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Abstract Cesàro spaces. Duality ☆

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ABSTRACT

We study abstract Cesàro spaces CX, which may be regarded as generalizations of Cesàro sequence spaces ces_p and Cesàro function spaces $Ces_p(I)$ on $I = [0, 1]$ or $I = [0, \infty)$, and also as the description of optimal domain from which Cesàro operator acts to X. We find the dual of such spaces in a very general situation. What is however even more important, we do it in the simplest possible way. Our proofs are more elementary than the known ones for ces_p and $Ces_p(I)$. This is the point how our paper should be seen, i.e. not as a generalization of known results, but rather like grasping and exhibiting the general nature of the problem, which is not so easily visible in previous publications. Our results show also an interesting phenomenon that there is a big difference between duality in the cases of finite and infinite intervals.

1. Introduction

In 1968 the Dutch Mathematical Society posted a problem to find the Köthe dual of Cesàro sequence spaces ces_p and Cesàro function spaces $Ces_p[0, \infty)$. In 1974 the problem was solved (isometrically) by Jagers [12] even for weighted Cesàro sequence spaces, but the proof is far from being easy and elementary. Before, in 1966 Luxemburg and Zaanen [22] have found the Köthe dual of $Ces_\infty[0, 1]$ space (known as the Korenblyum–Kreĭn–Levin space – cf. [17]). Already in 1957 Alexiewicz, in his overlooked paper [1], has found implicitly the Köthe dual of weighted ces_∞-spaces (see Section 5 for more information). Later on some number of papers appeared concerning the case of sequence spaces as well as function spaces. Bennett [4] found another isomorphic representation of the dual $(ces_p)^*$ for $1 < p < \infty$ as the corollary from factorization theorems for Cesàro and l^p spaces. This description is simpler than the one given by Jagers [12]. On the one hand, his factorization method was universal enough to be adopted to the case

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of function spaces, which was done by Astashkin and Maligranda [2, Theorem 3]. Anyhow, this method is rather complicated and indirect, possibly valid only for power functions. A totally different approach appeared in the paper by Sy, Zhang and Lee [32], but the idea was based on Jagers' result. Also Kamińska and Kubiak in [13] were inspired by Jagers when they found the isometric representation of the Köthe dual of weighted Cesàro function spaces $Ces_{p,w}$. They used not so easy Jagers' idea of relative concavity and concave majorants. In 2007 Kerman, Milman and Sinnamon [15] defined abstract Cesàro spaces CX and the Köthe dual was found but only in the case of rearrangement invariant space X on $I = [0, \infty)$. The result comes from equivalence (in norm) of the Cesàro operator with the so-called level function. This time again, one has to go through a very technical theory of down spaces and level functions to get the mentioned dual space. Recently, also Nekvinda and Pick in [25] have deduced a description of dual of some particular Cesàro function spaces from the so-called "block form".

Our goal in this paper is to grasp the general nature of the problem of duality of Cesàro spaces. We shall prove the duality theorem for abstract Cesàro spaces (isomorphic form) by a method as elementary as possible. Moreover, our method applies to a very general case and in the symmetric case our proof is easier and more comprehensive than any earlier one. Especially the function case on $[0, \infty)$ is instructive, but more delicate modification must be done in the case of interval $[0, 1]$. Generally, for one of the inclusions a result of Sinnamon, which is however very intuitive, elementary and avoids level functions, is crucial. The second inclusion follows from some kind of idempotency of the Cesàro operator, which was noticed firstly for the sequence case by Bennett in [4] and the proof was simplified by Curbera and Ricker in [6].

The paper is organized as follows. In Section 2 some necessary definitions and notations are collected, together with basic results on abstract Cesàro spaces. In particular, we can see when abstract Cesàro spaces CX are nontrivial.

Sections 3 and 4 contain results on the Köthe dual $(CX)'$ of abstract Cesàro spaces. There is a big difference between the cases on $[0, \infty)$ and on $[0, 1]$, as we can see in Theorems 3, 4 and 5. Important in our investigations were earlier results on the Köthe dual $(Ces_p[0, \infty))'$ due to Kerman, Milman and Sinnamon [15] and $(Ces_p(I))'$ due to Astashkin and Maligranda [2].

In Section 5, description of the Köthe dual of abstract Cesàro sequence spaces is presented in Theorem 6. We also collect here our knowledge about earlier results on Köthe duality of Cesàro sequence spaces ces_p and their weighted versions.

In Section 6 we give, in Theorem 7, a simple proof of a generalization of the Luxemburg–Zaanen [22] and Tandori [33] results on duality of weighted Cesàro spaces $Ces_{\infty,w}$. This proof also works for weighted Cesàro sequence spaces (implicitly proved by Alexiewicz [1]). Then in Theorem 8 we identify the Cesàro–Lorentz space $C\Lambda_\varphi$ with the weighted L^1-space, using Theorem 3, which simplifies the result of Delgado and Soria [8].

Finally, in connection to the proof of Theorem 5 we include Appendices A and B. The first one presents a proof of weighted version of the Calderón–Mitjagin interpolation theorem and the second one contains an improvement of Hardy inequality in weighted spaces $L^p(x^\alpha)$ on $[0, 1]$.

2. Definitions and basic facts

We recall some notations and definitions which will be needed later on. By $L^0 = L^0(I)$ we denote the set of all equivalence classes of real-valued Lebesgue measurable functions defined on $I = [0, 1]$ or $I = [0, \infty)$. A *Banach ideal space* $X = (X, \|\cdot\|)$ (on I) is understood to be a Banach space contained in $L^0(I)$, which satisfies the so-called ideal property: if $f, g \in L^0(I)$, $|f| \le |g|$ a.e. on I and $g \in X$, then $f \in X$ and $\|f\| \le \|g\|$. Sometimes we write $\|\cdot\|_X$ to be sure which norm is taken in the space. If it is not stated otherwise we understand that in a Banach ideal space there is $f \in X$ with $f(t) > 0$ for each $t \in I$ (such a function is called the *weak unit* in X), which means that $\operatorname{supp} X = I$.

Since the inclusion of two Banach ideal spaces is continuous, we should write $X \hookrightarrow Y$ rather that $X \subset Y$. Moreover, the symbol $X \overset{A}{\hookrightarrow} Y$ means $X \hookrightarrow Y$ with the norm of inclusion not bigger than A, i.e.,

$\|f\|_Y \le A\|f\|_X$ for all $f \in X$. Also $X = Y$ (and $X \equiv Y$) means that spaces have the same elements and norms are equivalent (equal).

For a Banach ideal space $X = (X, \|\cdot\|)$ on I the *Köthe dual space* (or *associated space*) X' is the space of all $f \in L^0(I)$ such that the *associated norm*

$$\|f\|' := \sup_{g \in X,\, \|g\|_X \le 1} \int_I \big|f(x)g(x)\big|\, dx \tag{2.1}$$

is finite. The Köthe dual $X' = (X', \|\cdot\|')$ is then a Banach ideal space. Moreover, $X \overset{1}{\hookrightarrow} X''$ and we have equality $X = X''$ with $\|f\| = \|f\|''$ if and only if the norm in X has the *Fatou property*, that is, if the conditions $0 \le f_n \nearrow f$ a.e. on I and $\sup_{n\in\mathbf{N}}\|f_n\| < \infty$ imply that $f \in X$ and $\|f_n\| \nearrow \|f\|$.

For a Banach ideal space $X = (X, \|\cdot\|)$ on I with the Köthe dual X' the following *generalized Hölder–Rogers inequality* holds: if $f \in X$ and $g \in X'$, then fg is integrable and

$$\int_I \big|f(x)g(x)\big|\, dx \le \|f\|_X\|g\|_{X'}. \tag{2.2}$$

A function f in a Banach ideal space X on I is said to have *order continuous norm* in X if for any decreasing sequence of Lebesgue measurable sets $A_n \subset I$ with $m(\bigcap_n A_n) = 0$, we have that $\|f\chi_{A_n}\| \to 0$ as $n \to \infty$. The set of all functions in X with order continuous norm is denoted by X_a. If $X_a = X$, then the space X is said to be *order continuous*. For an order continuous Banach ideal space X the Köthe dual X' and the dual space X^* coincide. Moreover, a Banach ideal space X with the Fatou property is reflexive if and only if both X and its associate space X' are order continuous.

For a weight $w(x)$, i.e. a measurable function on I with $0 < w(x) < \infty$ a.e. and for a Banach ideal space X on I, the *weighted Banach ideal space* $X(w)$ is defined as $X(w) = \{f \in L^0 : fw \in X\}$ with the norm $\|f\|_{X(w)} = \|fw\|_X$. Of course, $X(w)$ is also a Banach ideal space and

$$\big[X(w)\big]' \equiv X'\left(\frac{1}{w}\right). \tag{2.3}$$

By a *rearrangement invariant* or *symmetric space* on I with the Lebesgue measure m, we mean a Banach ideal space $X = (X, \|\cdot\|_X)$ with additional property that for any two equimeasurable functions $f \sim g$, $f, g \in L^0(I)$ (that is, they have the same distribution functions $d_f \equiv d_g$, where $d_f(\lambda) = m(\{x \in I : |f(x)| > \lambda\})$, $\lambda \ge 0$), and $f \in E$ we have $g \in E$ and $\|f\|_E = \|g\|_E$. In particular, $\|f\|_X = \|f^*\|_X$, where $f^*(t) = \inf\{\lambda > 0 : d_f(\lambda) < t\}$, $t \ge 0$.

For general properties of Banach ideal spaces and symmetric spaces we refer to the books [3,14,18,21,23].

In order to define and formulate results we need the continuous Cesàro operator C defined as

$$Cf(x) = \frac{1}{x}\int_0^x f(t)\, dt \quad \text{for } 0 < x \in I,$$

and also a nonincreasing majorant $\widetilde{f}$ of a given function f, which is defined for $x \in I$ as

$$\widetilde{f}(x) = \operatorname*{ess\,sup}_{t \in I,\, t \ge x} \big|f(t)\big|.$$

For a Banach ideal space X on I we define an *abstract Cesàro space* $CX = CX(I)$ as

$$CX = \big\{f \in L^0(I) : C|f| \in X\big\} \quad \text{with the norm } \|f\|_{CX} = \big\|C|f|\big\|_X, \tag{2.4}$$

and the space $\widetilde{X} = \widetilde{X}(I)$ as

$$\widetilde{X} = \{f \in L^0(I) : \widetilde{f} \in X\} \quad \text{with the norm } \|f\|_{\widetilde{X}} = \|\widetilde{f}\|_X. \tag{2.5}$$

The space CX for a Banach ideal space X on $[0,\infty)$ was defined already in [27] and spaces CX, $\widetilde{X}$ for X being a symmetric space on $[0,\infty)$ have appeared, for example, in [15] and [8].

The *dilation operators* σ_τ $(\tau > 0)$ defined on $L^0(I)$ by

$$\sigma_\tau f(x) = f(x/\tau)\chi_I(x/\tau) = f(x/\tau)\chi_{[0,\min(1,\tau)]}(x), \quad x \in I,$$

are bounded in any symmetric space X on I and $\|\sigma_\tau\|_{X\to X} \le \max(1,\tau)$ (see [3, p. 148] and [18, pp. 96–98]). Dilation operators are also bounded in some Banach ideal spaces which are not necessary symmetric. For example, if either $X = L^p(x^\alpha)$ or $X = CL^p(x^\alpha)$, then $\|\sigma_\tau\|_{X\to X} = \tau^{1/p+\alpha}$ (see [27] for more examples).

Let us collect some basic properties of CX and $\widetilde{X}$ spaces.

Theorem 1. *Let X be a Banach ideal space on I. Then both CX and $\widetilde{X}$ are also Banach ideal spaces on I (not necessarily with a weak unit). Moreover,*

(*a*) *$CX[0,\infty) \ne \{0\}$ if and only if $\frac{1}{x}\chi_{[a,\infty)}(x) \in X$ for some $a > 0$.*
(*b*) *$CX[0,1] \ne \{0\}$ if and only if $\chi_{[a,1]} \in X$ for some $0 < a < 1$.*
(*c*) *$\widetilde{X} \ne \{0\}$ if and only if X contains a nonzero, nonincreasing function on I.*
(*d*) *If X has the Fatou property, then CX and $\widetilde{X}$ have the Fatou property.*
(*e*) *$(\widetilde{X})_a = \{0\}$.*

Proof. (a) Suppose that $\frac{1}{x}\chi_{[a,\infty)}(x) \in X$ for some $a > 0$. Then for any $b > a$ we have

$$\begin{aligned}\|\chi_{[a,b]}\|_{CX} &= \left\| \frac{1}{x}\int_0^x \chi_{[a,b]}(t)\,dt \right\|_X = \left\| \frac{x-a}{x}\chi_{[a,b]}(x) + \frac{b-a}{x}\chi_{(b,\infty)}(x) \right\|_X \\ &\le \left\| \frac{b-a}{x}\chi_{[a,\infty)}(x) \right\|_X = (b-a)\left\| \frac{1}{x}\chi_{[a,\infty)}(x) \right\|_X < \infty,\end{aligned}$$

whence $\chi_{[a,b]} \in CX$.

If $CX \ne \{0\}$, then there exists $0 \ne f \in CX$, that is, $|f(x)| > 0$ for $x \in A$ with $0 < m(A) < \infty$, and we can find $a > 0$ such that $b = \int_0^a |f(t)|\,dt > 0$. Thus,

$$\begin{aligned}\frac{b}{x}\chi_{[a,\infty)}(x) &\le \frac{1}{x}\int_0^a |f(t)|\,dt\,\chi_{[a,\infty)}(x) \\ &\le \frac{1}{x}\int_0^x |f(t)|\,dt\,\chi_{[a,\infty)}(x) \le C|f|(x) \in X,\end{aligned}$$

and so $\frac{1}{x}\chi_{[a,\infty)}(x) \in X$.

(b) Proof is similar as in (a). However, observe that the condition $\chi_{[a,1]} \in X$ has no weight $w(x) = 1/x$ as in (a). Proof of (c) is clear. Proof of (d) follows from the fact that if $f_n, f \in L^0(I)$ and $0 \le f_n \nearrow f$ pointwise on I, then $Cf_n \nearrow Cf$ pointwise on I. Also $f_n\chi_{[x,\infty)\cap I} \nearrow f\chi_{[x,\infty)\cap I}$ for every $x \in I$, and so $\widetilde{f_n} \nearrow \widetilde{f}$ pointwise on I, which implies, by the Fatou property of X, that $\|f_n\|_{CX} = \|Cf_n\|_X \nearrow \|Cf\|_X = \|f\|_{CX}$ and also $\|f_n\|_{\widetilde{X}} = \|\widetilde{f_n}\|_X \nearrow \|\widetilde{f}\|_X = \|f\|_{\widetilde{X}}$.

(e) Suppose $0 \neq f \in \widetilde{X}$. Then $\operatorname{ess\,sup}_{x\in I}|f(x)| = a > 0$. It means

$$m\big(\big\{x \in I : \big|f(x)\big| > a/2\big\}\big) = b > 0.$$

In particular, for $A = \{x \in I : |f(x)| > a/2\}\setminus[0, b/2]$ we have $m(A) \geq b/2$. Now, choose a sequence of sets of positive measure (A_n) such that $m(\bigcap_{n=1}^{\infty} A_n) = 0$ and $A_{n+1} \subset A_n \subset A$ $(n = 1, 2, 3, \ldots)$. Then

$$\frac{a}{2}\chi_{[0,b/2]} \leq \widetilde{f\chi_{A_n}} \quad \text{for all } n = 1, 2, 3, \ldots,$$

and consequently

$$\left\|\frac{a}{2}\chi_{[0,b/2]}\right\|_X = \left\|\frac{a}{2}\chi_{[0,b/2]}\right\|_{\widetilde{X}} \leq \|\widetilde{f\chi_{A_n}}\|_X = \|f\chi_{A_n}\|_{\widetilde{X}},$$

which means that $f \notin (\widetilde{X})_a$. □

Remark 1. It is important to notice that there are Banach ideal spaces X for which Cesàro space $CX \neq \{0\}$ but they do not contain a weak unit (see Example 2 below). Using the notion of support of the space (more information about this notion can be found, for example, in [14, p. 137], [23, pp. 169–170] and [16, pp. 879–880]) we can say even more, namely that $\operatorname{supp} CX \subset \operatorname{supp} X$ and the inclusion can be essentially strict. On the other hand, in our investigations of the duality of Cesàro spaces the natural assumption is that Cesàro operator is bounded in a given Banach ideal space X which ensures that $\operatorname{supp} CX = \operatorname{supp} X = I$.

Example 2. Consider the Banach ideal space $X = L^p(w)$ with $1 < p < \infty$ and the weight $w(x) = \max(\frac{1}{1-x}, 1)$ on $I = [0, \infty)$. Then $\operatorname{supp} X = I$ but $\operatorname{supp} CX = [1, \infty)$.

Remark 3. Cesàro spaces CX on I are not symmetric spaces even when X is a symmetric Banach space on I and Cesàro operator C is bounded on X. In fact, it was proved in [8, Theorem 2.1] that $CX \not\hookrightarrow L^1 + L^\infty$ for $I = [0, \infty)$ from which it follows that $CX[0, \infty)$ cannot be symmetric and in [2] it was shown that $CL^p[0, 1]$ is not symmetric. It seems that CX is never symmetric (even cannot be renormed to be symmetric) but this is only our conjecture.

3. Duality on $[0, \infty)$

The description of Köthe dual of $Ces_p[0, \infty)$ spaces for $1 < p \leq \infty$ appeared as remark in Bennett [4, p. 124], but it was proved by Astashkin and Maligranda [2]. For more general spaces CX, where X is a symmetric space having additional properties, it was proved by Kerman, Milman and Sinnamon [15, Theorem D] and they used in the proof some of Sinnamon results; [30, Theorem 2.1] and [29, Proposition 2.1 and Lemma 3.2].

In the case of symmetric spaces on $[0, \infty)$ one can simplify the proof of duality theorem from [15, Theorem D], using results of Sinnamon (cf. [29–31]). The proof below seems to be simpler, although uses the same ideas.

Theorem 2. *Let X be a symmetric space on $[0, \infty)$ with the Fatou property. If C is a bounded operator on X, then*

$$(CX)' = \widetilde{X'}. \tag{3.1}$$

Proof. By Theorem 1, spaces CX and $\widetilde{X'}$ have the Fatou property. Therefore it is enough to show that

$$CX = (CX)'' = \big(\widetilde{X'}\big)'.$$

For $f \in (\widetilde{X'})'$ we have

$$\begin{aligned}\|f\|_{(\widetilde{X'})'} &= \sup\Big\{\int_I \big|f(x)g(x)\big|\,dx : g \in \widetilde{X'},\ \|g\|_{\widetilde{X'}} \le 1\Big\} \\ &= \sup\Big\{\int_I \big|f(x)g(x)\big|\,dt : \widetilde{g} \in X',\ \|\widetilde{g}\|_{X'} \le 1\Big\} := I(f).\end{aligned}$$

From one side

$$I(f) \le \sup\Big\{\int_I \big|f(x)\big|\widetilde{g}(x)\,dx : \widetilde{g} \in X',\ \|\widetilde{g}\|_{X'} \le 1\Big\},$$

and on the other hand

$$I(f) \ge \sup\Big\{\int_I \big|f(x)g(x)\big|\,dx : g = \widetilde{g} \in X',\ \|\widetilde{g}\|_{X'} \le 1\Big\},$$

because the supremum on the right is taken over smaller set of functions. Therefore,

$$\|f\|_{(\widetilde{X'})'} = \sup\Big\{\int_I \big|f(x)\big|h(x)\,dt : 0 \le h\downarrow,\ \|h\|_{X'} \le 1\Big\} =: \|f\|_{X^{\downarrow}}. \tag{3.2}$$

The last supremum defines the so-called "down space" $X^{\downarrow}$ (cf. [28] and [29]) and so

$$\|f\|_{(\widetilde{X'})'} = \|f\|_{X^{\downarrow}}. \tag{3.3}$$

For $I = [0, \infty)$ the identification of the down space $X^{\downarrow}$ with the Cesàro space CX, provided that the operator C is bounded on the symmetric space X, is a consequence of Sinnamon's result [29, Theorem 3.1]. Thus,

$$\|f\|_{(\widetilde{X'})'} = \|f\|_{X^{\downarrow}} \approx \big\|C|f|\big\|_X = \|f\|_{CX}. \qquad \square \tag{3.4}$$

Remark 4. Let us mention that the duality of down space $X^{\downarrow}$ for symmetric space X with the help of level functions was investigated by Sinnamon in his papers [28, Theorem 6.7], [29, Theorem 5.7] and [31, Theorem 2.1]. Another proof of (3.3) can be found in [29, Theorem 5.6].

Let us generalize the above theorem to a wider class than symmetric spaces, which will include corresponding isomorphic versions from [13] and [12]. Our method here is more direct and does not need neither the notion of level functions nor down spaces. However, one result of Sinnamon, namely [30, Theorem 2.1] (see also [31] for a very nice intuitive graphical explanation of this equality) will be necessary. Since the result is given for a general measure on $\mathbb{R}$, we need to reformulate it slightly to make it compatible with our notion.

Proposition 5 *(Sinnamon, 2003). Let either $I = [0, \infty)$ or $I = [0, 1]$. For a measurable $f, g, h \ge 0$ on I we have*

$$\int_I f(x)\widetilde{g}(x)\,dx = \sup_{h \prec f} \int_I h(x)g(x)\,dx, \tag{3.5}$$

where $h \prec f$ means that $\int_0^u h(x)dx \le \int_0^u f(x)dx$ for all $u \in I$.

Proof. From [30, Theorem 2.1] we have for a λ-measurable $f, g, h \ge 0$ on $\mathbb{R}$

$$\int_{\mathbb{R}} f\widetilde{g}\,d\lambda = \sup_{h \prec_\lambda f} \int_{\mathbb{R}} hg\,d\lambda,$$

where $h \prec_\lambda f$ means that $\int_{-\infty}^u hd\lambda \le \int_{-\infty}^u fd\lambda$ for all $u \in \mathbb{R}$. In the case $I = [0,\infty)$ we put λ to be just the Lebesgue measure on $[0,\infty)$ and zero elsewhere. Then $h \prec_\lambda f$ if and only if $h \prec f$ because for each $u \ge 0$

$$\int_{-\infty}^{u} hd\lambda \le \int_{-\infty}^{u} fd\lambda \iff \int_0^u h(x)dx \le \int_0^u f(x)dx.$$

Then

$$\int_0^\infty f(x)\widetilde{g}(x)\,dx = \int_{\mathbb{R}} f\widetilde{g}\,d\lambda = \sup_{h \prec_\lambda f} \int_{\mathbb{R}} hg\,d\lambda = \sup_{h \prec f} \int_0^\infty h(x)g(x)\,dx.$$

In the case of interval $[0,1]$ we put λ to be the Lebesgue measure on $[0,1]$ and zero elsewhere. Then $\widetilde{g}(x) = \widetilde{g\chi_{[0,1]}}(x)$ for $x \in [0,1]$ and the remaining part of proof works in the same way as before. □

Theorem 3. *Let X be a Banach ideal space on $I = [0,\infty)$ such that both the Cesàro operator C and the dilation operator σ_τ, for some $0 < \tau < 1$, are bounded on X. Then*

$$(CX)' = \widetilde{X'} \quad \textit{with equivalent norms.} \tag{3.6}$$

We start with a continuous version of inequality proved for sequences by Curbera and Ricker [6, Proposition 2].

Lemma 6. *If $0 \le f \in L^1_{loc}[0,\infty)$ and $a > 1$ is arbitrary, then*

$$\int_0^{x/a} f(t)\,dt \le \frac{1}{\ln a} \int_0^x \left(\frac{1}{t} \int_0^t f(s)\,ds \right) dt \quad \textit{for all } x > 0, \tag{3.7}$$

that is, $Cf(x/a) \le \frac{a}{\ln a} CCf(x)$ for all $x > 0$.

Proof. For $x > 0$, by the Fubini theorem, we have

$$\begin{aligned}\int_0^x \left(\frac{1}{t} \int_0^t f(s)\,ds \right) dt &= \int_0^x f(s) \left(\int_s^x \frac{1}{t}\,dt \right) ds = \int_0^x f(s) \ln \frac{x}{s}\,ds \\ &= \int_0^{x/a} f(s) \ln \frac{x}{s}\,ds + \int_{x/a}^x f(s) \ln \frac{x}{s}\,ds \\ &\ge \ln a \int_0^{x/a} f(s)\,ds,\end{aligned}$$

and so

$$CCf(x) = \frac{1}{x}\int_0^x Cf(t)\,dt \geq \frac{\ln a}{a} Cf(x/a). \qquad \square$$

Remark 7. From the classical Hardy inequality and (3.7) we obtain that if $1 < p \leq \infty$, then

$$C^2 L^p = C\big(CL^p\big) = CL^p.$$

Such an equality for Cesàro sequence spaces ces_p $(1 < p < \infty)$ was proved by Bennett (cf. [4, Theorem 20.31]) and simplified in [6]. In fact, by the Hardy inequality we have $L^p \overset{p'}{\hookrightarrow} CL^p$ (cf. [19]) and so $CL^p \overset{p'}{\hookrightarrow} CCL^p$. On the other hand, (3.7) shows that

$$a^{1/p}\big\|C|f|\big\|_{L^p} = \big\|\sigma_a\big(C|f|\big)\big\|_{L^p} \leq \frac{a}{\ln a}\big\|CC|f|\big\|_{L^p}.$$

Since $\inf_{a>1} \frac{a^{1-1/p}}{\ln a} = \frac{e}{p'}$ it follows that $CCL^p \overset{e/p'}{\hookrightarrow} CL^p$.

Proof of Theorem 3. We start with usually simpler inclusion $\widetilde{X'} \hookrightarrow (CX)'$. By substitution $t = au$ in the right integral of (3.7) we obtain

$$\int_0^{x/a} \big|f(t)\big|\,dt \leq \frac{a}{\ln a}\int_0^{x/a}\left(\frac{1}{au}\int_0^{au}\big|f(s)\big|\,ds\right)du = \frac{a}{\ln a}\int_0^{x/a} C|f|(au)\,du \quad \text{for all } x > 0.$$

Let $g \in \widetilde{X'}$ and $f \in CX$, then applying the above estimate to property 18^o from page 72 in [18] and by the Hölder–Rogers inequality (2.2) we obtain

$$\begin{aligned}\int_0^\infty \big|f(x)g(x)\big|\,dx &\leq \int_0^\infty \big|f(x)\big|\widetilde{g}(x)\,dx \leq \frac{a}{\ln a}\int_0^\infty C|f|(ax)\widetilde{g}(x)\,dx \\ &\leq \frac{a}{\ln a}\big\|C|f|(ax)\big\|_X \|\widetilde{g}\|_{X'} \leq \frac{a}{\ln a}\|\sigma_{1/a}\|_{X\to X}\big\|C|f|\big\|_X\|\widetilde{g}\|_{X'} \\ &= \frac{a}{\ln a}\|\sigma_{1/a}\|_{X\to X}\|f\|_{CX}\|g\|_{\widetilde{X'}},\end{aligned}$$

which means that

$$\|g\|_{(CX)'} \leq \frac{a}{\ln a}\|\sigma_{1/a}\|_{X\to X}\|g\|_{\widetilde{X'}}$$

and so $\widetilde{X'} \overset{A}{\hookrightarrow} (CX)'$ with $A = \frac{a}{\ln a}\|\sigma_{1/a}\|_{X\to X}$ and $a > 1$.

We can now turn our attention into, usually more difficult, the second inclusion $(CX)' \hookrightarrow \widetilde{X'}$. Let $g \in (CX)'$ and $f \in X$, then by (3.5) in Proposition 5, the generalized Hölder–Rogers inequality and using assumption that operator C is bounded on X we obtain

$$\begin{aligned}\int_0^\infty \big|f(x)\big|\widetilde{g}(x)\,dx &= \sup_{|h|\prec|f|}\int_0^\infty \big|h(x)g(x)\big|\,dx \leq \sup_{|h|\prec|f|}\|h\|_{CX}\|g\|_{(CX)'} \\ &\leq \|f\|_{CX}\|g\|_{(CX)'} = \big\|C|f|\big\|_X\|g\|_{(CX)'} \leq B\|f\|_X\|g\|_{(CX)'},\end{aligned}$$

where $B = \|C\|_{X \to X}$. Therefore,

$$\|g\|_{\widetilde{X'}} = \|\widetilde{g}\|_{X'} = \sup_{\|f\|_X \leq 1} \int_0^\infty |f(x)| \widetilde{g}(x)\, dx \leq B \|g\|_{(CX)'}$$

and thus $(CX)' \overset{B}{\hookrightarrow} \widetilde{X'}$. $\square$

Remark 8. It is worth to notice that in Theorem 3, the assumption on boundedness of the dilation operator was used only for the inclusion $\widetilde{X'} \hookrightarrow (CX)'$. On the other hand, the proof of inclusion $(CX)' \hookrightarrow \widetilde{X'}$ requires only boundedness of C on X.

Several authors investigated weighted Cesàro operators or Cesàro operator C in weighted $L^p(w)$ spaces which leads to weighted Cesàro spaces $Ces_{p,w}$. These spaces are particular examples of abstract Cesàro spaces CX. In fact, $Ces_{p,w} = C(L^p(w))$. From Theorem 3 we obtain the following duality result even for more general weighted Cesàro spaces $C(X(w))$.

Corollary 9. *Let X be a symmetric space on $[0, \infty)$ and w be a weight on $[0, \infty)$ such that the dilation operator σ_a (for some $0 < a < 1$) and Cesàro operator C are bounded on $X(w)$. Then*

$$\big[C\big(X(w)\big)\big]' = \left(\widetilde{X'\left(\frac{1}{w}\right)}\right).$$

It seems that our approach does not include all weights from [13] but for power weights $w(x) = x^\alpha$ with $\alpha < 1 - 1/p$ and $1 < p < \infty$ we obtain from Corollary 9

$$[Ces_{p,x^\alpha}]' = \big[C\big(L^p(x^\alpha)\big)\big]' = \widetilde{L^{p'}(x^{-\alpha})},$$

since C is bounded in $L^p(x^\alpha)$ with the norm $\|C\| = (1 - \alpha - 1/p)^{-p}$ (cf. [11, p. 245] or [19, p. 23]) and σ_τ has norm $\|\sigma_\tau\| = \tau^{1/p+\alpha}$.

4. Duality on $[0, 1]$

The duality of Cesàro spaces on $I = [0, 1]$ is more delicate and less known. Astashkin and Maligranda [2, Theorem 3] proved that for $1 < p < \infty$ we have $(Ces_p)' = U(p') := \widetilde{L^{p'}(\frac{1}{1-x})}$, where $f \in \widetilde{L^{p'}(\frac{1}{1-x})}$ means that $\widetilde{f} \in L^{p'}(\frac{1}{1-x})$ with the norm

$$\|f\|_{U(p')} = \left[\int_0^1 \left(\frac{\widetilde{f}(x)}{1 - x}\right)^{p'} dx\right]^{1/p}.$$

The proof of inclusion $U(p') \hookrightarrow (Ces_p)'$ requires improvement of the Hardy inequality, which they gave in [2, inequality (21)]: if $1 < p < \infty$, then $C : L^p(1 - x) \to L^p$ is bounded, that is,

$$\|Cf\|_{L^p} \leq A_p \big\|(1 - x)f(x)\big\|_{L^p} \quad \text{for all } f \in L^p(1 - x), \tag{4.1}$$

with $A_p \leq 2(p' + 2p)$. In fact, their proof gives even more general result, which we will use later on. Let us present the proof for case of symmetric spaces on $[0, 1]$. We will need in the proof Copson operator which is formally defined as $C^* f(x) = \int_x^1 \frac{f(t)}{t}\, dt$.

Lemma 10 *(Astashkin–Maligranda, 2009). If X is a symmetric space on $I = [0, 1]$ and both operators $C, C^* : X \to X$ are bounded, then $C : X(1-x) \to X$ is also bounded.*

Proof. Let $w(x) = 1 - x$, $f(x) \geq 0$ for $x \in I$. Then for $0 < x \leq 1/2$,

$$Cf(x) = \frac{1}{x}\int_0^x f(t)\,dt \leq \frac{2}{x}\int_0^x f(t)w(t)\,dt = 2C(fw)(x),$$

and for $1/2 \leq x \leq 1$,

$$\begin{aligned} Cf(x) &\leq 2\int_0^x f(t)\,dt = 2\int_{1-x}^1 f(1-t)\,dt \\ &= 2\int_{1-x}^1 \frac{f(1-t)w(1-t)}{t}\,dt = 2\,C^*(\overline{fw})(1-x), \end{aligned}$$

where $\overline{fw}(t) = f(1-t)w(1-t)$. Therefore,

$$\|Cf\|_X \leq 2\big\|C(fw)\big\|_X + 2\big\|C^*(\overline{fw})\big\|_X \leq 2\big(\|C\|_{X\to X} + \big\|C^*\big\|_{X\to X}\big)\|fw\|_X. \qquad \square$$

We will prove equality $(CX)' = \widetilde{X'(\frac{1}{1-x})}$ under some assumptions on the space X but each inclusion will be proved separately.

Theorem 4. *Let X be a Banach ideal space on $I = [0, 1]$ such that the operator $C : X(1-x) \to X$ is bounded. Then*

$$(CX)' \hookrightarrow \widetilde{X'\left(\frac{1}{1-x}\right)}. \tag{4.2}$$

Proof. Let $g \in (CX)'$ and $f \in CX$, then using Proposition 5 and applying our assumption in the last inequality we get

$$\begin{aligned} \int_0^1 \big|f(x)\big|\widetilde{g}(x)\,dx &= \sup_{|h|\prec|f|}\int_0^1 \big|h(x)g(x)\big|\,dx \leq \sup_{|h|\prec|f|}\|h\|_{CX}\|g\|_{(CX)'} \leq \|f\|_{CX}\|g\|_{(CX)'} \\ &= \|Cf\|_X\|g\|_{(CX)'} \leq D\big\|(1-x)f(x)\big\|_X\|g\|_{(CX)'}, \end{aligned}$$

where $D = \|C\|_{X(1-x)\to X}$. Since $[X(1-x)]' \equiv X'(\frac{1}{1-x})$ it follows that

$$\|g\|_{\widetilde{X'(\frac{1}{1-x})}} = \sup_{\|f\|_{X(1-x)}\leq 1}\int_0^1 \big|f(x)\widetilde{g}(x)\big|\,dx \leq D\|g\|_{(CX)'},$$

thus $(CX)' \overset{D}{\hookrightarrow} \widetilde{X'(\frac{1}{1-x})}$ with $D = \|C\|_{X(1-x)\to X}$. $\square$

Theorem 5. *If X is a symmetric space on $I = [0, 1]$ with the Fatou property, then*

$$\widetilde{X'\left(\frac{1}{1-x}\right)} \hookrightarrow (CX)'. \tag{4.3}$$

In the proof of Theorem 5 we will need the following result, similar to Lemma 6.

Lemma 11. *If* $\int_0^t |f(x)|dx < \infty$ *for each* $0 < t < 1$*, then*

$$\int_0^{t/d(t)} |f(x)|dx \le \int_0^t \frac{1}{1-x}\left(\frac{1}{x}\int_0^x |f(s)|ds\right)dx = \int_0^t \frac{C|f|(x)}{1-x}dt \quad \text{for all } 0 < t < 1,$$

where $d(t) := t + e - et$.

Proof. Observe that $1 < d(t) < e$ for $0 < t < 1$. By the Fubini theorem we obtain

$$\begin{aligned}\int_0^t \frac{C|f|(x)}{1-x}\,dx &= \int_0^t \frac{\int_0^x |f(s)|\,ds}{x(1-x)}\,dx = \int_0^t |f(s)|\left(\int_s^t \frac{dx}{x(1-x)}\right)ds \\ &= \int_0^t |f(s)|\left[\int_s^t \left(\frac{1}{x} + \frac{1}{1-x}\right)dx\right]ds = \int_0^t |f(s)|\ln\frac{t(1-s)}{s(1-t)}\,ds \\ &= \int_0^{t/d(t)} |f(s)|\ln\frac{t(1-s)}{s(1-t)}\,ds + \int_{t/d(t)}^t |f(s)|\ln\frac{t(1-s)}{s(1-t)}\,ds.\end{aligned}$$

It is easy to see that for $0 < s \le t/(t + e - et)$ we have $\frac{t(1-s)}{s(1-t)} \ge e$ and of course $\ln\frac{t(1-s)}{s(1-t)} \ge 1$ for each $0 < s < 1$. Thus we get

$$\int_0^{t/d(t)} |f(x)|\,dx \le \int_0^t \frac{C|f|(x)}{1-x}\,dx$$

as required. □

Proof of Theorem 5. Let $0 \le g = \widetilde{g} \in X'(\frac{1}{1-x})$ be a simple function. Since g is nonincreasing, $X' \hookrightarrow L^1$ and $\frac{1}{1-x} \notin L^1$, we can see that g may be written as

$$g = \sum_{k=1}^n a_k \chi_{[0,t_k)},$$

for $0 \le a_k$ and $0 < t_k < 1$, $k = 1, 2, \ldots, n$. Define S as the class of all g of the above form. We need to show that there is a constant $M > 0$ such that for each $f \in CX$ and $g \in S$

$$\int_0^1 |g(x)f(x)|\,dx \le M\left\|\frac{g(x)}{1-x}\right\|_{X'}\big\|C|f|\big\|_X = M\|g\|_{X'(\frac{1}{1-x})}\|f\|_{CX}. \tag{4.4}$$

For $d(t) = t + e - et$ denote

$$g_d = \sum_{k=1}^n a_k \chi_{[0,t_k/d(t_k))}.$$

Then, by Lemma 11, we get

$$\int_0^1 g_d(x)\big|f(x)\big|\,dx = \sum_{k=1}^n a_k \int_0^{t_k/d(t_k)} \big|f(x)\big|\,dx$$
$$\leq \sum_{k=1}^n a_k \int_0^{t_k} \frac{C|f|(x)}{1-x}\,dx = \int_0^1 g(x)\frac{C|f|(x)}{1-x}\,dx.$$

Applying generalized Hölder–Rogers inequality (2.2) one has

$$\int_0^1 g_d(x)\big|f(x)\big|\,dx \leq \int_0^1 g(x)\frac{C|f|(x)}{1-x}\,dx \leq \left\|\frac{g(x)}{1-x}\right\|_{X'} \big\|C|f|\big\|_X.$$

Therefore, to get (4.4) we need to find a constant $M > 0$, independent of the choice of g, such that $\|g\|_{X'(\frac{1}{1-x})} \leq M\|g_d\|_{X'(\frac{1}{1-x})}$. To do this, let us consider a function $\sigma : [0,1] \to [0,1]$ given by $\sigma(t) = \frac{t}{d(t)} = \frac{t}{t+e-et}$. Then $\sigma^{-1}(t) = \frac{et}{1-t+et}$. Define the composition operator T on $L^0[0,1]$ by

$$Th(t) = h\big(\sigma(t)\big).$$

The key now is to notice that

$$T\chi_{[0,a)}(t) = \chi_{[0,a)}\big(\sigma(t)\big) = \chi_{[0,\sigma^{-1}(a))}(t),$$

where the last equality is a consequence of equivalence $\sigma(t) = a \Leftrightarrow \sigma^{-1}(a) = t$. Therefore, if $a = x/d(x)$, then

$$T\chi_{[0,x/d(x))} = \chi_{[0,x)}$$

and consequently for g and g_d like above

$$Tg_d = g.$$

To complete the proof it is enough to show that T is bounded on $X'(\frac{1}{1-x})$. Of course, using weighted version of the Calderón–Mitjagin interpolation theorem (cf. Appendix A) it suffices to prove its boundedness only on $L^\infty(\frac{1}{1-x})$ and on $L^1(\frac{1}{1-x})$. Since $\varphi(x) = 1-x$ belongs to $L^\infty(\frac{1}{1-x})$ and T preserves lattice structure, we only need to show that $T\varphi \in L^\infty(\frac{1}{1-x})$. We have

$$T\varphi(x) = \varphi\big(\sigma(x)\big) = 1-\sigma(x) = 1 - \frac{x}{x+e-ex} = \frac{(1-x)e}{x+e-ex}$$

and so

$$\frac{T\varphi(x)}{1-x} = \frac{e}{x+e-ex} \leq e \quad \text{for each } x \in [0,1],$$

which means that

$$\|T\|_{L^\infty(\frac{1}{1-x})\to L^\infty(\frac{1}{1-x})} \leq e.$$

On the other hand, for $h \in L^1(\frac{1}{1-x})$

$$\|Th\|_{L^1(\frac{1}{1-x})} = \int_0^1 \frac{h(\sigma(x))}{1-x} dx$$

and changing variables $\sigma(x) = u$, $dx = \frac{e}{(1-u+eu)^2} du$ we obtain

$$\|Th\|_{L^1(\frac{1}{1-x})} = \int_0^1 \frac{h(u)}{1-u} \frac{e}{1-u+eu} du \leq e \int_0^1 \frac{h(u)}{1-u} du.$$

Thus, once again

$$\|T\|_{L^1(\frac{1}{1-x}) \to L^1(\frac{1}{1-x})} \leq e.$$

To finish the proof notice that T is a bijection and so, in particular, for each $g \in S$, there is $h \in S$ such that $h_d = g$. □

Remark 12. The proof of Theorem 5 is true also for Banach ideal spaces X with the Fatou property in which the above composition operator T is bounded, for example, in weighted $L^p(w)$ spaces when the weight w is nondecreasing or power function on $I = (0, 1]$. Lemma 10 and so Theorem 4 are true in some non-symmetric spaces like weighted L^p-spaces (see Appendix B). In particular, Theorems 4 and 5 together with this remark give that

$$(Ces_{p,x^\alpha})' = \left[C\left(L^p(x^\alpha)\right)\right]' = \left(L^{p'}\left(\widetilde{\frac{1}{x^\alpha(1-x)}}\right)\right),$$

provided $\alpha < 1 - 1/p$ and $1 \leq p < \infty$.

Corollary 13. *If X is a symmetric space on $[0, 1]$ with the Fatou property such that $C, C^* : X \to X$ are bounded, then*

$$(CX)' = X'\left(\widetilde{\frac{1}{1-x}}\right).$$

5. Duality in the sequence case

Using analogous method we can prove the duality theorem also for Cesàro sequence spaces. Only in this section C will stand for the *discrete Cesàro operator* C, which is defined on a sequence $x = (x_n)$ of real numbers by

$$(Cx)_n = \frac{1}{n} \sum_{k=1}^{n} x_k, \quad n \in \mathbb{N}.$$

We define also the nonincreasing majorant $\widetilde{x}$ of a given sequence x by

$$(\widetilde{x})_n = \sup_{k \in \mathbb{N},\, k \geq n} |x_k|, \quad n \in \mathbb{N}.$$

If X is a Banach ideal sequence space, we define an *abstract Cesàro sequence space* CX as

$$CX = \left\{x \in \mathbb{R}^{\mathbb{N}} : C|x| \in X\right\} \quad \text{with the norm } \|x\|_{CX} = \big\|C|x|\big\|_X.$$

The space $\widetilde{X}$ is defined as before with evident modification. It is not difficult to see that $CX \neq \{0\}$ if and only if $(\frac{1}{n}) \in X$. Note that if $(\frac{1}{n}) \in X_a$, then $\|e_k\|_{CX} = \|C(e_k)\|_X = \|(\frac{1}{n})\chi_{[k,\infty)}\|_X \to 0$ as $k \to \infty$, which means that CX is not a symmetric space.

If $x = (x_n)$ and $m \in \mathbb{N}$, then the dilations $\sigma_m x$ are defined by (cf. [21, p. 131] and [18, p. 165]):

$$\sigma_m x = \left((\sigma_m x)_n\right)_{n=1}^{\infty} = (x_{[\frac{m-1+n}{m}]})_{n=1}^{\infty} = \big(\overbrace{x_1, x_1, \ldots, x_1}^{m}, \overbrace{x_2, x_2, \ldots, x_2}^{m}, \ldots\big).$$

We have the following duality result.

Theorem 6. *Let X be an ideal Banach sequence space such that the Cesàro operator C is bounded on X and the dilation operator σ_3 is bounded on X'. Then*

$$(CX)' = \widetilde{X'} \quad \textit{with equivalent norms.} \tag{5.1}$$

Proof. For the inclusion $(CX)' \hookrightarrow \widetilde{X'}$ it is enough to follow the proof of Theorem 3. Let us only notice that the corresponding result of Sinnamon, which is the main contribution there, holds also for counting measure on $\mathbb{N}$. As before, in this part of the proof we need C to be bounded on X.

We will comment the inclusion $(CX)' \hookleftarrow \widetilde{X'}$ a little more careful just because one cannot make a corresponding substitution in the case when an integral is replaced by a series. First of all we can reformulate just slightly the inequality from [6]. Proving exactly like there one has that for $0 \leq x \in \mathbb{R}^{\mathbb{N}}$,

$$\sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{k} x_j \geq \sum_{j=1}^{[\frac{n+1}{2}]} x_j \sum_{k=j}^{n} \frac{1}{k} \geq \frac{1}{4} \sum_{j=1}^{[\frac{n+1}{2}]} x_j,$$

just because $\sum_{k=[\frac{n+1}{2}]}^{n} \frac{1}{k} \geq \frac{1}{4}$ for $n \in \mathbb{N}$. Then

$$\sum_{j=1}^{n} x_{[\frac{j+2}{3}]} \leq 3 \sum_{j=1}^{[\frac{n+1}{2}]} x_j \leq 12 \sum_{j=1}^{n} (Cx)_j, \tag{5.2}$$

where details of the first estimation are left for the reader. Finally, for $y \in \widetilde{X'}$ and $0 \leq x \in CX$, we put $\widetilde{y} = b$ and by (5.2) and Hölder–Rogers inequality (2.2) we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} |y_n x_n| \leq \sum_{n=1}^{\infty} \widetilde{y}_n x_n = \frac{1}{3} \sum_{n=1}^{\infty} b_{[\frac{n+2}{3}]} x_{[\frac{n+2}{3}]} &\leq 4 \sum_{n=1}^{\infty} b_{[\frac{n+2}{3}]} (Cx)_n \\
&\leq 4\|\sigma_3 b\|_{X'} \|Cx\|_X \leq 4\|\sigma_3\|_{X' \to X'} \|b\|_{X'} \|Cx\|_X \\
&= 4\|\sigma_3\|_{X' \to X'} \|y\|_{\widetilde{X'}} \|x\|_{CX}.
\end{aligned}$$

Thus $\widetilde{X'} \overset{D}{\hookrightarrow} (CX)'$ with $D = 4\|\sigma_3\|_{X' \to X'}$. $\square$

Particular duality results for Cesàro sequence spaces ces_p and weighted Cesàro sequence spaces $ces_{p,w}$ were proved by several authors. Already in 1957 Alexiewicz [1] showed that for weight $w = (w_n)$ with $w_n \geq 0$, $w_1 > 0$ we have $(\widetilde{l^1(w)})' \equiv ces_{\infty,v}$, where $v(n) = \frac{n}{\sum_{k=1}^{n} w_k}$. Using the Fatou property of the space $\widetilde{l^1(w)}$ we obtain

$$(ces_{\infty,v})' \equiv \big(\widetilde{l^1(w)}\big)'' \equiv \widetilde{l^1(w)}. \tag{5.3}$$

Jagers [12] has presented the isometric description of $(ces_{p,w})'$ for $1 < p < \infty$, which is not so easy to present shortly. Ng and Lee [26] extended Jagers result on duality to the case $(ces_{\infty,w})'$ under additional assumption that $w_n \geq w_{n+1}$ for all $n \in \mathbb{N}$. Let us notice that the result in (5.3) is simpler and precisely described.

Bennett [4], using factorization technique (technique where we replace the classical inequalities by identities), showed that for $1 < p < \infty$ we have $(ces_p)' = \widetilde{(l^{p'})}$ with equivalent norms. This identification follows from his Theorems 4.5 and 12.3 in [4]. Moreover, on page 62, he proved (5.3) with the equality of norms.

Grosse-Erdmann [9, Corollary 7.5], using the blocking technique, was able to show Bennett's result on duality $(ces_p)' = \widetilde{(l^{p'})}$ with equivalent norms (for $1 < p < \infty$). He also has some weighted generalizations and duality results for more general sequence spaces. Blocking technique allows to replace sequence space which is quasi-normed by an expression in the section form with equivalent quasi-norm in block form and vice versa:

$$\left(\sum_{n=1}^{\infty}\left[a_n\left(\sum_{k=1}^{n}|x_k|^p\right)^{1/p}\right]^q\right)^{1/q} \approx \left(\sum_{\nu=0}^{\infty}\left[2^{-\nu\alpha}\left(\sum_{k\in I_\nu}|x_k|^p\right)^{1/p}\right]^q\right)^{1/q},$$

where $\{I_\nu\}$ is a partition of the natural numbers into disjoint intervals. Often, but not always, I_ν may be taken as the dyadic block $[2^\nu, 2^{\nu+1})$. The disadvantage of blocking technique is loosing the control over constants (it does not convey the best-possible constants) and can be used only for l^p or weighted l^p spaces, not for more general spaces.

For example, if $1 \leq p < \infty$ and $\alpha < 1 - 1/p$, then the discrete Cesàro operator C is bounded in weighted spaces $l^p(n^\alpha)$ (see Hardy–Littlewood [10] and Leindler [20] with more general weights and $\|C\|_{l^p(n^\alpha)\to l^p(n^\alpha)} \leq p\frac{(1-\alpha)p}{(1-\alpha)p-1}$). Moreover, we can easily prove that $\|\sigma_3\|_{l^p(n^\alpha)\to l^p(n^\alpha)} \leq 3^{1/p}\max(1, 3^\alpha)$ and from Theorem 6 we obtain duality

$$(ces_{p,\alpha})' = \left(Cl^p\left(n^\alpha\right)\right)' = \widetilde{l^{p'}\left(n^{-\alpha}\right)} \tag{5.4}$$

with equivalent norms. This result was also proved in [9, Theorem 7.2] by use of the blocking technique. Our method gives these results as well, but it is much simpler. Moreover, our Theorem 6 covers for example Cesàro–Orlicz sequence spaces (cf. [24]) or weighted Cesàro–Orlicz sequence spaces, where the blocking technique, Jagers' method or Bennett's factorization seem to be not applicable.

6. Extreme case and applications

First, we give a simple proof of a generalization of the Luxemburg–Zaanen [22] and Tandori [33] duality result to weighted L^∞-spaces. They proved that $(Ces_\infty[0,1])' \equiv (CL^\infty[0,1])' \equiv \widetilde{L^1[0,1]}$.

Theorem 7. *Let either $I = [0,1]$ or $I = [0,\infty)$. If a weight w on I is such that $W(x) = \int_0^x w(t)dt < \infty$ for any $x \in I$ and $v(x) = \frac{x}{W(x)}$, then*

$$(Ces_{\infty,v})' \equiv \left[C\left(L^\infty(v)\right)\right]' \equiv \widetilde{L^1(w)}. \tag{6.1}$$

Proof. Let $g \in \widetilde{L^1(w)}$ and $f \in X := C(L^\infty(v))$ with the norm $\|f\|_X = 1$. Then

$$\int_0^u \left|f(x)\right|dx \leq W(u)\|f\|_X = \int_0^u w(x)dx \quad \text{for all } u \in I,$$

and since $\widetilde{g}$ is nonincreasing on I we obtain

$$\int_I |f(x)|\widetilde{g}(x)dx \le \int_I w(x)\widetilde{g}(x)dx,$$

and so

$$\int_I |f(x)g(x)|dx \le \int_I |f(x)|\widetilde{g}(x)dx \le \int_I w(x)\widetilde{g}(x)dx.$$

Thus $\|g\|_{X'} \le \|g\|_{\widetilde{L^1(w)}}$ and $\widetilde{L^1(w)} \overset{1}{\hookrightarrow} X'$.

On the other hand, for $g_{W,t}(x) = \frac{1}{W(t)}\chi_{[0,t]}(x)$ with $t, x \in I$ and $f \in (\widetilde{L^1(w)})'$ we have

$$\|g_{W,t}\|_{\widetilde{L^1(w)}} = \|\widetilde{g_{W,t}}\|_{L^1(w)} = \|g_{W,t}\|_{L^1(w)} = \frac{1}{W(t)}\|\chi_{[0,t]}\|_{L^1(w)} = \frac{\int_0^t w(x)dx}{W(t)} = 1$$

and

$$\begin{aligned}\|f\|_{(\widetilde{L^1(w)})'} &= \sup_{\|g\|_{\widetilde{L^1(w)}}=1} \int_I |f(x)g(x)|dx \ge \sup_{t\in I} \int_I |f(x)g_{W,t}(x)|dx \\ &= \sup_{t\in I} \frac{\int_0^t |f(x)|dx}{W(t)} = \sup_{t\in I} C|f|(t)\frac{t}{W(t)} = \big\|C|f|\big\|_{L^\infty(v)} = \|f\|_X.\end{aligned}$$

This means $(\widetilde{L^1(w)})' \overset{1}{\hookrightarrow} X$ or

$$X' \overset{1}{\hookrightarrow} \big(\widetilde{L^1(w)}\big)'' \equiv \widetilde{L^1(w)}. \qquad \square$$

The above proof works as well in the case of sequence spaces and gives (5.3).

Section 4 of the paper [8] was devoted to the identification of Cesàro spaces CX with X being the Lorentz space Λ_φ defined on $I = [0, \infty)$ by

$$\Lambda_\varphi = \Big\{f \in L^0 : \|f\|_{\Lambda_\varphi} = \int_I f^*(t)d\varphi(t) < \infty\Big\},$$

where φ is a concave, positive and increasing function on I with $\varphi(0) = 0$. We shall demonstrate, that the result proved in [8, Theorem 4.4] is a straightforward consequence of our duality result in Theorem 3.

Theorem 8. *For a Lorentz space Λ_φ on $I = [0, \infty)$ with φ satisfying $\varphi(0^+) = 0$ and for which there are constants $c_1, c_2 > 0$ such that*

$$\int_0^t \frac{\varphi(s)}{s}ds \le c_1\varphi(t), \qquad \int_t^\infty \frac{\varphi(s)}{s^2}ds \le c_2\frac{\varphi(t)}{t} \quad \textit{for all } t > 0, \tag{6.2}$$

we have

$$C\Lambda_\varphi = L^1\big(\varphi(t)/t\big).$$

Proof. It is known that $(\Lambda_\varphi)' = M_{t/\varphi(t)}$ (see [18, Theorem 5.2, p. 112]), where $M_{t/\varphi(t)}$ is the Marcinkiewicz space given by the norm

$$\|f\|_{M_{t/\varphi(t)}} = \sup_{t>0} \frac{tf^{**}(t)}{\varphi(t)} = \sup_{t>0} \frac{\int_0^t f^*(s)\,ds}{\varphi(t)}.$$

Since $\|f\|_{M^*_{t/\varphi(t)}} \le \|f\|_{M_{t/\varphi(t)}} \le c_1 \|f\|_{M^*_{t/\varphi(t)}}$, where $\|f\|_{M^*_{t/\varphi(t)}} = \sup_{t>0} \frac{tf^*(t)}{\varphi(t)}$, it follows from Theorem 3 (the second estimate in (6.2) ensures boundedness of operator C in Λ_φ) that

$$(C\Lambda_\varphi)' = \widetilde{\Lambda'_\varphi} = \widetilde{M_{t/\varphi(t)}} = \widetilde{M^*_{t/\varphi(t)}}.$$

Also $(\widetilde{f})^* = \widetilde{f}$ gives that

$$\begin{aligned}\|f\|_{\widetilde{M^*_{t/\varphi(t)}}} &= \sup_{t>0} \frac{t\widetilde{f}(t)}{\varphi(t)} = \sup_{t>0} \operatorname*{ess\,sup}_{s\ge t} \frac{tf(s)}{\varphi(t)} = \operatorname*{ess\,sup}_{s>0} \sup_{t\le s} \frac{tf(s)}{\varphi(t)} \\ &= \operatorname*{ess\,sup}_{s>0} \frac{s|f(s)|}{\varphi(s)} = \|f\|_{L^\infty(t/\varphi(t))},\end{aligned}$$

because $t/\varphi(t)$ is once again nondecreasing. Therefore, $\widetilde{M^*_{t/\varphi(t)}} \equiv L^\infty(t/\varphi(t))$ and the result follows by duality $[L^\infty(\varphi(t)/t)]' \equiv L^1(t/\varphi(t))$. In fact,

$$C\Lambda_\varphi \equiv (C\Lambda_\varphi)'' = (\widetilde{M_{t/\varphi(t)}})' = \big(\widetilde{M^*_{t/\varphi(t)}}\big)' \equiv \big[L^\infty\big(t/\varphi(t)\big)\big]' \equiv L^1\big(\varphi(t)/t\big). \qquad \square$$

Appendix A

We present here a simple proof of weighted version of the Calderón–Mitjagin interpolation theorem. We will use notations from [18] and [3].

Proposition 14. *Let weight w and all symmetric spaces X, L^1, L^∞ be on I. If X is an interpolation space between L^1 and L^∞ with $\|T\|_{X\to X} \le C\max(\|T\|_{L^1\to L^1}, \|T\|_{L^\infty\to L^\infty})$, then $X(w)$ is an interpolation space between $L^1(w)$ and $L^\infty(w)$ and*

$$\|T\|_{X(w)\to X(w)} \le C\max\big(\|T\|_{L^1(w)\to L^1(w)}, \|T\|_{L^\infty(w)\to L^\infty(w)}\big). \tag{A.1}$$

Proof. First of all, note that for $f \in L^1(w) + L^\infty(w)$ we have

$$K\big(t, f; L^1(w), L^\infty(w)\big) = K\big(t, fw; L^1, L^\infty\big). \tag{A.2}$$

In fact, if $f = g + h$ is an arbitrary decomposition of f with $g \in L^1(w)$ and $h \in L^\infty(w)$, then $gw \in L^1$, $hw \in L^\infty$ and so

$$K\big(t, fw; L^1, L^\infty\big) \le \|gw\|_{L^1} + t\|hw\|_{L^\infty} = \|g\|_{L^1(w)} + t\|h\|_{L^\infty(w)},$$

which gives that $fw \in L^1 + L^\infty$ or $f \in (L^1 + L^\infty)(w)$ and

$$K\big(t, fw; L^1, L^\infty\big) \le K\big(t, f; L^1(w), L^\infty(w)\big).$$

On the other hand, if $f \in (L^1 + L^\infty)(w)$ or $fw \in L^1 + L^\infty$, then for arbitrary decomposition $fw = g_1 + g_2$ with $g_1 \in L^1$, $g_2 \in L^\infty$ we take for $i = 1, 2$

$$f_i = \frac{g_i}{w} \quad \text{on the support of } w \quad \text{and} \quad f_i = 0 \quad \text{elsewhere}.$$

Then

$$f = \frac{g_1}{w} + \frac{g_2}{w} = f_1 + f_2 \quad \text{on the support of } w \quad \text{and} \quad f = 0 \quad \text{elsewhere,}$$

and so $f_1 \in L^1(w)$, $f_2 \in L^\infty(w)$. Therefore,

$$\begin{aligned} K\big(t, f; L^1(w), L^\infty(w)\big) &\le \|f_1\|_{L^1(w)} + t\|f_2\|_{L^\infty(w)} \\ &= \|f_1 w\|_{L^1} + t\|f_2 w\|_{L^\infty} = \|g_1\|_{L^1} + t\|g_2\|_{L^\infty} \end{aligned}$$

for arbitrary decomposition $fw = g_1 + g_2$, which gives

$$K\big(t, f; L^1(w), L^\infty(w)\big) \le K\big(t, fw; L^1, L^\infty\big),$$

and (A.2) is proved. Secondly, if $T : (L^1(w), L^\infty(w)) \to (L^1(w), L^\infty(w))$ is a bounded linear operator, then

$$K\big(t, Tf; L^1(w), L^\infty(w)\big) \le \max(C_1, C_\infty) K\big(t, f; L^1(w), L^\infty(w)\big)$$

for any $fw \in L^1 + L^\infty$, where $C_i = \|T\|_{L^i(w) \to L^i(w)}$, $i = 1, \infty$. Therefore, by (A.2), we obtain

$$K\big(t, (Tf)w; L^1, L^\infty\big) \le \max(C_1, C_\infty) K\big(t, fw; L^1, L^\infty\big) \quad \text{for any } fw \in L^1 + L^\infty.$$

If now $fw \in X$, then by the Calderón–Mitjagin interpolation theorem (cf. [5, Theorem 3], [18, Theorem 4.3 on p. 95] and [3, Theorem 2.12]) we have $(Tf)w \in X$ and

$$\big\|(Tf)w\big\|_X \le C \max(C_1, C_\infty)\|fw\|_X \quad \text{or} \quad \|Tf\|_{X(w)} \le C \max(C_1, C_\infty)\|f\|_{X(w)}.$$

Thus, estimate (A.1) is proved. □

Appendix B

We give an improvement of the Hardy inequality on $[0, 1]$.

Theorem 9. *If $1 \le p < \infty$ and $\alpha < 1 - 1/p$, then*

$$\int_0^1 \big[Cf(x)x^\alpha\big]^p \, dx \le (C_{p,\alpha})^p \int_0^1 \big[(1 - x)f(x)x^\alpha\big]^p \, dx \tag{B.1}$$

for all $0 \le f \in L^p((1 - x)x^\alpha)$, where $C_{p,\alpha} = \frac{p}{p - \alpha p - 1} \max(1, p - \alpha p - 1)^{1/p}$.

Proof. For $p = 1$, $\alpha < 0$ and $0 \le f \in L^1(x^\alpha)$ we have by the Fubini theorem

$$\begin{aligned} \int_0^1 \big[x^\alpha Cf(x)\big] \, dx &= \int_0^1 x^{\alpha - 1}\left(\int_0^x f(t) \, dt\right) dx = \int_0^1 \left(\int_t^1 x^{\alpha - 1} \, dx\right) f(t) \, dt \\ &= \frac{1}{-\alpha} \int_0^1 \big(1 - t^{-\alpha}\big) f(t)t^\alpha \, dt \le \frac{\max(1, -\alpha)}{-\alpha} \int_0^1 (1 - t)f(t)t^\alpha \, dt. \end{aligned}$$

Let $1 < p < \infty$ and $0 \le f \in L^1[0,1]$. Simple differentiation of $F(x) = (\int_0^x f(t)\,dt)^p$ gives equality (sometimes referred as Davis–Petersen's lemma – see [7, Lemma 2])

$$\left(\int_0^x f(t)\,dt\right)^p = p\int_0^x f(t)\left[\int_0^t f(s)\,ds\right]^{p-1} dt. \tag{B.2}$$

Let $0 \le f \in L^p(x^\alpha)$. Of course, $f \in L^1$ because $L^p(x^\alpha) \overset{A}{\hookrightarrow} L^1$ with $A = (1-\alpha p')^{-1/p'}$. We have

$$\begin{aligned} I = \int_0^1 \left[x^\alpha Cf(x)\right]^p dx &= \int_0^1 x^{(\alpha-1)p}\left(\int_0^x f(t)\,dt\right)^p dx \\ &= p\int_0^1 x^{(\alpha-1)p}\left(\int_0^x g(t)t^{p-1}dt\right)dx, \end{aligned}$$

where $g(t) = f(t)[Cf(t)]^{p-1}$. By the Fubini theorem and the Hölder–Rogers inequality the last integral is

$$\begin{aligned} I &= p\int_0^1\left(\int_t^1 x^{(\alpha-1)p}dx\right)g(t)t^{p-1}dt = p\int_0^1 \frac{1-t^{(\alpha-1)p+1}}{(\alpha-1)p+1}g(t)t^{p-1}dt \\ &= \frac{p}{(1-\alpha)p-1}\int_0^1 \left(t^{(\alpha-1)p+1}-1\right)t^{-\alpha p+p-1}g(t)t^{\alpha p}dt \\ &= \frac{p}{p-\alpha p-1}\int_0^1 \left(1-t^{p-\alpha p-1}\right)f(t)\left[Cf(t)\right]^{p-1}t^{\alpha p}dt \\ &\le \frac{p}{p-\alpha p-1}\left(\int_0^1\left(1-t^{p-\alpha p-1}\right)^p f(t)^p t^{\alpha p}dt\right)^{1/p}\left(\int_0^1 Cf(t)^p t^{\alpha p}dt\right)^{1/p'}. \end{aligned}$$

Observe that $p - \alpha p - 1 > 0$ and so

$$1 - t^{p-\alpha p-1} \le \max(1, p-\alpha p-1)(1-t) \quad \text{for } t \in I.$$

Really, if $p - \alpha p - 1 \le 1$, then it is clear and if $p - \alpha p - 1 \ge 1$, then by the Bernoulli inequality

$$t^{p-\alpha p-1} = (1+t-1)^{p-\alpha p-1} \ge 1 + (p-\alpha p-1)(t-1),$$

that is, $1 - t^{p-\alpha p-1} \le (p-\alpha p-1)(1-t)$. Moreover, note that if $0 \le f \in L^p(x^\alpha)$ and $\alpha < 1 - 1/p$, then by the classical Hardy inequality (cf. [10,11,19]) $Cf \in L^p(x^\alpha)$. Hence,

$$\begin{aligned} I &= \int_0^1 \left[Cf(x)x^\alpha\right]^p dx \\ &\le \frac{p}{p-\alpha p-1}\max(1, p-\alpha p-1)^{1/p}\left(\int_0^1 (1-t)^p f(t)^p t^{\alpha p}dt\right)^{1/p} I^{1/p'}, \end{aligned}$$

and dividing by $I^{1/p'}$ we obtain

$$\left(\int_0^1 \left[Cf(x)x^\alpha\right]^p dx\right)^{1/p} \leq C_{p,\alpha}\left(\int_0^1 (1-x)^p f(x)^p x^{\alpha p} dx\right)^{1/p},$$

which is (B.1) for all $0 \leq f \in L^p(x^\alpha)$. Since subspace $L^p(x^\alpha)$ is dense in $L^p((1-x)x^\alpha)$ we can extend estimate (B.1) to all $f \in L^p((1-x)x^\alpha)$, which finishes the proof. $\square$

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