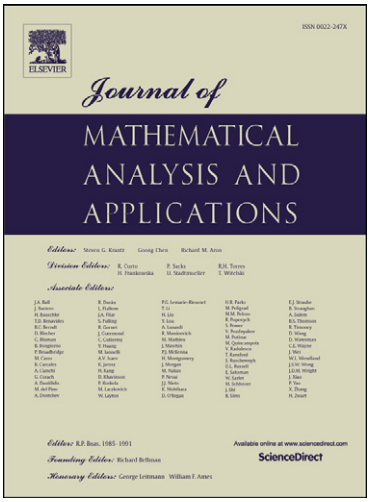
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Riesz potentials, Bessel potentials and fractional derivatives on Triebel-Lizorkin spaces for the Gaussian measure

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RIESZ POTENTIALS, BESSEL POTENTIALS AND FRACTIONAL DERIVATIVES ON TRIEBEL-LIZORKIN SPACES FOR THE GAUSSIAN MEASURE.

A. EDUARDO GATTO, EBNER PINEDA, AND WILFREDO O. URBINA

Abstract. In [7] the boundedness properties of Riesz Potentials, Bessel potentials and Fractional Derivatives were studied in detail on Gaussian Besov-Lipschitz spaces $B^{\alpha}_{p,q}(\gamma_d)$. In this paper we will continue our study proving the boundedness of those operators on Gaussian Triebel-Lizorkin spaces $F^{\alpha}_{p,q}(\gamma_d)$. Also these results can be extended to the case of Laguerre or Jacobi expansions and even further to the general framework of diffusions semigroups.

1. Introduction and Preliminaries

On $\mathbb{R}^d$ let us consider the Gaussian measure

$$\gamma_d(dx) = \frac{e^{-\|x\|^2}}{\pi^{d/2}} dx,\ x \in \mathbb{R}^d \tag{1.1}$$

and the Ornstein-Uhlenbeck differential operator

$$L = \frac{1}{2}\triangle_x - \langle x, \nabla_x \rangle . \tag{1.2}$$

Let $\nu = (\nu_1, ..., \nu_d)$ be a multi-index such that $\nu_i \geq 0, i = 1, \cdots, d$, let $\nu! = \prod_{i=1}^{d} \nu_i!$, $|\nu| = \sum_{i=1}^{d} \nu_i$, $\partial_i = \frac{\partial}{\partial x_i}$, for each $1 \leq i \leq d$ and $\partial^{\nu} = \partial_1^{\nu_1} ... \partial_d^{\nu_d}$, consider the normalized Hermite polynomials of order ν in d variables,

$$h_{\nu}(x) = \frac{1}{\left(2^{|\nu|}\nu!\right)^{1/2}} \prod_{i=1}^{d} (-1)^{\nu_i} e^{x_i^2} \partial_i^{\nu_i}(e^{-x_i^2}). \tag{1.3}$$

It is well known, that the Hermite polynomials are eigenfunctions of the operator L,

$$Lh_{\nu}(x) = -\,|\nu|\, h_{\nu}(x). \tag{1.4}$$

Given a function $f \in L^1(\gamma_d)$ its ν-Fourier-Hermite coefficient is defined by

$$\hat{f}(\nu) = < f, h_{\nu} >_{\gamma_d} = \int_{\mathbb{R}^d} f(x) h_{\nu}(x) \gamma_d(dx).$$

Let C_n be the closed subspace of $L^2(\gamma_d)$ generated by the linear combinations of $\{h_{\nu} : |\nu| = n\}$. By the orthogonality of the Hermite polynomials with

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respect to γ_d it is easy to see that $\{C_n\}$ is an orthogonal decomposition of $L^2(\gamma_d)$,

$$L^2(\gamma_d) = \bigoplus_{n=0}^{\infty} C_n,$$

this decomposition is called the Wiener chaos.

Let J_n be the orthogonal projection of $L^2(\gamma_d)$ onto C_n, then if $f \in L^2(\gamma_d)$

$$J_n f = \sum_{|\nu|=n} \hat{f}(\nu) h_\nu.$$

Let us define the Ornstein-Uhlenbeck semigroup $\{T_t\}_{t\geq 0}$ as

$$\begin{aligned} T_t f(x) &= \frac{1}{(1-e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{e^{-2t}(|x|^2+|y|^2)-2e^{-t}\langle x,y\rangle}{1-e^{-2t}}} f(y)\gamma_d(dy) \\ &= \frac{1}{\pi^{d/2}(1-e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|y-e^{-t}x|^2}{1-e^{-2t}}} f(y)dy \end{aligned} \tag{1.5}$$

The family $\{T_t\}_{t\geq 0}$ is a strongly continuous Markov semigroup on $L^p(\gamma_d)$, $1 \leq p \leq \infty$, with infinitesimal generator L.

Now, by Bochner subordination formula, see Stein [12], we define the Poisson-Hermite semigroup $\{P_t\}_{t\geq 0}$ as

$$P_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u} f(x) du \tag{1.6}$$

By the change of variables $s = t^2/4u$ we have,

$$P_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u} f(x) du = \int_0^\infty T_s f(x) \mu_t^{(1/2)}(ds), \tag{1.7}$$

where the measure

$$\mu_t^{(1/2)}(ds) = \frac{t}{2\sqrt{\pi}} \frac{e^{-t^2/4s}}{s^{3/2}} ds, \tag{1.8}$$

is called the one-side stable measure on $(0,\infty)$ of order $1/2$.

The family $\{P_t\}_{t\geq 0}$ is a strongly continuous semigroup on $L^p(\gamma_d)$, $1 \leq p \leq \infty$, with infinitesimal generator $-(-L)^{1/2}$, furthermore $\{P_t\}$ is an analytic semigroup in t. In what follows, often we are going to use the notation

$$u(x,t) = P_t f(x) \text{ and } u^{(k)}(x,t) = \frac{\partial^k}{\partial t^k} P_t f(x).$$

Observe that by (1.4) we have that

$$T_t h_\nu(x) = e^{-t|\nu|} h_\nu(x), \tag{1.9}$$

and

$$P_t h_\nu(x) = e^{-t\sqrt{|\nu|}} h_\nu(x), \tag{1.10}$$

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i.e. the Hermite polynomials are eigenfunctions of T_t and P_t for any $t \geq 0$.

The operators that we are going to consider in this paper are the following:

- For $\beta > 0$, the Fractional Integral or Riesz potential of order β, I_β, with respect to the Gaussian measure is defined formally as

$$I_\beta = (-L)^{-\beta/2}\Pi_0, \tag{1.11}$$

where, $\Pi_0 f = f - \int_{\mathbb{R}^d} f(y)\gamma_d(dy)$, for $f \in L^2(\gamma_d)$. That means that for the Hermite polynomials $\{h_\nu\}$, for $|\nu| > 0$,

$$I_\beta h_\nu(x) = \frac{1}{|\nu|^{\beta/2}} h_\nu(x), \tag{1.12}$$

and for $\nu = 0 = (0, ..., 0)$, $I_\beta(h_0) = 0$. Then by linearity, I_β^γ can be defined to any polynomial.

In this paper the following representation of I_β^γ is crucial to get the results. If f is a polynomial,

$$I_\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1}(P_t f(x) - P_\infty f(x))\, dt, \tag{1.13}$$

where $P_\infty f(x) = \int_{\mathbb{R}^d} f(y)\gamma_d(dy)$. The formula is immediate for constant polynomials, it is easy to check for Hermite polynomials and then it follows for any polynomial.

Observation 1.1. *P. A. Meyer's multiplier theorem shows that I_β has a continuous extension to $L^p(\gamma_d)$, $1 < p < \infty$, and then (1.13) can be extended for $f \in L^p(\gamma_d)$, by the density of the polynomials, see* [1].

- The Bessel Potential of order $\beta > 0$, $\mathcal{J}_\beta$, associated to the Gaussian measure is defined formally as

$$\mathcal{J}_\beta = (I + \sqrt{-L})^{-\beta}, \tag{1.14}$$

meaning that for the Hermite polynomials we have,

$$\mathcal{J}_\beta h_\nu(x) = \frac{1}{(1 + \sqrt{|\nu|})^\beta} h_\nu(x).$$

Again by linearity $\mathcal{J}_\beta$ can be extended to any polynomial. By similar argument as above (1.13), the Bessel potentials can be represented as

$$\mathcal{J}_\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} t^\beta e^{-t} P_t f(x) \frac{dt}{t}, \tag{1.15}$$

Observation 1.2. *P. A. Meyer's theorem shows that $\mathcal{J}_\beta$ is a continuous operator on $L^p(\gamma_d)$, $1 < p < \infty$, and again (1.15) can be extended for $f \in L^p(\gamma_d)$, by the density of the polynomials there.*

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- The Riesz fractional derivative of order $\beta > 0$ with respect to the Gaussian measure D^β, is defined formally as $D^\beta = (-L)^{\beta/2}$, meaning that for the Hermite polynomials, we have

$$D^\beta h_\nu(x) = |\nu|^{\beta/2}\, h_\nu(x), \tag{1.16}$$

thus by linearity can be extended to any polynomial, see [8] and [9].

Now, if f is a polynomial, by the linearity of the operators I_β and D^β, (1.12) and (1.16), we get

$$\Pi_0 f = I_\beta(D^\beta f) = D^\beta(I_\beta f). \tag{1.17}$$

In the case of $0 < \beta < 1$ we have the following integral representation, for f a polynomial,

$$D^\beta f(x) = \frac{1}{c_\beta}\int_0^\infty t^{-\beta-1}(I - P_t)\, f(x)dt, \tag{1.18}$$

where $c_\beta = \int_0^\infty u^{-\beta-1}(1-e^{-u})du$.

Now, if $\beta \geq 1$, let k be the smallest integer greater than β i.e. $k-1 \leq \beta < k$, then the fractional derivative D^β can be represented as

$$D^\beta f = \frac{1}{c_\beta^k}\int_0^\infty t^{-\beta-1}(I - P_t)^k f\, dt, \tag{1.19}$$

where $c_\beta^k = \int_0^\infty u^{-\beta-1}(1-e^{-u})^k\, du$ and f a polynomial, see [11].

- We also define a Bessel fractional derivative $\mathcal{D}^\beta$, defined formally as $\mathcal{D}^\beta = (I + \sqrt{-L})^\beta$, which means that for the Hermite polynomials, we have

$$\mathcal{D}^\beta h_\nu(x) = (1 + \sqrt{|\nu|})^\beta h_\nu(x), \tag{1.20}$$

In the case of $0 < \beta < 1$ we have the following integral representation,

$$\mathcal{D}^\beta f = \frac{1}{c_\beta}\int_0^\infty t^{-\beta-1}(I - e^{-t}P_t)\, f dt, \tag{1.21}$$

where, as before, $c_\beta = \int_0^\infty u^{-\beta-1}(1-e^{-u})du$ and f a polynomial.

Moreover, if $\beta \geq 1$ let k be the smallest integer greater than β i.e. $k-1 \leq \beta < k$, we have the following representation of $\mathcal{D}^\beta f$

$$\mathcal{D}^\beta f = \frac{1}{c_\beta^k}\int_0^\infty t^{-\beta-1}(I - e^{-t}P_t)^k\, f\, dt, \tag{1.22}$$

where $c_\beta^k = \int_0^\infty u^{-\beta-1}(1-e^{-u})^k du$ and f a polynomial, see [11].

The Gaussian Triebel-Lizorkin $F_{p,q}^\alpha(\gamma_d)$ spaces were introduced in [10], see also [9], as follows

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Definition 1.1. *Let $\alpha \geq 0$, k be the smallest integer greater than α, and $1 \leq p, q < \infty$. The Gaussian Triebel-Lizorkin space $F^{\alpha}_{p,q}(\gamma_d)$ is the set of functions $f \in L^p(\gamma_d)$ for which*

$$\left\| \left(\int_0^{\infty} (t^{k-\alpha} \left| \frac{\partial^k P_t f}{\partial t^k} \right|)^q \frac{dt}{t} \right)^{1/q} \right\|_{p,\gamma_d} < \infty. \tag{1.23}$$

The norm of $f \in F^{\alpha}_{p,q}(\gamma_d)$ is defined as

$$\|f\|_{F^{\alpha}_{p,q}} := \|f\|_{p,\gamma_d} + \left\| \left(\int_0^{\infty} (t^{k-\alpha} \left| \frac{\partial^k P_t f}{\partial t^k} \right|)^q \frac{dt}{t} \right)^{1/q} \right\|_{p,\gamma_d}. \tag{1.24}$$

The definition of $F^{\alpha}_{p,q}(\gamma_d)$ does not depend on which $k > \alpha$ is chosen and the resulting norms are equivalent, for the proof of this result and other properties of these spaces see [10].

Observe that, by Minkowski's inequality, the Besov-Lipschitz space $B^{\alpha}_{p,q}(\gamma_d)$ defined in [7] is contained in $F^{\alpha}_{p,q}(\gamma_d)$. Thus the results obtained in this paper are more general that those in [7].

For Gaussian Triebel-Lizorkin spaces we have the following inclusion result,

Proposition 1.1. *The inclusion $F^{\alpha_1}_{p,q_1}(\gamma_d) \subset F^{\alpha_2}_{p,q_2}(\gamma_d)$ holds for $\alpha_1 > \alpha_2 > 0$ and $q_1 \geq q_2$.*

For the proof of this see [10], Proposition 2.2

In what follows we will need Hardy's inequalities, so for completeness we will write them here, see [12] page 272,

$$\int_0^{+\infty} \Big(\int_0^x f(y)dy \Big)^p x^{-r-1} dx \leq \frac{p}{r} \int_0^{+\infty} (yf(y))^p y^{-r-1} dy, \tag{1.25}$$

and

$$\int_0^{+\infty} \Big(\int_x^{\infty} f(y)dy \Big)^p x^{r-1} dx \leq \frac{p}{r} \int_0^{+\infty} (yf(y))^p y^{r-1} dy, \tag{1.26}$$

where $f \geq 0$, $p \geq 1$ and $r > 0$.

Finally, in [6] Gaussian Lipchitz spaces $Lip_{\alpha}(\gamma_d)$ were considered and the boundedness of Riesz Potentials, Bessel potentials and Fractional Derivatives on them and in [7] the boundedness of those operators were studied on Gaussian Besov-Lipschitz spaces $B^{\alpha}_{p,q}(\gamma_d)$. In the next section we are going to study the boundedness properties of those operators for Gaussian Triebel-Lizorkin spaces $F^{\alpha}_{p,q}(\gamma_d)$ which as we have already said contain those previous results. As usual in what follows C represents a constant that is not necessarily the same in each occurrence.

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2. Main results and Proofs

Theorem 2.1. *Let $\alpha \geq 0, \beta > 0$, $1 < p < \infty, 1 \leq q < \infty$ then I_β is bounded from $F^{\alpha}_{p,q}(\gamma_d)$ into $F^{\alpha+\beta}_{p,q}(\gamma_d)$.*

Proof.

Let $k > \alpha + \beta + 1$ an integer fixed and $f \in F^{\alpha}_{p,q}(\gamma_d)$. Using the integral representation of Riesz Potentials (1.13), the semigroup property of $\{P_t\}$ and the fact that $P_\infty f(x)$ is a constant, we get

$$\begin{aligned} P_t(I_\beta f)(x) &= \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} P_t(P_s f(x) - P_\infty f(x)) ds \\ &= \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} (P_{t+s} f(x) - P_\infty f(x)) ds. \end{aligned} \tag{2.1}$$

Then using again that $P_\infty f(x)$ is a constant and the chain rule,

$$\begin{aligned} \frac{\partial^k}{\partial t^k} P_t(I_\beta f)(x) &= \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} \frac{\partial^k}{\partial t^k} (P_{t+s} f(x) - P_\infty f(x)) ds \\ &= \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^{\beta-1} u^{(k)}(x, t+s) ds. \end{aligned} \tag{2.2}$$

i) Case $\beta \geq 1$: Using (2.2), the change of variables $r = t + s$, $dr = ds$ and Hardy's inequality (1.26), we have

$$\begin{aligned} &\Big(\int_0^{+\infty} \big(t^{k-(\alpha+\beta)} \big| \frac{\partial^k P_t(I_\beta f)(x)}{\partial t^k} \big| \big)^q \frac{dt}{t} \Big)^{1/q} \\ &\quad \leq \frac{1}{\Gamma(\beta)} \Big(\int_0^{+\infty} t^{q(k-(\alpha+\beta))} \Big(\int_0^{+\infty} s^{\beta-1} |u^{(k)}(x, t+s)| ds \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \\ &\quad = \frac{1}{\Gamma(\beta)} \Big(\int_0^{+\infty} t^{q(k-(\alpha+\beta))} \Big(\int_t^{+\infty} (r-t)^{\beta-1} |u^{(k)}(x, r)| dr \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \\ &\quad \leq \frac{1}{\Gamma(\beta)} \Big(\int_0^{+\infty} t^{q(k-(\alpha+\beta))} \Big(\int_t^{+\infty} r^{\beta-1} |u^{(k)}(x, r)| dr \Big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \\ &\quad \leq \frac{1}{\Gamma(\beta)} \frac{1}{(k - (\alpha+\beta))^{1/q}} \Big(\int_0^{+\infty} \big(r^{k-\alpha} |u^{(k)}(x, r)| \big)^q \frac{dr}{r} \Big)^{\frac{1}{q}}, \end{aligned}$$

and therefore

$$\begin{aligned} &\| \Big(\int_0^{+\infty} \big(t^{k-(\alpha+\beta)} \big| \frac{\partial^k P_t(I_\beta f)}{\partial t^k} \big| \big)^q \frac{dt}{t} \Big)^{\frac{1}{q}} \|_{p,\gamma} \\ &\quad \leq C_{k,\alpha,\beta} \| \Big(\int_0^{+\infty} \big(r^{k-\alpha} \big| \frac{\partial^k P_r f}{\partial r^k} \big| \big)^q \frac{dr}{r} \Big)^{\frac{1}{q}} \|_{p,\gamma} < \infty, \end{aligned}$$

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since $f \in F^{\alpha}_{p,q}$. By Observation 1.1 and the previous estimate,

$$\|I_\beta f\|_{F^{\alpha+\beta}_{p,q}} \leq C\|f\|_{F^{\alpha}_{p,q}}.$$

ii) Case $0 < \beta < 1$: Again using (2.2),

$$\begin{aligned}
&\Big(\int_0^{+\infty} \big(t^{k-(\alpha+\beta)}|\frac{\partial^k P_t(I_\beta f)(x)}{\partial t^k}|\big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \\
&\qquad \leq \frac{1}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{q(k-(\alpha+\beta))}\Big(\int_0^{+\infty} s^{\beta}|u^{(k)}(x,t+s)|\frac{ds}{s}\Big)^q \frac{dt}{t}\Big)^{\frac{1}{q}} \\
&\qquad \leq \frac{C}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{q(k-(\alpha+\beta))-1}\Big(\int_0^{t} s^{\beta-1}|u^{(k)}(x,t+s)|ds\Big)^q dt\Big)^{\frac{1}{q}} \\
&\qquad\quad + \frac{C}{\Gamma(\beta)}\Big(\int_0^{+\infty} t^{q(k-(\alpha+\beta))-1}\Big(\int_t^{+\infty} s^{\beta-1}|u^{(k)}(x,t+s)|ds\Big)^q dt\Big)^{\frac{1}{q}} \\
&\qquad = (I) + (II).
\end{aligned}$$

Writing $s^{\beta-1} = s^{\frac{\beta-1}{q}+\frac{\beta-1}{q'}}$, $\frac{1}{q}+\frac{1}{q'}=1$, and using Hölder's inequality in the internal integral,

$$(I) \quad \leq \quad \frac{C_\beta}{\beta^{\frac{q-1}{q}}}\Big(\int_0^{+\infty} t^{q(k-\alpha)-\beta-1}\int_0^{t} s^{\beta-1}|u^{(k)}(x,t+s)|^q ds\,dt\Big)^{1/q}.$$

By Fubini- Tonelli's theorem and using that $q(k-\alpha)-\beta-1>0$, as $k > \alpha+\beta+1$, we get

$$\begin{aligned}
(I) \quad &\leq \quad \frac{C_\beta}{\beta^{\frac{q-1}{q}}}\Big(\int_0^{+\infty} s^{\beta-1}\int_s^{+\infty} t^{q(k-\alpha)-\beta-1}|u^{(k)}(x,t+s)|^q dt\,ds\Big)^{1/q} \\
&\leq \quad \frac{C_\beta}{\beta^{\frac{q-1}{q}}}\Big(\int_0^{+\infty} s^{\beta-1}\int_s^{+\infty} (t+s)^{q(k-\alpha)-\beta-1}|u^{(k)}(x,t+s)|^q dt\,ds\Big)^{1/q}.
\end{aligned}$$

Then, by the change of variables $r = t+s$ and Hardy's inequality (1.26),

$$\begin{aligned}
(I) \quad &\leq \quad \frac{C_\beta}{\beta^{\frac{q-1}{q}}}\Big(\int_0^{+\infty} s^{\beta-1}\int_{2s}^{+\infty} r^{q(k-\alpha)-\beta-1}|u^{(k)}(x,r)|^q dr\,ds\Big)^{1/q} \\
&\leq \quad \frac{C_\beta}{\beta^{\frac{q-1}{q}}}\Big(\int_0^{+\infty} s^{\beta-1}\int_{s}^{+\infty} r^{q(k-\alpha)-\beta-1}|u^{(k)}(x,r)|^q dr\,ds\Big)^{1/q} \\
&\leq \quad \frac{C_\beta}{\beta}\Big(\int_0^{+\infty} \big(r^{k-\alpha}|u^{(k)}(x,r)|\big)^q \frac{dr}{r}\Big)^{1/q}.
\end{aligned}$$

On the other hand, since $\beta < 1$ then $t < s$ implies that $s^{\beta-1} < t^{\beta-1}$ and by the change of variables $r = t+s$ and Hardy's inequality

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(1.26), as $k > \alpha + \beta + 1 > \alpha + 1$, we obtain

$$\begin{aligned}(II) &\leq C_\beta\Big(\int_0^{+\infty} t^{q(k-\alpha-1)-1}\Big(\int_t^{+\infty}|u^{(k)}(x,t+s)|ds\Big)^q dt\Big)^{\frac{1}{q}}\\ &\leq C_\beta\Big(\int_0^{+\infty} t^{q(k-\alpha-1)-1}\Big(\int_{2t}^{+\infty}|u^{(k)}(x,r)|dr\Big)^q dt\Big)^{\frac{1}{q}}\\ &\leq C_\beta\frac{1}{(k-\alpha-1)^{1/q}}\Big(\int_0^{+\infty}\Big(r^{k-\alpha}|\frac{\partial^k P_r f(x)}{\partial r^k}|\Big)^q\frac{dr}{r}\Big)^{\frac{1}{q}}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|\Big(\int_0^{+\infty}\Big(t^{k-(\alpha+\beta)}|\frac{\partial^k P_t(I_\beta f)(x)}{\partial t^k}|\Big)^q\frac{dt}{t}\Big)^{\frac{1}{q}}\|_{p,\gamma}\\ &\qquad\leq C_{k,\alpha,\beta}\|\Big(\int_0^{+\infty}\Big(r^{k-\alpha}|\frac{\partial^k P_r f}{\partial r^k}|\Big)^q\frac{dr}{r}\Big)^{\frac{1}{q}}\|_{p,\gamma}<\infty.\end{aligned}$$

as $f \in F^\alpha_{p,q}$. By Observation 1.1 and the previous estimate, we get

$$\|I_\beta f\|_{F^{\alpha+\beta}_{p,q}} \leq C\|f\|_{F^\alpha_{p,q}}.$$

□

Next, we want to study the boundedness properties of the Bessel potentials on Triebel-Lizorkin spaces. In [10], Theorem 4, the following result was proved, for completeness the proof will be given here too.

Theorem 2.2. *Let $\alpha \geq 0, \beta > 0$ then for $1 < p < \infty$, $1 \leq q < \infty$ $\mathcal{J}_\beta$ is bounded from $F^\alpha_{p,q}(\gamma_d)$ into $F^{\alpha+\beta}_{p,q}(\gamma_d)$.*

Proof.

Let $k > \alpha + \beta + 1$ a fixed integer and $f \in F^\alpha_{p,q}(\gamma_d)$. Observe that

$$\begin{aligned}|\frac{\partial^k P_t}{\partial t^k}(\mathcal{J}_\beta f)(x)| &\leq \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^\beta e^{-s}|u^{(k)}(x,t+s)|\frac{ds}{s}\\ &\leq \frac{1}{\Gamma(\beta)}\int_0^{+\infty} s^{\beta-1}|u^{(k)}(x,t+s)|ds,\end{aligned}$$

which is like (2.2) of Theorem 2.1. Thus the conclusion follows from the proof of that theorem. □

We will study next the boundedness of the Riesz fractional derivative D^β on Triebel-Lizorkin spaces.

Theorem 2.3. *Let $1 \leq p, q < \infty$, for $0 < \beta < \alpha < 1$, D^β is bounded from $F^\alpha_{p,q}(\gamma_d)$ into $F^{\alpha-\beta}_{p,q}(\gamma_d)$.*

Proof.

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Let $f \in F^{\alpha}_{p,q}(\gamma_d)$, using the Fundamental Theorem of Calculus, and Hardy's inequality (1.25) with $p = 1$,

$$\begin{aligned} |D^{\beta} f(x)| &\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |P_s f(x) - f(x)| ds \\ &\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^s |\frac{\partial}{\partial r} P_r f(x)| dr\, ds \\ &\leq \frac{1}{c_\beta \beta} \int_0^{+\infty} r^{1-\beta} |\frac{\partial}{\partial r} P_r f(x)| \frac{dr}{r}. \end{aligned} \tag{2.3}$$

Thus,

$$\begin{aligned} \|D^{\beta} f\|_{p,\gamma} &\leq C_\beta \| \int_0^{+\infty} r^{1-\beta} |\frac{\partial}{\partial r} P_r f| \frac{dr}{r} \|_{p,\gamma} \\ &\leq C_\beta \|f\|_{F^{\alpha}_{p,q}} < \infty, \end{aligned} \tag{2.4}$$

because $F^{\alpha}_{p,q}(\gamma_d) \subset F^{\beta}_{p,1}(\gamma_d)$ ($\alpha > \beta$ and $q \geq 1$). Now, by analogous argument using Hardy's inequality (1.25) with $p = 1$,

$$\begin{aligned} |\frac{\partial}{\partial t} P_t (D^{\beta} f)(x)| &\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |\frac{\partial}{\partial t} P_{t+s} f(x) - \frac{\partial}{\partial t} P_t f(x)| ds \\ &\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^s |u^{(2)}(x, t+r)| dr\, ds \\ &\leq \frac{1}{c_\beta \beta} \int_0^{+\infty} r^{-\beta} |u^{(2)}(x, t+r)| dr \end{aligned}$$

This implies that

$$\begin{aligned} \int_0^{\infty} \big(t^{1-(\alpha-\beta)} |\frac{\partial}{\partial t} P_t (D_\beta f)(x)|\big)^q \frac{dt}{t} &\leq \frac{1}{c_\beta \beta} \int_0^{\infty} \big(t^{1-(\alpha-\beta)} \int_0^{+\infty} r^{-\beta} |u^{(2)}(x, t+r)| dr\big)^q \frac{dt}{t} \\ &\leq C_\beta \int_0^{\infty} \big(t^{1-(\alpha-\beta)} \int_0^{t} r^{-\beta} |u^{(2)}(x, t+r)| dr\big)^q \frac{dt}{t} \\ &\quad + C_\beta \int_0^{\infty} \big(t^{1-(\alpha-\beta)} \int_t^{+\infty} r^{-\beta} |u^{(2)}(x, t+r)| dr\big)^q \frac{dt}{t} \\ &= (I) + (II). \end{aligned} \tag{2.5}$$

Writing $r^{-\beta} = r^{\frac{-\beta}{q} + \frac{-\beta}{q'}}$, $\frac{1}{q} + \frac{1}{q'} = 1$, and using Hölder's inequality in the internal integral, we have

$$(I) \leq C_\beta (1-\beta)^{1-q} \int_0^{\infty} t^{(2-\alpha)q-2+\beta} \int_0^t r^{-\beta} |u^{(2)}(x, t+r)|^q dr\, dt,$$

then by Fubini-Tonelli's theorem, we get

$$(I) \leq C_\beta (1-\beta)^{1-q} \int_0^{\infty} r^{-\beta} \int_r^{\infty} t^{(2-\alpha)q+\beta-2} |u^{(2)}(x, t+r)|^q dt\, dr.$$

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It is easy to prove that $(2-\alpha)q+\beta-2>-1$. We need to study two cases: Case #1: If $(2-\alpha)q+\beta-2<0$: as $r<t$ and by the change of variable $w=t+r$, we have

$$\begin{aligned}(I) &\leq C_\beta(1-\beta)^{1-q}\int_0^\infty r^{(2-\alpha)q-2}\int_r^\infty |u^{(2)}(x,t+r)|^q dt\,dr\\ &\leq C_\beta(1-\beta)^{1-q}\int_0^\infty r^{[(2-\alpha)q-1]-1}\int_{2r}^\infty |u^{(2)}(x,w)|^q dw\,dr\\ &\leq C_\beta(1-\beta)^{1-q}\int_0^\infty r^{[(2-\alpha)q-1]-1}\int_r^\infty |u^{(2)}(x,w)|^q dw\,dr.\end{aligned}$$

Then by Hardy's inequality (1.26) as $(2-\alpha)q-1>0$

$$(I) \leq C_\beta(1-\beta)^{1-q}\frac{1}{(2-\alpha)q-1}\int_0^\infty \big(w^{2-\alpha}|u^{(2)}(x,w)|\big)^q\frac{dw}{w}.$$

Case #2: If $(2-\alpha)q+\beta-2\geq 0$: By the change of variable $w=t+r$, we get

$$\begin{aligned}(I) &\leq C_\beta(1-\beta)^{1-q}\int_0^\infty r^{-\beta}\int_r^\infty (t+r)^{(2-\alpha)q+\beta-2}|u^{(2)}(x,t+r)|^q dt\,dr\\ &= C_\beta(1-\beta)^{1-q}\int_0^\infty r^{-\beta}\int_{2r}^\infty w^{(2-\alpha)q+\beta-2}|u^{(2)}(x,w)|^q dw\,dr\\ &\leq C_\beta(1-\beta)^{1-q}\int_0^\infty r^{-\beta}\int_r^\infty w^{(2-\alpha)q+\beta-2}|u^{(2)}(x,w)|^q dw\,dr,\end{aligned}$$

and by Hardy's inequality (1.26),

$$(I) \leq \frac{C_\beta}{(1-\beta)^q}\int_0^\infty \big(w^{2-\alpha}|u^{(2)}(x,w)|\big)^q\frac{dw}{w}.$$

Therefore, in both cases we have

$$(I) \leq C_\beta\int_0^\infty \big(w^{2-\alpha}|u^{(2)}(x,w)|\big)^q\frac{dw}{w}.$$

To estimate (II) observe that $r^{-\beta}<t^{-\beta}$, for $r>t$ and $\beta>0$, then we use the same argument used before to estimate (I) case #1, doing the change of variables $w=t+r$, and using Hardy's inequality (1.26) so that

$$(II) \leq \frac{C_\beta}{1-\alpha}\int_0^\infty \big(w^{2-\alpha}|u^{(2)}(x,w)|\big)^q\frac{dw}{w}.$$

Finally,

$$\begin{aligned}&\|\Big(\int_0^\infty \big(t^{1-(\alpha-\beta)}|\frac{\partial}{\partial t}P_t(D^\beta f)|\big)^q\frac{dt}{t}\Big)^{1/q}\|_{p,\gamma}\\ &\leq C\|\Big(\int_0^\infty \big(t^{1-(\alpha-\beta)}\int_0^{+\infty} r^{-\beta}|u^{(2)}(\cdot,t+r)|dr\big)^q\frac{dt}{t}\Big)^{1/q}\|_{p,\gamma}\\ &\leq C\|\Big(\int_0^\infty \big(w^{2-\alpha}|u^{(2)}(\cdot,w)|\big)^q\frac{dw}{w}\Big)^{1/q}\|_{p,\gamma}<\infty,\end{aligned}\tag{2.6}$$

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as $f \in F^{\alpha}_{p,q}(\gamma_d)$. By the previous estimate and (2.4)

$$\|D^{\beta} f\|_{F^{\alpha-\beta}_{p,q}} \leq C \|f\|_{F^{\alpha}_{p,q}}.$$

□

In the following theorem we will study the boundedness of the Bessel fractional derivative on Triebel-Lizorkin spaces, for the case $0 < \beta < \alpha < 1$.

Theorem 2.4. *Let $0 < \beta < \alpha < 1$, $1 \leq p, q < \infty$ then $\mathcal{D}^{\beta}$ is bounded from $F^{\alpha}_{p,q}(\gamma_d)$ into $F^{\alpha-\beta}_{p,q}(\gamma_d)$.*

Proof.

Let $f \in L^p(\gamma_d)$, using the Fundamental Theorem of Calculus we can write,

$$\begin{aligned}
|\mathcal{D}^{\beta} f(x)| &\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |e^{-s} P_s f(x) - f(x)| ds \\
&\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} e^{-s} |P_s f(x) - f(x)| ds + \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |e^{-s} - 1| |f(x)| ds \\
&\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^{s} |\frac{\partial}{\partial r} P_r f(x)| dr\, ds + \frac{1}{c_\beta} |f(x)| \int_0^{+\infty} s^{-\beta-1} \int_0^{s} e^{-r} dr\, ds.
\end{aligned}$$

Now, using Hardy's inequality (1.25), with $p = 1$, in both integrals, we have

$$|\mathcal{D}^{\beta} f(x)| \leq \frac{1}{\beta c_\beta} \int_0^{+\infty} r^{1-\beta} |\frac{\partial}{\partial r} P_r f(x)| \frac{dr}{r} + \frac{1}{\beta c_\beta} \Gamma(1-\beta) |f(x)|.$$

Thus,

$$|\mathcal{D}^{\beta} f(x)| \leq \frac{1}{\beta c_\beta} \int_0^{+\infty} r^{1-\beta} |u^{(1)}(x, r)| \frac{dr}{r} + \frac{1}{\beta c_\beta} \Gamma(1-\beta) |f(x)|.$$

Therefore, if $f \in F^{\alpha}_{p,q}(\gamma_d)$, we get

$$\begin{aligned}
\|\mathcal{D}^{\beta} f\|_{p,\gamma} &\leq \frac{1}{\beta c_\beta} \| \int_0^{+\infty} r^{1-\beta} |u^{(1)}(\cdot, r)| \frac{dr}{r} \|_{p,\gamma} + \frac{1}{\beta c_\beta} \Gamma(1-\beta) \|f\|_{p,\gamma} \\
&\leq C_\beta \|f\|_{F^{\beta}_{p,1}} \leq C'_\beta \|f\|_{F^{\alpha}_{p,q}}, \qquad (2.7)
\end{aligned}$$

since $F^{\alpha}_{p,q}(\gamma_d) \subset F^{\beta}_{p,1}(\gamma_d)$, as $\alpha > \beta$, and $q \geq 1$.

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On the other hand, using a similar argument as above (the Fundamental Theorem of Calculus and Hardy's inequality (1.25) with $p=1$), we have,

$$\begin{aligned}
|\frac{\partial}{\partial t}P_t(\mathcal{D}^\beta f)(x)| &\leq \frac{1}{c_\beta}\int_0^\infty s^{-\beta-1}|e^{-s}\frac{\partial}{\partial t}P_{t+s}f(x)-\frac{\partial}{\partial t}P_tf(x)|ds\\
&\leq \frac{1}{c_\beta}\int_0^\infty s^{-\beta-1}e^{-s}|\frac{\partial}{\partial t}P_{t+s}f(x)-\frac{\partial}{\partial t}P_tf(x)|ds\\
&\quad +\frac{1}{c_\beta}\int_0^\infty s^{-\beta-1}|e^{-s}-1||\frac{\partial}{\partial t}P_tf(x)|ds\\
&\leq \frac{1}{c_\beta}\int_0^\infty s^{-\beta-1}\int_0^s |u^{(2)}(x,t+r)|dr\,ds\\
&\quad +\frac{1}{c_\beta}|u^{(1)}(x,t)|\int_0^\infty s^{-\beta-1}\int_0^s e^{-r}dr\,ds,\\
&\leq \frac{1}{\beta c_\beta}\int_0^\infty r^{-\beta}|u^{(2)}(x,t+r)|dr+\frac{\Gamma(1-\beta)}{\beta c_\beta}|u^{(1)}(x,t)|.
\end{aligned}$$

Therefore,

$$|\frac{\partial}{\partial t}P_t(\mathcal{D}^\beta f(x))|\leq \frac{1}{\beta c_\beta}\int_0^\infty r^{-\beta}|u^{(2)}(x,t+r)|dr+\frac{\Gamma(1-\beta)}{\beta c_\beta}|u^{(1)}(x,t)|,$$

and then, we have

$$\begin{aligned}
&\|\Big(\int_0^\infty \big(t^{1-(\alpha-\beta)}|\frac{\partial}{\partial t}P_t(\mathcal{D}^\beta f)|\big)^q\frac{dt}{t}\Big)^{1/q}\|_{p,\gamma}\\
&\leq \frac{C}{\beta c_\beta}\|\Big(\int_0^\infty \big(t^{1-(\alpha-\beta)}\int_0^\infty r^{-\beta}|u^{(2)}(\cdot,t+r)|dr\big)^q\frac{dt}{t}\Big)^{1/q}\|_{p,\gamma}\\
&\qquad +\frac{C}{\beta c_\beta}\Gamma(1-\beta)\|\Big(\int_0^\infty \big(t^{1-(\alpha-\beta)}|u^{(1)}(\cdot,t)|\big)^q\frac{dt}{t}\Big)^{1/q}\|_{p,\gamma}.
\end{aligned}$$

Now the first term can be estimated as in the proof of Theorem 3, estimates (2.5) and (2.6) so that,

$$\|\Big(\int_0^\infty \big(t^{1-(\alpha-\beta)}\int_0^\infty r^{-\beta}|u^{(2)}(\cdot,t+r)|dr\big)^q\frac{dt}{t}\Big)^{1/q}\|_{p,\gamma} \leq C\|\int_0^\infty \big(t^{2-\alpha}|\frac{\partial^2}{\partial t^2}P_tf|\big)^q\frac{dt}{t}\Big)^{1/q}\|_{p,\gamma}$$

which is finite as $f\in F^\alpha_{p,q}(\gamma_d)$, and for the second term, we have

$$\|\Big(\int_0^\infty \big(t^{1-(\alpha-\beta)}|u^{(1)}(x,t)|\big)^q\frac{dt}{t}\Big)^{1/q}\|_{p,\gamma} \leq C\|f\|_{F^{\alpha-\beta}_{p,q}}\leq C\|f\|_{F^\alpha_{p,q}}$$

as $F^\alpha_{p,q}(\gamma_d)\subset F^{\alpha-\beta}_{p,q}(\gamma_d)$, thus,

$$\|\Big(\int_0^\infty \big(t^{1-(\alpha-\beta)}|\frac{\partial}{\partial t}P_t(\mathcal{D}^\beta f)|\big)^q\frac{dt}{t}\Big)^{1/q}\|_{p,\gamma} \leq C\|f\|_{F^\alpha_{p,q}}.$$

Therefore, $\mathcal{D}^\beta f\in F^{\alpha-\beta}_{p,q}(\gamma_d)$ and moreover, by previous estimate and (2.7)

$$\|\mathcal{D}^\beta f\|_{F^{\alpha-\beta}_{p,q}} \leq C\|f\|_{F^\alpha_{p,q}}.$$

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□

To study the general case for fractional derivatives (removing the condition that the indexes must be less than 1), we need to consider forward differences. For a given function f, the k-th order forward difference of f starting at t with increment s is defined as,

$$\Delta_s^k(f,t) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j f(t + (k-j)s).$$

We will need the following technical results

Lemma 2.1. *For any positive integer k*

i) $\Delta_s^k(f,t) = \Delta_s^{k-1}(\Delta_s(f,\cdot),t) = \Delta_s(\Delta_s^{k-1}(f,\cdot),t)$

ii) $\Delta_s^k(f,t) = \int_t^{t+s} \int_{v_1}^{v_1+s} ... \int_{v_{k-2}}^{v_{k-2}+s} \int_{v_{k-1}}^{v_{k-1}+s} f^{(k)}(v_k) dv_k dv_{k-1} ... dv_2 dv_1$

iii) *For any positive integer k,*

$$\frac{\partial}{\partial s}(\Delta_s^k(f,t)) = k\, \Delta_s^{k-1}(f', t+s), \tag{2.8}$$

and for any integer $j > 0$,

$$\frac{\partial^j}{\partial t^j}(\Delta_s^k(f,t)) = \Delta_s^k(f^{(j)}, t). \tag{2.9}$$

The proof of this known lemma can also be found in an appendix in [6].

Lemma 2.2. *(Hardy's type innequality) Let $f \geq 0, r > 0, p \geq 1$ and $k \in \mathbb{N}$ then*

$$\begin{aligned} &\Big(\int_0^{+\infty} \Big(\int_0^x ... \int_0^x f(r_1,...,r_k)\, dr_1 ... dr_k\Big)\Big)^p x^{-r-1} dx\Big)^{1/p} \\ &\leq \int_0^1 ... \int_0^1 \Big(\int_0^{+\infty} (x^k f(xv_1,...,xv_k))^p x^{-r-1} dx\Big)^{1/p} dv_1 ... dv_k \end{aligned}$$

Proof. Taking $r_1 = xv_1, ..., r_k = xv_k$, we get

$$\begin{aligned} &\Big(\int_0^{+\infty} \Big(\int_0^x ... \int_0^x f(r_1,...,r_k) dr_1 ... dr_k)\Big)^p x^{-r-1} dx\Big)^{1/p} \\ = &\Big(\int_0^{+\infty} \Big(\int_0^1 ... \int_0^1 f(xv_1,...,xv_k) x^k dv_1 ... dv_k)\Big)^p x^{-r-1} dx\Big)^{1/p} \\ = &\Big(\int_0^{+\infty} \Big(\int_{(0,1)^k} f(xv_1,...,xv_k) x^k dv_1 ... dv_k)\Big)^p x^{-r-1} dx\Big)^{1/p} \end{aligned}$$

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Now, consider the spaces $L^p((0,+\infty), x^{-r-1})$ and $L^p((0,1)^k)$; then by Minkowski's inequality,

$$\begin{aligned}
&\Big(\int_0^{+\infty}\Big(\int_{(0,1)^k} f(xv_1,...,xv_k)x^k dv_1...dv_k)\Big)^p x^{-r-1}dx\Big)^{1/p}\\
\leq\ &\int_{(0,1)^k}\Big(\int_0^{+\infty}\big(f(xv_1,...,xv_k)x^k\big)^p x^{-r-1}dx\Big)^{1/p}dv_1...dv_k\\
=\ &\int_0^1...\int_0^1\Big(\int_0^{+\infty}\big(x^k f(xv_1,...,xv_k)\big)^p x^{-r-1}dx\Big)^{1/p}dv_1...dv_k.
\end{aligned}$$

□

Lemma 2.3. *For any positive integer k,*

$$\begin{aligned}
\Delta_s^k(f,t) &= \int_t^{t+s}\int_{v_1}^{v_1+s}...\int_{v_{k-1}}^{v_{k-1}+s} f^{(k)}(v_k)dv_k...dv_2dv_1\\
&= \int_0^s...\int_0^s f^{(k)}(t+v_1+...+v_k)dv_k...dv_1
\end{aligned}$$

For the proof of this result see [6] Lemma 3.1, ii).

Lemma 2.4. *Let $t\geq 0, \beta>0$ and let k be the smallest integer greater than β and let f differentiable up to order k, then*

$$\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(f,t)|ds \leq C_{\beta,k}\int_0^{+\infty} w^{k-\beta-1}|f^{(k)}(t+w)|dw$$

where $C_{\beta,k}=\int_0^1...\int_0^1 (v_1+...+v_k)^{\beta-k}dv_1...dv_k$

Proof. From Lemmas 2.2 with $p=1$ and 2.3 we have,

$$\begin{aligned}
\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(f,t)|ds &\leq \int_0^{+\infty} s^{-\beta-1}\int_0^s...\int_0^s |f^{(k)}(t+v_1+...+v_k)|dv_1...dv_kds\\
&\leq \int_0^1...\int_0^1\Big(\int_0^{+\infty}(s^k|f^{(k)}(t+s(v_1+...+v_k)|)s^{-\beta-1}ds\Big)dv_1...dv_k\\
&= \int_0^1...\int_0^1\Big(\int_0^{+\infty}(s^{k-\beta-1}|f^{(k)}(t+s(v_1+...+v_k)|)ds\Big)dv_1...dv_k
\end{aligned}$$

taking $r=s(v_1+...+v_k)$ then $dr=(v_1+...+v_k)ds$,

$$\begin{aligned}
\int_0^{+\infty} s^{k-\beta-1}|f^{(k)}(t+s(v_1+...+v_k)|ds &= \int_0^{+\infty} r^{k-\beta}|f^{(k)}(t+r)|\frac{dr}{r}(v_1+...+v_k)^{\beta-k}\\
&= \int_0^{+\infty} r^{k-\beta-1}|f^{(k)}(t+r)|dr(v_1+...+v_k)^{\beta-k}.
\end{aligned}$$

Therefore,

$$\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(f,t)|ds$$

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$$\begin{aligned}
&\leq \int_0^1 ... \int_0^1 \big(\int_0^{+\infty} r^{k-\beta-1} |f^{(k)}(t+r)| dr (v_1 + ... + v_k)^{\beta-k} \big) dv_1 ... dv_k \\
&= \big(\int_0^{+\infty} r^{k-\beta-1} |f^{(k)}(t+r)| dr \big) \int_0^1 ... \int_0^1 (v_1 + ... + v_k)^{\beta-k} dv_1 ... dv_k \\
&= C_{\beta,k} \big(\int_0^{+\infty} r^{k-\beta-1} |f^{(k)}(t+r)| dr,
\end{aligned}$$

where $C_{\beta,k} = \int_0^1 ... \int_0^1 (v_1 + ... + v_k)^{\beta-k} dv_1 ... dv_k < \infty$. □

Observation 2.1. *Using the Binomial Theorem and the semigroup property of $\{P_s\}$, we have*

$$\begin{aligned}
(P_s - I)^k f(x) &= \sum_{j=0}^{k} \binom{k}{j} P_s^{k-j} (-I)^j f(x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j P_s^{k-j} f(x) \\
&= \sum_{j=0}^{k} \binom{k}{j} (-1)^j P_{(k-j)s} f(x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j u(x, (k-j)s) \\
&= \Delta_s^k (u(x, \cdot), 0), \qquad (2.10)
\end{aligned}$$

where as usual, $u(x,s) = P_s f(x)$ and furthermore,

$$P_t (P_s - I)^k f(x) = \Delta_s^k (u(x, \cdot), t). \qquad (2.11)$$

The following result extends Theorem 2.3 to the general case $0 < \beta < \alpha$,

Theorem 2.5. *Let $0 < \beta < \alpha$, $1 \leq p, q < \infty$ then D^β is bounded from $F_{p,q}^{\alpha}(\gamma_d)$ into $F_{p,q}^{\alpha-\beta}(\gamma_d)$.*

Proof. Let $f \in F_{p,q}^{\alpha}(\gamma_d)$, using Observation 2.1 and Lemma 2.4

$$\begin{aligned}
|D_\beta f(x)| &\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |(P_s - I)^k f(x)| ds \\
&= \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |\Delta_s^k (u(x, \cdot), 0)| ds \\
&\leq C_{\beta,k} \int_0^{+\infty} r^{k-\beta-1} |u^{(k)}(x, r)| dr,
\end{aligned}$$

then

$$\|D_\beta f\|_{p,\gamma} \leq C_{\beta,k} \| \int_0^{+\infty} r^{k-\beta} |u^{(k)}(\cdot, r)| \frac{dr}{r} \|_{p,\gamma} < \infty,$$

since $F_{p,q}^{\alpha}(\gamma_d) \subset F_{p,1}^{\beta}(\gamma_d)$, ($\alpha > \beta$ and $1 \leq q < \infty$).

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On the other hand, let $n \in \mathbb{N}, n > \alpha$; by Observation 2.1, Lemma 2.1 (2.9) and Lemma 2.4, we get,

$$\begin{aligned} |\frac{\partial^n}{\partial t^n} P_t(D_\beta f)(x)| &\leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(u^{(n)}(x,\cdot),t)|ds \\ &\leq \frac{1}{c_\beta} C_{\beta,k}\int_0^{+\infty} r^{k-\beta-1}|u^{(n+k)}(x,t+r)|dr. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_0^{\infty} \big(t^{n-(\alpha-\beta)}|\frac{\partial^n}{\partial t^n} P_t(D_\beta f)(x)|\big)^q \frac{dt}{t} \\ &\quad \leq C_{\beta,k}\int_0^{\infty} \big(t^{n-(\alpha-\beta)}\int_0^{+\infty} r^{k-\beta-1}|u^{(n+k)}(x,t+r)|dr\big)^q \frac{dt}{t} \end{aligned}$$

which is inequality (2.5) for $n = k = 1$. The rest of the proof follows the argument used in Theorem 2.3, so that

$$\begin{aligned} (\int_0^{\infty} \big(t^{n-(\alpha-\beta)}\int_0^{+\infty} r^{k-\beta-1}|u^{(n+k)}(x,t+r)|dr\big)^q \frac{dt}{t})^{1/q} \\ \leq C\big(\int_0^{\infty} \big(s^{n+k-\alpha}|u^{(n+k)}(x,s)|\big)^q \frac{ds}{s}\big)^{1/q}. \end{aligned} \tag{2.12}$$

Taking $L^p(\gamma)$-norm both side of the inequality we get the result. □

Finally, the following result extends Theorem 2.4 to the general case $0 < \beta < \alpha$,

Theorem 2.6. *Let $0 < \beta < \alpha$, $1 < p < \infty$ and $1 \leq q < \infty$, then $\mathcal{D}^\beta$ is bounded from $F_{p,q}^{\alpha}(\gamma_d)$ into $F_{p,q}^{\alpha-\beta}(\gamma_d)$.*

Proof. Let $f \in F_{p,q}^{\alpha}(\gamma_d)$, k be an integer such that $k-1 \leq \beta < k$ and $v(x,r) = e^{-r}u(x,r)$, using Lemma 2.4 and Leibnitz's differentiation rule for the product

$$\begin{aligned} |\mathcal{D}^\beta f(x)| &\leq \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|(e^{-s}P_s - I)^k f(x)|ds = \frac{1}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|\Delta_s^k(v(x,\cdot),0)|ds \\ &\leq C_{\beta,k}\int_0^{+\infty} r^{k-\beta}|v^{(k)}(x,r)|\frac{dr}{r} \leq C_{\beta,k}\big(\sum_{j=0}^{k}\binom{k}{j}\int_0^{+\infty} r^{k-\beta}e^{-r}|u^{(k-j)}(x,r)|\frac{dr}{r}\big) \\ &= C_{\beta,k}\big(\sum_{j=0}^{k-1}\binom{k}{j}\int_0^{+\infty} r^{k-\beta}e^{-r}|u^{(k-j)}(x,r)|\frac{dr}{r}\big) + C_{\beta,k}\int_0^{+\infty} r^{k-\beta}e^{-r}|u(x,r)|\frac{dr}{r}, \end{aligned}$$

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then

$$\begin{aligned}
\|\mathcal{D}^\beta f\|_{p,\gamma} &\leq C_{\beta,k}\Big(\sum_{j=0}^{k-1}\binom{k}{j}\|\int_0^{+\infty} r^{k-\beta}|u^{(k-j)}(\cdot,r)|\frac{dr}{r}\|_{p,\gamma}\Big) + C_{\beta,k}\|\int_0^{+\infty} r^{k-\beta}e^{-r}|u(\cdot,r)|\frac{dr}{r}\|_{p,\gamma} \\
&\leq C_{\beta,k}\Big(\sum_{j=0}^{k-1}\binom{k}{j}\|\int_0^{+\infty} r^{k-\beta}|u^{(k-j)}(\cdot,r)|\frac{dr}{r}\|_{p,\gamma}\Big) + C_{\beta,k}\int_0^{+\infty} r^{k-\beta}e^{-r}\|u(\cdot,r)\|_{p,\gamma}\frac{dr}{r} \\
&\leq C_{\beta,k}\Big(\sum_{j=0}^{k-1}\binom{k}{j}\|\int_0^{+\infty} r^{k-j-(\beta-j)}|u^{(k-j)}(\cdot,r)|\frac{dr}{r}\|_{p,\gamma}\Big) + C_{\beta,k}\|f\|_{p,\gamma}\Gamma(k-\beta) \\
&\leq C\|f\|_{F^\alpha_{p,q}},
\end{aligned}$$

because $F^\alpha_{p,q}(\gamma_d) \subset F^{\beta-j}_{p,1}(\gamma_d)$, as $\alpha > \beta \geq \beta - j \geq 0$, for $j = 0, ..., k-1$ and $q \geq 1$.

On the other hand, let $n \in \mathbb{N}, n > \alpha$ and $w(x,t) = e^{-t}u^{(n)}(x,t)$, by Lemma 2.4 we get

$$\begin{aligned}
|\frac{\partial^n}{\partial t^n}P_t(\mathcal{D}^\beta f)(x)| &\leq \frac{e^t}{c_\beta}\int_0^{+\infty} s^{-\beta-1}|\Delta^k_s(w(x,\cdot),t)|ds \\
&\leq e^t C_{\beta,k}\int_0^{+\infty} s^{k-\beta-1}|w^{(k)}(x,t+s)|ds.
\end{aligned}$$

Now, by Leibnitz rule, $w^{(k)}(x,r) = \sum_{j=0}^{k}\binom{k}{j}(-1)^j e^{-r}u^{(k+n-j)}(x,r)$ and then

$$|w^{(k)}(x,r)| \leq \sum_{j=0}^{k}\binom{k}{j}e^{-r}|u^{(k+n-j)}(x,r)|,$$

for all $r > 0$. Thus

$$|\frac{\partial^n}{\partial t^n}P_t(\mathcal{D}^\beta f)(x)| \leq C_{\beta,k}\sum_{j=0}^{k}\binom{k}{j}\int_0^{+\infty} s^{k-\beta-1}e^{-s}|u^{(k+n-j)}(x,t+s)|ds.$$

Therefore,

$$\begin{aligned}
&\Big(\int_0^\infty \big(t^{n-(\alpha-\beta)}|\frac{\partial^n}{\partial t^n}P_t(\mathcal{D}^\beta f)(x)|\big)^q\frac{dt}{t}\Big)^{1/q} \\
\leq\ & C_{\beta,k}\sum_{j=0}^{k}\binom{k}{j}\Big(\int_0^\infty \big(t^{n-(\alpha-\beta)}\int_0^{+\infty} s^{k-j-(\beta-j)-1}e^{-s}|u^{(k-j+n)}(x,t+s)|ds\big)^q\frac{dt}{t}\Big)^{1/q}
\end{aligned}$$

For $0 \leq j \leq k-1$, we have $\beta - j \geq \beta - (k-1) \geq 0$ and taking into account that each term of the above sum is bounded by the left side of the inequality (2.12), with k replaced by $k-j$ and β replaced by $\beta - j$, we get that

$$\|\Big(\int_0^\infty \big(t^{n-(\alpha-\beta)}\int_0^{+\infty} s^{k-j-(\beta-j)-1}e^{-s}|u^{(k+n-j)}(x\cdot,t+s)|ds\big)^q\frac{dt}{t}\Big)^{1/q}\|_{p,\gamma} < C\|f\|_{F^\alpha_{p,q}}$$

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for $0 \leq j \leq k-1$.
Unfortunately the remaining case $j = k$ requires a special argument, that uses the following known estimate for the Poisson-Hermite semigroup

$$|\frac{\partial^n}{\partial t^n} P_t f(x)| \leq C T^* f(x) t^{-n}, \tag{2.13}$$

where T^*f is the maximal function of the Ornstein-Uhlenbeck semigroup. The proof of this estimate can be found in [10], Lemma 1 (for more details, see [9]), then

$$\begin{aligned}
&\Big(\int_0^\infty \big(t^{n-(\alpha-\beta)} \int_0^{+\infty} s^{k-\beta-1} e^{-s} |u^{(n)}(\cdot, t+s)| ds\big)^q \frac{dt}{t}\Big)^{1/q} \\
\leq\ & C\Big(\int_0^\infty \big(t^{n-(\alpha-\beta)} \int_0^{t} s^{k-\beta-1} e^{-s} |u^{(n)}(x, t+s)| ds\big)^q \frac{dt}{t}\Big)^{1/q} \\
&+ C\Big(\int_0^\infty \big(t^{n-(\alpha-\beta)} \int_t^{+\infty} s^{k-\beta-1} e^{-s} |u^{(n)}(x, t+s)| ds\big)^q \frac{dt}{t}\Big)^{1/q} \\
=\ & (I) + (II).
\end{aligned}$$

We consider first the case $k \leq \alpha$. The term (I) is estimated as term (I) in the proof of Theorem 2.3.

$$(I) \ \leq \ C\Big(\int_0^\infty \big(v^{n-(\alpha-k)} |u^{(n)}(x,v)|\big)^q \frac{dv}{v}\Big)^{1/q}.$$

Since $\beta \geq k-1$, making the change of variable $v = t+s$ we get

$$\begin{aligned}
(II) \ &\leq \ C\Big(\int_0^\infty t^{(n+k-\alpha-1)q-1} \Big(\int_t^{+\infty} |u^{(n)}(x, t+s)| ds\Big)^q dt\Big)^{1/q} \\
&= \ C\Big(\int_0^\infty t^{(n+k-\alpha-1)q-1} \Big(\int_{2t}^{+\infty} |u^{(n)}(x, r)| dr\Big)^q dt\Big)^{1/q} \\
&\leq \ C\Big(\int_0^\infty t^{(n+k-\alpha-1)q-1} \Big(\int_t^{+\infty} |u^{(n)}(x, r)| dr\Big)^q dt\Big)^{1/q}.
\end{aligned}$$

Therefore, by Hardy's inequality (1.26),

$$(II) \ \leq \ \frac{C}{(n+k-\alpha-1)^{1/q}} \Big(\int_0^\infty \big(r^{n-(\alpha-k)} |u^{(n)}(x,r)|\big)^q \frac{dr}{r}\Big)^{1/q},$$

Next consider the case $k > \alpha$. In this case, using inequality (2.13) and Hardy's inequality (1.25), we have

$$\begin{aligned}
(I) \ &\leq \ C_n |T^* f(x)| \Big(\int_0^\infty t^{-(\alpha-\beta)q-1} \Big(\int_0^t s^{k-\beta-1} e^{-s} ds\Big)^q dt\Big)^{1/q} \\
&\leq \ C_n |T^* f(x)| \frac{1}{(\alpha-\beta)^{1/q}} \Big(\int_0^\infty s^{(k-\alpha)q-1} e^{-sq} ds\Big)^{1/q} \\
&= \ C_n |T^* f(x)| \frac{1}{(\alpha-\beta))^{1/q} q^{k-\alpha}} \big(\Gamma((k-\alpha)q\big)^{1/q}.
\end{aligned}$$

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On the other hand,

$$\begin{aligned}(II) &\leq \big(\int_0^1 t^{(n+k-\alpha-1)q-1}\big(\int_t^{+\infty} e^{-s}|u^{(n)}(x,t+s)|ds\big)^q dt\big)^{1/q}\\ &\quad+\big(\int_1^{\infty} t^{(n+k-\alpha-1)q-1}\big(\int_t^{+\infty} e^{-s}|u^{(n)}(x,t+s)|ds\big)^q dt\big)^{1/q}\\ &= (III)+(IV).\end{aligned}$$

By the usual argument using the change of variable $v = t + s$ and Hardy's inequality (1.26), we get

$$\begin{aligned}(III) &\leq \big(\int_0^1 t^{(n-1)q-1}\big(\int_t^{+\infty} |u^{(n)}(x,t+s)|ds\big)^q dt\big)^{1/q}\\ &\leq \big(\int_0^{\infty} t^{(n-1)q-1}\big(\int_t^{+\infty} |u^{(n)}(x,t+s)|ds\big)^q dt\big)^{1/q}\\ &= \big(\int_0^{\infty} t^{(n-1)q-1}\big(\int_{2t}^{+\infty} |u^{(n)}(x,r)|dr\big)^q dt\big)^{1/q}\\ &\leq \big(\int_0^{\infty} t^{(n-1)q-1}\big(\int_t^{+\infty} |u^{(n)}(x,r)|dr\big)^q dt\big)^{1/q}\\ &\leq \frac{1}{n-1}\big(\int_0^{\infty}\big(r^n|u^{(n)}(x,r)|\big)^q\frac{dr}{r}\big)^{1/q}.\end{aligned}$$

Finally using again inequality (2.13), we get

$$\begin{aligned}(IV) &\leq \big(\int_1^{\infty} t^{(n+k-\alpha-1)q-1}\big(\int_t^{+\infty} e^{-s}C_n|T^*f(x)|t^{-n}ds\big)^q dt\big)^{1/q}\\ &= C_n|T^*f(x)|\big(\int_1^{\infty} t^{(k-\alpha-1)q-1}e^{-tq}dt\big)^{1/q}\\ &\leq C_n|T^*f(x)|\big(\int_1^{\infty} t^{(k-\alpha-1)q-1}dt\big)^{1/q} = C_n|T^*f(x)|\big(\frac{1}{(\alpha+1-k)q}\big)^{1/q}.\end{aligned}$$

Hence, in both cases, we get that

$$\|\big(\int_0^{\infty}\big(t^{n-(\alpha-\beta)}|\frac{\partial^n}{\partial t^n}P_t(\mathcal{D}^\beta f)|\big)^q\frac{dt}{t}\big)^{1/q}\|_{p,\gamma} \quad < \quad \infty,$$

as $f \in F^{\alpha}_{p,q}(\gamma_d)$. Therefore, $\mathcal{D}^\beta f \in F^{\alpha-\beta}_{p,q}(\gamma_d)$ and moreover,

$$\|\mathcal{D}^\beta f\|_{F^{\alpha-\beta}_{p,q}} \quad \leq \quad C\|f\|_{F^{\alpha}_{p,q}}.$$

□

Observation Let us observe that if instead of considering the *Ornstein-Uhlenbeck operator* (1.2) and the *Poisson-Hermite semigroup* (1.6) we consider the *Laguerre differential operator* in $\mathbb{R}^d_+$,

$$\mathcal{L}^\alpha = \sum_{i=1}^{d}\Big[x_i\frac{\partial^2}{\partial x_i^2} + (\alpha_i + 1 - x_i)\frac{\partial}{\partial x_i}\Big], \tag{2.14}$$

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and the corresponding *Poisson-Laguerre semigroup*, or if we consider the *Jacobi differential operator* in $(-1,1)^d$,

$$\mathcal{L}^{\alpha,\beta} = -\sum_{i=1}^{d}\left[(1-x_i^2)\frac{\partial^2}{\partial x_i^2} + (\beta_i - \alpha_i - (\alpha_i+\beta_i+2)\,x_i)\frac{\partial}{\partial x_i}\right], \tag{2.15}$$

and the corresponding *Poisson-Jacobi semigroup* (for details we refer to [16]), the arguments are analogous. That is to say, we can define in analogous manner *Laguerre-Triebel-Lizorkin spaces,* and *Jacobi-Triebel-Lizorkin spaces* then prove that the corresponding notions of Fractional Integrals and Fractional Derivatives behave similarly. In order to see this it is more convenient to use the representation (1.6) of P_t in terms of the one-sided stable measure $\mu_t^{(1/2)}(ds)$, see [10].

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