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A new proof of I. Guerra's results concerning nonlinear biharmonic equations with negative exponents

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ABSTRACT

Here we give a self-contained new proof of the asymptotic behavior of the radially symmetric entire solution of

$$\Delta^2 u = -u^{-p}, \quad \text{in } \mathbb{R}^3.$$

These results were obtained by I. Guerra in [4]. Our proof is much more direct and simpler.

1. Introduction

Consider solutions to the following equation with negative exponent:

$$\Delta^2 u = -u^{-p}, \quad \text{in } \mathbb{R}^3, \text{ where } p > 1. \tag{1.1}$$

This problem has its root in Riemannian geometry. Let us briefly describe the background of this equation. Let $g = (g_{ij})$ be the standard Euclidean metric on $\mathbb{R}^N, N \geqslant 3$, with $g_{ij} = \delta_{ij}$. Let $\bar{g} = u^{\frac{4}{N-4}} g$ $(N \neq 4)$ be a second metric derived from g by the positive conformal factor $u : \mathbb{R}^N \to \mathbb{R}$. Then u satisfies

$$\Delta^2 u = \frac{N-4}{2} Q_{\bar{g}} u^{\frac{N+4}{N-4}},$$

where $Q_{\bar{g}}$ is the scalar curvature of $\bar{g}$. If we assume that $Q_{\bar{g}} > 0$ is a constant, we can obtain (1.1) via scaling. The existence and properties of solution have been considered by various authors, see [1,2,4,5,7] and the references therein. Here we recall some results which are related to the present paper:

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- (See [1].) If $1 < p \leqslant 7$, $u \in C^4(\mathbb{R}^3)$ is a positive solution of (1.1) such that

$$\lim_{|x|\to\infty} \frac{u(x)}{|x|} = \alpha_1 \quad \text{uniformly for some constant } \alpha_1 > 0,$$

  then $p = 7$.
- (See [7].) There is no radial solution with linear growth for $4 < p < 7$.

However, I. Guerra in [4] pointed out that the above results were incorrect and gave a detail explanation for mistakes of the proof, and used the involved phase–space analysis to obtain the following results:

Theorem 1.1.

(i) *For $p = 3$ there exists a radial solution of (1.1) such that*

$$\lim_{r\to\infty} \frac{u(r)}{r(\log r)^{\frac{1}{4}}} = 2^{\frac{1}{4}}.$$

(ii) *For $p > 3$ there exists a radial solution of (1.1) such that for any $\alpha > 0$ its asymptotic behavior is given by*

$$\lim_{r\to\infty} \frac{u(r)}{r} = \alpha.$$

Remark 1.2. For $1 < p < 3$, due to the technique developed by [3], the asymptotic of radial solutions of (1.1) have been studied extensively, for details see [2,5]. However, this technique is invalid for $p \geqslant 3$, and not until the work of I. Guerra, could one completely understand the asymptotic of radial solutions.

In the present paper, using the technique of simple ordinary differential equations, we give a self-contained new proof for I. Guerra's result (Theorem 1.1). Our proof is much more direct and simpler.

2. Proof of Theorem 1.1

In this section, we give the new proof of Theorem 1.1. We first consider the radial version of (1.1)

$$\begin{cases} \Delta^2 u(r) = -u^{-p}, & \text{for } r \in \big(0, R_{\max}(\beta)\big), \\ u(0) = 1, \quad \Delta u(0) = \beta, & u'(0) = (\Delta u)'(0) = 0. \end{cases} \tag{2.1}$$

Here $[0, R_{\max}(\beta))$ is the interval of existence of the solution. We say that the solution of (2.1) is entire (resp., local) if $R_{\max}(\beta) = \infty$ (resp., $R_{\max}(\beta) < \infty$). Now we recall some results in the following Lemma 1 for Eq. (2.1) (for details see [4,5,7]), which will be used for our proofs.

Lemma 1. *Assume $p > 1$. Then there is a unique $\beta_0 > 0$ such that:*

a) *If $\beta < \beta_0$ then $R_{max}(\beta) < \infty$;*
b) *If $\beta \geqslant \beta_0$ then $R_{max}(\beta) = \infty$ and $\lim_{r\to\infty} \Delta u_\beta(r) \geqslant 0$;*
c) *We have $\beta = \beta_0$ if and only if $\lim_{r\to\infty} \Delta u_\beta(r) = 0$.*

Now, inspired by the arguments of [6], we use an Emden–Fowler transformation to transform (2.1) into an ODE whose linear part has constant coefficients.

Lemma 2. *Let $p > 1$. If u is a positive radial solution of (2.1), then*

$$W(s) = e^{ms}u\big(e^s\big) = r^m u(r), \quad s = \log r = \log|x|, \ m = -\frac{4}{p+1},$$

is such that

$$Q_4(m - \partial_s)W(s) := (\partial_s - m + N - 4)(\partial_s - m + N - 2)(\partial_s - m - 2)(\partial_s - m) = -W^{-p}(s).$$

Proof. Since $\partial_r = e^{-s}\partial_s$, a short computation allows us to write the radial Laplacian as

$$\Delta = \partial_r^2 + (N-1)r^{-1}\partial_r = e^{-2s}(n - 2 + \partial_s).$$

Using the operator identity $\partial_s e^{-ks} = e^{-ks}(\partial_s - k)$, one can then easily check that

$$\Delta^2 e^{-ms} = e^{-4s-ms}Q_4(m - \partial_s) = e^{-mps}Q_4(m - \partial_s).$$

This also implies that $Q_4(m - \partial_s)W(s) = e^{mps}\Delta^2 U(e^s) = -W^{-p}(s)$, as required. □

Now, we prove the variation of parameters formula by integral method.

Lemma 3. *For a given f, let $Z(t)$ be the solution of*

$$(\partial_t - \lambda_1)(\partial_t - \lambda_2)(\partial_t - \lambda_3)(\partial_t - \lambda_4)Z(t) = f(Z)(t) \quad \text{for } t \in \mathbb{R}. \tag{2.2}$$

Given any $t_0 \in R$, then there exist some constants α_i, d_i such that

$$Z(t) = \sum_{i=1}^{i=4}\left(\alpha_i e^{\lambda_i s} + d_i \int_{t_0}^{t} e^{\lambda_i(t-\tau)} \cdot f(Z)(\tau)d\tau\right) \tag{2.3}$$

in the case $\lambda_i \neq \lambda_j$ $(i, j = 1...4)$ and

$$Z(t) = \sum_{i=1}^{i=3}\left(\alpha_i e^{\lambda_i t} + d_i \int_{t_0}^{t} e^{\lambda_i(t-\tau)} \cdot f(Z)(\tau)d\tau\right) + \alpha_4 t e^{\lambda_4 t} + d_4 \int_{t_0}^{t} (t-\tau)e^{\lambda_4(t-\tau)} \cdot f(Z)(\tau)d\tau \tag{2.4}$$

in the case $\lambda_1 \neq \lambda_2 \neq \lambda_3$ and $\lambda_3 = \lambda_4$. Moreover, each α_i depends on t_0 and λ_i $(i = 1, ..., 4)$, whereas each d_i depends solely on the λ_i $(i = 1, ..., 4)$.

Proof. We multiply Eq. (2.2) by $e^{-\lambda_1 s}$ and then integrate to get

$$(\partial_t - \lambda_2)(\partial_t - \lambda_3)(\partial_t - \lambda_4)Z(t) = A_1 e^{\lambda_1 t} + \int_{t_0}^{t} e^{\lambda_1(t-\tau)} \cdot f(Z)(\tau)d\tau$$

for some constant A_1. Repeating the same argument once again, we arrive at

$$(\partial_t - \lambda_3)(\partial_t - \lambda_4)Z(t) = B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} + \int_{t_0}^{t}\int_{t_0}^{\rho} e^{\lambda_2(t-\rho)} e^{\lambda_1(\rho-\tau)} \cdot f(Z)(\tau)d\tau$$

because $\lambda_1 \neq \lambda_2$. We now switch the order of integration to get

$$(\partial_t - \lambda_3)(\partial_t - \lambda_4)Z(t) = B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} + \int_{t_0}^{t} \frac{e^{\lambda_1(t-\tau)} - e^{\lambda_2(t-\tau)}}{\lambda_1 - \lambda_2} \cdot f(Z)(\tau)d\tau.$$

In the first case, we can repeat our approach two times to deduce (2.3). In the second case, our approach also leads to (2.4). □

Remark 2.1. For the first case, we have, by simple calculation,

$$d_1 = \frac{1}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)}; \qquad d_2 = \frac{1}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)};$$
$$d_3 = \frac{1}{(\lambda_3 - \lambda_4)(\lambda_3 - \lambda_2)(\lambda_3 - \lambda_1)}; \qquad d_4 = \frac{1}{(\lambda_4 - \lambda_1)(\lambda_4 - \lambda_2)(\lambda_4 - \lambda_3)}.$$

Proof of Theorem 1.1. Let $W(s) = e^{ms}u_{\beta_0}(r)$, where β_0 is defined as in Lemma 1 and $r = e^s$. We, by the following Lemma 2, have

$$Q_4(m - \partial_s)W(s) = W^{(4)}(s) + K_3 W(s)''' + K_2 W(s)'' + K_1 W' + Q_4(m)W(s) = -W^{-p}(s), \tag{2.5}$$

where K_3, K_2, K_1 are fixed constants.

Step 1. We claim $W(s)$ is unbounded for $N = 3$, $p \geqslant 3$. Suppose by contradiction that $W(s)$ is bounded. Indeed, since $N = 3$, and $p \in [3, +\infty)$ then $Q_4(m) \geqslant 0$, and there exist $\varepsilon > 0$ and $s_0 \gg 1$ such that

$$-Q_4(m)W(s) - W^{-p}(s) \leqslant -W^{-p}(s) < -\varepsilon, \quad \forall s \in (s_0, +\infty). \tag{2.6}$$

After integration in (2.5), by (2.6) we obtain as $s \to +\infty$

$$W'''(s) + K_3 W''(s) + K_2 W'(s) < -\varepsilon(s - s_0) + O(1).$$

Two further integrations yield

$$W'(s) < -\frac{\varepsilon}{6}(s - s_0)^3 + O(s^2) \quad \text{as } s \to +\infty.$$

This contradicts the fact that $W(s) > 0$ for any $s \in (s_0, +\infty)$.

Step 2. We claim that $W(s) \to \infty$ as $s \to \infty$ for $p = 3$, $N = 3$. Indeed, $W(s)$ satisfies

$$(\partial_s + 2)(\partial_s + 1)(\partial_s - 1)W'(s) = -W^{-p}(s) < 0. \tag{2.7}$$

Multiplying by e^{2s} and integrating over $(-\infty, s)$, we get

$$e^{2s}(\partial_s + 1)(\partial_s - 1)W'(s) \leqslant 0.$$

We now ignore the exponential factor and use the same argument to get

$$(\partial_s - 1)W'(s) \leqslant 0, \quad \text{i.e.,} \quad \left(e^{-s}W_s(s)\right)' \leqslant 0.$$

Since $\lim_{r\to\infty} \Delta u_{\beta_0}(r) = 0$, we have

$$\lim_{s\to+\infty} e^{-s}W_s = \lim_{s\to+\infty}\left(e^{-2s}W(s) + \frac{u_r}{r}\right) = 0,$$

and then $W'(s) > 0$. Combining with step 1, we prove our claim.

Step 3. $\lim_{r\to\infty} \frac{u_{\beta_0}}{r(\log r)^{\frac{1}{4}}} = 2^{\frac{1}{4}}$, for $N = 3$, $p = 3$.

Combining with step 2, we can take (2.5) as

$$Q_4(m - \partial_s)W = -W^{-3} = o(1) \quad \text{as } s \to +\infty.$$

The linearly independent solutions of the homogeneous equation $Q_4(m - \partial_s)z = 0$ are $e^{\lambda_i s}$ with

$$\lambda_1 = 0, \qquad \lambda_2 = -2, \qquad \lambda_3 = -1, \qquad \lambda_4 = 1.$$

And then we have

$$W(s) = \sum_{i=1}^{i=4} C_i e^{\lambda_i s} + \sum_{i=1}^{i=4} d_i \int_{s_0}^{s} e^{\lambda_i(s-\tau)}\bigl(-W^{-3}(\tau)\bigr)d\tau \tag{2.8}$$

where $d_1 = -2^{-1}$, d_i $(i = 2, 3, 4)$ are fixed constants and $s \geqslant s_0$. Since $\lambda_4 = 1 > 0$, and using the fact that $\int_{s_0}^{s} = \int_{s_0}^{+\infty} - \int_{s}^{+\infty}$, we have

$$W(s) = \sum_{i=1}^{i=4} C_i' e^{\lambda_i s} + 2\int_{s_0}^{s} W^{-3}(\tau)d\tau + \sum_{i=2}^{i=3} d_i \int_{s_0}^{s} e^{\lambda_i(s-\tau)}W^{-3}(\tau)d\tau - d_4 \int_{s}^{+\infty} e^{(s-\tau)}W^{-3}(\tau)d\tau.$$

Obviously, we have

$$\int_{s}^{+\infty} e^{\lambda_4(s-\tau)}W^{-3}(\tau)d\tau, \int_{s_0}^{s} e^{\lambda_i(s-\tau)}W^{-3}(\tau)d\tau \to 0, \quad \text{as } s \to +\infty\ (i = 2, 3).$$

Since $u_{\beta_0}(r) = o(r^2)$ as $r \to \infty$, we conclude that

$$W(s) = o\bigl(e^s\bigr) \quad \text{as } s \to +\infty.$$

And so we obtain that

$$C_4' = 0,$$

and then

$$W(s) = C + 2^{-1}\int_{s_0}^{s} W^{-3}(\tau)d\tau + r(s) \tag{2.9}$$

where

$$r(s) = \sum_{i=2}^{i=3} d_i \int_{s_0}^{s} e^{\lambda_i(s-\tau)}W^{-1}(\tau)d\tau - d_4 \int_{s}^{+\infty} e^{\lambda_4(s-\tau)}W^{-3}(\tau)d\tau = o(1) \quad \text{as } s \to +\infty. \tag{2.10}$$

Moreover, we have

$$r'(s) = o(1) \quad \text{as } s \to +\infty. \tag{2.11}$$

Now using (2.9), we obtain

$$W'W^3 = 2^{-1} + r'(t)W^3(s) \quad \text{and} \quad W'(s) \to 0 \quad \text{as } s \to +\infty.$$

Now, we claim $r'(s)W^3(s) \to 0$ as $s \to +\infty$. Indeed,

$$r'(s)W^3(s) = \sum_{i=2}^{i=3}\left(d_i W^3(s)\lambda_i \int_{s_0}^{t} e^{\lambda_i(s-\tau)}W^{-3}(\tau)d\tau + d_i\right) + d_4\lambda_4 W^3 \int_{s}^{+\infty} e^{\lambda_i(s-\tau)}W^{-3}(\tau)d\tau + d_4.$$

By l'Hôspital's rule we obtain our claim, where we have used the fact that $W'(s) \to 0$ as $s \to +\infty$. Integrating over (s_0, s) and using (2.10), we conclude that

$$\frac{1}{4}W^4(s) = 2^{-1}(s - s_0) + o(s - s_0) \quad \text{for } s \gg 1.$$

From this, we immediately have

$$\lim_{s\to+\infty} \frac{W(s)}{\sqrt[4]{s}} = 2^{\frac{1}{4}}, \tag{2.12}$$

which gives the proof of Theorem 1.1(i).

Step 4. $\lim_{r\to\infty} \frac{u_{\beta_0}}{r} = \alpha > 0$ for $N = 3$, $p > 3$. Indeed, by

$$W(s) = \sum_{i=1}^{i=4} C_i e^{\lambda_i s} + \sum_{i=1}^{i=2} d_i \int_{s}^{+\infty} e^{\lambda_i(s-\tau)}W^{-p}(\tau)d\tau + \sum_{i=3}^{i=4} d_i \int_{s_0}^{s} e^{\lambda_i(s-\tau)}W^{-p}(\tau)d\tau$$

where d_i $(i = 1, 2, 3, 4)$ are fixed constants and $s \geqslant s_0$, and $\lambda_1 = m+1 > 0$, $\lambda_2 = m+2 > 1$, $\lambda_3 = m-1 < 0$, $\lambda_4 = m < 0$. Since by Lemma 13 of [2], we have

$$\liminf_{s\to+\infty} W(s) > 0,$$

and then

$$\sum_{i=1}^{i=2} d_i \int_{s}^{+\infty} e^{\lambda_i(s-\tau)}W^{-p}(\tau)d\tau + \sum_{i=3}^{i=4} d_i \int_{s_0}^{s} e^{\lambda_i(s-\tau)}W^{-p}(\tau)d\tau = O(1).$$

Besides, $u(s) = o(e^{2s})$, as $s \to +\infty$, so $C_2 = 0$, and then we have, by step 1,

$$\lim_{s\to+\infty} e^{-(m+1)s}W(s) = C,$$

which is a desired result. $\square$

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