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Convergence of nonlocal diffusion models on lattices

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ABSTRACT

In this paper we look at models of nonlocal (or anomalous) diffusion which are defined on subsets of the lattice $\epsilon\mathbb{Z}^n$, for some $\epsilon > 0$, and ask if they can be approximated by continuum models. The answer is given by an operator semigroup convergence theorem. As an application, we establish hypotheses under which a discrete model of nonlocal diffusion satisfying an absorbing boundary condition has a continuum limit which is conservative.

1. Introduction

The study of nonlocal diffusion (also called anomalous diffusion) has recently emerged as an important area of scientific research, with applications in such disparate areas as groundwater hydrology (see Meerschaert and Sikorskii [11]), optimal search theory (Raposo et al. [12]), and financial market modeling (Mantegna [10]). Roughly speaking, nonlocal diffusion occurs when a “particle” moves in a way similar to a simple random walk but has different asymptotic properties because it occasionally takes very large jumps. It is well known that, under appropriate hypotheses, simple random walks can be approximated by continuum models governed by the heat equation (see Burdzy and Chen [3], Lin and Segel [9]). The aim of this paper is to prove some related results for models of nonlocal diffusion.

As a starting point, consider the system of equations

$$\frac{d}{dt}p(x,t) = \sum_{y\in\mathbb{Z}^n\setminus\{x\}} \frac{\mathcal{C}(p(y,t)-p(x,t))}{|y-x|^{n+\alpha}}, \quad x\in\mathbb{Z}^n \tag{1}$$

where $\mathcal{C} > 0$ and $\alpha \in (0,2)$. The solution to this system gives the probability $p(x,t)$ that a randomly moving particle is at the point x at time t, given an appropriate initial condition and given that the position X_t of

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the particle at time t is a continuous time Markov chain with transition probabilities given by

$$\mathrm{P}(X_{t+s} = y | X_t = x) = \frac{s\mathcal{C}}{|y-x|^{n+\alpha}} + o(s).$$

For a given $\epsilon > 0$, if we rescale space and time so that the movement of the particle is described by $X^{(\epsilon)}$, where $X_t^{(\epsilon)} = \epsilon X_{\epsilon^{-\alpha}t}$, then one can easily verify that the probability $p^\epsilon(x,t)$ of finding the particle at x at time t satisfies

$$\frac{d}{dt}p^\epsilon(x,t) = \sum_{y\in\epsilon\mathbb{Z}^n\setminus\{x\}} \frac{\mathcal{C}(p^\epsilon(y,t) - p^\epsilon(x,t))\epsilon^n}{|y-x|^{n+\alpha}}, \quad x \in \epsilon\mathbb{Z}^n. \tag{2}$$

The results of Husseini and Kassmann [7] show that as $\epsilon \to 0$, $X^{(\epsilon)}$ converges to a stochastic process governed by a fractional diffusion equation.

A natural generalization of (2) for subsets E of $\epsilon\mathbb{Z}^n$ is the system

$$\frac{d}{dt}p(x,t) = \sum_{y\in E\setminus\{x\}} \frac{\mathcal{C}(p(y,t) - p(x,t))\epsilon^n}{|y-x|^{n+\alpha}}, \quad x \in E. \tag{3}$$

This model has appeared in certain applied contexts (as a special case of the model of human mobility in Brockmann [2] and as a model of anomalous diffusion in Condat, Rangel and Lamberti [4]). In what follows we will study this model, as well as an altered version involving an absorbing boundary condition. We will give hypotheses under which they converge, in a sense to be made precise in the next section, as $\epsilon \to 0$. This is a continuation of previous work with Seidman in [15].

2. Formal construction of the models and statement of the main results

To motivate all the definitions below, let us briefly summarize the elements of the argument to follow. Given a lattice $\epsilon\mathbb{Z}^n$ and a bounded open set $U \subset \mathbb{R}^n$, we consider a family of n-dimensional cubes $S_1, \ldots, S_m$ which cover U. We choose the cubes so that each one is centered at a lattice point in $\epsilon\mathbb{Z}^n$, has non-empty intersection with U, and has volume ϵ^n. Letting z_i denote the lattice point at the center of S_i, δ_{z_i} denote the Dirac measure centered at z_i, and $\mathbb{1}_{S_i}$ denote the characteristic function for S_i, we see that each probability measure $\mu = \sum_{i=1}^m c_i\delta_{z_i}$ on $\{z_1, \ldots, z_m\}$ can be associated with a probability density $v = \frac{1}{\epsilon^n}\sum_{i=1}^m c_i\mathbb{1}_{S_i}$ on $\bigcup_{i=1}^m S_i$, which satisfies $\mu(\{z_i\}) = \int_{S_i} v(x)\,dx$ for all i. Using this correspondence, we can identify the transition semigroup of a given Markov chain on $\{z_1, \ldots, z_m\}$ with a semigroup acting on a space of piecewise constant functions $g : \bigcup_{i=1}^m S_i \to \mathbb{R}$. For sufficiently small ϵ, the latter semigroup will approximate some limiting semigroup acting on $L^2(U)$, and this is the continuum-limit model.

We can now proceed with the detailed construction of the models. In everything that follows, $\alpha \in (0,2)$, $\mathcal{C}$ is a positive constant and $(\epsilon_k)_{k\in\mathbb{N}}$ is a sequence of positive real numbers such that $\epsilon_k \downarrow 0$. We will assume U is a bounded open subset of $\mathbb{R}^n$ satisfying the *segment property*: for each x contained in the boundary ∂U of U, there is a neighborhood N_x of x in $\mathbb{R}^n$ and a vector y_x, distinct from the zero vector $\mathbf{0}$, such that $z + ty_x \in U$ for every $z \in \overline{U} \cap N_x$ and $t \in (0,1)$. (All bounded Lipschitz open sets satisfy the segment property, see Grisvard [6, Theorem 1.2.2.2].) We assume in addition that

$$\lim_{\xi\downarrow 0} \lambda\big(\big\{x \in \mathbb{R}^n\colon d(x,\partial U) < \xi\big\}\big) = 0$$

where λ denotes the Lebesgue measure on $\mathbb{R}^n$. For each k, fix a bijection $\mathbb{N} \to \epsilon_k\mathbb{Z}^n : i \mapsto z_{ki}$. Define the cube

$$S=\left\{\left(r^1,\dots,r^n\right)\in\mathbb{R}^n\colon \left|r^i\right|<\frac{1}{2}\ \text{for}\ i=1,\dots,n\right\}$$

and for each $i,k\in\mathbb{N}$ let

$$S_{ki}=z_{ki}+\epsilon_k S$$

(so S_{ki} is a cube centered at z_{ki}) and

$$S^*_{ki}=S_{ki}\cap U.$$

To construct the conservative model, we now define

$$\begin{aligned}\mathcal{I}_k&=\{i\in\mathbb{N}\colon S_{ki}\cap U\neq\varnothing\},\\ E_k&=\left\{z_{ki}\in\epsilon_k\mathbb{Z}^n\colon i\in\mathcal{I}_k\right\},\\ \mathcal{V}_k&=\left\{\sum_{i\in\mathcal{I}_k}a_i\mathbb{1}_{S_{ki}}\colon a_i\in\mathbb{R}\ \forall i\in\mathcal{I}_k\right\}.\end{aligned}$$

One can see that $\mathcal{I}_k$ is an index of cubes intersecting U, E_k consists of the lattice points at the centers of these cubes, and $\mathcal{V}_k$ denotes the set of functions on $U_k:=\bigcup_{i\in\mathcal{I}_k}S_{ki}$ which are constant when restricted to a cube S_{ki}. We make $\mathcal{V}_k$ a Hilbert space by giving it the $L^2(U_k)$ inner product. Define the maps $\pi_k:L^2(U)\to\mathcal{V}_k$ and $F_k:\mathcal{V}_k\to L^2(U)$ by

$$\pi_k: f\mapsto\sum_{i\in\mathcal{I}_k}a_i\mathbb{1}_{S_{ki}},\quad a_i=\frac{1}{\lambda(S^*_{ki})}\int_{S^*_{ki}}f\,d\lambda,$$

$$F_k:\sum_{i\in\mathcal{I}_k}a_i\mathbb{1}_{S_{ki}}\mapsto\sum_{i\in\mathcal{I}_k}a_i\mathbb{1}_{S^*_{ki}}.$$

For every $g=\sum_{i\in\mathcal{I}_k}a_i\mathbb{1}_{S_{ki}}\in\mathcal{V}_k$, let

$$T_k(t)g=\frac{1}{\epsilon_k^n}\sum_{i\in\mathcal{I}_k}p_k(z_{ki},t)\mathbb{1}_{S_{ki}}$$

where p_k is the unique solution to

$$\frac{d}{dt}p_k(z_{ki},t)=\sum_{z_{kj}\in E_k\backslash\{z_{ki}\}}\frac{\mathcal{C}(p_k(z_{kj},t)-p_k(z_{ki},t))\epsilon_k^n}{|z_{kj}-z_{ki}|^{n+\alpha}},\quad z_{ki}\in E_k \tag{4}$$

such that $p_k(z_{ki},0)=a_i\epsilon_k^n$ for all $i\in\mathcal{I}_k$. Given $v\in L^2(U)$, we may interpret $\|v-F_kT_k(t)g\|_{L^2(U)}$ as the distance between v and $p_k(.,t)$. It is easy to see that $(T_k(t))_{t\geqslant 0}$ is a C_0-semigroup on $\mathcal{V}_k$. By a continuum-limit for p_k, we mean a C_0-semigroup $(T(t))_{t\geqslant 0}$ on $L^2(U)$ such that $F_kT_k(t)\pi_kf\to T(t)f$ (in $L^2(U)$) for every $f\in L^2(U)$ and $t\geqslant 0$.

The other discrete model we shall consider is one with an absorbing boundary condition. To construct it, let M be a positive real number and define

$$\begin{aligned}\widehat{\mathcal{I}}_k&=\left\{i\in\mathcal{I}_k\colon d\left(z_{ki},\mathbb{R}^n\backslash U\right)>M\epsilon_k\right\},\\ \widehat{E}_k&=\{z_{ki}\in E_k\colon i\in\widehat{\mathcal{I}}_k\}.\end{aligned}$$

In the above, $\widehat{\mathcal{I}}_k$ is an index of lattice points inside of U, and $E_k \backslash \widehat{E}_k$ can be thought of as the boundary of E_k. Notice that if we choose $M > \sqrt{n}/2$, then we have $S_{ki} \subset U$ for all $i \in \widehat{\mathcal{I}}_k$. (For some results below we will need the stronger condition $M > \sqrt{n}$.) Also define

$$\theta_{ki} = \begin{cases} 1 & \text{if } z_{ki} \in \widehat{E}_k, \\ 0 & \text{otherwise}, \end{cases}$$

$$\widehat{\mathcal{V}}_k = \left\{ \sum_{i \in \mathcal{I}_k} a_i \mathbb{1}_{S_{ki}} \in \mathcal{V}_k \colon \theta_{ki} = 0 \Longrightarrow a_i = 0 \right\},$$

$$\widehat{\pi}_k : f \mapsto \sum_{i \in \mathcal{I}_k} a_i \mathbb{1}_{S_{ki}}, \quad a_i := \theta_{ki} \frac{1}{\lambda(S_{ki}^*)} \int_{S_{ki}^*} f \, d\lambda.$$

Notice that $\widehat{\mathcal{V}}_k$ is simply the space of functions in $\mathcal{V}_k$ which vanish on the boundary cubes S_{ki}, $i \in \mathcal{I}_k \backslash \widehat{\mathcal{I}}_k$. If we take the particle described by (4) and "kill" it (and immediately remove it from E_k) upon its fist entrance to the set $E_k \backslash \widehat{E}_k$, then the probability of finding the particle at z_{ki} at time t is $\widehat{p}_k(z_{ki}, t)$, where $\widehat{p}_k$ solves

$$\frac{d}{dt} \widehat{p}_k(z_{ki}, t) = \sum_{z_{kj} \in E_k \backslash \{z_{ki}\}} \frac{\mathcal{C}(\widehat{p}_k(z_{kj}, t) - \widehat{p}_k(z_{ki}, t)) \epsilon_k^n}{|z_{kj} - z_{ki}|^{n+\alpha}}, \quad z_{ki} \in \widehat{E}_k, \tag{5}$$

$$\widehat{p}_k(z_{ki}, t) = 0, \quad z_{ki} \in E_k \backslash \widehat{E}_k. \tag{6}$$

We define a C_0-semigroup $(\widehat{T}_k(t))_{t \geqslant 0}$ on $\widehat{\mathcal{V}}_k$ by setting, for every $g = \sum_{i \in \mathcal{I}_k} a_i \mathbb{1}_{S_{ki}} \in \widehat{\mathcal{V}}_k$,

$$\widehat{T}_k(t) g = \frac{1}{\epsilon_k^n} \sum_{i \in \mathcal{I}_k} \widehat{p}_k(z_{ki}, t) \mathbb{1}_{S_{ki}}$$

where $\widehat{p}_k$ is the unique solution to (5)–(6) such that $\widehat{p}_k(z_{ki}, 0) = a_i \epsilon_k^n$ for every $z_{ki} \in E_k$. Thus we seek a C_0-semigroup $(\widehat{T}(t))_{t \geqslant 0}$ on $L^2(U)$ such that $F_k \widehat{T}_k(t) \widehat{\pi}_k f \to \widehat{T}(t) f$ for $f \in L^2(U)$ and $t \geqslant 0$.

Let us now construct the limiting semigroup models. For a Hilbert space $\mathcal{H}$, a dense linear subspace $\mathcal{D}[\mathcal{F}]$ and a symmetric bilinear form $\mathcal{F}$ defined on $\mathcal{D}[\mathcal{F}]$, we say that $(\mathcal{F}, \mathcal{D}[\mathcal{F}])$ is a closed symmetric bilinear form on $\mathcal{H}$ if $\mathcal{D}[\mathcal{F}]$ is a Hilbert space with respect to the inner product $(.,.)_{\mathcal{D}[\mathcal{F}]} := (.,.)_{\mathcal{H}} + \mathcal{F}(.,.)$. (From this point forward, for any bilinear form $(\mathcal{F}, \mathcal{D}[\mathcal{F}])$, we will always give $\mathcal{D}[\mathcal{F}]$ the topology defined by $(.,.)_{\mathcal{D}[\mathcal{F}]}$.) It is known that (see Fukushima, Oshima, and Masayoshi [5, Theorem 1.3.1]) for any such closed symmetric bilinear form $(\mathcal{F}, \mathcal{D}[\mathcal{F}])$, there is a unique C_0-semigroup $(S(t))_{t \geqslant 0}$ of symmetric contraction operators on $\mathcal{H}$ such that

$$\mathcal{F}(f, g) = (\sqrt{-A} f, \sqrt{-A} g)_{\mathcal{H}}$$

and

$$\mathcal{D}[\mathcal{F}] = \mathcal{D}(\sqrt{-A})$$

where $(A, \mathcal{D}(A))$ is the generator of $(S(t))_{t \geqslant 0}$ and $(\sqrt{-A}, \mathcal{D}(\sqrt{-A}))$ denotes the square root of $-A$. We call $(S(t))_{t \geqslant 0}$ the semigroup associated with $(\mathcal{F}, \mathcal{D}[\mathcal{F}])$. Let $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ be the bilinear form

$$\mathcal{E}(f, g) = \frac{1}{2} \mathcal{C} \iint_{U \times U} \frac{(f(y) - f(x))(g(y) - g(x)) \, dx \, dy}{|y - x|^{n+\alpha}},$$

$$\mathcal{D}[\mathcal{E}] = \left\{ f \in L^2(U)\colon \iint\limits_{U\times U} \frac{(f(y)-f(x))^2\,dx\,dy}{|y-x|^{n+\alpha}} < \infty \right\}.$$

This is a closed symmetric bilinear form, and in fact the space $\mathcal{D}[\mathcal{E}]$ is equivalent to the fractional-order Sobolev space $H^{\alpha/2}(U)$ (see Wloka [16]). Define $(\widehat{\mathcal{E}}, \mathcal{D}[\widehat{\mathcal{E}}])$ so that $\mathcal{D}[\widehat{\mathcal{E}}]$ is the closure of $C_0^\infty(U)$ in $\mathcal{D}[\mathcal{E}]$ and $\widehat{\mathcal{E}}$ is the restriction of $\mathcal{E}$ to $\mathcal{D}[\widehat{\mathcal{E}}]$. Let $(T(t))_{t\geqslant 0}$ and $(\widehat{T}(t))_{t\geqslant 0}$ denote the semigroups associated with $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ and $(\widehat{\mathcal{E}}, \mathcal{D}[\widehat{\mathcal{E}}])$, respectively. The generator of $(T(t))_{t\geqslant 0}$ is known as the Neumann fractional Laplacian (see Siudeja [13]). The underlying stochastic models for these semigroups, and the relationships between them, were studied by Bogdan, Burdzy and Chen [1].

Here is the main theorem of this paper, which we will prove in the next section:

Theorem 2.1. *Let $f \in L^2(U)$ and $\tau > 0$.*

1. *We have*

$$\lim_{k\to\infty} \max_{0\leqslant t\leqslant\tau} \left\| F_k T_k(t)\pi_k f - T(t)f \right\|_{L^2(U)} = 0.$$

2. *If $M > \sqrt{n}$, then*

$$\lim_{k\to\infty} \max_{0\leqslant t\leqslant\tau} \left\| F_k \widehat{T}_k(t)\widehat{\pi}_k f - \widehat{T}(t)f \right\|_{L^2(U)} = 0.$$

Corollary 2.2. *Let $f \in L^2(U)$ be a probability density and let $\widehat{\pi}_k f = \sum_{i\in\mathcal{I}_k} a_{ki}\mathbb{1}_{S_{ki}}$. For every k, let $\widehat{p}_k$ be the solution to (5)–(6) such that $\widehat{p}_k(z_{ki}, 0) = a_{ki}\epsilon_k^n$ for all $i \in \mathcal{I}_k$. If $C_0^\infty(U)$ is dense in $\mathcal{D}[\mathcal{E}]$, then for every $t \geqslant 0$,*

$$\lim_{k\to\infty} \sum_{i\in\mathcal{I}_k} \widehat{p}_k(z_{ki}, t) = 1. \tag{7}$$

Remark 2.3. Under the assumptions we have made so far, if U has a Lipschitz boundary, then $C_0^\infty(U)$ is dense in $\mathcal{D}[\mathcal{E}]$ if and only if $\alpha \in (0, 1]$ (see [1]). Thus the corollary above is not vacuous. It also should be emphasized that when $C_0^\infty(U)$ is not dense in $\mathcal{D}[\mathcal{E}]$, the model $(\widehat{T}(t))_{t\geqslant 0}$ is *not* conservative (see [1, Corollary 2.6]).

The corollary states that the model (5)–(6) becomes conservative in the continuum limit, provided that $C_0^\infty(U)$ is dense in $\mathcal{D}[\mathcal{E}]$. This somewhat odd result is closely related to Theorem 1.1 of [1], where the authors investigated (among other things) the boundary behavior for the stochastic models associated with the semigroups $(T(t))_{t\geqslant 0}$ and $(\widehat{T}(t))_{t\geqslant 0}$. However, Corollary 2.2 above is of interest in its own right because it connects the results of [1] with the behavior of discrete models.

3. Convergence of the operator semigroups

In this section we prove Theorem 2.1. We will do this by proving convergence for certain sequences of bilinear forms, in the following sense:

Definition 3.1. Suppose $(\mathcal{H}_k)_{k\in\mathbb{N}}$ is a sequence of Hilbert spaces, $\mathcal{H}$ is a Hilbert space, $\Phi_k : \mathcal{H}_k \to \mathcal{H}$ is a bounded linear operator for every k, $\Pi_k : \mathcal{H} \to \mathcal{H}_k$ is a bounded linear operator such that $\Pi_k\Phi_k$ is the identity on $\mathcal{H}_k$ for every k, and $\lim_{k\to\infty} \|\Pi_k f\|_{\mathcal{H}_k} = \|f\|_{\mathcal{H}}$ for every $f \in \mathcal{H}$. Let $(\mathcal{F}, \mathcal{D}[\mathcal{F}])$ be a closed symmetric bilinear form on $\mathcal{H}$, and for every k, let $(\mathcal{F}_k, \mathcal{D}[\mathcal{F}_k])$ be a closed symmetric bilinear form on $\mathcal{H}_k$. We say that $(\mathcal{F}_k, \mathcal{D}[\mathcal{F}_k])$ *converges to* $(\mathcal{F}, \mathcal{D}[\mathcal{F}])$ if the following two conditions hold.

1. For every sequence $(h_k)_{k\in\mathbb{N}}$ in $\mathcal{H}$ and $h \in \mathcal{H}$ such that $\Phi_k \Pi_k h_k$ converges weakly to h in $\mathcal{H}$, we have

$$\liminf_{k\to\infty} \mathcal{F}_k(\Pi_k h_k, \Pi_k h_k) \geqslant \mathcal{F}(h,h),$$

where for any bilinear form $(\mathcal{G}, \mathcal{D}[\mathcal{G}])$ we define $\mathcal{G}(f,f) = \infty$ for all $f \notin \mathcal{D}[\mathcal{G}]$.
2. For every $h \in \mathcal{D}[\mathcal{F}]$, there is a sequence $(h_k)_{k\in\mathbb{N}}$ such that $h_k \in \mathcal{D}[\mathcal{F}_k]$ for every k, $h_k \to h$ in $\mathcal{H}$, and

$$\limsup_{k\to\infty} \mathcal{F}_k(h_k, h_k) \leqslant \mathcal{F}(h,h).$$

The extension of the Trotter–Kato theorem given below is from Kim [8].

Theorem 3.2. *With the notation and hypotheses of the previous definition, if $(S_k(t))_{t\geqslant 0}$ is the C_0-semigroup associated with $(\mathcal{F}_k, \mathcal{D}[\mathcal{F}_k])$ and $(S(t))_{t\geqslant 0}$ is the C_0-semigroup associated with $(\mathcal{F}, \mathcal{D}[\mathcal{F}])$, then the following are equivalent*:

1. *The sequence $(\mathcal{F}_k, \mathcal{D}[\mathcal{F}_k])$ converges to $(\mathcal{F}, \mathcal{D}[\mathcal{F}])$.*
2. *For every $h \in \mathcal{H}$ and $\tau > 0$,*

$$\lim_{k\to\infty} \max_{0\leqslant t\leqslant\tau} \left\| \Phi_k S_k(t) \Pi_k h - S(t)h \right\|_{\mathcal{H}} = 0.$$

In order to prove the first part of Theorem 2.1, define the bilinear form $(\mathcal{E}_k, \mathcal{D}[\mathcal{E}_k])$ by $\mathcal{D}[\mathcal{E}_k] = \mathcal{V}_k$ and

$$\mathcal{E}_k\left(\sum_{i\in\mathcal{I}_k} a_i \mathbb{1}_{S_{ki}}, \sum_{i\in\mathcal{I}_k} b_i \mathbb{1}_{S_{ki}}\right) = \frac{1}{2}\mathcal{C} \sum_{\substack{i,j\in\mathcal{I}_k \\ i\neq j}} \frac{(a_j - a_i)(b_j - b_i)\epsilon_k^{2n}}{|z_{kj} - z_{ki}|^{n+\alpha}}.$$

It is easy to see that $(T_k(t))_{t\geqslant 0}$ is the semigroup associated with $(\mathcal{E}_k, \mathcal{D}[\mathcal{E}_k])$. Thus the following lemma, which states that $(\mathcal{E}_k, \mathcal{D}[\mathcal{E}_k])$ converges to $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$, establishes the first part of Theorem 2.1. In the proof we will use Lemmas 4.4 and 4.5 from [15]. Although it is assumed in [15] that the domain has a Lipschitz boundary, in the proofs of Lemmas 4.4 and 4.5 this assumption was only used in order to guarantee the existence of a dense subset of $\mathcal{D}[\mathcal{E}]$ consisting of Lipschitz functions. Thus by [16, Theorem 3.6], Lemmas 4.4 and 4.5 of [15] are still valid in the present context because we have assumed that U satisfies the segment property.

Lemma 3.3. *The bilinear forms $(\mathcal{E}_k, \mathcal{D}[\mathcal{E}_k])$ converge to $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$. If $f \in C_0^\infty(U)$, then*

$$\lim_{k\to\infty} \mathcal{E}_k(\pi_k f, \pi_k f) = \mathcal{E}(f,f).$$

Proof. Suppose $F_k \pi_k f_k$ converges weakly to f in $L^2(U)$, and let $F_k \pi_k f_k = \sum_{i\in\mathcal{I}_k} a_i \mathbb{1}_{S_{ki}^*}$. Then since

$$\mathcal{E}_k(\pi_k f_k, \pi_k f_k) \geqslant \frac{1}{2}\mathcal{C} \sum_{\substack{i,j\in\mathcal{I}_k \\ i\neq j}} \frac{(a_{kj} - a_{ki})^2 \lambda(S_{ki}^*)\lambda(S_{kj}^*)}{|z_{kj} - z_{ki}|^{n+\alpha}},$$

we have

$$\liminf_{k\to\infty} \mathcal{E}_k(\pi_k f_k, \pi_k f_k) \geqslant \mathcal{E}(f,f)$$

by [15, Lemma 4.4].

Now let f be the restriction to U of a function in $C_0^\infty(\mathbb{R}^n)$. The set of such functions, which we shall denote by $C_0^\infty(\mathbb{R}^n)|_U$, is dense in $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ because U satisfies the segment property (see [16, Theorem 3.6]). Let $\pi_k f = \sum_{i\in\mathcal{I}_k} a_{ki}\mathbb{1}_{S^*_{ki}}$. By [15, Lemma 4.5],

$$\lim_{k\to\infty} \frac{1}{2}\mathcal{C} \sum_{\substack{i,j\in\mathcal{I}_k \\ i\neq j}} \frac{(a_{kj}-a_{ki})^2\lambda(S^*_{ki})\lambda(S^*_{kj})}{|z_{kj}-z_{ki}|^{n+\alpha}} = \mathcal{E}(f,f).$$

Because f is Lipschitz, for any γ such that $0 < n+\alpha-\gamma < n$, we may choose a positive constant C_1 independent of i, j and k such that

$$(a_{ki}-a_{kj})^2 \leqslant C_1|z_{kj}-z_{ki}|^\gamma.$$

Additionally, for any $x \in S_{ki}$ and $y \in S_{kj}$ such that $i \neq j$, we have

$$\frac{1}{|z_{kj}-z_{ki}|^{n+\alpha-\gamma}} \leqslant \frac{C_2}{|y-x|^{n+\alpha-\gamma}}$$

where C_2 is independent of x, y, i, j, k. Consequently,

$$\sum_{\substack{i,j\in\mathcal{I}_k \\ i\neq j,\, S^*_{kj}\neq S_{kj}}} \frac{(a_{kj}-a_{ki})^2\epsilon_k^{2n}}{|z_{kj}-z_{ki}|^{n+\alpha}} \leqslant C\lambda\bigl(\bigl\{x\in\mathbb{R}^n\colon d(x,\partial U) < \sqrt{n}\epsilon_k\bigr\}\bigr)\int_{B(\mathbf{0},R)} \frac{dx}{|x|^{n+\alpha-\gamma}}$$

for some positive constants C and R independent of k. Since the right-hand side goes to 0 as $k \to \infty$, and

$$\left|\mathcal{E}_k(\pi_k f, \pi_k f) - \frac{1}{2}\mathcal{C}\sum_{\substack{i,j\in\mathcal{I}_k \\ i\neq j}} \frac{(a_{kj}-a_{ki})^2\lambda(S^*_{ki})\lambda(S^*_{kj})}{|z_{kj}-z_{ki}|^{n+\alpha}}\right| \leqslant 4\mathcal{C}\sum_{\substack{i,j\in\mathcal{I}_k \\ i\neq j,\, S^*_{kj}\neq S_{kj}}} \frac{(a_{kj}-a_{ki})^2\epsilon_k^{2n}}{|z_{kj}-z_{ki}|^{n+\alpha}},$$

it follows that

$$\lim_{k\to\infty} \mathcal{E}_k(\pi_k f, \pi_k f) = \mathcal{E}(f,f).$$

Using the density of $C_0^\infty(\mathbb{R}^n)|_U$ in $\mathcal{D}[\mathcal{E}]$ and the fact that $\pi_k f \to f$, we are finished. $\square$

Let us now look at the bilinear forms associated with the model $(\widehat{T}_k(t))_{t\geqslant 0}$. Define $(\widehat{\mathcal{E}}_k, \mathcal{D}[\widehat{\mathcal{E}}_k])$ to be the restriction of $\mathcal{E}_k$ to the space $\mathcal{D}[\widehat{\mathcal{E}}_k] := \widehat{\mathcal{V}}_k$.

Lemma 3.4. *Assume $M > \sqrt{n}$, $g \in L^2(U)$, and $(g_k)_{k\in\mathbb{N}}$ is a sequence such that $g_k \in \widehat{\mathcal{V}}_k$ for every k. Suppose that there is a subsequence $(g_{k(i)})_{i\in\mathbb{N}}$ such that $F_{k(i)}g_{k(i)}$ converges weakly to g in $L^2(U)$ and*

$$\sup_i \widehat{\mathcal{E}}_{k(i)}(g_{k(i)}, g_{k(i)}) < \infty.$$

Then $g \in \mathcal{D}[\widehat{\mathcal{E}}]$.

Proof. It will be clear from the proof that, given our other assumptions, we can assume without loss of generality that $(k(i))_{i\in\mathbb{N}}$ is really just the sequence $(k)_{k\in\mathbb{N}}$. Thus we will assume the hypotheses of the lemma hold with $(k(i))_{i\in\mathbb{N}}$ replaced by $(k)_{k\in\mathbb{N}}$.

Letting $\mathbf{1}$ be the vector in $\mathbb{R}^n$ with each component equal to 1, we see that each set $\frac{1}{2}\epsilon_k\mathbf{1} + \overline{S_{ki}}$ is an n-cube such that all vertices are elements of the set $\epsilon_k\mathbb{Z}^n$. This n-cube can be subdivided into $n!$ simplices such that the vertices of each simplex are vertices of $\frac{1}{2}\epsilon_k\mathbf{1} + \overline{S_{ki}}$. We choose some such subdivision for each $\frac{1}{2}\epsilon_k\mathbf{1}+\overline{S_{ki}}$, and denote the resulting simplices by $\Delta_{ki}^1, \dots, \Delta_{ki}^{n!}$. For each simplex Δ_{ki}^ℓ, the maximum distance between any two points in Δ_{ki}^ℓ is bounded by $\epsilon_k\sqrt{n}$. Since $M > \sqrt{n}$, if $z_{ki} \in \widehat{E}_k$, then it must be the case that $\Delta_{ki}^\ell \cap (\mathbb{R}^n \backslash U) = \varnothing$ for all $\ell = 1, \dots, n!$.

Any point $x \in U$ is contained in some simplex Δ_{ki}^ℓ with vertices $z_{ki_0}, \dots, z_{ki_n}$. As a convex combination of those vertices, x has a unique representation

$$x = \sum_{j=0}^{n} \gamma_j z_{ki_j}.$$

We define

$$g_k^\Delta(x) = \sum_{j=0}^{n} \gamma_j a_{ki_j}$$

where

$$a_{ki_j} = \frac{1}{\lambda(S_{ki_j})} \int\limits_{S_{ki_j}} g_k(y)\,dy.$$

(Notice that since $g_k \in \widehat{\mathcal{V}}_k$, $a_{ki_j} = 0$ if $z_{ki_j} \notin \widehat{E}_k$.) Since any two simplices (as constructed in the previous paragraph) can only intersect on their boundaries, g_k^Δ is well defined and continuous as a function $U \to \mathbb{R}$. Also notice that if $\Delta_{ki}^\ell \cap (\mathbb{R}^n \backslash U) \neq \varnothing$, then all the vertices of Δ_{ki}^ℓ are outside of $\widehat{E}_k$, and consequently the restriction of g_k^Δ to Δ_{ki}^ℓ is identically zero; hence g_k^Δ has compact support in U. In addition, it is easy to see that each g_k^Δ is a Lipschitz function. Thus, $g_k^\Delta \in \mathcal{D}[\widehat{\mathcal{E}}]$ by [16, Theorem 3.3].

We will now show that

$$\sup_k \mathcal{E}\left(g_k^\Delta, g_k^\Delta\right) < \infty. \tag{8}$$

Clearly, it suffices to show that

$$\sup_k \iint\limits_{U\times U} \frac{(g_k^\Delta(y) - g_k^\Delta(x))^2 h_k(x,y)\,dx\,dy}{|y-x|^{n+\alpha}} < \infty \tag{9}$$

and

$$\sup_k \iint\limits_{U\times U} \frac{(g_k^\Delta(y) - g_k^\Delta(x))^2 (1 - h_k(x,y))\,dx\,dy}{|y-x|^{n+\alpha}} < \infty \tag{10}$$

where h_k is defined by

$$h_k(x,y) = \sum_{i\in\mathcal{I}_k} \sum_{j\in\mathcal{I}_k\backslash\mathcal{A}_{ki}} \mathbb{1}_{S_{ki}\times S_{kj}}(x,y)$$

and

$$\mathcal{A}_{ki} := \{m \in \widehat{\mathcal{I}}_k\colon \overline{S_{km}} \cap \overline{S_{ki}} \neq \varnothing\}.$$

We will start by proving (9). We have

$$\iint\limits_{U\times U} \frac{(g_k^{\Delta}(y) - g_k^{\Delta}(x))^2 h_k(x,y)\,dx\,dy}{|y-x|^{n+\alpha}}$$
$$= \iint\limits_{U\times U} \frac{([g_k(y) - g_k(x)] + [g_k(x) - g_k^{\Delta}(x)] + [g_k^{\Delta}(y) - g_k(y)])^2 h_k(x,y)\,dx\,dy}{|y-x|^{n+\alpha}}. \tag{11}$$

If we expand $([g_k(y) - g_k(x)] + [g_k(x) - g_k^{\Delta}(x)] + [g_k^{\Delta}(y) - g_k(y)])^2$ by multiplying without separating terms in brackets, we get nine integrals

$$\iint\limits_{U\times U} \frac{(g_k(y) - g_k(x))^2 h_k(x,y)\,dx\,dy}{|y-x|^{n+\alpha}}, \tag{12}$$

$$\iint\limits_{U\times U} \frac{(g_k(y) - g_k(x))(g_k(x) - g_k^{\Delta}(x)) h_k(x,y)\,dx\,dy}{|y-x|^{n+\alpha}}, \tag{13}$$

$$\iint\limits_{U\times U} \frac{(g_k(y) - g_k(x))(g_k(y) - g_k^{\Delta}(y)) h_k(x,y)\,dx\,dy}{|y-x|^{n+\alpha}}, \tag{14}$$

and so on. We will show that each of these nine integrals stays bounded as $k \to \infty$, which will prove (9). For every $x \in S_{ki}$ we have

$$\big(g_k(x) - g_k^{\Delta}(x)\big)^2 = \Big(a_{ki} - \sum_{j\in\mathcal{A}_{ki}} \gamma_j a_{kj}\Big)^2 \leqslant \sum_{j\in\mathcal{A}_{ki}} (a_{ki} - a_{kj})^2$$

by Jensen's inequality. Also, for some positive constants C_i which are independent of k,

$$\begin{aligned}
\sum_{i\in\mathcal{I}_k}\sum_{j\in\mathcal{A}_{ki}} (a_{kj} - a_{ki})^2 \lambda(S_{ki}) &\leqslant C_1 \sum_{i\in\mathcal{I}_k}\sum_{j\in\mathcal{A}_{ki}\setminus\{i\}} \frac{(a_{kj} - a_{ki})^2 \epsilon_k^{2n}}{|z_{kj} - z_{ki}|^{n+\alpha}} \epsilon_k^{\alpha} \\
&\leqslant C_2 \widehat{\mathcal{E}}_k(g_k, g_k) \epsilon_k^{\alpha} \\
&\leqslant C_3 \epsilon_k^{\alpha}
\end{aligned}$$

because $\sup_k \widehat{\mathcal{E}}_k(g_k, g_k) < \infty$. Putting all this together, it follows that, for some positive C_i, R independent of k,

$$\begin{aligned}
\iint\limits_{U\times U} \frac{(g_k(x) - g_k^{\Delta}(x))^2 h_k(x,y)\,dx\,dy}{|y-x|^{n+\alpha}} &\leqslant \sum_{i\in\mathcal{I}_k} \iint\limits_{U\times S_{ki}} \frac{\sum_{j\in\mathcal{A}_{ki}} (a_{kj} - a_{ki})^2 h_k(x,y)\,dx\,dy}{|y-x|^{n+\alpha}} \\
&\leqslant \sum_{i\in\mathcal{I}_k}\sum_{j\in\mathcal{A}_{ki}} (a_{kj} - a_{ki})^2 \lambda(S_{ki}) \int\limits_{B(\mathbf{0},R)\setminus B(\mathbf{0},C_1\epsilon_k)} \frac{dx}{|x|^{n+\alpha}} \\
&\leqslant C_2 \epsilon_k^{\alpha} \int\limits_{B(\mathbf{0},R)\setminus B(\mathbf{0},C_1\epsilon_k)} \frac{dx}{|x|^{n+\alpha}} \\
&\leqslant C_3 \epsilon_k^{\alpha} \epsilon_k^{-\alpha} \\
&= C_3.
\end{aligned}$$

Thus

$$\sup_k \iint\limits_{U\times U} \frac{(g_k(x) - g_k^{\Delta}(x))^2 h_k(x,y)\,dx\,dy}{|y-x|^{n+\alpha}} < \infty. \tag{15}$$

Now, let

$$\sigma_k(x,y) = \sum_{i,j\in\mathcal{I}_k} \frac{h_k(x,y)}{|z_{kj} - z_{ki}|^{n+\alpha}}.$$

For some real $C > 0$ independent of k, x, y, we have

$$\frac{h_k(x,y)}{|y-x|^{n+\alpha}} \leqslant C\sigma_k(x,y),$$

from which it follows that

$$\iint\limits_{U\times U} \frac{(g_k(y) - g_k(x))^2 h_k(x,y)\,dx\,dy}{|y-x|^{n+\alpha}} \leqslant C\widehat{\mathcal{E}}_k(g_k, g_k),$$

so

$$\sup_k \iint\limits_{U\times U} \frac{(g_k(y) - g_k(x))^2 h_k(x,y)\,dx\,dy}{|y-x|^{n+\alpha}} < \infty. \tag{16}$$

Eqs. (15) and (16) together imply that

$$\sup_k \iint\limits_{U\times U} \frac{(g_k(y) - g_k(x))(g_k(x) - g_k^{\Delta}(x))h_k(x,y)\,dx\,dy}{|y-x|^{n+\alpha}} < \infty.$$

The convergence of the other six integrals can be proved by using these facts, along with arguments similar to those given above. Thus (9) holds.

We will now prove (10). If Δ_{ki}^{ℓ} is a simplex as constructed above with vertices $z_{ki_0}, \dots, z_{ki_n}$, and $x, y \in \Delta_{ki}^{\ell}$, then for some convex combinations $\sum_j \eta_j z_{ki_j}$ and $\sum_j \gamma_j z_{ki_j}$

$$\begin{aligned}
\frac{|g_k^{\Delta}(y) - g_k^{\Delta}(x)|}{|y-x|} &= \frac{|\sum_{j=0}^n \eta_j a_{ki_j} - \sum_{j=0}^n \gamma_j a_{ki_j}|}{|\sum_{j=0}^n \eta_j z_{ki_j} - \sum_{j=0}^n \gamma_j z_{ki_j}|} \\
&= \frac{|\sum_{j=0}^n \eta_j (a_{ki_j} - a_{ki_0}) - \sum_{j=0}^n \gamma_j (a_{ki_j} - a_{ki_0})|}{|\sum_{j=0}^n \eta_j (z_{ki_j} - z_{ki_0}) - \sum_{j=0}^n \gamma_j (z_{ki_j} - z_{ki_0})|} \\
&= \frac{|\sum_{j=0}^n (\eta_j - \gamma_j)(a_{ki_j} - a_{ki_0})|}{|\sum_{j=0}^n (\eta_j - \gamma_j)(z_{ki_j} - z_{ki_0})} \\
&\leqslant C\frac{|\sum_{j=0}^n (\eta_j - \gamma_j)(a_{ki_j} - a_{ki_0})|}{|\sum_{j=0}^n (\eta_j - \gamma_j)\epsilon_k e_j|} \\
&\leqslant C\frac{\max_{j=1,\dots,n} |a_{ki_j} - a_{ki_0}|}{\epsilon_k}
\end{aligned}$$

where C is independent of x, y, i and k, and $\{e_1, \dots, e_n\}$ is the standard basis for $\mathbb{R}^n$. Now let $x \in S_{ki}^*$ and $y \in S_{kj}^*$ be such that $h_k(x,y) = 0$. Consequently, for some C independent of x, y, i, j and k,

$$\left|g_k^{\triangle}(y) - g_k^{\triangle}(x)\right| \leqslant \frac{1}{\epsilon_k} C|y-x| \max_{l,m\in\mathcal{A}_{ki}\cup\mathcal{A}_{kj}} |a_{kl} - a_{km}|.$$

It follows that for some positive numbers C_i independent of k,

$$\begin{aligned}
\iint\limits_{U\times U} \frac{(g_k^{\triangle}(y) - g_k^{\triangle}(x))^2(1-h_k(x,y))\,dx\,dy}{|y-x|^{n+\alpha}} &\leqslant C_1 \frac{1}{\epsilon_k^2} \sum_{i\in\mathcal{I}_k} \sum_{j\neq i} (a_j - a_i)^2 \lambda(S_{ki}) \int\limits_{B(\mathbf{0},2\sqrt{n}\epsilon_k)} \frac{dx}{|x|^{n+\alpha-2}} \\
&\leqslant C_2 \frac{\epsilon_k^{\alpha}}{\epsilon_k^2} \sum_{i\in\mathcal{I}_k} \sum_{j\neq i} \frac{(a_j-a_i)^2\epsilon_k^n}{|z_{kj}-z_{ki}|^{n+\alpha}} \int\limits_{B(\mathbf{0},2\sqrt{n}\epsilon_k)} \frac{dx}{|x|^{n+\alpha-2}} \\
&\leqslant C_3 \frac{\epsilon_k^{\alpha}}{\epsilon_k^2} \sum_{i\in\mathcal{I}_k} \sum_{j\neq i} \frac{(a_j-a_i)^2\epsilon_k^{2n}}{|z_{kj}-z_{ki}|^{n+\alpha}} \int\limits_{B(\mathbf{0},2\sqrt{n}\epsilon_k)} \frac{dx}{|x|^{n+\alpha-2}} \\
&\leqslant C_4 \frac{\epsilon_k^{\alpha}}{\epsilon_k^2} \sum_{i\in\mathcal{I}_k} \sum_{j\neq i} \frac{(a_j-a_i)^2\epsilon_k^{2n}}{|z_{kj}-z_{ki}|^{n+\alpha}} \epsilon_k^{2-\alpha} \\
&\leqslant C_5 \widehat{\mathcal{E}}_k(g_k, g_k)
\end{aligned}$$

which stays bounded as $k\to\infty$. This proves (10), and thus establishes (8).

Now by the Banach–Saks theorem, there is a subsequence (g_{k_l}) of (g_k) such that $\bar{g}_m \to g$ in $L^2(U)$, where

$$\bar{g}_m := \frac{1}{m}\sum_{l=1}^{m} g_{k_l}.$$

Also define

$$\bar{g}_m^{\triangle} := \frac{1}{m}\sum_{l=1}^{m} g_{k_l}^{\triangle}.$$

We have

$$\begin{aligned}
\|\bar{g}_m - \bar{g}_m^{\triangle}\|_{L^2(U)}^2 &\leqslant \frac{1}{m}\sum_{l=1}^{m} \int\limits_U \left(g_{k_l}^{\triangle} - g_{k_l}\right)^2 d\lambda \\
&\leqslant \frac{1}{m}\sum_{l=1}^{m} \sum_{j\in\mathcal{I}_{k_l}} \sum_{q\in\mathcal{A}_{kj}} (a_{kq} - a_{kj})^2 \lambda(S_{kj}) \\
&\leqslant \frac{1}{m}\sum_{l=1}^{m} C\epsilon_{k_l}^{\alpha} \widehat{\mathcal{E}}_{k_l}(g_{k_l}, g_{k_l}) \\
&\to 0.
\end{aligned}$$

Thus $\bar{g}_m^{\triangle} \to g$ in $L^2(U)$. In addition, by Jensen's inequality and (8),

$$\sup_m \widehat{\mathcal{E}}\left(\bar{g}_m^{\triangle}, \bar{g}_m^{\triangle}\right) \leqslant \sup_m \frac{1}{m}\sum_{l=1}^{m} \widehat{\mathcal{E}}\left(g_{k_l}^{\triangle}, g_{k_l}^{\triangle}\right) < \infty.$$

Thus we have a sequence $\bar{g}_m^{\triangle}$ which is bounded in $\mathcal{D}[\widehat{\mathcal{E}}]$, and is such that $\bar{g}_m^{\triangle} \to g$ in $L^2(U)$. Applying the Banach–Saks theorem a second time, we obtain a Cauchy sequence in $\mathcal{D}[\widehat{\mathcal{E}}]$ with limit equal to g. $\square$

Lemma 3.5. *Suppose $M > \sqrt{n}$. Then $(\widehat{\mathcal{E}}_k, \mathcal{D}[\widehat{\mathcal{E}}_k])$ converges to $(\widehat{\mathcal{E}}, \mathcal{D}[\widehat{\mathcal{E}}])$.*

Proof. First let $f \in C_0^\infty(U)$. Then for all sufficiently large k, $\widehat{\pi}_k f = \pi_k f$, and consequently $\widehat{\mathcal{E}}_k(\widehat{\pi}_k f, \widehat{\pi}_k f) = \mathcal{E}(\pi_k f, \pi_k f)$ for sufficiently large k. Thus by Lemma 3.3, $\widehat{\mathcal{E}}_k(\widehat{\pi}_k f, \widehat{\pi}_k f) \to \widehat{\mathcal{E}}(f, f)$ as $k \to \infty$. Since $C_0^\infty(U)$ is dense in $\mathcal{D}[\widehat{\mathcal{E}}]$, this proves the second condition in Definition 3.1.

If $(f_k)_{k\in\mathbb{N}}$ is a sequence such that $F_k\widehat{\pi}_k f_k$ converges weakly to f in $L^2(U)$, then setting $g_k := F_k\widehat{\pi}_k f_k$, we see that $F_k\pi_k g_k$ converges weakly to f in $L^2(U)$ because $\pi_k g_k = \widehat{\pi}_k f_k$ for every k. Consequently,

$$\liminf_k \widehat{\mathcal{E}}_k(\widehat{\pi}_k f_k, \widehat{\pi}_k f_k) = \liminf_k \mathcal{E}_k(\pi_k g_k, \pi_k g_k) \geqslant \mathcal{E}(f, f)$$

by Lemma 3.3. Thus

$$\liminf_{k\to\infty} \mathcal{E}_k(\widehat{\pi}_k f_k, \widehat{\pi}_k f_k) \geqslant \widehat{\mathcal{E}}(f, f) \quad \text{if } f \in \mathcal{D}[\widehat{\mathcal{E}}].$$

It remains to show that if $f \in L^2(U)\backslash\mathcal{D}[\widehat{\mathcal{E}}]$, then

$$\liminf_k \widehat{\mathcal{E}}_k(\widehat{\pi}_k f_k, \widehat{\pi}_k f_k) = \infty. \tag{17}$$

Suppose there is some $f \in L^2(U)\backslash\mathcal{D}[\widehat{\mathcal{E}}]$ such that $F_k\widehat{\pi}_k f_k$ converges weakly to f but (17) is not satisfied. Then there is a subsequence $(f_{k_i})_{i\in\mathbb{N}}$ satisfying the hypotheses of Lemma 3.4, from which we may conclude that $f \in \mathcal{D}[\widehat{\mathcal{E}}]$, which is a contradiction. $\square$

Lemmas 3.3 and 3.5, combined with Theorem 3.2, prove Theorem 2.1. We now finish the paper by proving the corollary.

Proof of Corollary 2.2. If $C_0^\infty(U)$ is dense in $\mathcal{D}[\mathcal{E}]$, then $(\mathcal{E}, \mathcal{D}[\mathcal{E}]) = (\widehat{\mathcal{E}}, \mathcal{D}[\widehat{\mathcal{E}}])$ and thus $(T(t))_{t\geqslant 0} = (\widehat{T}(t))_{t\geqslant 0}$. Since f is a probability density, we have (e.g., by an argument using part 1 of Theorem 2.1)

$$\int_U T(t)f(x)\,dx = 1.$$

Since $M > \sqrt{n}$, if $S_{ki}^* \neq S_{ki}$, then $z_{ki} \notin \widehat{E}_k$ and $\widehat{p}_k(z_{ki}, t) = 0$. It follows that

$$\int_U F_k\widehat{T}_k(t)\widehat{\pi}_k f(x)\,dx = \sum_{i\in\mathcal{I}_k} \int_{S_{ki}} \widehat{T}_k(t)\widehat{\pi}_k f(x)\,dx = \sum_{i\in\mathcal{I}_k} \widehat{p}_k(z_{ki}, t).$$

This proves the corollary because by part 2 of Theorem 2.1,

$$\int_U F_k\widehat{T}_k(t)\widehat{\pi}_k f(x)\,dx \to \int_U T(t)f(x)\,dx$$

as $k \to \infty$. $\square$

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