J. Math. Anal. Appl. 415 (2014) 543–573

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Global stability and wavefronts in a cooperation model with state-dependent time delay ☆

Yunfei Lv, Rong Yuan *

School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing, 100875, People's Republic of China

ARTICLE INFO

Article history:
Received 8 August 2013
Available online 4 February 2014
Submitted by J. Shi

Keywords:
State-dependent delay
Cooperation
Stage structure
Comparison principle
Global stability
Wavefronts

ABSTRACT

This article deals with a diffusive cooperative model with state-dependent delay which is assumed to be an increasing function of the population density with lower and upper bounds. For the cooperative DDE system, the positivity and boundedness of solutions are firstly given. Using the comparison principle of the state-dependent delay equations obtained, the stability criterion of model is analyzed both from local and global points of view. When the diffusion is properly introduced, the existence of traveling waves is obtained by constructing a pair of upper–lower solutions and Schauder's fixed point theorem. Calculating the minimum wave speed shows that the wave is slowed down by the state-dependent delay. Finally, the traveling wavefront solutions for large wave speed are also discussed, and the fronts appear to be all monotone, regardless of the state dependent time delay. This is an interesting property, since many findings are frequently reported that delay causes a loss of monotonicity, with the front developing a prominent hump in some other delay models.

1. Introduction

In a natural ecosystem, the maturity of a species individuals is not an instantaneous process but is mediated by some time lag which can be viewed as the time taken from birth to maturity. Systems with time lag (or time delay) lead to delay differential equations (DDE), which have been studied intensively and systematically [9,23,16,41]. The theory and applications of DDEs are emerging as an important area of investigation. Previously, some models of population growth with time delay (discrete and distributed time delays, stochastic, etc.) were discussed in literature [7,15,37,24,40].

But, in these above systems, only the constant time delay is considered. In 1992, Aiello et al. [2] have already considered a system with a state dependent delay, where the time delay is taken to be an increasing

☆ This work is supported by the National Nature Science Foundation of China (11031002, 11371058), RFDP (20110003110004), and the Grant of Beijing Education Committee Key Project (Z201310028031).

* Corresponding author.
E-mail address: ryuan@bnu.edu.cn (R. Yuan).

http://dx.doi.org/10.1016/j.jmaa.2014.01.086

function of the total populations. This assumption is believed to be realistic in the example of Antarctic whale and seal populations [11]. It is observed that individual of a small seal species takes three to four years to mature and of large seals takes five years to mature, of small whales takes seven to ten years and of large whale species takes twelve to fifteen years to reach maturity. Besides, Andrewartha and Birch [6] considered how the duration of larval development of flies is viewed as a nonlinear increasing function of larval density. For the interesting phenomenon, many authors investigated state-dependent time delay population model in literature [43,4,5,1,21]. In fact, the state-dependent delay $\tau(u_1)$ measures the intraspecific competition effects of a species u_1. Since the limited food resources made the species individuals devote more energy and time to finding food for their own survival and virtually none to reproduce, the time to maturity certainly becomes longer. That is, the period of maturity is longer if the number of species is larger, in return, it will lead to reduce the size of the population since the growth of the species is slowed down. Finally, the species will be equilibrium at some level u_1^*, and there is corresponding to an equilibrium delay $\tau(u_1^*)$. Our results imply that the stronger the intraspecific competition of the species, the smaller the equilibrium size u_1^* of the species and the lower the equilibrium delay $\tau(u_1^*)$. In this paper, we will deal with a diffusive cooperative model with state-dependent time delay. In biological terms, cooperation can be interpreted as that the presence of one species encourages the growth of the other species, which is one of the important interactions among species and is commonly seen in social animals and in human society (see, for example, [12,28,30]). Furthermore, we believe that such a diffusive model with state-dependent delay has not been discussed yet, and thus the work in this article is new.

As mentioned above, most species individuals have a life history that takes them through two stages: immature and mature, and species at two stages may have different behaviors. For example, for a number of mammals, the immature prey are concealed in the mountain cave and raised by their parents; they do not necessarily go out seeking food. When motion is allowed, then it is reasonable to suppose that the immature does not move (especially if the immature phase is a larval phase) and does not have a risk to contact with other species. Therefore, considering that stage structure in population is in accord with the natural phenomenon. Based on the fact that the amount of food available per biomass in a closed environment is a function of the consumer population, we propose the following diffusive cooperative model with a monotonically increasing, state-dependent delay

$$\begin{cases} \dfrac{dv_1}{dt} = \alpha_1 u_1 - \gamma_1 v_1 - \alpha_1 e^{-\gamma_1 \tau(u_1+v_1)} u_1\big(t-\tau(u_1+v_1)\big), \\ \dfrac{\partial u_1}{\partial t} = d_1 \dfrac{\partial^2 u_1}{\partial x^2} + \alpha_1 e^{-\gamma_1 \tau(u_1+v_1)} u_1\big(t-\tau(u_1+v_1)\big) - \beta_1 u_1^2 + \mu_1 u_1 u_2, \\ \dfrac{dv_2}{dt} = \alpha_2 u_2 - \gamma_2 v_2 - \alpha_2 e^{-\gamma_2 \tau(u_2+v_2)} u_2\big(t-\tau(u_2+v_2)\big), \\ \dfrac{\partial u_2}{\partial t} = d_2 \dfrac{\partial^2 u_2}{\partial x^2} + \alpha_2 e^{-\gamma_2 \tau(u_2+v_2)} u_2\big(t-\tau(u_2+v_2)\big) - \beta_2 u_2^2 + \mu_2 u_1 u_2, \end{cases} \quad t \in \mathbb{R}^+,\ x \in \mathbb{R}. \tag{1.1}$$

The following assumptions for model (1.1) are made:

- The variables $v_i(t,x)$ and $u_i(t,x)$ $(i=1,2)$ represent the densities of the cooperative immature and mature species at time t and at position x, respectively.
- The parameter $d_i(>0)$, $i=1,2$, is diffusion coefficient of population u_i. The delay τ is the time taken from birth to maturity. This paper considers the time delay to be state dependent, that is, the time delay is taken to be an increasing function of the total population u_i+v_i, so that $\tau'(u_i+v_i) \geqslant 0$, and we shall also assume that $0<\tau_m \leqslant \tau(u_i+v_i) \leqslant \tau_M$ with $\tau_m=\tau(0)$ and $\tau_M=\tau(\infty)$.
- The rate at which individuals are born is taken to be proportional to the number of matures at that time; this is the $\alpha_i u_i$ term. Death of immatures is modeled by the term $-\gamma_i v_i$. Death of matures is modeled by a quadratic term $\beta_i u_i^2$, as in the logistic equation. The term $\alpha_i e^{-\gamma_i \tau(u_i+v_i)} u_i(t-\tau(u_i+v_i),x)$ appearing in both equations represents the rate at time t and position x at which individuals leave the immature

and enter the mature class, having just reached maturity. These are individuals who were born at time $t-\tau(u_i+v_i)$. Therefore, the rate of entering the mature class is $\alpha_i u_i(t-\tau(u_i+v_i),x)$ times the fraction of those born at time $t-\tau(u_i+v_i)$ who are still alive now, where this fraction is $e^{-\gamma_i\tau(u_i+v_i)}$ follows from the assumption that the death of immatures is following a linear law given by the term $-\gamma_i u_i$ (on the basis of such a law, if $X(t)$ is any population, then the number that survive from $(t-\tau)$ to t is $e^{-\gamma_i\tau}X(t-\tau)$). Since the immature species do not move, the diffusion does not cause nonlocal response in the delay term.

• Assume that interspecific cooperative effects are of the classical Lotka–Volterra kind, and the effects of u_2 on u_1, and u_1 on u_2 are measured by $\mu_1>0$ and $\mu_2>0$, respectively.

Accompanied with (1.1), we take the initial conditions

$$\begin{aligned}
&u_1(\theta,x)=\varphi_1(\theta,x)\geqslant 0, \qquad u_2(\theta,x)=\varphi_2(\theta,x)\geqslant 0, \quad \theta\in[-\tau_M,0],\ x\in\mathbb{R},\\
&v_1(0,x)=v_{10}(x)>0, \qquad v_2(0,x)=v_{20}(x)>0, \qquad \varphi_1(0,x)>0, \qquad \varphi_2(0,x)>0, \quad x\in\mathbb{R},
\end{aligned}$$

with

$$v_{i0}(x)=\int_{-\tau_s}^{0}\alpha_i u_i(s,x)e^{\gamma_i s}\,ds, \quad i=1,2.$$

Thus, $v_{i0}(x)$ represents the number of the immature species i that have survived born from $-\tau_s$ to 0. For values of t, $-\tau_s\leqslant t\leqslant 0$ we understand that $u_i(t,x)=\varphi_i(t,x)$, and that $u_i(0,x)=\varphi_i(0,x)$, since anyone born before that time will have matured before time $t=0$. Note also that $\tau(u_i(0,x)+v_i(0,x))=\tau_s$, that is

$$\tau_s=\tau\big(u_i(0,x)+v_i(0,x)\big)=\tau\left(u_i(0,x)+\int_{-\tau_s}^{0}\alpha_i u_i(s,x)e^{\gamma_i s}\,ds\right).$$

Thus, τ_s is determined implicitly. From this, we can conclude that the initial conditions of v_i is dependent on the initial conditions of u_i. Solving the first Eq. (1.1), it follows that

$$v_i(s)=e^{-\gamma_i s}\left(v_i(0)+\alpha_i\int_0^s e^{\gamma_i t}\big(u_i(t)-e^{-\gamma_i\tau(u_i(t)+v_i(t))}u_i\big(t-\tau\big(u_i(t)+v_i(t)\big)\big)\big)\,dt\right).$$

Therefore, the solution $v_i(s,x)$ is dependent on the solution $u_i(s,x)$. For convenience, we consider that the delay τ is only the function of u_i not u_i+v_i. Thus, (1.1) is not a fully coupled system in that the second and fourth equations, for the mature populations u_1 and u_2, can be solved independent of the first and third, respectively. Consideration of this second and fourth equations alone is an interesting and non-trivial mathematical problem in its own right

$$\begin{cases}
\dfrac{\partial u_1}{\partial t}=d_1\dfrac{\partial^2 u_1}{\partial x^2}+\alpha_1 e^{-\gamma_1\tau(u_1)}u_1\big(t-\tau(u_1)\big)-\beta_1 u_1^2+\mu_1 u_1u_2, \quad t\in\mathbb{R}^+,\ x\in\mathbb{R},\\[2mm]
\dfrac{\partial u_2}{\partial t}=d_2\dfrac{\partial^2 u_2}{\partial x^2}+\alpha_2 e^{-\gamma_2\tau(u_2)}u_2\big(t-\tau(u_2)\big)-\beta_2 u_2^2+\mu_2 u_1u_2, \quad t\in\mathbb{R}^+,\ x\in\mathbb{R},
\end{cases} \tag{1.2}$$

with initial conditions

$$\begin{aligned}
&u_1(\theta,x)=\varphi_1(\theta,x)\geqslant 0, \qquad u_2(\theta,x)=\varphi_2(\theta,x)\geqslant 0, \quad \theta\in[-\tau_M,0],\ x\in\mathbb{R},\\
&\varphi_1(0,x)>0, \qquad \varphi_2(0,x)>0, \quad x\in\mathbb{R}.
\end{aligned} \tag{1.3}$$

• For our model to make sense, i.e., excluding the possibility of adults becoming immatures except by birth, the function $t-\tau(u_i+v_i)$ must be a strictly increasing function of t [2]. So, we need $(\partial/\partial t)\tau(u_i+v_i) = \tau'(u_i+v_i)(u_i+v_i)' < 1$. This is necessary to find conditions on $\tau(u_i+v_i)$ such that this assumption holds. In the absence of diffusion ($d_i = 0$), it follows that

$$u_i' + v_i' = \alpha_i u_i - \gamma_i v_i - \beta_i u_i^2 + \mu_i u_1 u_2 \leqslant (\alpha_i + \mu_i \Delta_i)u_i - \beta_i u_i^2 \leqslant \frac{(\alpha_i + \mu_i \Delta_i)^2}{4\beta_i},$$

here we used the positivity and boundedness of the solution u_i (which will be proved in Theorem 2.1) and v_i (which has been shown in [2]), this means that the assumption holds if $\tau' < \min\{4\beta_1/(\alpha_1+\mu_1\Delta_1)^2, 4\beta_2/(\alpha_2+\mu_2\Delta_2)^2\}$. In the presence of diffusion ($d_i \neq 0$), we mainly study the existence of traveling wave solutions connecting two equilibria which will be obtained in the region $\Gamma := \{(u_1,u_2) \mid 0 < u_1 < B_1,\ 0 < u_2 < B_2\}$, ($B_1$, B_2 see Proposition 3.1) by constructing upper–lower solutions in Section 3. For the case $d_i \neq 0$, we are only interested in the dynamics of model (1.2) in Γ. According to Lemma 3.3.1 in [42], similarly, we can impose the conditions on τ' such that the assumption holds, i.e., there exists a constant L which is dependent on B_1 and B_2 such that $\tau' < L$.

Without diffusion, the state-dependent delay differential model (1.2) extends the classical two-species Lotka–Volterra model. For the Lotka–Volterra ODE competition or cooperation system, such system generates a monotone dynamical system with respect to the standard ordering. The global dynamics is natural and can be obtained by applying the powerful monotone dynamical systems theory [34]. This flow monotonicity with respect to the standard ordering relation has also made it possible to establish the existence of traveling waves connecting equilibria for the corresponding reaction–diffusion monotone model. General results about monostable traveling waves for the above reaction–diffusion equations admitting comparison principles can be found in literature [39,25,27,22,10]. Some authors in [3,45] have studied the competitive model with stage structure. Note that in the two species model (1.2) with age stages, delay is indispensable. However, in the presence of state-dependent delay, it is easy to see that the system (1.2) is no longer order preserving and the equilibrium solutions depend on the delay. Since a Hopf bifurcation of stable periodic solutions may occur when time delay is large. There some partial answers for global stability and monotony of the competitive model with constant delay are given in [3,45,35,36]. It is difficulty to study the monotony of solutions and the global dynamics of system with state dependent delay. In this work, we will consider the global behavior, the existence and the monotonicity of traveling wave solutions of the cooperative model with state-dependent delay.

Furthermore, we will deal with a diffusive cooperative model with state-dependent delay, to allow for individuals moving around. When different species individuals inhabit the same environment, how species move, distribute, and persist is an important biological and mathematical question. The existence of traveling wave solutions for spatial systems provide a good answer to this question. From biological point of view, traveling wave solutions describe the species invasion, which is called the "wave of invasion". Some authors [3,45,20,44,14] investigated the stability and traveling waves in a competitive model with stage structure and delay. Gourley and Kuang [14] studied wavefronts and global stability for the well known stage structure model proposed by Aiello and Freedman, to allow for individuals moving around. One may naturally ask if a similar conclusion holds for the cooperative system with state-dependent delay. It is our main aim in this paper to study the existence of traveling waves and their properties, especially the monotonicity, for a diffusive cooperative model with state-dependent delay.

Firstly, we analyze the stability of the equilibria of the DDE system in Section 2. By constructing a pair of upper–lower solutions, we employ the cross iteration method and Schauder's fixed point theorem in a profile set to obtain the existence of traveling wave solutions, and further discuss the wavefronts for large enough speed in Section 3. Section 4 is devoted to some conclusions.

2. Stability analysis

In this section, we shall discuss the stability of the equilibria of the DDE system

$$\begin{cases} \dfrac{du_1}{dt} = \alpha_1 e^{-\gamma_1 \tau(u_1)} u_1\big(t-\tau(u_1)\big) - \beta_1 u_1^2 + \mu_1 u_1 u_2, \\ \dfrac{du_2}{dt} = \alpha_2 e^{-\gamma_2 \tau(u_2)} u_2\big(t-\tau(u_2)\big) - \beta_2 u_2^2 + \mu_2 u_1 u_2, \\ u_1(\theta) = \varphi_1(\theta) \geqslant 0, \qquad u_2(\theta) = \varphi_2(\theta) \geqslant 0, \quad \theta \in [-\tau_M, 0]. \end{cases} \tag{2.1}$$

Firstly, we show the positivity and boundedness of the solution. From the standpoint of biology, positivity means that the system persists, i.e., the populations may survive. Boundedness may be viewed as a natural restriction to growth as a result of limited resources in a closed environment. Based on these considerations, we have the following theorem.

Theorem 2.1. *Let $\varphi_i(\theta) > 0$, $i = 1, 2$, for $\theta \in [-\tau_M, 0]$. Then*

(a) $u_i(t) > 0$ *for* $t > 0$;
(b) *there exists* $\delta_i = \delta_i(\varphi_i) > 0$ *such that* $u_i(t) > \delta_i$ *for all* $t \geqslant 0$, *where*

$$\delta_i(\varphi_i) = \frac{1}{2}\min\Big\{\inf_{-\tau_M \leqslant \theta \leqslant 0} \varphi_i(\theta), \beta_i^{-1}\alpha_i e^{-\gamma_i \tau_M}\Big\};$$

(c) *there exists* $\Delta_i = \Delta_i(\varphi_i) > 0$ *such that* $u_i(t) < \Delta_i$ *for all* $t \geqslant 0$ *if* $\beta_1\beta_2 > \mu_1\mu_2$, *where*

$$\begin{aligned} \Delta_1 &= \max\Big\{\sup_{-\tau_M \leqslant \theta \leqslant 0} \varphi_1(\theta), \frac{\beta_2 \alpha_1 e^{-\gamma_1 \tau_m} + \mu_1 \alpha_2 e^{-\gamma_2 \tau_m}}{\beta_1\beta_2 - \mu_1\mu_2}\Big\}, \\ \Delta_2 &= \max\Big\{\sup_{-\tau_M \leqslant \theta \leqslant 0} \varphi_2(\theta), \frac{\mu_2 \alpha_1 e^{-\gamma_1 \tau_m} + \beta_1 \alpha_2 e^{-\gamma_2 \tau_m}}{\beta_1\beta_2 - \mu_1\mu_2}\Big\}. \end{aligned} \tag{2.2}$$

Proof. First, we show the positivity of u_i. Otherwise, there would be t_1 so that $u_i(t_1) = 0$. By continuity of solutions, it follows from $u_i(0) > 0$ that such t_1 must be strictly greater than zero. Let $t^* := \inf\{t\colon\ t > 0, u_i(t) = 0\}$. Then from the ith Eq. (2.1), we have

$$u_i'(t^*) = \alpha_i e^{-\gamma_i \tau_m} u_i\big(t^* - \tau_m\big).$$

From assumption $t^* - \tau_m < t^*$, then $u_i(t^* - \tau_m) > 0$ by the definition of t^*. This, in turn, implies $u_i'(t^*) > 0$, giving us a contradiction. So no such t^* exists, and we obtain the results (a).

Second, we show u_i is uniformly bounded away from zero for a given positive initial function. Set $\delta_i(\varphi_i) = \frac{1}{2}\min\{\inf_{-\tau_M \leqslant \theta \leqslant 0} \varphi_i(\theta), \beta_i^{-1}\alpha_i e^{-\gamma_i \tau_M}\}$. Otherwise, there exists an s_1 such that $s_1 = \inf\{t\colon\ t \geqslant 0, u_i(t) = \delta_i\}$ and $u_i'(s_1) \leqslant 0$. By the definition of δ_i, we have $u_i(0) = \varphi_i(0) \geqslant 2\delta_i$. It follows from the continuity that $s_1 > 0$. Thus,

$$\begin{aligned} u_i'(s_1) &= \alpha_i e^{-\gamma_i \tau(u_i)} u_i\big(s_1 - \tau(u_i)\big) - \beta_i u_i^2(s_1) + \mu_i u_1(s_1) u_2(s_1) \\ &\geqslant \alpha_i e^{-\gamma_i \tau_M}\delta_i - \beta_i \delta_i^2 \geqslant \alpha_i e^{-\gamma_i \tau_M}\delta_i - \frac{1}{2}\alpha_i e^{-\gamma_i \tau_M}\delta_i = \frac{1}{2}\alpha_i e^{-\gamma_i \tau_M}\delta_i > 0. \end{aligned}$$

We have a contradiction. So, such s_1 does not exist and $u_i(t) > \delta_i$ for all $t > 0$.

Next, we show that the ith population is bounded above by Δ_i. Our proof is divided into two steps.

(i) Suppose that $u_i'(t) \geqslant 0$ for all $t > T$ for some $T \geqslant 0$. Then for $t > T + \tau$,

$$\begin{aligned} 0 \leqslant u_i'(t) &= \alpha_i e^{-\gamma_i \tau(u_i)} u_i\big(t - \tau(u_i)\big) - \beta_i u_i^2(t) + \mu_i u_1(t) u_2(t) \\ &\leqslant \alpha_i e^{-\gamma_i \tau_m} u_i(t) - \beta_i u_i^2(t) + \mu_i u_1(t) u_2(t), \end{aligned}$$

since $u_i(t - \tau(u_i)) \leqslant u_i(t)$. This means that

$$\beta_1 u_1 - \mu_1 u_2 \leqslant \alpha_1 e^{-\gamma_1 \tau_m}, \qquad \beta_2 u_2 - \mu_2 u_1 \leqslant \alpha_2 e^{-\gamma_2 \tau_m}, \quad t > T.$$

Thus, it follows from $u_i(t) > 0$ that

$$u_1(t) \leqslant \frac{\beta_2 \alpha_1 e^{-\gamma_1 \tau_m} + \mu_1 \alpha_2 e^{-\gamma_2 \tau_m}}{\beta_1 \beta_2 - \mu_1 \mu_2}, \qquad u_2(t) \leqslant \frac{\beta_1 \alpha_2 e^{-\gamma_2 \tau_m} + \mu_2 \alpha_1 e^{-\gamma_1 \tau_m}}{\beta_1 \beta_2 - \mu_1 \mu_2},$$

if $\beta_1 \beta_2 > \mu_1 \mu_2$, giving us our desired result.

(ii) Assume that $\beta_1 \beta_2 > \mu_1 \mu_2$. Now, if there are two sequences $\{t_n\}_{n=1}^{\infty}$ and $\{s_m\}_{m=1}^{\infty}$ such that $u_1'(t_n) = 0$, $u_2'(s_m) = 0$, and $u_1(t_n)$, $u_2(s_m)$ is a local maximum, where $u_1(t) \leqslant u_1(t_n)$, $0 < t < t_n$ for all n, and $u_2(t) \leqslant u_2(s_m)$, $0 < t < s_m$ for all m, then by a similar analysis at $t = t_n$ and $t = s_m$, it follows that

$$\beta_1 u_1(t_n) - \mu_1 u_2(t_n) \leqslant \alpha_1 e^{-\gamma_1 \tau_m}, \qquad \beta_2 u_2(s_m) - \mu_2 u_1(s_m) \leqslant \alpha_2 e^{-\gamma_2 \tau_m}. \tag{2.3}$$

For any given t_n, we take $s_n = \max\{s_m\colon s_m \leqslant t_n\}$. If $s_n = t_n$, it follows from (i) that the solutions $u_1(t)$ and $u_2(t)$ are bounded above by a bound. If $s_n < t_n$ and $u_2(s_n) < u_2(t_n)$, then $u_2'(t_n) > 0$ and $u_2(t) \leqslant u_2(t_n)$ for all $t \leqslant t_n$. Otherwise, there is an $s_n < t < t_n$ such that $u_2'(t) = 0$, which contradicts the definition of s_n. Thus, we have

$$\begin{aligned} 0 < u_2'(t_n) &= \alpha_2 e^{-\gamma_2 \tau(u_2)} u_2\big(t_n - \tau(u_2)\big) - \beta_2 u_2^2(t_n) + \mu_2 u_1(t_n) u_2(t_n) \\ &\leqslant \alpha_2 e^{-\gamma_2 \tau_m} u_2(t_n) - \beta_2 u_2^2(t_n) + \mu_2 u_1(t_n) u_2(t_n). \end{aligned}$$

In a similar way of (i), we get the same results $u_1(t)$ and $u_2(t)$ are bounded above by a bound. If $s_n < t_n$ and $u_2(t_n) < u_2(s_n)$, then it follows from the first inequality of (2.3) that

$$\beta_1 u_1(s_m) - \mu_1 u_2(s_m) \leqslant \beta_1 u_1(t_n) - \mu_1 u_2(t_n) \leqslant \alpha_1 e^{-\gamma_1 \tau_m}.$$

Combined with the second inequality of (2.3), we obtain the same results $u_1(t)$ and $u_2(t)$ are bounded above by a bound.

Choosing $\Delta_i(\varphi_i)$ in (2.2), this completes the theorem. □

Next, we discuss steady states of system (2.1). We first examine the nullclines of the system

$$\begin{cases} \alpha_1 e^{-\gamma_1 \tau(u_1)} - \beta_1 u_1 + \mu_1 u_2 = 0, \\ \alpha_2 e^{-\gamma_2 \tau(u_2)} - \beta_2 u_2 + \mu_2 u_1 = 0. \end{cases} \tag{2.4}$$

It is easy to show that system (2.1) has the extinction equilibrium $E_0 = (0, 0)$, the boundary equilibria $E_1 = (\widehat{u}_1, 0)$ and $E_2 = (0, \widehat{u}_2)$ where $\widehat{u}_i$ satisfies the following equation

$$\alpha_i e^{-\gamma_i \tau(u_i)} - \beta_i u_i = 0, \quad i = 1, 2. \tag{2.5}$$

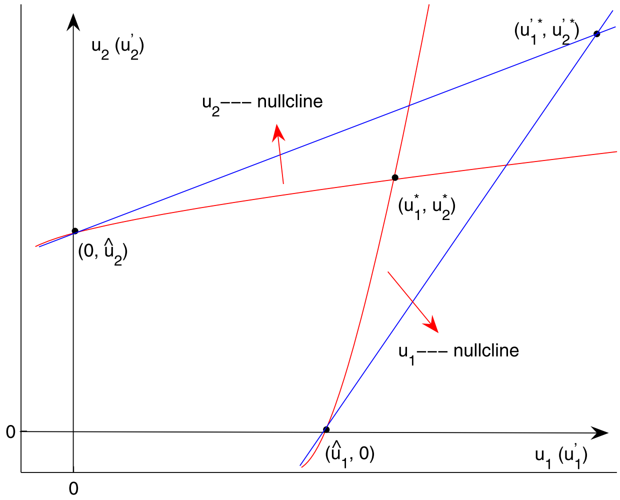

Fig. 1. Graphs of systems (a) and (b).

We observe that $\alpha_i e^{-\gamma_i \tau(u_i)}$ is a decreasing function with respect to u_i, and $\alpha_i e^{-\gamma_i \tau(0)} = \alpha_i e^{-\gamma_i \tau_m} > 0$, $\lim_{u_i \to \infty} \alpha_i e^{-\gamma_i \tau(u_i)} = \alpha_i e^{-\gamma_i \tau_M}$. Furthermore, $\lim_{u_i \to \infty} \beta_i u_i = \infty$, there exists a unique boundary equilibria $E_1 = (\widehat{u}_1, 0)$ or $E_2 = (0, \widehat{u}_2)$. Therefore, if the boundary equilibria exists then $\widehat{u}_i < \beta_i^{-1} \alpha_i e^{-\gamma_i \tau_m}$ is unique. For the well-extended logistic model (1.2), we can denote by $\widehat{u}_i$ the carrying capacity of the species i.

Eqs. (2.4) can be rewritten as the following equations:

$$\begin{cases} u_2 = \mu_1^{-1}(\beta_1 u_1 - \alpha_1 e^{-\gamma_1 \tau(u_1)}), \\ u_1 = \mu_2^{-1}(\beta_2 u_2 - \alpha_2 e^{-\gamma_2 \tau(u_2)}), \end{cases} \quad \text{(a)} \qquad \begin{cases} u_2' = \mu_1^{-1}(\beta_1 u_1' - \alpha_1 e^{-\gamma_1 \tau(\widehat{u}_1)}), \\ u_1' = \mu_2^{-1}(\beta_2 u_2' - \alpha_2 e^{-\gamma_2 \tau(\widehat{u}_2)}). \end{cases} \quad \text{(b)}$$

It is easy to verify that the system (b) with constant delays $\tau(\widehat{u}_1)$ and $\tau(\widehat{u}_2)$ has a positive solution $({u_1'}^*, {u_2'}^*)$ if $\mu_1\mu_2 < \beta_1\beta_2$. Note that both lines which are represented by the first equation of (a) and (b) go through the point $E_1 = (\widehat{u}_1, 0)$; and both lines which are represented by second equation of (a) and (b) go through the point $E_2 = (0, \widehat{u}_2)$; the first line of (a) is always above the first line of (b) for $u_1 > \widehat{u}_1$; the second line of (a) is always below the second line of (b) for $u_2 > \widehat{u}_2$. We illustrate the case in Fig. 1. Thus, the system (a) has always a positive solution (u_1^*, u_2^*) if $\mu_1\mu_2 < \beta_1\beta_2$.

Besides, solving (2.4) gives

$$u_1^* = \frac{\alpha_1\beta_2 e^{-\gamma_1\tau(u_1^*)} + \alpha_2\mu_1 e^{-\gamma_2\tau(u_2^*)}}{\beta_1\beta_2 - \mu_1\mu_2}, \qquad u_2^* = \frac{\alpha_2\beta_1 e^{-\gamma_2\tau(u_2^*)} + \alpha_1\mu_2 e^{-\gamma_1\tau(u_1^*)}}{\beta_1\beta_2 - \mu_1\mu_2}, \tag{2.6}$$

provided $u_1^* > 0$, $u_2^* > 0$ if and only if $\mu_1\mu_2 < \beta_1\beta_2$. In order to discuss the criteria for which there exists a unique equilibrium, both u_1-nullcline and u_2-nullcline define u_2 in Eq. (2.4) as a function of u_1, $u_2 = g_1(u_1)$ and $u_2 = g_2(u_1)$, respectively. Then E^* will be unique if $g_1'(u_1^*) > g_2'(u_1^*)$ for every such E^*, otherwise there were more than one equilibrium E^*, then the reverse inequality must hold for alternate equilibria and this is a contradiction.

Now, it follows from Eq. (2.4) that $u_2 = g_1(u_1) = \mu_1^{-1}(\beta_1 u_1 - \alpha_1 e^{-\gamma_1\tau(u_1)})$ and $u_2 = g_2(u_1)$. Therefore, taking derivatives along g_1 and g_2 with respect to u_1 gives us

$$\begin{aligned} g_1'(u_1) &= \mu_1^{-1}\bigl(\gamma_1\tau'(u_1)\alpha_1 e^{-\gamma_1\tau(u_1)} + \beta_1\bigr), \\ g_2'(u_1) &= \mu_2\bigl(\gamma_2\tau'(u_2)\alpha_2 e^{-\gamma_2\tau(u_2)} + \beta_2\bigr)^{-1}. \end{aligned} \tag{2.7}$$

From (2.4) we get the relations

$$\alpha_1 e^{-\gamma_1\tau(u_1^*)} = \beta_1 u_1^* - \mu_1 u_2^* > 0, \qquad \alpha_2 e^{-\gamma_2\tau(u_2^*)} = \beta_2 u_2^* - \mu_2 u_1^* > 0, \tag{2.8}$$

and also

$$g_1'(u_1^*) = \mu_1^{-1}\big(\gamma_1\tau'(u_1^*)\big(\beta_1 u_1^* - \mu_1 u_2^*\big) + \beta_1\big),$$
$$g_2'(u_1^*) = \mu_2\big(\gamma_2\tau'(u_2^*)\big(\beta_2 u_2^* - \mu_2 u_1^*\big) + \beta_2\big)^{-1}.$$

The uniqueness of coexistence equilibrium is obtained providing $g_1'(u_1^*) > g_2'(u_1^*)$, that is

$$\mu_1\mu_2 < \big(\gamma_1\tau'(u_1^*)\big(\beta_1 u_1^* - \mu_1 u_2^*\big) + \beta_1\big)\big(\gamma_2\tau'(u_2^*)\big(\beta_2 u_2^* - \mu_2 u_1^*\big) + \beta_2\big). \tag{2.9}$$

Then, the coexistence equilibrium exists and is unique if $\mu_1\mu_2 < \beta_1\beta_2$, since the right part of the inequality (2.9) is always above $\beta_1\beta_2$.

Therefore, we have the following theorem.

Theorem 2.2. *The system (2.1) has a unique extinction equilibrium $E_0 = (0,0)$, boundary equilibrium $E_1 = (\widehat{u}_1, 0)$ and $E_2 = (0, \widehat{u}_2)$, and coexistence equilibrium $E^* = (u_1^*, u_2^*)$ if and only if $\mu_1\mu_2 < \beta_1\beta_2$.*

Remark 2.1. From above analysis, we can conclude that if the coexistence equilibria exists, then $u_1^* > \widehat{u}_1$ and $u_2^* > \widehat{u}_2$, which implies the mutualistic effects raises the equilibrium levels of each species. Since, the coexistence equilibrium values u_i^* is greater than the level $\widehat{u}_i$ (the carrying capacities for each species) in the absence of cooperative interaction.

Furthermore, we will discuss the local asymptotic stability of equilibria by studying the sign of the real parts of eigenvalues of the associated characteristic equations (see [18,38] for more details about linearization and stability of state-dependent delay differential equations). Let $E = (u_1^0, u_2^0)$ be an arbitrary equilibrium. Using Taylor expansions, and neglecting all nonlinear terms in u_1 and u_2, the linearized system (2.1) about E is given by

$$\begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \begin{pmatrix} -2\beta_1 u_1^0 - \xi_1^* + \mu_1 u_2^0 & \mu_1 u_1^0 \\ \mu_2 u_2^0 & -2\beta_2 u_2^0 - \xi_2^* + \mu_2 u_1^0 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$
$$+ \begin{pmatrix} \alpha_1 e^{-\gamma_1\tau(u_1^0)} & 0 \\ 0 & \alpha_2 e^{-\gamma_2\tau(u_2^0)} \end{pmatrix} \begin{pmatrix} u_1(t-\tau(u_1)) \\ u_2(t-\tau(u_2)) \end{pmatrix}, \tag{2.10}$$

where

$$\xi_i^* = \alpha_i\gamma_i u_i^0 e^{-\gamma_i\tau(u_i^0)}\tau'(u_i^0), \quad i = 1, 2. \tag{2.11}$$

Trial solutions proportional to $(c_1, c_2)\exp(\lambda t)$ leads to the characteristic equation

$$\begin{vmatrix} a_{11} - \xi_1^* + \mu_1 u_2^0 - \lambda & \mu_1 u_1^0 \\ \mu_2 u_2^0 & a_{22} - \xi_2^* + \mu_2 u_1^0 - \lambda \end{vmatrix} = 0, \tag{2.12}$$

where $a_{ii} = \alpha_i e^{-\tau(u_i^0)(\gamma_i+\lambda)} - 2\beta_i u_i^0$.

For the extinction equilibrium $E_0 = (0,0)$, Eq. (2.12) reduces to

$$\big(\alpha_1 e^{-\tau_m(\gamma_1+\lambda)} - \lambda\big)\big(\alpha_2 e^{-\tau_m(\gamma_2+\lambda)} - \lambda\big) = 0.$$

All eigenvalues are given by solutions of

$$\lambda_i = \alpha_i e^{-\tau_m(\gamma_i+\lambda_i)}, \quad i = 1, 2.$$

So, the above equation always has a real positive solution and E_0 is an unstable point.

For the boundary equilibrium $E_1 = (\widehat{u}_1, 0)$, the eigenvalues are the roots of the equation

$$\bigl(\alpha_1 e^{-\tau(\widehat{u}_1)(\gamma_1+\lambda)} - 2\beta_1\widehat{u}_1 - \xi_1^* - \lambda\bigr)\bigl(\alpha_2 e^{-\tau_m(\gamma_2+\lambda)} + \mu_2\widehat{u}_1 - \lambda\bigr) = 0.$$

Clearly, some of the eigenvalues are given by the equation

$$\lambda + 2\beta_1\widehat{u}_1 + \xi_1^* = \alpha_1 e^{-\tau(\widehat{u}_1)(\gamma_1+\lambda)}.$$

We claim that all eigenvalues have negative real parts. Suppose that $\operatorname{Re}\lambda \geqslant 0$, then from the above equation we compute the real parts of λ and have

$$\operatorname{Re}\lambda + 2\beta_1\widehat{u}_1 + \beta_1\gamma_1\tau'(\widehat{u}_1)\widehat{u}_1^2 = \beta_1\widehat{u}_1 e^{-\tau_1 \operatorname{Re}\lambda}\cos(\gamma_1\tau_1 \operatorname{Im}\lambda) \leqslant \beta_1\widehat{u}_1.$$

Hence $\operatorname{Re}\lambda \leqslant -\beta_1\widehat{u}_1(1 + \gamma_1\tau'(\widehat{u}_1)\widehat{u}_1) < 0$, a contradiction proving the claim.

The others λ are given by the equation

$$\alpha_2 e^{-\tau_m(\gamma_2+\lambda)} = \lambda - \mu_2\widehat{u}_1.$$

It is easy to see that the above equation has always one positive real root, by plotting the left- and right-hand sides of the above equation against λ. Therefore, the boundary equilibrium $E_1 = (\widehat{u}_1, 0)$ is unstable.

In a similar way, we can show that $E_2 = (0, \widehat{u}_2)$ is unstable.

For the coexistence equilibrium state $E^* = (u_1^*, u_2^*)$, all eigenvalues satisfy the equation

$$\bigl(\lambda - a_{11} + \xi_1^* - \mu_1 u_2^*\bigr)\bigl(\lambda - a_{22} + \xi_2^* - \mu_2 u_1^*\bigr) - \mu_1\mu_2 u_1^* u_2^* = 0. \tag{2.13}$$

In order to show that E^* is locally asymptotically stable, we just need to prove the roots of the above characteristic equation have negative real parts. Let $\lambda = a + ib$, where a and b are real numbers. Let

$$\begin{aligned}
&D_1 = a + \beta_1 u_1^* + \alpha_1 e^{-\gamma_1\tau(u_1^*)}\bigl(1 + \gamma_1 u_1^*\tau'\bigl(u_1^*\bigr) - e^{-a\tau(u_1^*)}\cos\bigl(b\tau\bigl(u_1^*\bigr)\bigr)\bigr),\\
&D_2 = a + \beta_2 u_2^* + \alpha_2 e^{-\gamma_2\tau(u_2^*)}\bigl(1 + \gamma_2 u_2^*\tau'\bigl(u_2^*\bigr) - e^{-a\tau(u_2^*)}\cos\bigl(b\tau\bigl(u_2^*\bigr)\bigr)\bigr),\\
&E_1 = b + \alpha_1 e^{-\tau(u_1^*)(\gamma_1+a)}\sin\bigl(b\tau\bigl(u_1^*\bigr)\bigr), \qquad E_2 = b + \alpha_2 e^{-\tau(u_2^*)(\gamma_2+a)}\sin\bigl(b\tau\bigl(u_2^*\bigr)\bigr).
\end{aligned}$$

Substituting $\lambda = a + ib$ into Eq. (2.13), we get

$$D_1 D_2 - E_1 E_2 = \mu_1\mu_2 u_1^* u_2^* \quad \text{and} \quad D_1 E_2 + D_2 E_1 = 0.$$

Then

$$\bigl(\mu_1\mu_2 u_1^* u_2^*\bigr)^2 = (D_1 D_2)^2 + (E_1 E_2)^2 - 2 D_1 D_2 E_1 E_2.$$

Now, we assume that $\operatorname{Re}\lambda = a \geqslant 0$. By $1 - e^{-a\tau(u_i^*)}\cos(b\tau(u_i^*)) \geqslant 0$, it follows that

$$D_i \geqslant \beta_i u_i^*, \quad \text{and} \quad D_1 D_2 \geqslant \mu_1\mu_2 u_1^* u_2^* \quad \text{if } \beta_1\beta_2 \geqslant \mu_1\mu_2.$$

But, using $D_1E_2 + D_2E_1 = 0$, we have

$$\left(\mu_1\mu_2 u_1^* u_2^*\right)^2 = (D_1D_2)^2 + (E_1E_2)^2 + (D_1E_2)^2 + (D_2E_1)^2.$$

And so $(\mu_1\mu_2 u_1^* u_2^*)^2 > (D_1D_2)^2$, which contradicts the assumption. Therefore, the coexistence equilibrium state E^* is locally asymptotically stable if $\beta_1\beta_2 \geqslant \mu_1\mu_2$.

The next two lemmas are elementary but useful in the following discussion, which can be found in Gopalsamy [13] and Hirsch et al. [19].

Lemma 2.1 *(Barbălat lemma). Let a be a finite number and $f : [a,\infty) \to \mathbb{R}$ be a differentiable function. If $\lim_{t\to\infty} f(t)$ exists (finite) and f' is uniformly continuous on $[a,\infty)$, then $\lim_{t\to\infty} f'(t) = 0$.*

Lemma 2.2 *(Fluctuation lemma). Let a be a finite number and $f : [a,\infty) \to \mathbb{R}$ be a differentiable function. If $\liminf_{t\to\infty} f(t) < \limsup_{t\to\infty} f(t)$, then there exist sequences $\{t_n\} \uparrow \infty$ and $\{s_n\} \uparrow \infty$ such that $\lim_{n\to\infty} f(t_n) = \limsup_{t\to\infty} f(t)$, $f'(t_n) = 0$ and $\lim_{n\to\infty} f(s_n) = \liminf_{t\to\infty} f(t)$, $f'(s_n) = 0$.*

Next, we are mainly interested in the global asymptotic stability of the coexistence equilibria of our model. Before proceeding, we will need the following theorem.

Theorem 2.3. *Let $u_1(t)$ be the solution of*

$$\frac{du_1}{dt} = \alpha_1 e^{-\gamma_1\tau(u_1)} u_1\big(t-\tau(u_1)\big) - \beta_1 u_1^2 + Au_1, \tag{2.14}$$

where the initial data $u_1(\theta) = \varphi_1(\theta) > 0$, for $\theta \in [-\tau_M, 0]$. Then $\lim_{t\to\infty} u_1(t) = \tilde{u}_1$ where $\tilde{u}_1 = \beta_1^{-1}(\alpha_1 e^{-\gamma_1\tau(\tilde{u}_1)} + A)$.

Proof. It is easy to verify that $u_1(t)$ is positive and bounded (the proof is same as that of Theorem 2.1). Let us first deal with the case when $u_1(t)$ is eventually monotonic. For this case, there exists $0 \leqslant \tilde{u}_1 < \infty$ such that $\lim_{t\to\infty} u_1(t) = \tilde{u}_1$ and $\lim_{t\to\infty} u_1'(t) = 0$. Hence from system (2.14), taking the limit as $t \to \infty$, we get that

$$0 = \lim_{t\to\infty} u_1'(t) = \tilde{u}_1\big(\alpha_1 e^{-\gamma_1\tau(\tilde{u}_1)} - \beta_1\tilde{u}_1 + A\big).$$

Thus $\tilde{u}_1 = 0$ and $\tilde{u}_1 = \beta_1^{-1}(\alpha_1 e^{-\gamma_1\tau(\tilde{u}_1)} + A)$. So, this limit must be an equilibrium of (2.14) and is therefore either zero or the value stated. Zero is ruled out since a standard linearized analysis yields that the zero solution of (2.14) is linearly unstable. Therefore, $\tilde{u}_1 = \beta_1^{-1}(\alpha_1 e^{-\gamma_1\tau(\tilde{u}_1)} + A)$.

The rest of case to discuss is that $u_1(t)$ is neither eventually monotonically increasing nor decreasing. Thus, we assume that $u_1(t)$ is oscillatory. Then $u_1(t)$ has an infinite sequence of local maxima and define the sequence $\{t_j\}$ as those times for which $u_1'(t_j) = 0$ and $u_1''(t_j) < 0$. Here, we will only discuss in detail the case of the local maximum $u_1(t_j) > \tilde{u}_1$ for all $j = 1,2,3,\dots$, and other cases can be dealt with analogously.

Now, we prove that $\sup_{t\geqslant t_1} u_1(t) = u_1(t_k)$ for some integer k. Otherwise, after every local maximum $u_1(t_j)$ there is another that is higher, and a subsequence of $\{t_j\}$ (still relabeled $\{t_j\}$) therefore can be chosen with the property that $u_1(t) < u_1(t_j)$ for all $t_1 \leqslant t < t_j$ and each j. The subsequence is selected by including each local maximum which is higher than every one before it. By assumption $t_j - \tau(u_1(t_j)) < t_j$, for each j

$$\begin{aligned} 0 = \dot{u}_1(t_j) &= \alpha_1 e^{-\gamma_1\tau(u_1)} u_1\big(t_j-\tau(u_1)\big) - \beta_1 u_1^2(t_j) + Au_1(t_j) \\ &< u_1(t_j)\big(\alpha_1 e^{-\gamma_1\tau(u_1)} - \beta_1 u_1(t_j) + A\big) < u_1(t_j)\big(\alpha_1 e^{-\gamma_1\tau(\tilde{u}_1)} - \beta_1\tilde{u}_1 + A\big) \\ &= 0, \end{aligned}$$

this is a contradiction. So, $\sup_{t\geqslant t_1} u_1(t) = u_1(t_k)$ for some integer k and we let $s_1 = t_k$. Now, by applying this same analysis to the interval $t \geqslant t_{k+1}$, the existence of a t_l $(l > k)$ with $\sup_{t\geqslant t_{k+1}} u_1(t) = u_1(t_l)$ can be obtained, and we set $s_2 = t_l$. Continuing this process, we obtain an infinite sequence $\{s_j\}$ of times such that $s_{j+1} > s_j$, $s_j \to \infty$, $u_1(t) \leqslant u_1(s_j)$ for all $t > s_j$, and $u_1'(s_j) = 0$.

Let $y(t) = u_1(t) - \widetilde{u}_1$. Next, we will prove that $y(t) \to 0$ as $t \to \infty$. We have got a sequence $y(s_j) > y(s_{j+l}) > 0$ (since $u_1(s_j) \geqslant u_1(s_{j+l})$ and $u_1(s_j) > \widetilde{u}_1$), and it is now enough to show that $y(s_j) \to 0$ as $j \to \infty$. In terms of y, Eq. (2.14) becomes, at $t = s_j$,

$$\begin{aligned}0 = \dot y(s_j) &= \alpha_1 e^{-\gamma_1\tau(y+\widetilde{u}_1)} y\bigl(s_j - \tau(y+\widetilde{u}_1)\bigr) - \beta_1 y^2(s_j) - 2\beta_1 y(s_j)\widetilde{u}_1 + Ay(s_j)\\ &\quad + \bigl(\alpha_1 e^{-\gamma_1\tau(y+\widetilde{u}_1)}\widetilde{u}_1 - \beta_1\widetilde{u}_1^2 + A\widetilde{u}_1\bigr)\end{aligned}$$

so that

$$\begin{aligned}&\alpha_1 e^{-\gamma_1\tau(y+\widetilde{u}_1)} y\bigl(s_j - \tau(y+\widetilde{u}_1)\bigr)\\ &\quad = \bigl(2\beta_1\widetilde{u}_1 + \beta_1 y(s_j) - A\bigr)y(s_j) - \alpha_1\widetilde{u}_1\bigl(e^{-\gamma_1\tau(y(s_j)+\widetilde{u}_1)} - e^{-\gamma_1\tau(\widetilde{u}_1)}\bigr)\\ &\quad = \bigl(2\beta_1\widetilde{u}_1 + \beta_1 y(s_j) + \alpha_1 e^{-\gamma_1\tau(\widetilde{u}_1)} - \beta_1\widetilde{u}_1\bigr)y(s_j) - \alpha_1\widetilde{u}_1\bigl(e^{-\gamma_1\tau(y(s_j)+\widetilde{u}_1)} - e^{-\gamma_1\tau(\widetilde{u}_1)}\bigr)\\ &\quad = \bigl(\beta_1\widetilde{u}_1 + \beta_1 y(s_j) + \alpha_1 e^{-\gamma_1\tau(\widetilde{u}_1)}\bigr)y(s_j) + \alpha_1\widetilde{u}_1\bigl(e^{-\gamma_1\tau(\widetilde{u}_1)} - e^{-\gamma_1\tau(y(s_j)+\widetilde{u}_1)}\bigr)\\ &\quad \geqslant \bigl(\alpha_1 e^{-\gamma_1\tau(\widetilde{u}_1)} + \beta_1\widetilde{u}_1\bigr)y(s_j)\end{aligned}$$

since $\tau(\widetilde{u}_1) < \tau(y(s_j) + \widetilde{u}_1)$. By the sequence $\{s_j\}$, we choose a final subsequence, once again denoted $\{s_j\}$, so that $s_j - \tau_M \geqslant s_{j-1}$. Then $y(s_j - s) < y(s_{j-1})$ for all $s \in [0, \tau_M]$ and therefore

$$y(s_j) \leqslant \frac{\alpha_1 e^{-\gamma_1\tau(y+\widetilde{u}_1)} y(s_j - \tau(y+\widetilde{u}_1))}{\alpha_1 e^{-\gamma_1\tau(\widetilde{u}_1)} + \beta_1\widetilde{u}_1} \leqslant S y(s_{j-1}), \quad \text{where } S = \frac{\alpha_1 e^{-\gamma_1\tau(\widetilde{u}_1)}}{\alpha_1 e^{-\gamma_1\tau(\widetilde{u}_1)} + \beta_1\widetilde{u}_1}.$$

Now, $S < 1$ and S is independent of j. Therefore, $y(s_j) \to 0$ as $j \to \infty$. We summarize that $\lim_{t\to\infty} u_1(t) = \widetilde{u}_1$ and complete the proof of Theorem 2.3. □

Besides, in the proof of the following theorem, a comparison principle will be used. As we know, the comparison principles does not always hold for delay equation, let alone the state-dependent equation, which depends very much on how the delay term appears in the equations. For example, the essential requirement for a comparison principle to hold is that the reaction term be a nondecreasing function of the delayed variable in scalar equation [29]. Now, we give the proof of the comparison principles of state-dependent delay equations.

Theorem 2.4. *Let $u_1(t)$ be the solution of*

$$u_1'(t) = \alpha_1 e^{-\gamma_1\tau(u_1)} u_1\bigl(t - \tau(u_1)\bigr) - \beta_1 u_1^2 + \lambda u_1, \quad t > 0,$$

and $u_2(t)$ some function satisfying

$$u_2'(t) \geqslant \alpha_1 e^{-\gamma_1\tau(u_2)} u_2\bigl(t - \tau(u_2)\bigr) - \beta_1 u_2^2 + \lambda u_2, \quad t > 0. \tag{2.15}$$

Assume also that $u_2(\theta) \geqslant u_1(\theta)$ for all $\theta \in [-\tau_M, 0]$. Then $u_2(t) \geqslant u_1(t)$ for all $t > 0$.

Proof. First suppose that the inequality in (2.15) and $u_2(\theta) > u_1(\theta)$ for all $\theta \in [-\tau_M, 0]$ are strict. We claim that $u_2(t) > u_1(t)$ for all $t > 0$. If it is false there would exist $t_0 > 0$ such that $u_2(t) > u_1(t)$, $t \in [-\tau_M, t_0)$ and $u_2(t_0) = u_1(t_0)$. It follows that $u_2'(t_0) \leqslant u_1'(t_0)$. But

$$\begin{aligned} u_1'(t_0) &= \alpha_1 e^{-\gamma_1\tau(u_1)} u_1\big(t_0-\tau(u_1)\big) - \beta_1 u_1^2(t_0) + \lambda u_1(t_0) \\ &< \alpha_1 e^{-\gamma_1\tau(u_2)} u_2\big(t_0-\tau(u_2)\big) - \beta_1 u_2^2(t_0) + \lambda u_2(t_0) \\ &< u_2'(t_0) \end{aligned}$$

because $u_2(t_0-\tau(u_2(t_0))) > u_1(t_0-\tau(u_1(t_0)))$ and $u_2(t_0)=u_1(t_0)$. This contradiction proves the result in this case.

For the general case, let $\varepsilon>0$ and $u_\varepsilon(t)$ be the solution of

$$u_1'(t) = \alpha_1 e^{-\gamma_1\tau(u_1)} u_1\big(t-\tau(u_1)\big) - \beta_1 u_1^2 + \lambda u_1 - \varepsilon$$

corresponding to initial data $u_\varepsilon(\theta)=u_1(\theta)-\varepsilon$, $\theta\in[-\tau_M,0)$. By the results of the previous paragraph, we may conclude that $u_\varepsilon(t)<u_2(t)$ for all $t>0$ for which $u_\varepsilon(t)$ is defined. It can be shown that for sufficiently small $\varepsilon>0$, the solution $u_\varepsilon(t)\to u_1(t)$ as $\varepsilon\to 0$ for all $t\geqslant -\tau_M$. See, for example, Theorem 2.2 of Chapter 2 in [17]. Consequently, $u_1(t)=\lim_{\varepsilon\to 0}u_\varepsilon(t)\leqslant u_2(t)$. This proves the general case. □

For the comparison principles, the other reversed inequalities follow analogously, and will be used in the next discussions. Furthermore, a differential inequality of the form (2.15), which holds only for t above some value, say t_1, and not for all $t>0$, will be often used in applications of these comparison results. That is the initial time is simply thought of as t_1 rather than 0, and $u_2(t)\geqslant u_1(t)$ is arranged to hold for $t\leqslant t_1$ by appropriate definition of $u_1(t)$ for values of $t\leqslant t_1$. In the interests of clarity, this latter case in detail will not be always explained.

As well as we know, the consequence of global stability is that the state-dependent effects will not irreversibly change the system. As long as one of two species does not extinct, the system is able to recover. So, it is possible and necessary to investigate the global stability of the equilibrium.

Theorem 2.5. *The coexistence equilibrium E^* is globally asymptotically stable if $\beta_1\beta_2>\mu_1\mu_2$.*

Proof. We first give some definitions $\bar u_1=\limsup_{t\to\infty}u_1(t)$, $\underline{u}_1=\liminf_{t\to\infty}u_1(t)$, $\bar u_2=\limsup_{t\to\infty}u_2(t)$, $\underline{u}_2=\liminf_{t\to\infty}u_2(t)$. Since

$$\begin{aligned} u_i'(t) &= \alpha_i e^{-\gamma_i\tau(u_i)} u_i\big(t-\tau(u_i)\big) - \beta_i u_i^2(t) + \mu_i u_1(t)u_2(t) \\ &\geqslant \alpha_i e^{-\gamma_i\tau(u_i)} u_i\big(t-\tau(u_i)\big) - \beta_i u_i^2(t), \end{aligned}$$

we can obtain from Theorem 2.3 that $\underline{u}_i\geqslant \widehat{u}_i=\beta_i^{-1}\alpha_i e^{-\gamma_i\tau(\widehat{u}_i)}$, where $\widehat{u}_i$ is the positive component of the equilibrium E_i.

From the condition of $\beta_1\beta_2>\mu_1\mu_2$, it follows that E^* is linearly stable. Thus, we only need to prove the global attractiveness of the equilibrium. We shall construct four sequences, $M_n^{u_1}$, $M_n^{u_2}$, $N_n^{u_1}$ and $N_n^{u_2}$ with the properties that $\bar u_i\leqslant M_n^{u_i}$ for each n with $M_n^{u_i}\to u_i^*$ as $n\to\infty$, and $\underline{u}_i\geqslant N_n^{u_i}$ for each n with $N_n^{u_i}\to u_i^*$ as $n\to\infty$ (so that $\underline{u}_i\leqslant u_i^*$). It is useful to know that M_n denotes an upper bound and N_n a lower bound on the lim inf and lim sup, respectively, as $t\to\infty$, of the variable in the superscript. We will derive recursion formulae for these bounds and use them to prove the result. Firstly, let $v_{i1}(t)$ satisfy

$$v_{i1}'(t) = \alpha_i e^{-\gamma_i\tau(v_{i1})} v_{i1}\big(t-\tau(v_{i1})\big) - \beta_i v_{i1}^2, \quad t>0,$$

with, for $-\tau_M\leqslant t\leqslant 0$, $v_{i1}(t)\equiv\max\{u_i(t),\ t\in[-\tau_M,0]\}>0$. Since $u_i(t)$ is nonnegative, by comparison principle, $u_i(t)\geqslant v_{i1}(t)$ and therefore

$$\underline{u}_i=\limsup_{t\to\infty}u_i(t)\geqslant\lim_{t\to\infty}v_{i1}(t)=v_{i1}^*=\beta_i^{-1}\alpha_i e^{-\gamma_i\tau(v_{i1}^*)}=:N_1^{u_i}.$$

It is easy to see that $N_1^{u_i} < u_i^*$. From the assumption of the theorem, we can choose $\varepsilon > 0$ small enough so that

$$0 < \varepsilon < \beta_2\beta_1 - \mu_1\mu_2. \tag{2.16}$$

Then there exists $t_1 > \tau_M$ so that $u_2(t) \geqslant N_1^{u_2} - \varepsilon$ for every $t \geqslant t_1$. For $t > t_1$, let $v_{12}(t)$ be the solution of

$$v_{12}'(t) = \alpha_1 e^{-\gamma_1\tau(v_{12})} v_{12}\big(t - \tau(v_{12})\big) - \beta_1 v_{12}^2 + \mu_1 v_{12}\big(N_1^{u_2} - \varepsilon\big),$$

and let

$$v_{12}(t) \equiv \max\big\{u_1(t),\ t \in [t_1 - \tau_M, t_1]\big\} \quad \text{for } t \in [t_1 - \tau_M, t_1],$$

which is strictly positive, since $u_1(t) > 0$ on $(0, \infty)$. As mentioned in above, Theorem 2.4 is now being applied with initial time t_1 rather than 0, therefore, it is not necessary to define $v_{12}(t)$ for $t < t_1 - \tau_M$. By the definition ε in (2.16), Theorem 2.3 tells us that $\lim_{t\to\infty} v_{12}(t) = v_{12}^*$, where v_{12}^* satisfies

$$v_{12}^* = \beta_1^{-1}\big(\alpha_1 e^{-\gamma_1\tau(v_{12}^*)} + \mu_1\big(N_1^{u_2} - \varepsilon\big)\big).$$

Now, since $u_2(t) \geqslant N_1^{u_2} - \varepsilon$ for $t \geqslant t_1$, we have, for such t,

$$\begin{aligned} u_1'(t) &= \alpha_1 e^{-\gamma_1\tau(u_1)} u_1\big(t - \tau(u_1)\big) - \beta_1 u_1^2 + \mu_1 u_1 u_2 \\ &\geqslant \alpha_1 e^{-\gamma_1\tau(u_1)} u_1\big(t - \tau(u_1)\big) - \beta_1 u_1^2 + \mu_1 u_1\big(N_1^{u_2} - \varepsilon\big). \end{aligned}$$

By comparison principle, $u_1(t) \geqslant v_{12}(t)$ and therefore

$$\underline{u}_1 = \liminf_{t\to\infty} u_1(t) \geqslant \lim_{t\to\infty} v_{12}(t) = v_{12}^*.$$

Since this is true for any $\varepsilon > 0$ satisfying (2.16), it follows that

$$\underline{u}_1 \geqslant \beta_1^{-1}\big(\alpha_1 e^{-\gamma_1\tau(N_2^{u_1})} + \mu_1 N_1^{u_2}\big) =: N_2^{u_1}, \quad \text{and} \quad N_2^{u_1} < u_1^*.$$

In a similar way, we have

$$\underline{u}_2 \geqslant \beta_2^{-1}\big(\alpha_2 e^{-\gamma_2\tau(N_2^{u_2})} + \mu_2 N_1^{u_1}\big) =: N_2^{u_2}, \quad \text{and} \quad N_2^{u_2} < u_2^*.$$

This process can be continued to generated two sequences $N_n^{u_1}$, $N_n^{u_2}$, $n = 1, 2, 3, \dots,$ such that, for $n \geqslant 2$, $N_n^{u_i} < u_i^*$, $i = 1, 2$, and

$$N_n^{u_2} = \beta_2^{-1}\big(\alpha_2 e^{-\gamma_2\tau(N_n^{u_2})} + \mu_2 N_{n-1}^{u_1}\big), \qquad N_n^{u_1} = \beta_1^{-1}\big(\alpha_1 e^{-\gamma_1\tau(N_n^{u_1})} + \mu_1 N_{n-1}^{u_2}\big). \tag{2.17}$$

Now, we consider the other two sequences $M_n^{u_1}$ and $M_n^{u_2}$. From Theorem 2.1, it follows that the two populations are bounded from above under the condition of $\beta_1\beta_2 > \mu_1\mu_2$. That is there exist Δ_i which is defined in (2.2) such that $u_i(t) < \Delta_i$ and $\Delta_i \geqslant u_i^*$ for all $t \geqslant -\tau_M$. It follows that

$$u_1'(t) = \alpha_1 e^{-\gamma_1\tau(u_1)} u_1\big(t - \tau(u_1)\big) - \beta_1 u_1^2 + \mu_1 u_1 u_2 \leqslant \alpha_1 e^{-\gamma_1\tau(u_1)} u_1\big(t - \tau(u_1)\big) - \beta_1 u_1^2 + \mu_1 u_1 \Delta_2.$$

Set

$$M_1^{u_1} := \beta_1^{-1}\big(\alpha_1 e^{-\gamma_1\tau(M_1^{u_1})} + \mu_1\Delta_2\big).$$

Clearly, $M_1^{u_1} \geqslant u_1^*$. Hence, $\bar{u}_1 = \limsup_{t\to\infty} u_1(t) \leqslant M_1^{u_1}$. In a similar way, we have

$$M_1^{u_2} := \beta_2^{-1}\left(\alpha_2 e^{-\gamma_2 \tau(M_1^{u_2})} + \mu_2 \Delta_1\right), \qquad \bar{u}_2 = \limsup_{t\to\infty} u_2(t) \leqslant M_1^{u_2},$$

and $M_1^{u_2} \geqslant u_2^*$. Let $\varepsilon > 0$. We have $t_2 > 0$ so that $u_2(t) \leqslant M_1^{u_2} + \varepsilon$ for every $t \geqslant t_2$. Then

$$u_1'(t) \leqslant \alpha_1 e^{-\gamma_1 \tau(u_1)} u_1\left(t - \tau(u_1)\right) - \beta_1 u_1^2 + \mu_1 u_1\left(M_1^{u_2} + \varepsilon\right) \quad \text{for } t \geqslant t_2.$$

Thus, let $n_1^{u_1}(t)$ be the solution of

$$n_1^{u_1\prime}(t) = \alpha_1 e^{-\gamma_1 \tau(n_1^{u_1})} u_1\left(t - \tau\left(n_1^{u_1}\right)\right) - \beta_1\left(n_1^{u_1}\right)^2 + \mu_1 n_1^{u_1}\left(M_1^{u_2} + \varepsilon\right) \quad \text{for } t \geqslant t_2$$

with appropriate initial data, then $u_1(t) \leqslant n_1^{u_1}(t)$ and therefore

$$\bar{u}_1 \leqslant \lim_{t\to\infty} n_1^{u_1}(t) = \beta_1^{-1}\left(\alpha_1 e^{-\gamma_1 \tau(\lim_{t\to\infty} n_1^{u_1}(t))} + \mu_1\left(M_1^{u_2} + \varepsilon\right)\right).$$

In fact, we have used assumption of Theorem 2.5 to infer that $n_1^{u_1}(t)$ has this limiting behavior. So, for any ε, we have

$$M_2^{u_1} := \beta_1^{-1}\left(\alpha_1 e^{-\gamma_1 \tau(M_2^{u_1})} + \mu_1 M_1^{u_2}\right) \quad \text{and } \bar{u}_1 \leqslant M_2^{u_1},\ M_2^{u_1} \geqslant u_1^*.$$

Similarly, we deduce the following estimated for $\bar{u}_2$:

$$M_2^{u_2} := \beta_2^{-1}\left(\alpha_2 e^{-\gamma_2 \tau(M_2^{u_2})} + \mu_2 M_1^{u_1}\right) \quad \text{and } \bar{u}_2 \leqslant M_2^{u_2},\ M_2^{u_2} \geqslant u_2^*.$$

If follows that the transition from the $(n-1)^{th}$ to the n^{th} step in this iterative process

$$M_n^{u_2} = \beta_2^{-1}\left(\alpha_2 e^{-\gamma_2 \tau(M_n^{u_2})} + \mu_2 M_{n-1}^{u_1}\right) \quad \text{and} \quad M_n^{u_1} = \beta_1^{-1}\left(\alpha_1 e^{-\gamma_1 \tau(M_n^{u_1})} + \mu_1 M_{n-1}^{u_2}\right),$$

and $M_n^{u_1} \geqslant u_1^*$, $M_n^{u_2} \geqslant u_2^*$. It necessary to show that both $M_n^{u_1}$ and $N_n^{u_1}$ approach u_1^* as $n \to \infty$ and that both $M_n^{u_2}$ and $N_n^{u_2}$ approach u_2^*.

We see at once that

$$N_n^{u_1} = \frac{\beta_2 \alpha_1 e^{-\gamma_1 \tau(N_n^{u_1})} + \mu_1 \alpha_2 e^{-\gamma_2 \tau(N_{n-1}^{u_2})}}{\beta_1 \beta_2} + \frac{\mu_1 \mu_2}{\beta_1 \beta_2} N_{n-2}^{u_1}. \tag{2.18}$$

We claim that $N_n^{u_1}$ is a monotonically increasing sequence that is bounded above by u_1^*. The boundedness below by u_1^* follows immediately from (2.18) by induction. Then, by (2.17) and (2.18),

$$\begin{aligned} \frac{N_n^{u_1}}{N_{n-2}^{u_1}} &= \frac{\beta_2 \alpha_1 e^{-\gamma_1 \tau(N_n^{u_1})} + \mu_1 \alpha_2 e^{-\gamma_2 \tau(N_{n-1}^{u_2})}}{\beta_1 \beta_2 N_{n-2}^{u_1}} + \frac{\mu_1 \mu_2}{\beta_1 \beta_2} \\ &\geqslant \frac{\beta_2 \alpha_1 e^{-\gamma_1 \tau(u_1^*)} + \mu_1 \alpha_2 e^{-\gamma_2 \tau(u_2^*)}}{\beta_1 \beta_2 u_1^*} + \frac{\mu_1 \mu_2}{\beta_1 \beta_2} = 1 \end{aligned} \tag{2.19}$$

so that $N_n^{u_1}$ is a monotonically increasing. Hence $N_n^{u_1}$ converges to a limit which, by (2.17)–(2.19), equals u_1^*. Clearly, convergence of $N_n^{u_1}$ means convergence of $N_n^{u_2}$, and it is easy to verify that $N_n^{u_2}$ has the limit u_2^*.

The analysis for the remaining two sequences $M_n^{u_1}$ and $M_n^{u_2}$ is similar. $\square$

Our focus so far has been on the dynamic behaviors of the system (2.1). To facilitate the interpretation of our mathematical results in model (2.1), we give a summary of the dynamic behavior in Table 1.

Table 1
Conditions and dynamic behavior for system (2.1).

	Solutions of (2.1)	E_0	E_1	E_2	E^*
$\beta_1\beta_2 > \mu_1\mu_2$	Bounded	Unstable	Unstable	Unstable	GAS
$\beta_1\beta_2 < \mu_1\mu_2$	Unbounded	Unstable	Unstable	Unstable	Does not exist

GAS: Globally asymptotical stability.

Our conclusion, therefore, is that the dynamics depends on the values of the two quantities $\beta_1\beta_2$ and $\mu_1\mu_2$. From biological point of view, the term $\beta_i u_i^2$ represents the death of population u_i, which can be also illustrated the death caused by crowding effects or the intraspecific competition effects; the term $\mu_i u_1 u_2$ represents strongly mutualistic effects between both species. The conditions $\beta_1\beta_2 \geqslant \mu_1\mu_2$ means that the intraspecific competition of species is stronger than the mutualistic effects between both species. Therefore, the intraspecific competition effects is the main factor which affects the boundedness of solutions and the existence, stability of the coexistence equilibrium E^*. As long as the intraspecific competition of one species do not result in the extinction of the species and its effects is stronger than mutualistic effects between both species, the system is able to stable at the coexistence equilibrium. However, when the intraspecific competition of species is lower than the mutualistic effects between both species, the mutualistic effects would result in the growth unlimited of the two populations, i.e., both populations grow unboundedly.

3. Existence of traveling waves

Motivated by the results of the linearized analysis we have just carried out, it is interesting to inquire into the possibility of traveling wave solutions connecting the extinct equilibrium state E_0 to the coexistence equilibrium state E^* when the diffusion is introduced. A traveling wave solution of (1.2) is a special translation invariant solution.

To obtain a traveling wave solution of the system (1.2), we set $u_1(x,t) = \phi_1(s)$, $u_2(x,t) = \phi_2(s)$, $s = x+ct$ where $(\phi_1, \phi_2) \in C^2(\mathbb{R}, \mathbb{R}^2)$ are the profiles of the wave that propagates through one-dimensional spatial domain at a constant speed $c \geqslant 0$. Substituting $u_1(x,t) = \phi_1(s)$ and $u_2(x,t) = \phi_2(s)$ into (1.2), we obtain the wave equations

$$\begin{cases} d_1\phi_1''(s) - c\phi_1'(s) + \alpha_1 e^{-\gamma_1\tau(\phi_1)}\phi_1\big(s - c\tau(\phi_1)\big) - \beta_1\phi_1^2(s) + \mu_1\phi_1(s)\phi_2(s) = 0, \\ d_2\phi_2''(s) - c\phi_2'(s) + \alpha_2 e^{-\gamma_2\tau(\phi_2)}\phi_2\big(s - c\tau(\phi_2)\big) - \beta_2\phi_2^2(s) + \mu_2\phi_1(s)\phi_2(s) = 0, \end{cases} \tag{3.1}$$

with

$$\big(\phi_1(-\infty), \phi_2(-\infty)\big) = (0,0) \quad \text{and} \quad \big(\phi_1(\infty), \phi_2(\infty)\big) = \big(u_1^*, u_2^*\big). \tag{3.2}$$

A solution of (3.1), (3.2) corresponds to a leftward-moving traveling wave solution moving with speed c. Based on the ecological considerations, we are only concentrated on solutions that are non-negative for all s. Rewrite system (3.1) as an equivalent system in $\mathbb{R}^4$

$$\begin{cases} \phi_1'(s) = \psi_1(s), \\ \psi_1'(s) = \dfrac{1}{d_1}\big(c\psi_1(s) - \alpha_1 e^{-\gamma_1\tau(\phi_1)}\phi_1\big(s - c\tau(\phi_1)\big) + \beta_1\phi_1^2(s) - \mu_1\phi_1(s)\phi_2(s)\big), \\ \phi_2'(s) = \psi_2(s), \\ \psi_2'(s) = \dfrac{1}{d_2}\big(c\psi_2(s) - \alpha_2 e^{-\gamma_2\tau(\phi_2)}\phi_2\big(s - c\tau(\phi_2)\big) + \beta_2\phi_2^2(s) - \mu_2\phi_1(s)\phi_2(s)\big). \end{cases} \tag{3.3}$$

We can obtain a necessary condition (under which the solutions are non-negative) on the front speed c by linearizing (3.3) ahead of the front, i.e. for $s \to -\infty$. The linearized equations at $(0,0,0,0)$ will be

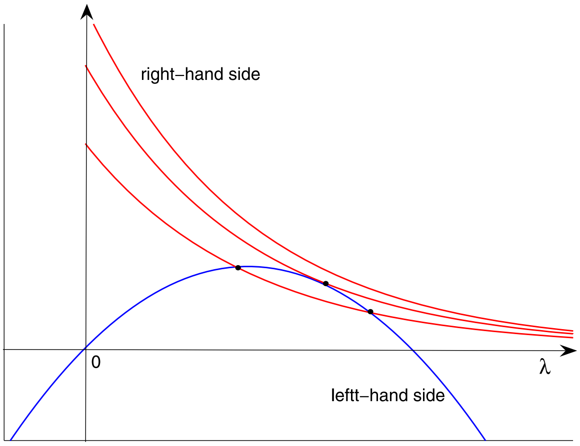

Fig. 2. Graphs of the left- and right-hand sides of (3.6) as function of λ.

$$\begin{cases} \phi_1'(s) = \psi_1(s), \\ \psi_1'(s) = \dfrac{1}{d_1}\big(c\psi_1(s) - \alpha_1 e^{-\gamma_1\tau_m}\phi_1(s - c\tau_m)\big), \\ \phi_2'(s) = \psi_2(s), \\ \psi_2'(s) = \dfrac{1}{d_2}\big(c\psi_2(s) - \alpha_2 e^{-\gamma_2\tau_m}\phi_2(s - c\tau_m)\big). \end{cases} \tag{3.4}$$

Seeking solutions of this proportional to $(c_1, c_2, c_3, c_4)\exp(\lambda s)$, we find λ satisfies

$$\big(d_1\lambda^2 - c\lambda + \alpha_1 e^{-(\gamma_1 + c\lambda)\tau_m}\big)\big(d_2\lambda^2 - c\lambda + \alpha_2 e^{-(\gamma_2 + c\lambda)\tau_m}\big) = 0. \tag{3.5}$$

Certainly, when the delay $\tau_m = 0$, Eq. (3.5) reduces to two quadratic equations. Using standard phase-plane arguments, it is easy to compute that the minimum wave speed c for having a solution that is non-negative for all s is $\max\{2\sqrt{\alpha_1 d_1}, 2\sqrt{\alpha_2 d_2}\}$. Now we increase τ_m from zero, $\tau_m > 0$. To seek a front $(\phi_1(s), \phi_2(s))$ which approaches to $(0,0)$ as $s \to -\infty$ without oscillating, it will be necessary for (3.5) to have some real positive roots. The total loss of all real positive roots of (3.5) implies the onset of oscillations. In fact, Fig. 2 shows a plot of the left- and right-hand sides of the following equation as functions of λ

$$c\lambda - d_i\lambda^2 = \alpha_i e^{-(\gamma_i + c\lambda)\tau_m}, \tag{3.6}$$

and the situation tells us that there are two real positive roots.

In general, there exist either two real positive roots or none, and it is easy to see that the latter situation can be brought about through changing the values of certain parameters. The critical case which determine the minimum wave speed c is when the two curves touch, such that there is just one repeated root, and this happens when

$$\begin{cases} c\lambda_{i*} - d_i\lambda_{i*}^2 = \alpha_i e^{-(\gamma_i + c\lambda_{i*})\tau_m}, \\ c - 2d_i\lambda_{i*} = -c\tau_m\alpha_i e^{-(\gamma_i + c\lambda_{i*})\tau_m}, \end{cases} \tag{3.7}$$

where λ_{i*} is the single repeated root. Eliminating the exponential terms, it follows that λ_{i*} must satisfy the square equation

$$f(\lambda_{i*}) := c\tau_m\lambda_{i*}^2 + \left(2 - \frac{c^2\tau_m}{d_i}\right)\lambda_{i*} - \frac{c}{d_i} = 0. \tag{3.8}$$

It is immediately to find that this square equation always has one real positive root and one real negative root. Again, from the fact that $f(c/d_i) = c/d_i > 0$, we have that the positive root is less than c/d_i. The negative root cannot satisfy the second equation of (3.7). So, we conclude that λ_{i*} must be the positive root of (3.8). After knowing λ_{i*}, the value of c can be given implicitly by either equation of (3.7), which we then can denote by $c_i^* = c_{i,min}(\tau_m)$ the minimum wave speed. However, the minimum speed cannot be solved explicitly.

It is known that the minimum speed is $2\sqrt{\alpha_i d_i}$ when $\tau_m = 0$. We now want to know whether the minimum speed will decrease or increase when delay is introduced, and some useful information on this can be given using a perturbation analysis for small τ_m. Note that λ_{i*} depends on τ_m too. So, we have

$$\begin{cases} c_{i,min}(\tau_m) = c_{i,min}^{(0)} + \tau_m c_{i,min}^{(1)} + \tau_m^2 c_{i,min}^{(2)} + \cdots, \\ \lambda_{i*} = \lambda_{i*}^{(0)} + \tau_m \lambda_{i*}^{(1)} + \tau_m^2 \lambda_{i*}^{(2)} + \cdots, \end{cases}$$

where $c_{i,min}^{(0)} = 2\sqrt{\alpha_i d_i}$ and $\lambda_{i*}^{(0)} = c_{i,min}^{(0)}/2d_i$. Hence, we find that

$$c_{i,min}^{(1)} = -d_i\sqrt{\alpha_i d_i}(\gamma_i + 2\alpha_i) < 0,$$

such that, for small delays, the minimum speed is given by

$$c_{i,min}(\tau_m) = 2\sqrt{\alpha_i d_i} - \tau_m d_i \sqrt{\alpha_i d_i}(\gamma_i + 2\alpha_i) + \cdots. \tag{3.9}$$

Whether the speed is reduced or increased by a small delay τ_m depends on the sign of $c_{i,min}^{(1)}$. It is easy to see that $c_{i,min}^{(1)} < 0$, and therefore the speed is slowed down by the state-dependent delay. Here, we leave the nonexistence of nonnegative traveling waves for future study when $c < c_i^*$.

3.1. Existence of traveling waves

In this subsection, we use Schauder's fixed point theorem, the method of cross iteration scheme associated with upper–lower solutions to establish the existence of traveling wave solutions connecting the extinction equilibrium state $E_0 = (0, 0)$ to the coexistence equilibrium state $E^* = (u_1^*, u_2^*)$ with large wave speeds.

To seek such a traveling wave solution of (1.2), it is necessary to construct a pair of upper–lower solutions. The linearisation of the wave equation at E_0 is given by

$$\begin{cases} d_1\phi_1''(s) - c\phi_1'(s) + \alpha_1 e^{-\gamma_1\tau_m}\phi_1(s - c\tau_m) = 0, \\ d_2\phi_2''(s) - c\phi_2'(s) + \alpha_2 e^{-\gamma_2\tau_m}\phi_2(s - c\tau_m) = 0. \end{cases} \tag{3.10}$$

Substituting $(\phi_1, \phi_2)(s) = (c_1, c_2)e^{\lambda s}$ into the above equations yields the characteristic equations

$$\Delta_i(\lambda, c) := d_i\lambda^2 - c\lambda + \alpha_i e^{-(\gamma_i + c\lambda)\tau_m} = 0, \quad i = 1, 2. \tag{3.11}$$

Then it is easy to verify the following properties ($i = 1, 2$):

(i) $\Delta_i(0, c) = \alpha_i e^{-\gamma_i\tau_m} > 0$;
(ii) $\lim_{\lambda\to\infty} \Delta_i(\lambda, c) = \infty$ for all $c \geqslant 0$;
(iii) $\frac{\partial^2\Delta_i(\lambda,c)}{\partial\lambda^2} = 2d_i + c^2\tau_m^2\alpha_i e^{-(\gamma_i + c\lambda)\tau_m} > 0$ and

$$\frac{\partial\Delta_i(\lambda, c)}{\partial c} = -\lambda - \lambda\tau_m\alpha_i e^{-(\gamma_i + c\lambda)\tau_m} < 0 \quad \text{for all } \lambda > 0;$$

(iv) $\lim_{c\to\infty} \Delta_i(\lambda, c) = -\infty$ for all $\lambda > 0$ and $\Delta_i(\lambda, 0) > 0$.

Besides, we have discussed the minimum wave speed c_1^* and c_2^*. Clearly, $c_i^* > 0$, $i = 1, 2$, is well defined. As mentioned in [26], c_i^* may be viewed as the spreading speeds of one species u_i in the absence of the other species. These above properties lead to the following lemma.

Lemma 3.1. *Let c_i^* $(i = 1, 2)$ be defined as above, then the following statements hold.*

(a) *If $c \geqslant c_i^*$, then there exist two positive roots λ_{i1}, λ_{i2} with $\lambda_{i1} \leqslant \lambda_{i2}$ such that $\Delta_i(\lambda_{i1}, c) = \Delta_i(\lambda_{i2}, c) = 0$, $i = 1, 2$.*
(b) *If $c < c_i^*$, then $\Delta_i(\lambda, c) > 0$ for all $\lambda > 0$.*
(c) *If $c = c_i^*$, then $\lambda_{i1} = \lambda_{i2}$; and if $c > c_i^*$, then $\lambda_{i1} < \lambda_{i2}$, $\Delta_i(\lambda, c) < 0$ for all $\lambda \in (\lambda_{i1}, \lambda_{i2})$, and $\Delta_i(\lambda, c) \geqslant 0$ for all $\lambda \in [0, \infty) \backslash [\lambda_{i1}, \lambda_{i2}]$.*

For convenience, we need the following lemma.

Lemma 3.2. *Assume*

$$\beta_1 u_1^* \geqslant 3\mu_1 u_2^*, \qquad \beta_2 u_2^* \geqslant 3\mu_2 u_1^* \tag{3.12}$$

hold, there exist $\epsilon_i \in (0, u_i^*/2)$ $(i = 1, 2)$ such that

$$\begin{cases} -\beta_1\epsilon_1^2 + \beta_1 u_1^*\epsilon_1 + \mu_1\left(u_1^* - \epsilon_1 e^{-\lambda s}\right)\left(u_2^* - \epsilon_2 e^{-\lambda s}\right) - \mu_1 u_1^* u_2^* > \epsilon_0, \\ -\beta_2\epsilon_2^2 + \beta_2 u_2^*\epsilon_2 + \mu_2\left(u_2^* - \epsilon_2 e^{-\lambda s}\right)\left(u_1^* - \epsilon_1 e^{-\lambda s}\right) - \mu_2 u_1^* u_2^* > \epsilon_0, \end{cases}$$

where $\epsilon_0 > 0$ is a constant.

Proof. Let

$$h_1(\epsilon_1) = \beta_1 u_1^*\epsilon_1 - \beta_1\epsilon_1^2 \quad \text{and} \quad h_2(\epsilon_1) = \mu_1 u_1^* u_2^* - \mu_1\left(u_1^* - \epsilon_1\right)\left(u_2^* - \epsilon_2\right).$$

We have

$$\begin{aligned} &h_1(0) = 0, \qquad \max\left\{h_1(\epsilon_1)\right\} = h_1\left(\frac{u_1^*}{2}\right) = \frac{1}{4}\beta_1 {u_1^*}^2, \\ &h_2(0) = \mu_1 u_1^*\epsilon_2 > 0, \qquad h_2\left(\frac{u_1^*}{2}\right) = \mu_1 u_1^* u_2^* - \frac{\mu_1 u_1^*}{2}\left(u_2^* - \epsilon_2\right) \leqslant \frac{3}{4}\mu_1 u_1^* u_2^*, \quad \text{if } \epsilon_2 \in \left(0, u_2^*/2\right). \end{aligned}$$

If the first inequality of (3.12) holds, then $h_1(0) \leqslant h_2(0)$ and $h_1(\frac{u_1^*}{2}) \geqslant h_2(\frac{u_1^*}{2})$. Again, the function $h_2(\epsilon_1)$ is increasing with respect to ϵ_1. Thus, there exist $\epsilon_1^* \in (0, u_1^*/2)$ such that $h_1(\epsilon_1^*) = h_2(\epsilon_1^*)$ and

$$h_1(\epsilon_1) \leqslant h_2(\epsilon_1) \quad \text{for } 0 < \epsilon_1 \leqslant \epsilon_1^*, \qquad h_1(\epsilon_1) > h_2(\epsilon_1) \quad \text{for } \epsilon_1^* < \epsilon_1 < u_1^*/2.$$

So, the first result is obtained, and the other result can be gotten similarly. The proof is completed. □

From the conditions of Lemma 3.2, we derive that

$$\beta_1\beta_2 \geqslant 9\mu_1\mu_2.$$

That is, the condition in Theorem 2.5 is satisfied, thus the coexistence equilibrium (u_1^*, u_2^*) is globally asymptotically stable.

In order to give more clear parameter ranges such that (3.12) holds, it follows from (2.4), (2.6) and the assumptions of state dependent delay that

$$\alpha_1 e^{-\gamma_1\tau(u_1^*)} \geqslant \alpha_1 e^{-\gamma_1\tau_M} \geqslant 2\mu_1 \frac{\alpha_2\beta_1 e^{-\gamma_2\tau_m} + \alpha_1\mu_2 e^{-\gamma_1\tau_m}}{\beta_1\beta_2 - \mu_1\mu_2} \geqslant 2\mu_1 u_2^*,$$
$$\alpha_2 e^{-\gamma_2\tau(u_2^*)} \geqslant \alpha_2 e^{-\gamma_2\tau_M} \geqslant 2\mu_2 \frac{\alpha_1\beta_2 e^{-\gamma_1\tau_m} + \alpha_2\mu_1 e^{-\gamma_2\tau_m}}{\beta_1\beta_2 - \mu_1\mu_2} \geqslant 2\mu_1 u_1^*.$$

Thus, inequalities (3.12) hold providing $\mu_2 \leqslant \frac{\beta_1\alpha_2 e^{-\gamma_2\tau_M}}{2\alpha_1 e^{-\gamma_1\tau_m}} =: \mu_2^*$ and

$$\mu_1 \leqslant \min\left\{\frac{\beta_1\beta_2\alpha_1 e^{-\gamma_1\tau_M}}{2\alpha_2\beta_1 e^{-\gamma_2\tau_m} + \alpha_1\mu_2(2e^{-\gamma_1\tau_m} + e^{-\gamma_1\tau_M})}, \frac{\beta_1\beta_2\alpha_2 e^{-\gamma_2\tau_M} - 2\mu_2\beta_2\alpha_1 e^{-\gamma_1\tau_m}}{\mu_2\alpha_2(2e^{-\gamma_2\tau_m} + e^{-\gamma_2\tau_M})}\right\} =: \mu_1^*.$$

Remark 3.1. The coexistence equilibrium (u_1^*, u_2^*) is globally asymptotically stable, and the statement of Lemma 3.2 holds if $\mu_1 \leqslant \mu_1^*$ and $\mu_2 \leqslant \mu_2^*$.

Define

$$c^* = \max\{c_1^*, c_2^*\} \quad \text{and} \quad \lambda_0 = \max\{\lambda_{11}, \lambda_{21}\}.$$

We can state the following theorem of the existence of traveling wave solution, and will give the proof as follows.

Theorem 3.1. *For $c > c^*$, if $\mu_1 \leqslant \mu_1^*$, $\mu_2 \leqslant \mu_2^*$ and*

$$\Delta_1(\lambda_0, c) \leqslant 0, \qquad \Delta_2(\lambda_0, c) \leqslant 0, \tag{3.13}$$

then system (1.2) has traveling wave solutions connecting both equilibria E_0 and E^*.

Remark 3.2. It seems that the conditions (3.13) is inconvenient and probably not necessary for Theorem 3.1 to hold, however, it is needed in constructing the upper–lower solutions. The conditions (3.13) are equivalent to $(\lambda_{11}, \lambda_{12}) \cap (\lambda_{21}, \lambda_{22}) \neq \emptyset$ which guarantee the existence of $\lambda_0 \in (\lambda_{11}, \lambda_{12}) \cap (\lambda_{21}, \lambda_{22})$. Intuitively, it holds if two species have similar behavior. In fact, (3.13) can be certainly made (for example, by taking d_1 or d_2 sufficiently small).

A pair of function $(\overline{\phi}_1(s), \overline{\phi}_2(s))$ and $(\underline{\phi}_1(s), \underline{\phi}_2(s))$ can be constructed as follows:

$$\overline{\phi}_1(s) = \begin{cases} u_1^* e^{\lambda_0 s}, & s \leqslant 0,\\ u_1^*, & s \geqslant 0,\end{cases} \qquad \underline{\phi}_1(s) = \begin{cases} 0, & s \leqslant s_1,\\ u_1^* - \epsilon_1 e^{-\lambda s}, & s \geqslant s_1,\end{cases}$$
$$\overline{\phi}_2(s) = \begin{cases} u_2^* e^{\lambda_0 s}, & s \leqslant 0,\\ u_2^*, & s \geqslant 0,\end{cases} \qquad \underline{\phi}_2(s) = \begin{cases} 0, & s \leqslant s_2,\\ u_2^* - \epsilon_2 e^{-\lambda s}, & s \geqslant s_2,\end{cases} \tag{3.14}$$

where $\lambda > 0$ is sufficiently small. Clearly, $s_i = \frac{-1}{\lambda}\ln(\frac{u_i^*}{\epsilon_1}) < 0$.

It is straightforwardly to summarize some useful properties of $(\overline{\phi}_1, \overline{\phi}_2)$ and $(\underline{\phi}_1, \underline{\phi}_2)$ as follows.

Proposition 3.1. *Let $(\overline{\phi}_1(s), \overline{\phi}_2(s))$ and $(\underline{\phi}_1(s), \underline{\phi}_2(s))$ be constructed in (3.14), then the following statements are valid.*

(i) *There exist constants $B_1 > 0$ and $B_2 > 0$ such that $0 \leqslant \underline{\phi}_1(s) < \overline{\phi}_1(s) \leqslant B_1$ and $0 \leqslant \underline{\phi}_2(s) < \overline{\phi}_2(s) \leqslant B_2$.*
(ii) *$\overline{\phi}_1(s) \leqslant u_1^* e^{\lambda_0 s}$ and $\overline{\phi}_1(s) \leqslant u_1^*$ for $s \in \mathbb{R}$, and $\overline{\phi}_2(s) \leqslant u_2^* e^{\lambda_0 s}$ and $\overline{\phi}_2(s) \leqslant u_2^*$ for $s \in \mathbb{R}$.*
(iii) *$\underline{\phi}_1(s) \geqslant u_1^* - \epsilon_1 e^{-\lambda s} > 0$, and $\underline{\phi}_2(s) \geqslant u_2^* - \epsilon_2 e^{-\lambda s} > 0$ for $s \in \mathbb{R}$.*
(iv) *$\lim_{s\to-\infty}(\overline{\phi}_1(s), \overline{\phi}_2(s)) = (0,0)$ and $\lim_{s\to\infty}(\underline{\phi}_1(s), \underline{\phi}_2(s)) = (u_1^*, u_2^*)$.*

Throughout this paper, we adopt the standard ordering in $\mathbb{R}^2$. Thus, for $u = (u_1, u_2)^T$ and $v = (v_1, v_2)^T$, we denote $u \leqslant v$ if $u_i \leqslant v_i$, $i = 1, 2$; $u < v$ if $u \leqslant v$ but $u \neq v$; and $u \ll v$ if $u \leqslant v$ but $u_i \neq v_i$, $i = 1, 2$. If $u \leqslant v$, we also denote $(u, v] = \{w \in \mathbb{R}^2\colon\ u < w \leqslant v\}$, $[u, v) = \{w \in \mathbb{R}^2\colon\ u \leqslant w < v\}$ and $[u, v] = \{w \in \mathbb{R}^2\colon\ u \leqslant w \leqslant v\}$. Now, we introduce the concept of desirable pair of upper–lower solutions of system (3.1) as follows.

Definition 3.1. A pair of continuous functions $\bar{\rho} = (\overline{\phi}_1, \overline{\phi}_2)$ and $\underline{\rho} = (\underline{\phi}_1, \underline{\phi}_2)$ for $s \in \mathbb{R}$ is called a pair of upper–lower solutions of (3.1), if there exists a finite set of points $S = \{s_i \in \mathbb{R},\ i = 1, 2, \ldots, n\}$ with $s_1 < s_2 < \cdots < s_n$ such that $\bar{\rho}$ and $\underline{\rho}$ are twice continuously differentiable on $\mathbb{R} \setminus S$ and satisfy

$$\begin{aligned}
&d_1\overline{\phi}_1''(s) - c\overline{\phi}_1'(s) + \alpha_1 e^{-\gamma_1\tau(\overline{\phi}_1)}\overline{\phi}_1\big(s - c\tau(\overline{\phi}_1)\big) - \beta_1\overline{\phi}_1^2(s) + \mu_1\overline{\phi}_1(s)\overline{\phi}_2(s) \leqslant 0,\\
&d_2\overline{\phi}_2''(s) - c\overline{\phi}_2'(s) + \alpha_2 e^{-\gamma_2\tau(\overline{\phi}_2)}\overline{\phi}_2\big(s - c\tau(\overline{\phi}_2)\big) - \beta_2\overline{\phi}_2^2(s) + \mu_2\overline{\phi}_2(s)\overline{\phi}_1(s) \leqslant 0,\\
&d_1\underline{\phi}_1''(s) - c\underline{\phi}_1'(s) + \alpha_1 e^{-\gamma_1\tau(\underline{\phi}_1)}\underline{\phi}_1\big(s - c\tau(\underline{\phi}_1)\big) - \beta_1\underline{\phi}_1^2(s) + \mu_1\underline{\phi}_1(s)\underline{\phi}_2(s) \geqslant 0,\\
&d_2\underline{\phi}_2''(s) - c\underline{\phi}_2'(s) + \alpha_2 e^{-\gamma_2\tau(\underline{\phi}_2)}\underline{\phi}_2\big(s - c\tau(\underline{\phi}_2)\big) - \beta_2\underline{\phi}_2^2(s) + \mu_2\underline{\phi}_2(s)\underline{\phi}_1(s) \geqslant 0.
\end{aligned}\tag{3.15}$$

We shall prove that the continuous function $(\overline{\phi}_1(s), \overline{\phi}_2(s))$ and $(\underline{\phi}_1(s), \underline{\phi}_2(s))$ is an upper and a lower solutions of (3.1), respectively.

Lemma 3.3. *Suppose $\mu_1 \leqslant \mu_1^*$, $\mu_2 \leqslant \mu_2^*$ and (3.13). Then $(\overline{\phi}_1(s), \overline{\phi}_2(s))$ and $(\underline{\phi}_1(s), \underline{\phi}_2(s))$ which are constructed in (3.14) is a pair of upper–lower solutions of (3.1).*

Proof. It suffices to prove that $(\overline{\phi}_1(s), \overline{\phi}_2(s))$ and $(\underline{\phi}_1(s), \underline{\phi}_2(s))$ satisfy the definition of upper–lower solutions.

We first consider $\overline{\phi}_1$. It is easy to verify the case of $s > 0$ is trivial. If $s \leqslant 0$, then $\overline{\phi}_1(s) = u_1^* e^{\lambda_0 s}$. Proposition 3.1 (ii) shows that $\overline{\phi}_2(s) \leqslant u_2^* e^{\lambda_0 s}$ for $s \in \mathbb{R}$. It follows that

$$\begin{aligned}
&d_1\overline{\phi}_1''(s) - c\overline{\phi}_1'(s) + \alpha_1 e^{-\gamma_1\tau(\overline{\phi}_1)}\overline{\phi}_1\big(s - c\tau(\overline{\phi}_1)\big) - \beta_1\overline{\phi}_1^2(s) + \mu_1\overline{\phi}_1(s)\overline{\phi}_2(s)\\
&\quad\leqslant d_1\big(u_1^* e^{\lambda_0 s}\big)'' - c\big(u_1^* e^{\lambda_0 s}\big)' + \alpha_1 u_1^* e^{\lambda_0 s - \tau(\overline{\phi}_1)(\gamma_1 + c\lambda_0)} - \beta_1\big(u_1^* e^{\lambda_0 s}\big)^2 + \mu_1 u_1^* u_2^* e^{2\lambda_0 s}\\
&\quad\leqslant u_1^* e^{\lambda_0 s}\big(d_1\lambda_0^2 - c\lambda_0 + \alpha_1 e^{-\tau_m(\gamma_1 + c\lambda_0)} - \beta_1 u_1^* e^{\lambda_0 s} + \mu_1 u_2^* e^{\lambda_0 s}\big)\\
&\quad= u_1^* e^{\lambda_0 s}\big(\Delta_1(\lambda_0, c) - e^{\lambda_0 s}\big(\beta_1 u_1^* - \mu_1 u_2^*\big)\big) \leqslant 0.
\end{aligned}$$

Here, we have used some assumptions that $\Delta_1(\lambda_0, c) \leqslant 0$ and $\beta_1 u_1^* - \mu_1 u_2^* = \alpha_1 e^{-\gamma_1\tau(u_1^*)} > 0$. Thus, we have shown that $\overline{\phi}_1(s)$ is an upper solution. Similarly, we can also show that $\overline{\phi}_2(s)$ is an upper solution.

We now verify $(\underline{\phi}_1, \underline{\phi}_2)$. Let us consider the first case of lower solution $\underline{\phi}_1$ for the intervals $s > s_1$ and $s \leqslant s_1$. Obviously, it trivially holds for $s < s_1$ since in the case we have $\underline{\phi}_1(s) = 0$, and $\underline{\phi}_1(s) = u_1^* - \epsilon_1 e^{-\lambda s}$ for $s \geqslant s_1$. Furthermore, $\underline{\phi}_2(s) \geqslant u_2^* - \epsilon_2 e^{-\lambda s}$ for $s \in \mathbb{R}$. We thus obtain that

$$\begin{aligned}
&d_1\underline{\phi}_1''(s) - c\underline{\phi}_1'(s) + \alpha_1 e^{-\gamma_1\tau(\underline{\phi}_1)}\underline{\phi}_1\big(s - c\tau(\underline{\phi}_1)\big) - \beta_1\underline{\phi}_1^2(s) + \mu_1\underline{\phi}_1(s)\underline{\phi}_2(s)\\
&\quad\geqslant d_1\big(u_1^* - \epsilon_1 e^{-\lambda s}\big)'' - c\big(u_1^* - \epsilon_1 e^{-\lambda s}\big)' + \alpha_1 e^{-\gamma_1\tau(u_1^*)}\big(u_1^* - \epsilon_1 e^{-\lambda(s - c\tau(\underline{\phi}_1))}\big)
\end{aligned}$$

$$\begin{aligned}
&\quad - \beta_1\big(u_1^* - \epsilon_1 e^{-\lambda s}\big)^2 + \mu_1\big(u_1^* - \epsilon_1 e^{-\lambda s}\big)\big(u_2^* - \epsilon_2 e^{-\lambda s}\big) \\
&\geqslant \epsilon_1 e^{-\lambda s}\big(-d_1\lambda^2 - c\lambda - \alpha_1 e^{-(\gamma_1 - c\lambda)\tau(u_1^*)}\big) + \alpha_1 e^{-\gamma_1\tau(u_1^*)} u_1^* - \beta_1\big(u_1^* - \epsilon_1 e^{-\lambda s}\big)^2 \\
&\quad + \mu_1\big(u_1^* - \epsilon_1 e^{-\lambda s}\big)\big(u_2^* - \epsilon_2 e^{-\lambda s}\big) \\
&= \epsilon_1 e^{-\lambda s}\big(-\Delta(-\lambda, c) + \beta_1 u_1^*\big) + \beta_1 u_1^* \epsilon_1 e^{-\lambda s} - \beta_1 \epsilon_1^2 e^{-2\lambda s} - \mu_1 e^{-\lambda s}\big(u_1^*\epsilon_2 + u_2^*\epsilon_1 - \epsilon_1\epsilon_2 e^{-\lambda s}\big).
\end{aligned}$$

Noting that $-\Delta(0,c)+\beta_1 u_1^* = -\alpha_1 e^{-\gamma_1\tau(u_1^*)}+\beta_1 u_1^* = \mu_1 u_2^* > 0$, we can choose $\lambda_1^* > 0$ such that $-\Delta(-\lambda,c)+\beta_1 u_1^* > 0$ for $\lambda \in (0,\lambda_1^*)$. Besides, let

$$H_1(\lambda,s) := \beta_1 u_1^*\epsilon_1 e^{-\lambda s} - \beta_1\epsilon_1^2 e^{-2\lambda s} + \mu_1\big(u_1^* - \epsilon_1 e^{-\lambda s}\big)\big(u_2^* - \epsilon_2 e^{-\lambda s}\big) - \mu_1 u_1^* u_2^*.$$

By Lemma 3.2, we have that

$$H_1(\lambda,0) = \beta_1 u_1^*\epsilon_1 - \beta_1\epsilon_1^2 - \mu_1 u_1^* u_2^* + \mu_1\big(u_1^* - \epsilon_1\big)\big(u_2^* - \epsilon_2\big) > \epsilon_0 > 0.$$

Noting that we can choose $\delta_1, \delta_2 > 0$ to be small enough such that $\delta_1^* := \epsilon_1 + \delta_1$ and $\delta_2^* := \epsilon_2 + \delta_2$ satisfying

$$\beta_1 u_1^*\delta - \beta_1\delta^2 - \mu_1 u_1^* u_2^* + \mu_1\big(u_1^* - \delta\big)\big(u_2^* - \delta'\big) > \frac{\epsilon_0}{2} > 0 \quad \text{for } \delta \in \big[\epsilon_1, \delta_1^*\big],\ \delta' \in \big[\epsilon_2, \delta_2^*\big].$$

Let $\nu(s) := \epsilon_1 e^{-\lambda s}$, $\mu(s) := \epsilon_2 e^{-\lambda s}$. Choose $\lambda_1^* > 0$ small enough such that for any given $\lambda \in (0,\lambda_1^*)$, we have

$$\nu(s_1) = \delta_1^*, \qquad \mu(s_1) = \delta_2^* \quad (\text{noting that } s_1 < 0)$$

which leads to

$$\epsilon_1 \leqslant \nu(s) < \delta_1^*, \qquad \epsilon_2 \leqslant \mu(s) < \delta_2^* \quad \text{for } s \in (s_1, 0].$$

Therefore, it follows that $H_1(\lambda,s) > 0$ for $s \in (s_1,0]$.

Furthermore, the function

$$\mu_1\big(u_1^* - \epsilon_1 e^{-\lambda s}\big)\big(u_2^* - \epsilon_2 e^{-\lambda s}\big) - \mu_1 u_1^* u_2^* = \mu_1\epsilon_1\epsilon_2 e^{-2\lambda s} - \mu_1 e^{-\lambda s}\big(u_1^*\epsilon_2 + u_2^*\epsilon_1\big) < 0 \quad \text{for } s \geqslant 0$$

and is increasing on $s \geqslant 0$. The function $\beta_1 u_1^*\epsilon_1 e^{-\lambda s} - \beta_1\epsilon_1^2 e^{-2\lambda s} > 0$ for $s \geqslant 0$ and is decreasing for $s > -\frac{1}{\lambda}\ln(\frac{u_1^*}{2\epsilon_1})$. We obtain $H_1(\lambda,0) > \epsilon_0 > 0$ and $H_1(\lambda,\infty) = 0$. Thus, $H_1(\lambda,s) > 0$ uniformly for $s \geqslant s_1$.

Therefore, $\underline{\phi}_1$ satisfies the definition of the lower solution. That is

$$d_1\underline{\phi}_1''(s) - c\underline{\phi}_1'(s) + \alpha_1 e^{-\gamma_1\tau_1}\underline{\phi}_1(s - c\tau_1) - \beta_1\underline{\phi}_1^2(s) + \mu_1\underline{\phi}_1(s)\underline{\phi}_2(s) \geqslant 0 \quad \text{for } s \in \mathbb{R}.$$

Similarly, we also can show that there exists a constant $\lambda_2^* > 0$ such that for $\lambda \in (0,\lambda_2^*)$

$$d_2\underline{\phi}_2''(s) - c\underline{\phi}_2'(s) + \alpha_2 e^{-\gamma_2\tau_2}\underline{\phi}_2(s - c\tau_2) - \beta_2\underline{\phi}_2^2(s) + \mu_2\underline{\phi}_1(s)\underline{\phi}_2(s) \geqslant 0 \quad \text{for } s \in \mathbb{R}.$$

It follows that the statement of Lemma 3.3 holds for $\lambda \in (0, \min\{\lambda_1^*, \lambda_2^*\})$. Thus, $\underline{\phi}_1$ and $\underline{\phi}_2$ satisfy the definition of the lower solution. The proof of Lemma 3.3 is complete. □

Using the upper and lower solutions, we will show the existence of traveling wave solutions. Define a set of functions

$$\Gamma := \{(\phi_1,\phi_2) \in C(\mathbb{R},\mathbb{R}^2) \mid 0 < \underline{\phi}_1 \leqslant \phi_1 \leqslant \overline{\phi}_1 < B_1,\ 0 < \underline{\phi}_2 \leqslant \phi_2 \leqslant \overline{\phi}_2 < B_2\},$$

where B_i is as Proposition 3.1. Define a function $h_i(\phi_i) = \alpha_i e^{-\gamma_i\tau(\phi_i)}\phi_i(t - c\tau(\phi_i))$. Notice that $\tau'(\phi_i) \geqslant 0$, it is not difficult to verify the following equality

$$|h_i'(\phi_i)| = |\alpha_i - \alpha_i\gamma_i\tau'(\phi_i)\phi_i(t - c\tau(\phi_i))|e^{-\gamma_i\tau(\phi_i)} \leqslant \max\{\alpha_i, B_i L\alpha_i\gamma_i\} := A_i, \quad i = 1,2.$$

Define the operator $F = (F_1, F_2)$ from Γ to $C(\mathbb{R},\mathbb{R}^2)$ by

$$\begin{aligned} F_1(\phi_1,\phi_2)(s) &:= \alpha_1 e^{-\gamma_1\tau(\phi_1)}\phi_1(s - c\tau(\phi_1)) - \beta_1\phi_1^2(s) + \mu_1\phi_1(s)\phi_2(s) + \delta_1\phi_1(s),\\ F_2(\phi_1,\phi_2)(s) &:= \alpha_2 e^{-\gamma_2\tau(\phi_2)}\phi_2(s - c\tau(\phi_2)) - \beta_2\phi_2^2(s) + \mu_2\phi_1(s)\phi_2(s) + \delta_2\phi_2(s), \end{aligned} \tag{3.16}$$

where δ_1 and δ_2 are large positive numbers so that $\delta_1 > 2\beta_1 B_1 + A_1$ and $\delta_2 > 2\beta_2 B_2 + A_2$. Then system (3.1) now can be rewritten as

$$\begin{aligned} &d_1\phi_1''(s) - c\phi_1'(s) - \delta_1\phi_1(s) + F_1(\phi_1,\phi_2)(s) = 0,\\ &d_2\phi_2''(s) - c\phi_2'(s) - \delta_2\phi_2(s) + F_2(\phi_1,\phi_2)(s) = 0. \end{aligned} \tag{3.17}$$

Let

$$\begin{aligned} \Lambda_{11} &= \frac{c - \sqrt{c^2 + 4\delta_1 d_1}}{2d_1} < 0, \qquad \Lambda_{12} = \frac{c + \sqrt{c^2 + 4\delta_1 d_1}}{2d_1} > 0,\\ \Lambda_{21} &= \frac{c - \sqrt{c^2 + 4\delta_2 d_2}}{2d_2} < 0, \qquad \Lambda_{22} = \frac{c + \sqrt{c^2 + 4\delta_2 d_2}}{2d_2} > 0. \end{aligned}$$

Clearly, $\Lambda_{11} < 0 < \Lambda_{12}$, $\Lambda_{21} < 0 < \Lambda_{22}$, and $d_1\Lambda_{1i}^2 - c\Lambda_{1i} - \delta_1 = 0$ and $d_2\Lambda_{2i}^2 - c\Lambda_{2i} - \delta_2 = 0$, $i = 1,2$.

For $\mu > 0$, define

$$B_\mu := \left\{(\phi_1,\phi_2) \in C(\mathbb{R},\mathbb{R}^2) \;\middle|\; \sup_{s\in\mathbb{R}} |(\phi_1,\phi_2)(s)| e^{-\mu|s|} < \infty\right\}$$

and

$$|(\phi_1,\phi_2)|_\mu = \sup_{s\in\mathbb{R}} |(\phi_1,\phi_2)(s)| e^{-\mu|s|}.$$

Then it is easy to check that $(B_\mu(\mathbb{R},\mathbb{R}^2), |\cdot|_\mu)$ is a Banach space. For our purpose, we will take μ such that

$$0 < \mu < \min\{-\Lambda_{11}, \Lambda_{12}, -\Lambda_{21}, \Lambda_{22}\}.$$

Clearly, Γ is a bounded nonempty closed convex subset of B_μ.

Define the operator $Q = (Q_1, Q_2) : \Gamma \to B_\mu$ by

$$\begin{aligned} Q_1(\phi_1,\phi_2) &:= \frac{1}{d_1(\Lambda_{12} - \Lambda_{11})}\left(\int_{-\infty}^{s} e^{\Lambda_{11}(s-t)} F_1(\phi_1,\phi_2)(s)\,dt + \int_{s}^{\infty} e^{\Lambda_{12}(s-t)} F_1(\phi_1,\phi_2)(s)\,dt\right),\\ Q_2(\phi_1,\phi_2) &:= \frac{1}{d_2(\Lambda_{22} - \Lambda_{21})}\left(\int_{-\infty}^{s} e^{\Lambda_{21}(s-t)} F_2(\phi_1,\phi_2)(s)\,dt + \int_{s}^{\infty} e^{\Lambda_{22}(s-t)} F_2(\phi_1,\phi_2)(s)\,dt\right). \end{aligned} \tag{3.18}$$

It easily verify that the operator Q is well defined for $(\phi_1,\phi_2) \in \Gamma$ and

$$d_iQ_i(\phi_1,\phi_2)''(s) - cQ_i(\phi_1,\phi_2)'(s) - \delta_iQ_i(\phi_1,\phi_2)(s) + F_i(\phi_1,\phi_2)(s) = 0, \quad i = 1,2.$$

Thus the fixed point of Q is the solution of (3.17), which is the traveling solutions of (1.2).

Next, we explore some basic properties of F and Q. In view of the boundedness and continuity of $F_i(\phi_1,\phi_2) - \delta_i\phi_i$ on Γ, the following conclusion is obvious.

Lemma 3.4. *For $(\phi_1,\phi_2) \in \Gamma$, $Q_i(\phi_1,\phi_2) \in \Gamma$ $(i = 1,2)$ is nondecreasing in ϕ_1 and ϕ_2, respectively.*

Proof. It suffices to show that for $(\phi_1,\phi_2) \in \Gamma$, $F_i(\phi_1,\phi_2) \in \Gamma$ is nondecreasing in ϕ_1 and ϕ_2, respectively. In view of the definition of $Q(Q_1,Q_2)$, it is easy to see that Q also enjoys the same properties as those for F.

For $(\phi_{11},\phi_{21}) \geqslant (\phi_{12},\phi_{22})$ where $(\phi_{1i},\phi_{2i}) \in \Gamma$, we obtain from $\delta_1 > 2\beta_1B_1 + A_1$ that

$$\begin{aligned}
&F_1(\phi_{11},\phi_{21}) - F_1(\phi_{12},\phi_{21}) \\
&\quad = \alpha_1e^{-\gamma_1\tau(\phi_{11})}\phi_{11}\big(s - c\tau(\phi_{11})\big) - \alpha_1e^{-\gamma_1\tau(\phi_{12})}\phi_{12}\big(s - c\tau(\phi_{12})\big) \\
&\qquad - \beta_1\big(\phi_{11}^2 - \phi_{12}^2\big) + \mu_1\phi_{21}(\phi_{11} - \phi_{12}) + \delta_1(\phi_{11} - \phi_{12}) \\
&\quad \geqslant -A_1(\phi_{11} - \phi_{12}) - \beta_1\big(\phi_{11}^2 - \phi_{12}^2\big) + \mu_1\phi_{21}(\phi_{11} - \phi_{12}) + \delta_1(\phi_{11} - \phi_{12}) \\
&\quad \geqslant (\delta_1 - 2\beta_1B_1 - A_1 + \mu_1B_2)(\phi_{11} - \phi_{12}) \geqslant 0.
\end{aligned}$$

So, $F_1(\phi_1,\phi_2)$ is nondecreasing in ϕ_1. Furthermore, $F_1(\phi_1,\phi_2)$ is obvious nondecreasing in ϕ_2. The second result of Lemma 3.4 follows analogously. The proof is complete. □

Lemma 3.5. *$(\overline{\phi}_1,\overline{\phi}_2)$ is an upper solution and $(\underline{\phi}_1,\underline{\phi}_2)$ is a lower solution of the operator Q defined by (3.18) in the sense that*

$$Q_1(\overline{\phi}_1,\overline{\phi}_2) \leqslant \overline{\phi}_1, \qquad Q_1(\underline{\phi}_1,\underline{\phi}_2) \geqslant \underline{\phi}_1 \quad \textit{and} \quad Q_2(\overline{\phi}_1,\overline{\phi}_2) \leqslant \overline{\phi}_2, \qquad Q_2(\underline{\phi}_1,\underline{\phi}_2) \geqslant \underline{\phi}_2.$$

Proof. From Lemma 3.3 and the first equation of (3.17) we have that

$$F_1(\overline{\phi}_1,\overline{\phi}_2)(s) \leqslant -d_1\overline{\phi}_1''(s) + c\overline{\phi}_1'(s) + \delta_1\overline{\phi}_1(s),$$

which together with the equation of (3.18) gives

$$\begin{aligned}
&Q_1(\overline{\phi}_1,\overline{\phi}_2)(s) \\
&\quad = \frac{1}{d_1(\Lambda_{12} - \Lambda_{11})}\left(\int_{-\infty}^{s} e^{\Lambda_{11}(s-t)} + \int_{s}^{\infty} e^{\Lambda_{12}(s-t)}\right)F_1(\overline{\phi}_1,\overline{\phi}_2)(s)\,dt \\
&\quad \leqslant \frac{1}{d_1(\Lambda_{12} - \Lambda_{11})}\left(\int_{-\infty}^{s} e^{\Lambda_{11}(s-t)} + \int_{s}^{\infty} e^{\Lambda_{12}(s-t)}\right)\big(-d_1\overline{\phi}_1''(s) + c\overline{\phi}_1'(s) + \delta_1\overline{\phi}_1(s)\big)\,dt \\
&\quad = \overline{\phi}_1(s), \quad s \in \mathbb{R}.
\end{aligned}$$

An analogous argument shows $Q_2(\overline{\phi}_1,\overline{\phi}_2)(s) \leqslant \overline{\phi}_2$. Similarly, using Lemma 3.3, (3.17) and (3.18) one can also show that $Q_1(\underline{\phi}_1,\underline{\phi}_2) \geqslant \underline{\phi}_1$ and $Q_2(\underline{\phi}_1,\underline{\phi}_2) \geqslant \underline{\phi}_2$. The proof is complete. □

Lemma 3.6. *The operator $Q : \Gamma \to \Gamma$ is continuous with respect to the norm $|\cdot|_\mu$ in Γ.*

Proof. We first prove that $F : \Gamma \to \Gamma$ is continuous with respect to the norm $|\cdot|_\mu$. Let $\Phi_1 = (\phi_{11}, \phi_{21})$, $\Phi_2 = (\phi_{12}, \phi_{22}) \in \Gamma$ with $|\Phi_1 - \Phi_2|_\mu = \sup_{s\in\mathbb{R}} |\Phi_1 - \Phi_2| e^{-\mu|s|} < \delta$. It is easy to verify that

$$\begin{aligned}
&\big|F_1(\phi_{11}, \phi_{21}) - F_1(\phi_{12}, \phi_{22})\big| e^{-\mu|s|} \\
&\quad \leqslant \big|\alpha_1 e^{-\gamma_1\tau(\phi_{11})}\phi_{11}\big(s - c\tau(\phi_{11})\big) - \alpha_1 e^{-\gamma_1\tau(\phi_{12})}\phi_{12}\big(s - c\tau(\phi_{12})\big)\big| e^{-\mu|s|} + \delta_1|\phi_{11} - \phi_{12}|e^{-\mu|s|} \\
&\qquad + \beta_1(\phi_{11} + \phi_{12})|\phi_{11} - \phi_{12}|e^{-\mu|s|} + \mu_1\phi_{11}|\phi_{21} - \phi_{22}|e^{-\mu|s|} + \mu_1\phi_{22}|\phi_{11} - \phi_{12}|e^{-\mu|s|} \\
&\quad \leqslant \big(A_1 + 2\beta_1 B_1 + \mu_1(B_1 + B_2) + \delta_1\big)|\Phi_1 - \Phi_2|_\mu.
\end{aligned}$$

Define $C_1 := A_1 + 2\beta_1 B_1 + \mu_1(B_1 + B_2) + \delta_1$. Therefore, that implies $F_1 : \Gamma \to \Gamma$ is continuous with respect to the norm $|\cdot|_\mu$. Similarly, we can show that

$$\big|F_2(\phi_{11}, \phi_{21}) - F_2(\phi_{12}, \phi_{22})\big| e^{-\mu|s|} \leqslant C_2|\Phi_1 - \Phi_2|_\mu,$$

where $C_2 := A_2 + 2\beta_2 B_2 + \mu_2(B_1 + B_2) + \delta_2$, and thus $F_2 : \Gamma \to \Gamma$ is continuous with respect to the norm $|\cdot|_\mu$.

Now, we show that $Q_1 : \Gamma \to \Gamma$ is continuous with respect to the norm $|\cdot|_\mu$. For $s \geqslant 0$, we have from the choice of μ that

$$\begin{aligned}
&\big|Q_1(\phi_{11}, \phi_{21}) - Q_1(\phi_{12}, \phi_{22})\big| e^{-\mu|s|} \\
&\quad \leqslant \frac{e^{-\mu s}}{d_1(\Lambda_{12} - \Lambda_{11})}\left(\int_{-\infty}^{s} e^{\Lambda_{11}(s-t)} + \int_{s}^{\infty} e^{\Lambda_{12}(s-t)}\right)\big|F_1(\phi_{11}, \phi_{21})(s) - F_1(\phi_{11}, \phi_{21})(s)\big|\,dt \\
&\quad \leqslant \frac{C_1}{d_1(\Lambda_{12} - \Lambda_{11})}\left(\int_{-\infty}^{s} e^{\Lambda_{11}(s-t)}\,dt + \int_{s}^{\infty} e^{\Lambda_{12}(s-t)}\,dt\right)|\Phi_1 - \Phi_2|_\mu \\
&\quad \leqslant \frac{C_1}{d_1(\Lambda_{12} - \Lambda_{11})}\left(\int_{-\infty}^{0} e^{\Lambda_{11}(s-t)}\,dt + \int_{0}^{s} e^{\Lambda_{11}(s-t)}\,dt + \int_{s}^{\infty} e^{\Lambda_{12}(s-t)}\,dt\right)|\Phi_1 - \Phi_2|_\mu \\
&\quad = \frac{C_1}{d_1(\Lambda_{12} - \Lambda_{11})}\left(\frac{1}{\Lambda_{12}} - \frac{1}{\Lambda_{11}}\right)|\Phi_1 - \Phi_2|_\mu \\
&\quad = \frac{-C_1}{d_1\Lambda_{11}\Lambda_{12}}|\Phi_1 - \Phi_2|_\mu.
\end{aligned}$$

Similarly, for $s < 0$ we have

$$\big|Q_1(\phi_{11}, \phi_{21}) - Q_1(\phi_{12}, \phi_{22})\big| e^{-\mu|s|} \leqslant \frac{-C_1}{d_1\Lambda_{11}\Lambda_{12}}|\Phi_1 - \Phi_2|_\mu.$$

For any $\varepsilon > 0$, we set $\delta = \frac{-d_1\Lambda_{11}\Lambda_{12}}{C_1}\varepsilon$, if $|\Phi_1 - \Phi_2|_\mu < \delta$, then $|Q_1(\phi_{11}, \phi_{21}) - Q_1(\phi_{12}, \phi_{22})|e^{-\mu|s|} < \varepsilon$. This indicates that operator Q_1 is continuous with respect to the norm $|\cdot|_\mu$.

By using a similar argument as above, we can also prove that $Q_2 : \Gamma \to \Gamma$ is continuous with respect to the norm $|\cdot|_\mu$. The proof is complete. □

Proof of Theorem 3.1. By Lemmas 3.5 and 3.6, we obtain that for $(\phi_1(s), \phi_2(s)) \in \Gamma$

$$\begin{aligned}
\underline{\phi}_1 \leqslant Q_1(\underline{\phi}_1, \underline{\phi}_2) \leqslant Q_1(\phi_1, \phi_2) \leqslant Q_1(\overline{\phi}_1, \overline{\phi}_2) \leqslant \overline{\phi}_1, \\
\underline{\phi}_2 \leqslant Q_2(\underline{\phi}_1, \underline{\phi}_2) \leqslant Q_2(\phi_1, \phi_2) \leqslant Q_2(\overline{\phi}_1, \overline{\phi}_2) \leqslant \overline{\phi}_2.
\end{aligned}$$

This shows that if $(\phi_1, \phi_2) \in \Gamma$ then $Q(\phi_1, \phi_2)(s) \in \Gamma$. It follows that Γ is an invariant bounded nonempty closed convex subset of B_μ.

Let

$$M_i = \sup_{(\phi_1,\phi_2)\in\Gamma, s\in\mathbb{R}} \left|F_i(\phi_1, \phi_2)(s)\right| e^{-\mu|s|}.$$

For any given $\varepsilon > 0$ and $(\phi_1, \phi_2) \in \Gamma$, keeping in mind that $\Lambda_{11} < 0 < \Lambda_{12}$, by the definition of Q_1 we obtain that

$$\begin{aligned}
&\left|Q_1(\phi_1, \phi_2)(s+\varepsilon) - Q_1(\phi_1, \phi_2)(s)\right| e^{-\mu|s|} \\
&\quad = \frac{e^{-\mu|s|}}{d_1(\Lambda_{12} - \Lambda_{11})} \left| \left(\int_{-\infty}^{s+\varepsilon} e^{\Lambda_{11}(s+\varepsilon-t)} - \int_{-\infty}^{s} e^{\Lambda_{11}(s-t)} \right) F_1(\phi_1, \phi_2)(s)\, dt \right. \\
&\qquad \left. + \left(\int_{s+\varepsilon}^{\infty} e^{\Lambda_{12}(s+\varepsilon-t)} - \int_{s}^{\infty} e^{\Lambda_{12}(s-t)} \right) F_1(\phi_1, \phi_2)(s)\, dt \right| \\
&\quad \leqslant \frac{1}{d_1(\Lambda_{12} - \Lambda_{11})} \left(\left|e^{\Lambda_{11}\varepsilon} - 1\right| \int_{-\infty}^{s} e^{\Lambda_{11}(s-t)} \left|F_1(\phi_1, \phi_2)(s)\right| e^{-\mu|s|}\, dt \right. \\
&\qquad + \int_{s}^{s+\varepsilon} e^{\Lambda_{11}(s+\varepsilon-t)} \left|F_1(\phi_1, \phi_2)(t)\right| e^{-\mu|s|}\, dt + \left|e^{\Lambda_{12}\varepsilon} - 1\right| \int_{s+\varepsilon}^{\infty} e^{\Lambda_{12}(s-t)} \left|F_1(\phi_1, \phi_2)(s)\right| e^{-\mu|s|}\, dt \\
&\qquad \left. + \int_{s}^{s+\varepsilon} e^{\Lambda_{12}(s-t)} \left|F_1(\phi_1, \phi_2)(s)\right| e^{-\mu|s|}\, dt \right) \\
&\quad \leqslant \frac{M_1}{d_1(\Lambda_{12} - \Lambda_{11})} \left(e^{\Lambda_{11}s}\left(1 - e^{\Lambda_{11}\varepsilon}\right) \int_{-\infty}^{s} e^{-\Lambda_{11}t}\, dt + e^{\Lambda_{11}(s+\varepsilon)} \int_{s}^{s+\varepsilon} e^{-\Lambda_{11}t}\, dt \right. \\
&\qquad \left. + e^{\Lambda_{12}s}\left(e^{\Lambda_{12}\varepsilon} - 1\right) \int_{s+\varepsilon}^{\infty} e^{-\Lambda_{12}t}\, dt + \int_{s}^{s+\varepsilon} e^{\Lambda_{12}(s-t)}\, dt \right) \\
&\quad \leqslant \frac{M_1}{d_1(\Lambda_{12} - \Lambda_{11})} \left(\frac{2(e^{\Lambda_{11}\varepsilon} - 1)}{\Lambda_{11}} + \frac{2(1 - e^{-\Lambda_{12}\varepsilon})}{\Lambda_{12}} \right).
\end{aligned}$$

Similarly, one also can show that

$$\left|Q_2(\phi_1, \phi_2)(s+\varepsilon) - Q_2(\phi_1, \phi_2)(s)\right| e^{-\mu|s|} \leqslant \frac{M_2}{d_2(\Lambda_{22} - \Lambda_{21})} \left(\frac{2(e^{\Lambda_{21}\varepsilon} - 1)}{\Lambda_{21}} + \frac{2(1 - e^{-\Lambda_{22}\varepsilon})}{\Lambda_{22}} \right).$$

It follows that $\{Q(\phi_1, \phi_2)(s)\colon (\phi_1, \phi_2) \in \Gamma\}$ represents a family of equicontinuous functions. Then the Arzelà–Ascoli theorem implies that Q takes the bounded convex subset of Γ into a compact subset of Γ. An application of the Schauder–Tychonoff fixed point (see [32]) shows that Q has a fixed point (ϕ_1, ϕ_2) in Γ, which represents a traveling wave solution. Since $\underline{\phi}_1(s) \leqslant \phi_1(s) \leqslant \overline{\phi}_1(s)$ and $\underline{\phi}_2(s) \leqslant \phi_2(s) \leqslant \overline{\phi}_2(s)$, the properties of $(\underline{\phi}_1(s), \underline{\phi}_2(s))$ and $(\overline{\phi}_1(s), \overline{\phi}_2(s))$ show that $\lim_{s\to-\infty}(\phi_1(s), \phi_2(s)) \to (0, 0)$ and $\lim_{s\to\infty}(\phi_1(s), \phi_2(s)) \to (u_1^*, u_2^*)$. The proof is complete. □

From above discussion, we have shown system (2.1) has heteroclinic orbit connecting E_0 and E^* if $\beta_1\beta_2 > \mu_1\mu_2$.

3.2. Wavefronts

The existence of traveling wave solutions connecting E_0 and E^* has been established when $\beta_1\beta_2 > \mu_1\mu_2$. In this subsection, we are interesting in their properties, especially the monotonicity.

(1) In order to ensure that the solutions are non-negative, we have discussed the linearization at the start E_0 in the first part of Section 3. From this, the minimum wave speed is obtained and the wave is slowed down by the state-dependent delay.

(2) Another problem we concern with is whether the traveling wave solutions approaches the end E^* of the front ultimate monotonously as $s \to \infty$. Now, we consider the linearized system (3.3) at the rear of the front, where $s \to \infty$ and $(\phi_1, \psi_1, \phi_2, \psi_2)(s) \to (u_1^*, 0, u_2^*, 0)$, and obtain the following characteristic equation

$$\begin{aligned} &\bigl(d_1\lambda^2 - c\lambda + \alpha_1 e^{-\gamma_1\tau(u_1^*)}\bigl(e^{-c\lambda\tau(u_1^*)} - \gamma_1 u_1^*\tau'\bigl(u_1^*\bigr) - 1\bigr) - \beta_1 u_1^*\bigr)\bigl(d_2\lambda^2 - c\lambda \\ &\quad + \alpha_2 e^{-\gamma_2\tau(u_2^*)}\bigl(e^{-c\lambda\tau(u_2^*)} - \gamma_2 u_2^*\tau'\bigl(u_2^*\bigr) - 1\bigr) - \beta_2 u_2^*\bigr) - \mu_1\mu_2 u_1^* u_2^* = 0. \end{aligned} \tag{3.19}$$

We define four functions as follows

$$f_i(\lambda) = \alpha_i e^{-(\gamma_i + c\lambda)\tau(u_i^*)}, \qquad g_i(\lambda) = -d_i\lambda^2 + c\lambda + \alpha_i e^{-\gamma_i\tau(u_i^*)}\bigl(\gamma_i u_i^*\tau'\bigl(u_i^*\bigr) + 1\bigr) + \beta_i u_i^*, \quad i = 1, 2.$$

Obviously, $f_i(\lambda)$ is a decreasing function and $g_i(\lambda)$ is a quadratic function with $c/2d_i > 0$. It is easy to compute that

$$f_i(0) = \alpha_i e^{-\gamma_i\tau(u_i^*)} < \alpha_i e^{-\gamma_i\tau(u_i^*)}\bigl(1 + \gamma_i u_i^*\tau'\bigl(u_i^*\bigr)\bigr) + \beta_i u_i^* = g_i(0).$$

Therefore, $(f_i - g_i)(\lambda) = 0$ has two roots which are denoted by $\lambda_{i,1}$ and $\lambda_{i,2}$. Clearly $\lambda_{i,1} < 0 < \lambda_{i,2}$. Let $\lambda' = \min\{\lambda_{1,1}, \lambda_{2,1}\} < 0$. In addition, it can be easy to verify $(f_i - g_i)(\lambda)$ is a monotonically decreasing function of $\lambda \in (-\infty, \lambda']$, and $(f_1 - g_1)(f_2 - g_2)(\lambda)$ is also a decreasing function with $(f_1 - g_1)(f_2 - g_2)(-\infty) = \infty$ and $(f_1 - g_1)(f_2 - g_2)(\lambda') = 0$. By the continuity of function, there exists a $\lambda_0 \in (-\infty, \lambda']$ such that $(f_1 - g_1)(f_2 - g_2)(\lambda_0) = \mu_1\mu_2 u_1^* u_2^*$, therefore, Eq. (3.19) has real negative roots. Our results suggests that oscillations never set in and that the front probably approaches (u_1^*, u_2^*) monotonically as $s \to \infty$. Since monotonicity requires Eq. (3.19) to have real negative roots for $s \to \infty$, while, oscillations occurs in if all such roots are lost. For this situation, other investigators have observed 'humps' and oscillations in the traveling-front solutions of their constant delay models.

(3) We will consider the case of large wave speed c, by computing a uniformly valid asymptotic expression for the front. That is the speed is large enough for the front, $c \to \infty$. The approach is based on the idea of the constant time delay case studied by other investigators, for example, Canosa [8] obtained a similar such approximation to the traveling-front solution for the well-known Fisher equation when the speeds $c \to \infty$. Murray [31] got the same results when the speed c is given its lowest ecologically relevant value i.e., the minimum speed for positive solutions. Besides, Sherratt [33] also discussed the asymptotic approach for large wave-speeds in studying certain coupled systems.

By the approach of Canosa, we assume c is large, and introduce the small parameter

$$\varepsilon = c^{-2}.$$

We now seek a solution of (3.1) of the form $(\phi_1(s), \phi_2(s)) = (\widetilde{\phi}_1(\widetilde{s}), \widetilde{\phi}_2(\widetilde{s}))$, where $\widetilde{s} = \sqrt{\varepsilon}s = s/c$. By this transformation, system (3.1) is rewritten as

$$\begin{cases} \widetilde{\phi}_1'(\widetilde{s}) = \varepsilon d_1\widetilde{\phi}_1''(s) + \alpha_1 e^{-\gamma_1\tau(\widetilde{\phi}_1)}\widetilde{\phi}_1\bigl(\widetilde{s} - \tau(\widetilde{\phi}_1)\bigr) - \beta_1\widetilde{\phi}_1^2(\widetilde{s}) + \mu_1\widetilde{\phi}_1(\widetilde{s})\widetilde{\phi}_2(\widetilde{s}), \\ \widetilde{\phi}_2'(\widetilde{s}) = \varepsilon d_2\widetilde{\phi}_2''(s) + \alpha_2 e^{-\gamma_2\tau(\widetilde{\phi}_2)}\widetilde{\phi}_2\bigl(\widetilde{s} - \tau(\widetilde{\phi}_2)\bigr) - \beta_2\widetilde{\phi}_2^2(\widetilde{s}) + \mu_2\widetilde{\phi}_1(\widetilde{s})\widetilde{\phi}_2(\widetilde{s}). \end{cases} \tag{3.20}$$

The above equation admits a solution of the form

$$\big(\widetilde{\phi}_1(\widetilde{s}),\widetilde{\phi}_2(\widetilde{s})\big)=\big(\widetilde{\phi}_{10}(\widetilde{s})+\varepsilon\widetilde{\phi}_{11}(\widetilde{s})+\cdots,\widetilde{\phi}_{20}(\widetilde{s})+\varepsilon\widetilde{\phi}_{21}(\widetilde{s})+\cdots\big).\tag{3.21}$$

Substituting into (3.20) and comparing powers of ε^0 gives the finding that $(\widetilde{\phi}_{10},\widetilde{\phi}_{20})$ (we still denote $(\widetilde{\phi}_{10}(\widetilde{s}),\widetilde{\phi}_{20}(\widetilde{s}))$ as $(\phi_{10}(s),\phi_{20}(s))$) must satisfy

$$\begin{cases}\phi_{10}'(s)=\alpha_1e^{-\gamma_1\tau(\phi_{10})}\phi_{10}\big(s-\tau(\phi_{10})\big)-\beta_1\phi_{10}^2(s)+\mu_1\phi_{10}(s)\phi_{20}(s),\\ \phi_{20}'(s)=\alpha_2e^{-\gamma_2\tau(\phi_{20})}\phi_{20}\big(s-\tau(\phi_{20})\big)-\beta_2\phi_{20}^2(s)+\mu_2\phi_{10}(s)\phi_{20}(s),\end{cases}\tag{3.22}$$

with

$$(\phi_{10},\phi_{20})(-\infty)=(0,0),\qquad(\phi_{10},\phi_{20})(\infty)=\big(u_1^*,u_2^*\big).\tag{3.23}$$

We discuss the uniqueness of the solution for problem (3.22), (3.23) which is invariant to translations in s. In fact, the slight non-uniqueness can be solved. Since we are mainly interesting in the monotonicity, but we can, if we wish, eliminate the non-uniqueness by taking condition

$$\big(\phi_{10}(0),\phi_{20}(0)\big)=\left(\frac{u_1^*}{2},\frac{u_2^*}{2}\right).$$

The solution $(\phi_{10}(s),\phi_{20}(s))$ is the lowest-order term in the asymptotic expression (3.22) and (3.23), and what we shall now do is prove the following theorem concerning monotonicity of $(\phi_{10}(s),\phi_{20}(s))$. It is already known that some asymptotic analysis of the full nonlinear problem as $c\to\infty$ has been discussed in Section 2. From this, we know that all positive solutions of (3.22) tend to the state (u_1^*,u_2^*) as $s\to\infty$.

Theorem 3.2. *All positive solution of (3.22) subject to (3.23) is strictly increasing function of s for all $s\in\mathbb{R}$ providing $\beta_1\beta_2>\mu_1\mu_2$.*

Proof. We will prove the theorem in three steps:

(i) the decay of a positive solution of (3.22), (3.23) to zero as $s\to-\infty$ is strictly monotone;
(ii) a positive state solution $\phi_{i0}(s)\in[0,u_i^*]$, $i=1,2$;
(iii) a positive solution is strictly increasing.

The existence of traveling wave solutions connecting the equilibria E_0 and E^* has been obtained. In fact, by the results of linearized analysis, the extinction equilibrium $E_0=(0,0)$ is an unstable point. Thus, for any small $\epsilon>0$, there exists an s_ϵ such that $0<\phi_{10}(s)<\epsilon$ and $0<\phi_{20}(s)<\epsilon$ for $s\leqslant s_\epsilon$.

To prove (i), we let

$$\delta_1(\theta):=\alpha_1\beta_1^{-1}\exp\big\{-\tau_m\gamma_1-\tau_M\big(\alpha_1e^{\beta_1\theta\tau_M-\gamma_1\tau_m}+\mu_1\theta\big)\big\}$$

and further choose $\epsilon\in(0,\min\{1,\delta_1(1)\})$. For $s\leqslant s_\epsilon$, it follows from the first equation of (3.22) that

$$\phi_{10}'(s)\geqslant-\beta_1\phi_{10}^2(s)\geqslant-\beta_1\epsilon\phi_{10}(s),$$

which yields $\phi_{10}(s)\geqslant\phi_{10}(s-\tau(\phi_{10}))e^{-\beta_1\epsilon\tau(\phi_{10})}$ or, equivalently,

$$\phi_{10}\big(s-\tau(\phi_{10})\big)\leqslant\phi_{10}(s)e^{\beta_1\epsilon\tau(\phi_{10})}\leqslant\phi_{10}(s)e^{\beta_1\epsilon\tau_M}.$$

Substituting this into the first equation of (3.22) yields

$$\begin{aligned}\phi_{10}'(s) &\leqslant \alpha_1 e^{-\gamma_1\tau(\phi_{10})}\phi_{10}(s)e^{\beta_1\epsilon\tau_M} - \beta_1\phi_{10}^2(s) + \mu_1\phi_{10}(s)\phi_{20}(s)\\ &\leqslant \bigl(\alpha_1 e^{\beta_1\epsilon\tau_M-\gamma_1\tau_m} + \mu_1\epsilon\bigr)\phi_{10}(s)\end{aligned}$$

which leads to

$$\phi_{10}(s) \leqslant \phi_{10}\bigl(s-\tau(\phi_{10})\bigr)e^{\tau(\phi_{10})(\alpha_1 e^{\beta_1\epsilon\tau_M-\gamma_1\tau_m}+\mu_1\epsilon)}.$$

Thus, it follows, for $s \leqslant s_\epsilon$, that

$$\phi_{10}\bigl(s-\tau(\phi_{10})\bigr) \geqslant \phi_{10}(s)e^{-\tau(\phi_{10})(\alpha_1 e^{\beta_1\epsilon\tau_M-\gamma_1\tau_m}+\mu_1\epsilon)} \geqslant \phi_{10}(s)e^{-\tau_M(\alpha_1 e^{\beta_1\epsilon\tau_M-\gamma_1\tau_m}+\mu_1\epsilon)}$$

and

$$\phi_{10}'(s) \geqslant \alpha_1\phi_{10}(s)e^{-\tau(\phi_{10})\gamma_1-\tau_M(\alpha_1 e^{\beta_1\epsilon\tau_M-\gamma_1\tau_m}+\mu_1\epsilon)} - \beta_1\phi_{10}^2(s) + \mu_1\phi_{10}(s)\phi_{20}(s).$$

Therefore, $\phi_{10}'(s) > 0$ if $s \leqslant s_\epsilon$ and

$$\phi_{10}(s) < \alpha_1\beta_1^{-1}\exp\bigl\{-\tau_m\gamma_1 - \tau_M\bigl(\alpha_1 e^{\beta_1\epsilon\tau_M-\gamma_1\tau_m} + \mu_1\epsilon\bigr)\bigr\} = \delta_1(\varepsilon).$$

It is straightforward to see that $\phi_{10}(s) < \delta_1(\varepsilon)$ for $s \leqslant s_\epsilon$. Note that $\delta_1(\varepsilon)$ is strictly decreasing. Hence, $\delta_1(\epsilon) > \delta(1)$ for $\epsilon < 1$. So, we have $\phi_{10}(s) < \epsilon < \min\{1, \delta_1(1)\} < \delta_1(\varepsilon)$ providing $s \leqslant s_\epsilon$, as desired.

Similarly, we can obtain the same result on $\phi_{20}(s)$. If $s \leqslant s_\epsilon$, we have $\phi_{20}(s) < \epsilon < \min\{1, \delta_2(1)\} < \delta_2(\varepsilon)$, where

$$\delta_2(\theta) := \alpha_2\beta_2^{-1}\exp\bigl\{-\tau_m\gamma_2 - \tau_M\bigl(\alpha_2 e^{\beta_2\theta\tau_M-\gamma_2\tau_m} + \mu_2\theta\bigr)\bigr\}.$$

Thus $s \leqslant s_\epsilon$ implies $\phi_{10}'(s) > 0$ and $\phi_{20}'(s) > 0$, proving (i).

We now prove (ii), suppose that there exists a point where $\phi_{10}(s) > u_1^*$. Then $\phi_{10}(s)$ must attain a global maximum s_2. Without loss of generality, assume that $\phi_{10}'(s_2) = 0$ at s_2. So, we have the fact that $\phi_{10}'(s_2) = 0$ and $\phi_{10}(s_2) > u_1^*$. Then, it follows from the first equation of (3.22) that

$$\begin{aligned}0 = \phi_{10}'(s_2) &= \alpha_1 e^{-\gamma_1\tau(\phi_{10})}\phi_{10}\bigl(s_2-\tau(\phi_{10})\bigr) - \beta_1\phi_{10}^2(s_2) + \mu_1\phi_{10}(s_2)\phi_{20}(s_2)\\ &\leqslant \alpha_1 e^{-\gamma_1\tau(\phi_{10})}\phi_{10}(s_2) - \beta_1\phi_{10}^2(s_2) + \mu_1\phi_{10}(s_2)\phi_{20}(s_2)\\ &< \phi_{10}(s_2)\bigl(\alpha_1 e^{-\gamma_1\tau(u_1^*)} - \beta_1 u_1^* + \mu_1\phi_{20}(s_2)\bigr) = \phi_{10}(s_2)\bigl(-\mu_1 u_2^* + \mu_1\phi_{20}(s_2)\bigr).\end{aligned}$$

This means that $\phi_{20}(s_2) > u_2^*$. From the result of (i) and a similar proof of (c) in Theorem 2.1, we can obtain a contradiction. Thus, $\phi_{i0}(s) \in [0, u_i^*]$, $i = 1, 2$.

Next, we prove that the solution is strictly increasing. For the sake of contradiction and by translation invariance, assume that there exists a point such that $\phi_{10}'(s) = 0$. Let s_0 be the leftmost such point, which is well defined by the result (i). Furthermore, it follows from (i) that $\phi_{10}''(s_0) \leqslant 0$. Derivative of the first equation of (3.22) with respect to s takes the form

$$\begin{aligned}\phi_{10}''(s) = {}& \alpha_1 e^{-\gamma_1\tau(\phi_{10})}\bigl(\phi_{10}'\bigl(s-\tau(\phi_{10})\bigr) - \tau'(\phi_{10})\phi_{10}'(s)\bigl(\gamma_1\phi_{10}\bigl(s-\tau(\phi_{10})\bigr) + \phi_{10}'\bigl(s-\tau(\phi_{10})\bigr)\bigr)\bigr)\\ &- 2\beta_1\phi_{10}(s)\phi_{10}'(s) + \mu_1\phi_{10}'(s)\phi_{20}(s) + \mu_1\phi_{10}(s)\phi_{20}'(s).\end{aligned} \tag{3.24}$$

If the solution $\phi_{20}(s)$ is strictly increasing, then $\phi_{20}'(s_0) > 0$. When $s = s_0$, it follows from (3.24) that $\phi_{10}'(s_0 - \tau(\phi_{10})) < 0$, giving a contradiction.

If the solution $\phi_{20}(s)$ is not strictly increasing, then there exists at least one point such that $\phi'_{20}(s) = 0$, since the solution will ultimately tend to the state (u_1^*, u_2^*). Let s_1 be the leftmost such point, which is well defined by the result (i). Without loss of generality, we assume $s_1 > s_0$, otherwise, we consider the solution $\phi_{20}(s)$. Thus, the solutions $\phi'_{20}(s_0) > 0$ and $\phi'_{10}(s_0 - \tau(\phi_{10})) < 0$, giving a contradiction. The proof of the theorem is completed. □

4. Discussion

This article deals with a cooperative model with state-dependent delay which is the time taken from birth to maturity is directly related to the number of the species individuals. For the model to make sense, the delay is assumed to be an increasing function of the population density with lower and upper bound. Compared to the constant delay, the state-dependent delay makes the dynamic behavior more complex.

For the DDE system (2.1), the positivity and boundedness are firstly given, which implies the system persists and the populations is subjected to the natural restriction, the existence and the uniqueness of equilibria are then discussed. It is important to note that the comparison principle of the state-dependent delay equations are proved, which do not always hold even if for the constant delay equations. Using the comparison principle obtained, the stability criterion of the model is analyzed both from local and global points of view. For the PDE system (1.2), we mainly discuss the existence of traveling waves solutions. Firstly, we calculate the minimum speed c_i^* implicitly by linearizing the model at the origin, and shows that the wave is slowed down by the state-dependent delay. Then, the existence of traveling waves is obtain by constructing a pair of upper–lower solutions and using the cross iteration method and Schauder's fixed point theorem for $c > c^*$. As we know, the construction of upper and lower solutions is extremely skill, let alone the state-dependent delay system. In this article, the idea for the construction of upper and lower solutions is motivated from Zhang et al. [44], but we improved and developed the argument on verification of upper–lower solutions for our state-dependent delay cooperative model. Finally, the traveling wavefront solutions for large wave speed is also discussed. Our results implies that our fronts appear to be all monotone, regardless of the state-dependent time delay. This is an interesting property, since many findings are frequently reported that delay causes a loss of monotonicity, with the front developing a prominent hump in some other delay models.

Our results shows that the dynamics depends on the two quantities $\beta_1\beta_2$ and $\mu_1\mu_2$. The positivity and boundedness of solutions, the existence and global stability of the coexistence equilibrium, the existence and the monotonicity of the traveling wave solutions can be obtained if $\beta_1\beta_2 > \mu_1\mu_2$; such will lose if $\beta_1\beta_2 < \mu_1\mu_2$. From biological view of point, the term $\beta_i u_i^2$ represents the death of population u_i, which can be also illustrated the death caused by crowding effects or the intraspecific competition effects; the term $\mu_i u_1 u_2$ represents strongly mutualistic effects between both species. The conditions $\beta_1\beta_2 \geqslant \mu_1\mu_2$ means that the intraspecific competition of species is stronger than the mutualistic effects between both species. Therefore, the intraspecific competition effects is the main factor which affects the boundedness of solutions and the existence, stability of the coexistence equilibrium E^*. As long as the intraspecific competition of one species do not result in the extinction of the species and its effects is stronger than mutualistic effects between both species, the system is able to stable at the coexistence equilibrium. However, when the intraspecific competition of species is lower than the mutualistic effects between both species, the mutualistic effects would result in the growth unlimited of the two populations.

Acknowledgments

We are extremely grateful to the critical comments and invaluable suggestions made by anonymous honorable reviewers.

References

[1] M. Adimy, F. Crauste, M. Hbid, R. Qesmi, Stability and Hopf bifurcation for a cell population model with state-dependent delay, SIAM J. Appl. Math. 70 (2010) 1611–1633.
[2] W. Aiello, H. Freedman, J. Wu, Analysis of a model representing stage-structured population growth with state-dependent time delay, SIAM J. Appl. Math. 52 (1992) 855–869.
[3] J. Al-Omari, S. Gourley, Stability and traveling fronts in Lotka–Volterra competition models with stage structure, SIAM J. Appl. Math. 63 (2003) 2063–2086.
[4] J. Al-Omari, S. Gourley, Dynamics of a stage-structured population model incorporating a state-dependent maturation delay, Nonlinear Anal. Real World Appl. 6 (2005) 13–33.
[5] J. Al-Omari, A. Tallafha, Modelling and analysis of stage-structured population model with state-dependent maturation delay and harvesting, Int. J. Math. Anal. 1 (2007) 391–407.
[6] H. Andrewartha, L. Birch, The Distribution and Abundance of Animals, University of Chicago Press, Chicago, IL, 1954, p. 370.
[7] H. Barclay, P. van den Driessche, A model for a single species with two life history stages and added mortality, Ecol. Model. 11 (1980) 157–166.
[8] J. Canosa, On a nonlinear diffusion equation describing population growth, IBM J. Res. Develop. 17 (1973) 307–313.
[9] K. Das, S. Ray, Effect of delay on nutrient cycling in phytoplankton–zooplankton interactions in estuarine system, Ecol. Model. 215 (2008) 69–76.
[10] J. Fang, J. Wu, Monotone traveling waves for delayed Lotka–Volterra competition systems, Discrete Contin. Dyn. Syst. 32 (2012) 3043–3058.
[11] R. Gambell, Birds and mammals – Antarctic whales, in: W. Bonner, D. Walton (Eds.), Antarctica, Pergamon Press, New York, 1985, pp. 223–241.
[12] B. Goh, Global stability in two species interactions, J. Math. Biol. 3 (1976) 313–318.
[13] K. Gopalsamy, Stability and Oscillations in Delay Differential Equations of Population Dynamics, Kluwer, Dordrecht, The Netherlands, 1992.
[14] S. Gourley, Y. Kuang, Wavefronts and global stability in a time-delayed population model with stage structure, Proc. R. Soc. Lond. Ser. A 459 (2003) 1563–1579.
[15] W. Gurney, R. Nisbet, J. Lawton, The systematic formulation of tractible single species population models incorporating age structure, J. Anim. Ecol. 52 (1983) 479–485.
[16] J. Hale, Theory of Functional Differential Equations, Springer, New York, 1977.
[17] J. Hale, S. Verduyn Lunel, Introduction to Functional Differential Equations, New York, 1993.
[18] F. Hartung, T. Krisztin, H. Walther, J. Wu, Functional differential equations with state-dependent delay: Theory and applications, in: A. Canada, P. Drabek, A. Fonda (Eds.), Handbook of Differential Equations: Ordinary Differential Equations, vol. 3, Elsevier Science B.V., North-Holland, Amsterdam, 2006, pp. 435–545.
[19] W. Hirsch, H. Hanisch, J. Gabriel, Differential equations models of some parasitic infections, methods for the study of asymptotic behavior, Comm. Pure Appl. Math. 38 (1985) 733–753.
[20] K. Hong, P. Weng, Stability and traveling waves of a stage-structured predator–prey model with Holling type-II functional response and harvesting, Nonlinear Anal. Real World Appl. 14 (2013) 83–103.
[21] Q. Hu, X. Zhao, Global dynamics of a state-dependent delay model with unimodal feedback, J. Math. Anal. Appl. 399 (2013) 133–146.
[22] J. Huang, X. Zou, Traveling wavefronts in diffusive and cooperative Lotka–Volterra system with delays, J. Math. Anal. Appl. 271 (2002) 455–466.
[23] Y. Kuang, Delay Differential Equation with Applications in Population Dynamics, Academic, New York, 1993.
[24] H. Landahl, B. Hanson, A three stage population model with cannibalism, Bull. Math. Biol. 37 (1975) 11–17.
[25] M. Lewis, B. Li, H. Weinberger, Spreading speed and linear determinacy for two-species competition models, J. Math. Biol. 45 (2002) 219–233.
[26] X. Liang, X. Zhao, Asymptotic speeds of spread and traveling waves for monotone semiflows with applications, Comm. Pure Appl. Math. 60 (2007) 1–40; Comm. Pure Appl. Math. 61 (2008) 137–138 (Erratum).
[27] X. Liang, X. Zhao, Spreading speeds and traveling waves for abstract monostable evolution systems, J. Funct. Anal. 259 (2010) 857–903.
[28] Z. Lu, W. Wang, Permanence and global attractivity for Lotka–Volterra difference systems, J. Math. Biol. 39 (1999) 269–282.
[29] R. Martin, H. Smith, Abstract functional differential equations and reaction–diffusion systems, Trans. Amer. Math. Soc. 321 (1990) 1–44.
[30] Y. Muroya, Uniform persistence for Lotka–Volterra-type delay differential systems, Nonlinear Anal. Real World Appl. 4 (2003) 689–710.
[31] J. Murray, Mathematical Biology, Springer, 1989.
[32] W. Rudin, Functional Analysis, McGraw–Hill, 1991.
[33] J. Sherratt, Wavefront propagation in a competition equation with a new motility term modelling contact inhibition between cell populations, Proc. R. Soc. Lond. Ser. A 456 (2000) 2365–2386.
[34] H. Smith, Systems of ordinary differential equations which generate an order preserving flow. A survey of results, SIAM Rev. 30 (1988) 87–113.
[35] H. Smith, H. Thieme, Monotone semiflows in scalar non-quasi-monotone functional–differential equations, J. Math. Anal. Appl. 150 (1990) 289–306.
[36] H. Smith, H. Thieme, Strongly order preserving semiflows generated by functional–differential equations, J. Differential Equations 93 (1991) 332–363.

[37] K. Tognetti, The two stage stochastic model, Math. Biosci. 25 (1975) 195–204.

[38] H. Walther, The solution manifold and C^1-smoothness for differential equations with state dependent delay, J. Differential Equations 195 (2003) 46–65.

[39] H. Weinberger, On spreading speeds and traveling waves for growth and migration models in a periodic habitat, J. Math. Biol. 45 (2002) 511–548.

[40] S. Wood, S. Blythe, W. Gurney, R. Nisbet, Instability in mortality estimation schemes related to stage-structure population models, IMA J. Math. Appl. Med. Biol. 6 (1989) 47–68.

[41] Y. Yang, Hopf bifurcation in a two-competitor, one-prey system with time delay, Appl. Math. Comput. 214 (2009) 228–235.

[42] Q. Ye, Z. Li, M. Wang, Y. Wu, Introduction of Reaction–Diffusion Equations, second edition, China Science Publishing Group, 2011.

[43] A. Zaghrout, S. Attalah, Analysis of a model of stage-structured population dynamics growth with time state-dependent time delay, Appl. Math. Comput. 77 (1996) 185–194.

[44] G. Zhang, W. Li, G. Lin, Traveling waves in delayed predator–prey systems with nonlocal diffusion and stage structure, Math. Comput. Modelling 49 (2009) 1021–1029.

[45] L. Zhang, B. Li, J. Shang, Stability and travelling waves for a time-delayed population system with stage structure, Nonlinear Anal. Real World Appl. 13 (2012) 1429–1440.