co}}_{\mathbb{Q}}^{w^*}(A_0)$, by (2.2) we get $\langle u, x^* \rangle > \langle u, x^* \rangle$, a contradiction which proves the subclaim.

Therefore there exists $a \in \text{co}_{\mathbb{Q}}(A_0)$ such that $\|c_a - a\| \leq \epsilon/2$. Hence for $i = 1, \dots, p$ we have

$$|\langle c_a - u, x_i^* \rangle| \leq |\langle c_a - a, x_i^* \rangle| + |\langle a - u, x_i^* \rangle| \leq \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.$$

Thus $c_a \in W(u; x_1^*, \dots, x_p^*; \epsilon)$ and this proves the claim and the lemma. \square

Corollary 2.3. Let X be a Banach space and κ a cardinal.

(1) If $Y \subset X$ is a subspace, then $Y_{\kappa} = Y^{**} \cap X_{\kappa}$.

(2) If $X^{**} = X_{\kappa}$, then $Y^{**} = Y_{\kappa}$ for every Banach space Y that is either a subspace of X or a quotient of X .

Proof. (1) Clearly $Y_{\kappa} \subset Y^{**} \cap X_{\kappa}$. Let $z \in Y^{**} \cap X_{\kappa}$. If $z \in X$, then $z \in Y^{**} \cap X = Y \subset Y_{\kappa}$. Let $z \notin X$. By Lemma 2.2 we have $t_{w^*}(z, Y) \leq t_{w^*}(z, X) \leq \kappa$. Thus $z \in Y_{\kappa}$.

(2) If Y is a subspace of X , the statement follows from (1). Assume that the Banach space Y is a quotient of X and let $Q : X \rightarrow Y$ be the quotient mapping. Then $Q^{**} : X^{**} \rightarrow Y^{**}$ is also a quotient mapping that is w^* - w^* -continuous. Let $u \in Y^{**}$ and $v \in X^{**}$ be such that $Q^{**}v = u$. As $X^{**} = X_{\kappa}$, there exists a subset $A \subset X$ such that $|A| \leq \kappa$ and $v \in \bar{A}^{w^*}$. Since Q^{**} is w^* - w^* -continuous, we get $u \in \overline{Q(A)}^{w^*}$. Thus $Y^{**} = Y_{\kappa}$. \square

Let us study the following elementary results.

Proposition 2.4. Let $\kappa \geq \aleph_0$ be a cardinal, X a Banach space, A a subset of the dual X^* and $z \in X^*$. The following are equivalent:

(1) $z \in \bar{A}^{\mathcal{T}_{\kappa}}$.

(2) For every subspace $Y \subset X$ we have $i^*(z) \in \overline{i^*(A)}^{\mathcal{T}_{\kappa}}$, $i : Y \rightarrow X$ being the canonical inclusion mapping.

(3) For every subspace $Y \subset X$ with $\text{Dens}(Y) \leq \kappa$ we have $i^*(z) \in \overline{i^*(A)}^{\mathcal{T}_{\kappa}} = \overline{i^*(A)}^{w^*}$, $i : Y \rightarrow X$ being the canonical inclusion mapping.

Proof. (1) \Rightarrow (2) because the adjoint operator i^* is \mathcal{T}_{κ} - \mathcal{T}_{κ} -continuous. (2) \Rightarrow (3) is obvious. Let us see (3) \Rightarrow (1). Suppose that $z \notin \bar{A}^{\mathcal{T}_{\kappa}}$. Then there exist $\epsilon > 0$ and $u_i \in X_{\kappa}$, $i = 1, \dots, r$, such that $W(z; u_1, \dots, u_r; \epsilon) \cap A = \emptyset$, where

$$W(z; u_1, \dots, u_r; \epsilon) := \{x^* \in X^* : |\langle u_i, z - x^* \rangle| < \epsilon, i = 1, \dots, r\}$$

is a “standard” open neighborhood of z in $(X^*, \mathcal{T}_{\kappa})$. Let $Y \subset X$ be a subspace with $\text{Dens}(Y) \leq \kappa$ such that $u_i \in Y^{**} = Y_{\kappa}$, $i = 1, 2, \dots, r$, and $i : Y \rightarrow X$ be the canonical inclusion mapping. Then

$$W(i^*z; u_1, \dots, u_r; \epsilon) := \{y^* \in Y^* : |\langle u_i, i^*z - y^* \rangle| < \epsilon, i = 1, \dots, r\}$$

is an open neighborhood of i^*z in $(Y^*, \mathcal{T}_{\kappa})$, that satisfies

$$(i^*)^{-1}(W(i^*z; u_1, \dots, u_r; \epsilon)) = W(z; u_1, \dots, u_r; \epsilon).$$

Thus $W(i^*z; u_1, \dots, u_r; \epsilon) \cap i^*(A) = \emptyset$ because $W(z; u_1, \dots, u_r; \epsilon) \cap A = \emptyset$. This contradicts (3) and proves the statement. \square

If I is a set and $J \subset I$ a subset, $P_J : \ell_{\infty}(I) \rightarrow \ell_{\infty}(J)$ will be the restriction mapping to J , that is, $P_J(x) = x \upharpoonright J$, $\forall x \in \ell_{\infty}(I)$. Note that, if $i_J : \ell_1(J) \rightarrow \ell_1(I)$ is the inclusion mapping, then $P_J = i_J^*$. If $y \in \ell_1(I)$, let $\text{supp}(y) := \{i \in I : y_i \neq 0\}$.

Proposition 2.5. Let I be an infinite set, $X = \ell_1(I)$, $\kappa \geq \aleph_0$ a cardinal, A a convex subset of $X^* = \ell_\infty(I)$ and $z \in \ell_\infty(I)$. The following are equivalent:

- (1) $z \in \overline{A}^{\mathcal{T}_\kappa}$.
- (2) For every subset $J \subset I$ with $|J| \leq \kappa$, we have $P_J(z) \in \overline{P_J(A)}$.

Proof. (1) \Rightarrow (2). Let $i : \ell_1(J) \rightarrow \ell_1(I)$ be the canonical inclusion mapping. Then $i^* = P_J$ and $i^*z \in \overline{P_J(A)}^{\mathcal{T}_\kappa}$ by Proposition 2.4. Finally, note that $\overline{P_J(A)}^{\mathcal{T}_\kappa} = \overline{P_J(A)}$ because $\text{Dens}(\ell_1(J)) \leq \kappa$ and $P_J(A)$ is convex.

(2) \Rightarrow (1). Let $Y \subset \ell_1(I)$ be a subspace with $\text{Dens}(Y) \leq \kappa$. Choose a subset $J \subset I$ with $|J| \leq \kappa$ such that $\text{supp}(Y) := \bigcup \{\text{supp}(y) : y \in Y\} \subset J$. So actually Y is a subspace of $\ell_1(J)$. Let $i_2 : \ell_1(J) \rightarrow \ell_1(I)$, $i_1 : Y \rightarrow \ell_1(J)$ and $i : Y \rightarrow \ell_1(I)$ be the canonical inclusion mappings. Clearly, $i = i_2 \circ i_1$ and $P_J = i_2^* \circ i_1^*$. By hypothesis, $P_J(z) \in \overline{P_J(A)}$. Hence

$$i^*(z) = i_1^* \circ i_2^*(z) = i_1^*(P_J(z)) \in i_1^*(\overline{P_J(A)}) \subset \overline{i_1^*(P_J(A))} = \overline{i^*(A)}.$$

Now (1) follows from Proposition 2.4. \square

3. The main result: $\overline{\text{co}}(K) = \overline{\text{co}}^{\mathcal{T}_\kappa}(K)$ for $\kappa \geq \aleph_0$

If X is a Banach space, $\kappa \geq \aleph_0$ a cardinal and $K \subset X^*$ a w^* -compact subset, clearly

$$\overline{\text{co}}(K) \subset \overline{\text{co}}^{\mathcal{T}_\kappa}(K) \subset \overline{\text{co}}^{\mathcal{T}_{\aleph_0}}(K) \subset \overline{\text{co}}^{w^*}(K). \quad (3.1)$$

Here we show that actually $\overline{\text{co}}(K) = \overline{\text{co}}^{\mathcal{T}_\kappa}(K)$, $\forall \kappa \geq \aleph_0$. In order to prove this equality it is enough to consider the case $\kappa = \aleph_0$ by (3.1). In the first step of our approach the problem is reduced to looking into the dual spaces $\ell_1(I)^* = \ell_\infty(I)$, as we see in the following result.

Proposition 3.1. The following statements are equivalent:

- (1) There exist a dual Banach space X^* and a w^* -compact subset $K \subset X^*$ such that $\overline{\text{co}}^{\mathcal{T}_{\aleph_0}}(K) \neq \overline{\text{co}}(K)$.
- (2) There exist an uncountable set I and a w^* -compact subset $H \subset \ell_\infty(I)$ such that $\overline{\text{co}}^{\mathcal{T}_{\aleph_0}}(H) \neq \overline{\text{co}}(H)$.

Proof. (2) \Rightarrow (1) is obvious. Let us prove (1) \Rightarrow (2). Clearly $\text{Dens}(X) \geq \aleph_1$. Let I be a set with cardinality $|I| = \text{Dens}(X)$. It is well known that there exists a 1-quotient operator $Q : \ell_1(I) \rightarrow X$. Thus, if $H := Q^*(K)$, then H is a w^* -compact subset of $\ell_\infty(I)$ such that $\overline{\text{co}}^{\mathcal{T}_{\aleph_0}}(H) \neq \overline{\text{co}}(H)$. \square

Therefore, in the light of the previous proposition, Proposition 3.1, the natural place to investigate is the space $\ell_\infty(I)$ with I uncountable.

Lemma 3.2. Let I be a set, $K \subset [0, 1]^I$ a compact subset, $z \in [0, 1]^I$ and $\epsilon > 0$ such that for every finite subset $F \subset I$ there exists $k_F \in K$ fulfilling $|\pi_i(k_F - z)| \leq \epsilon$, $\forall i \in F$, $\pi_i : [0, 1]^I \rightarrow [0, 1]$ being the i -projection mapping. Then there exists $k_0 \in K$ such that $\|k_0 - z\|_\infty \leq \epsilon$.

Proof. By compactness, it is enough to note that the net $\{k_F : F \subset I \text{ finite}\}$ contains a subnet that converges to some $k_0 \in K$. Clearly, $\|k_0 - z\|_\infty \leq \epsilon$. \square

Proposition 3.3. Let I be a non-empty set. Then every w^* -compact subset $K \subset \ell_\infty(I)$ satisfies the equality $\overline{\text{co}}(K) = \overline{\text{co}}^{\mathcal{T}_{\aleph_0}}(K)$.

Proof. If $|I| \leq \aleph_0$ the statement is trivial because in this case $(\ell_1(I))_{\aleph_0} = \ell_\infty(I)^*$ and so $\mathcal{T}_{\aleph_0} = w$ in $\ell_\infty(I)$. Suppose that $|I| > \aleph_0$. Fix $z \in \overline{\text{co}}^{\mathcal{T}_{\aleph_0}}(K)$ and prove that $z \in \overline{\text{co}}(K)$. By Proposition 2.5 we have $P_J(z) \in \overline{\text{co}}(P_J(K))$ for every countable subset $J \subset I$.

Claim. Fix $\epsilon > 0$. There exist $p \in \mathbb{N}$ and $\lambda_i \in \mathbb{Q}^+$, $i = 1, 2, \dots, p$, with $\sum_{i=1}^p \lambda_i = 1$ such that, if $H_\epsilon := \sum_{i=1}^p \lambda_i \cdot K$, then there exists $k_\epsilon \in H_\epsilon$ such that $\|z - k_\epsilon\| \leq \epsilon$.

Indeed, consider I endowed with the discrete topology, which is a metric topology with the complete metric, $\forall i, j \in I$, $\delta(i, j) = 1$, if $i \neq j$, and $\delta(i, j) = 0$ if $i = j$. Consider $I^\mathbb{N}$ with the product topology τ_p , which is also a metric topology with the complete metric,

$$\forall \alpha = (\alpha_n)_n, \beta = (\beta_n)_n \in I^\mathbb{N}, \quad d(\alpha, \beta) := \sum_{n \geq 1} \frac{\delta(\alpha_n, \beta_n)}{2^n}.$$

The “standard” base \mathcal{B} for the topology τ_p is the following. Put $I^{<\mathbb{N}} := \bigcup_{n \geq 1} I^n$ and let $s, t \in I^{<\mathbb{N}}$ and $\sigma \in I^\mathbb{N}$.

- (i) If $s \in I^n$, we say that n is the length of s , for short, $|s| = n$.
- (ii) If $n \in \mathbb{N}$ then $\sigma \upharpoonright n = (\sigma(1), \dots, \sigma(n)) \in I^n$ and, if $n \leq |t|$, $t \upharpoonright n := (t(1), \dots, t(n)) \in I^n$.
- (iii) $s < t$ says that $|s| \leq |t|$ and $s = (t(1), \dots, t(n)) = t \upharpoonright n$.
- (iv) $s < \sigma$ says that $s = (\sigma(1), \dots, \sigma(n)) = \sigma \upharpoonright n$.

With this notation the base \mathcal{B} is $\mathcal{B} := \{N_s : s \in I^{<\mathbb{N}}\}$, N_s being $N_s := \{\sigma \in I^\mathbb{N} : s < \sigma\}$, $\forall s \in I^{<\mathbb{N}}$.

Remark. We consider each element $J \in I^{\mathbb{N}}$: (i) first, as a sequence of elements of I , namely, $J = (J(1), J(2), \dots)$; (ii) second, as a subset (finite or infinite) of I , namely, $J := \{J(n) : n \in \mathbb{N}\}$.

Let \mathcal{F} be the family of finite subsets $D \subset \mathbb{Q}^+$ such that $\sum_{d \in D} d = 1$. Clearly $|\mathcal{F}| = \aleph_0$. By Proposition 2.5 we have $P_J(z) \in \overline{\text{co}}(P_J(K))$ for each $J \in I^{\mathbb{N}}$. So we can choose a subset $D^J \in \mathcal{F}$; we say $D^J = \{q_1^J, \dots, q_{s_J}^J\}$, and $k_n^J \in K$, $n = 1, \dots, s_J$, such that $\|P_J(z) - \sum_{n=1}^{s_J} q_n^J \cdot k_n^J\|_{\ell_\infty(I)} \leq \epsilon$. We choose these elements in order to have $D^{J_1} = D^{J_2}$ if $J_1 = J_2$ as subsets of I .

For each $D \in \mathcal{F}$, let $W(D) = \{J \in I^{\mathbb{N}} : D^J = D\}$. Clearly $I^{\mathbb{N}} = \bigcup_{D \in \mathcal{F}} W(D)$. Since $I^{\mathbb{N}}$ is of second category by the theorem of Baire, there exists $\{\lambda_1, \dots, \lambda_p\} = D_0 \in \mathcal{F}$ such that $\text{int}(\overline{W(D_0)}) \neq \emptyset$. This means that there exists $s \in I^{<\mathbb{N}}$ such that $N_s \subset \overline{W(D_0)}$. Let $H_\epsilon := \sum_{d \in D_0} dK$. Let $F \subset I$ be an arbitrary finite subset and $t \in I^{<\mathbb{N}}$ such that $s \prec t$ and $F \subset t$ (t considered as a subset of I). As N_t is an open set with $N_t \subset N_s$, then $N_t \cap W(D_0) \neq \emptyset$. Thus there exists $J \in I^{\mathbb{N}}$ such that $t \prec J$ and $D^J = D_0$. Hence there exist $k_1, \dots, k_p \in K$ such that, if $k_J := \sum_{i=1}^p \lambda_i \cdot k_i \in H_\epsilon$, then $\|P_J(z) - P_J(k_J)\|_{\ell_\infty(I)} \leq \epsilon$. In particular, for each $i \in F \subset J$ we have $|\pi_i(z - k_J)| \leq \epsilon$. By Lemma 3.2 there exists $k_\epsilon \in H_\epsilon$ such that $\|z - k_\epsilon\| \leq \epsilon$.

Finally, note that $\epsilon > 0$ is arbitrary and that $H_\epsilon \subset \text{co}(K)$, $\forall \epsilon > 0$. Thus $z \in \overline{\text{co}}(K)$. \square

Theorem 3.4. Let X be a Banach space. Then $\overline{\text{co}}(K) = \overline{\text{co}}^{\mathcal{T}_K}(K)$ for every w^* -compact subset K of X^* and every cardinal $\kappa \geq \aleph_0$.

Proof. This follows from (3.1), and Propositions 3.1 and 3.3.

4. The distances dist_κ and the capacity for separating convex sets of X_κ

If $C, D \subset X^*$ are convex subsets and $z \in X^*$, by the Hahn–Banach separation theorem, the distances $\text{dist}(z, C)$ and $\text{dist}(D, C)$ defined in the Introduction satisfy (see [5, Lemma 2.1])

$$\begin{aligned} \text{dist}(z, C) &= \sup\{\inf\{\langle \varphi, z - C \rangle\} : \varphi \in S(X^{**})\} = \sup\{0 \vee \inf\langle \varphi, z - C \rangle : \varphi \in S(X^{**})\}, \\ \text{dist}(D, C) &= \sup\{0 \vee (\inf\langle \psi, C \rangle - \sup\langle \psi, D \rangle) : \psi \in S(X^{**})\} \\ &= \sup\{0 \vee (\inf\langle \psi, C - D \rangle) : \psi \in S(X^{**})\} = \text{dist}(0, C - D). \end{aligned}$$

Moreover, if $z \notin \overline{C}$, then

$$\text{dist}(z, C) = \sup\{\inf\langle \varphi, z - C \rangle : \varphi \in S(X^{**})\}.$$

In the light of the above facts, for every cardinal κ we define the distances $\text{dist}_\kappa(z, C)$ and $\text{dist}_\kappa(D, C)$ in the following way:

$$\text{dist}_\kappa(z, C) := \sup\{\inf\{\langle \psi, z - C \rangle\} : \psi \in S(X_\kappa)\}$$

and

$$\text{dist}_\kappa(D, C) := \sup\{0 \vee (\inf\langle \psi, C - D \rangle) : \psi \in S(X_\kappa)\}.$$

In this section we are concerned with the distances dist_κ and the capacity of the subspaces X_κ for separating convex subsets of X^* .

Remarks. Let $\kappa \geq 1$ be a cardinal, X a Banach space, and C and D two convex subsets of X^* and $z \in X^*$. Then:

(1) Obviously, $\text{dist}(z, C) \geq \text{dist}_\kappa(z, C) \geq 0$, $\text{dist}_\kappa(z, C) = \text{dist}_\kappa(z, \overline{C}^{\mathcal{T}_\kappa})$ and $\text{dist}_\kappa(z, C) = 0$ if and only if $z \in \overline{C}^{\mathcal{T}_\kappa}$. Thus, if $z \in \overline{C}^{\mathcal{T}_\kappa} \setminus \overline{C}$, then $\text{dist}_\kappa(z, C) = 0$ but $\text{dist}(z, C) > 0$.

(2) If $z \notin \overline{C}^{\mathcal{T}_\kappa}$ then $\text{dist}_\kappa(z, C) = \sup\{\inf\{\langle \psi, z - C \rangle\} : \psi \in S(X_\kappa)\}$.

(3) Obviously $\text{dist}(D, C) \geq \text{dist}_\kappa(D, C) \geq \text{dist}_\rho(D, C) \geq \text{dist}_{\aleph_0}(D, C)$ for cardinal numbers such that $\aleph_0 \leq \rho \leq \kappa$.

Moreover $\text{dist}_\kappa(D, C) = \text{dist}_\kappa(\overline{D}^{\mathcal{T}_\kappa}, \overline{C}^{\mathcal{T}_\kappa})$.

(4) $\text{dist}_\kappa = \text{dist}$ if and only if $X^{**} = X_\kappa$. Indeed, clearly, if $X^{**} = X_\kappa$, then $\text{dist}_\kappa = \text{dist}$. Suppose that $X^{**} \neq X_\kappa$. Then there exist $\|\cdot\|$ -closed convex subset $D \subset X^*$ and a vector $u \in X^*$ such that $u \notin D$ but $u \in \overline{D}^{\mathcal{T}_\kappa}$. Thus $\text{dist}(u, D) > 0$ but $\text{dist}_\kappa(u, D) = 0$.

(5) If $\kappa \geq \aleph_0$ and $\text{dist}_\kappa(D, C) = d_0 > 0$, there exists $\psi_0 \in S(X_\kappa)$ such that $\inf\langle \psi_0, D - C \rangle = d_0$. Indeed, by definition of dist_κ , for each $n \in \mathbb{N}$ there exists $\psi_n \in S(X_\kappa)$ such that $\inf\langle \psi_n, D - C \rangle > d_0 - \frac{1}{n}$. Now it is enough to take as ψ_0 any w^* -accumulation point of $\{\psi_n : n \geq 1\}$.

Although the distances $\text{dist} \neq \text{dist}_\kappa$ when $X^{**} \neq X_\kappa$, there are important pairs of convex subsets $D, C \subset X^*$ such that $\text{dist}(D, C) = \text{dist}_\kappa(D, C) > 0$. Next we see some of these pairs. Let us start with the following result.

Lemma 4.1. Let X be a Banach space, $K \subset X^*$ a w^* -compact subset and $z \in X^*$. Then $\text{dist}(z, \overline{\text{co}}(K)) = \text{dist}_\kappa(z, \overline{\text{co}}(K))$ for every cardinal $\kappa \geq \aleph_0$.

Proof. Since every cardinal $\kappa \geq \aleph_0$ satisfies

$$\text{dist}(z, \overline{\text{co}}(K)) \geq \text{dist}_\kappa(z, \overline{\text{co}}(K)) \geq \text{dist}_{\aleph_0}(z, \overline{\text{co}}(K)),$$

it is enough to consider the cardinal $\kappa = \aleph_0$. Without loss of generality we suppose that $z = 0$. First, assume that $\text{dist}(0, \overline{\text{co}}(K)) = 0$, that is, $0 \in \overline{\text{co}}(K)$. Then $0 \in \overline{\text{co}}^{\mathcal{T}_{\aleph_0}}(K)$ by [Theorem 3.4](#) and so $\text{dist}_{\aleph_0}(0, \overline{\text{co}}(K)) = 0$. Now assume that $0 \notin \overline{\text{co}}(K)$ and let $d_0 = \text{dist}(0, \overline{\text{co}}(K)) > 0$. Next we prove that $\text{dist}_{\aleph_0}(0, \overline{\text{co}}(K)) = d_0$. Fix $\epsilon > 0$ such that $d_0 > \epsilon > 0$. By [Theorem 3.4](#) clearly $0 \notin \overline{\text{co}}^{\mathcal{T}_{\aleph_0}}(K)$ and $\overline{\text{co}}(K + \epsilon B(X^*)) = \overline{\text{co}}^{\mathcal{T}_{\aleph_0}}(K + \epsilon B(X^*))$ whence

$$0 \notin \overline{\text{co}}(K) + \epsilon B(X^*) = \overline{\text{co}}(K + \epsilon B(X^*)) = \overline{\text{co}}^{\mathcal{T}_{\aleph_0}}(K + \epsilon B(X^*)).$$

Thus $\text{dist}_{\aleph_0}(0, \overline{\text{co}}(K + \epsilon B(X^*))) > 0$, whence

$$\begin{aligned} 0 &< \text{dist}_{\aleph_0}(0, \overline{\text{co}}(K + \epsilon B(X^*))) = \text{dist}_{\aleph_0}(0, \text{co}(K + \epsilon B(X^*))) \\ &= \sup\{\inf\langle \psi, K + \epsilon B(X^*) \rangle : \psi \in S(X_{\aleph_0})\} \\ &= \sup\{\inf\langle \psi, K \rangle : \psi \in S(X_{\aleph_0})\} - \epsilon = \text{dist}_{\aleph_0}(0, \overline{\text{co}}(K)) - \epsilon. \end{aligned}$$

So $\text{dist}_{\aleph_0}(0, \overline{\text{co}}(K)) > \epsilon$. As $\epsilon < d_0$ is arbitrary, we get

$$\text{dist}_{\aleph_0}(0, \overline{\text{co}}(K)) = d_0 = \text{dist}(0, \overline{\text{co}}(K)). \quad \square$$

Theorem 4.2. Let X be a Banach space and $K_1, K_2 \subset X^*$ be two w^* -compact subsets such that $\text{dist}(\overline{\text{co}}(K_1), \overline{\text{co}}(K_2)) = d_0 > 0$. If $\kappa \geq \aleph_0$ is a cardinal, there exists $\psi \in S(X_\kappa)$ such that

$$\inf\langle \psi, \overline{\text{co}}(K_2) \rangle = \sup\langle \psi, \overline{\text{co}}(K_1) \rangle + d_0. \quad (4.1)$$

In particular, $\text{dist}(\overline{\text{co}}(K_1), \overline{\text{co}}(K_2)) = \text{dist}_\kappa(\overline{\text{co}}(K_1), \overline{\text{co}}(K_2))$.

Proof. First note that

$$\text{co}(K_2 - K_1) = \text{co}(K_2) - \text{co}(K_1) \subset \text{co}(K_2) - \overline{\text{co}}(K_1) \subset \overline{\text{co}}(K_2 - K_1).$$

So by [Lemma 4.1](#)

$$\begin{aligned} \text{dist}_\kappa(0, \overline{\text{co}}(K_2 - K_1)) &= \text{dist}(0, \overline{\text{co}}(K_2 - K_1)) \\ &= \text{dist}(0, \overline{\text{co}}(K_2) - \overline{\text{co}}(K_1)) = \text{dist}(\overline{\text{co}}(K_2), \overline{\text{co}}(K_1)) = d_0 > 0. \end{aligned}$$

By (5) of the above remarks there exists $\psi \in S(X_\kappa)$ such that

$$d_0 = \inf\langle \psi, \overline{\text{co}}(K_2 - K_1) \rangle = \inf\langle \psi, \overline{\text{co}}(K_2) - \overline{\text{co}}(K_1) \rangle = \inf\langle \psi, \overline{\text{co}}(K_2) \rangle - \sup\langle \psi, \overline{\text{co}}(K_1) \rangle. \quad \square$$

Proposition 4.3. Let $\kappa \geq \aleph_0$ be a cardinal, X a Banach space, $K \subset X^*$ a w^* -compact subset and $D \subset X^*$ a convex subset such that $\text{Dens}(D, \mathcal{T}_\kappa) \leq \kappa$ and $\text{dist}(\overline{\text{co}}(K), D) = d_0 > 0$. Then there exists $\psi \in S(X_\kappa)$ such that

$$\inf\langle \psi, \overline{\text{co}}(K) \rangle = \sup\langle \psi, D \rangle + d_0,$$

and so $\text{dist}_\kappa(\overline{\text{co}}(K), D) = d_0$.

Proof. Let $D_0 \subset D$ be a \mathcal{T}_κ -dense subset of D with $|D_0| \leq \kappa$ and $F \subset D_0$ be a finite subset. As F is w^* -compact and

$$\text{dist}(\overline{\text{co}}(K), \text{co}(F)) \geq \text{dist}(\overline{\text{co}}(K), D) = d_0 > 0,$$

by [Theorem 4.2](#) there exists $\psi_F \in S(X_\kappa)$ such that

$$\langle \psi_F, h \rangle \geq \langle \psi_F, c \rangle + d_0, \quad \forall h \in \overline{\text{co}}(K), \forall c \in \text{co}(F).$$

The net $\{\psi_F : F \subset D_0\} \subset S(X_\kappa) \subset B(X^{**})$ has a subnet that converges to some $\varphi \in B(X^{**})$. Clearly

$$\inf\langle \varphi, \overline{\text{co}}(K) \rangle \geq \sup\langle \varphi, \text{co}(D_0) \rangle + d_0 = \sup\langle \varphi, D \rangle + d_0.$$

Actually $\inf\langle \varphi, \overline{\text{co}}(K) \rangle = \sup\langle \varphi, D \rangle + d_0$ and $\|\varphi\| = 1$ because $\text{dist}(\overline{\text{co}}(K), D) = d_0 > 0$. As $|D_0| \leq \kappa \geq \aleph_0$, then $|\{F \subset D_0 : F \text{ finite}\}| \leq \kappa$, whence $\varphi \in S(X_\kappa)$. \square

Proposition 4.4. Let $\kappa \geq \aleph_0$ be a cardinal, X a Banach space, $C, D \subset X^*$ two convex subsets such that $\text{Dens}(C, \mathcal{T}_\kappa) \leq \kappa \geq \text{Dens}(D, \mathcal{T}_\kappa)$ and $\text{dist}(C, D) = d_0 > 0$. Then there exists $\psi \in S(X_\kappa)$ such that

$$\inf\langle \psi, C \rangle = \sup\langle \psi, D \rangle + d_0,$$

and so $\text{dist}_\kappa(C, D) = d_0$.

Proof. Let $C_0 \subset C$ be a subset \mathcal{T}_κ -dense in C such that $|C_0| \leq \kappa$. Let $E \subset C_0$ be a finite subset. By Proposition 4.3 there exists $\psi_E \in S(X_\kappa)$ such that

$$\inf\langle \psi_E, \overline{\text{co}}(E) \rangle \geq \langle \psi_E, D \rangle + d_0.$$

The net $\mathcal{B} := \{\psi_E : E \subset C_0 \text{ is finite}\} \subset S(X_\kappa) \subset B(X^{**})$ has a subnet that converges to some $\psi \in B(X^{**})$. Since $|C_0| \leq \kappa \geq \aleph_0$, clearly $|\mathcal{B}| \leq \kappa$ and so $\psi \in X_\kappa$. Thus

$$\inf\langle \psi, \text{co}(C_0) \rangle = \inf\langle \psi, C \rangle \geq \sup\langle \psi, D \rangle + d_0.$$

Actually $\inf\langle \psi, C \rangle = \sup\langle \psi, D \rangle + d_0$ and $\|\psi\| = 1$ because $\text{dist}(C, D) = d_0 > 0$. And this completes the proof. \square

Acknowledgments

The first author was supported in part by DGICYT grant MTM2012-31286, grant UCM-910346 and UCM-BSCH grant PR27/05-14045.

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