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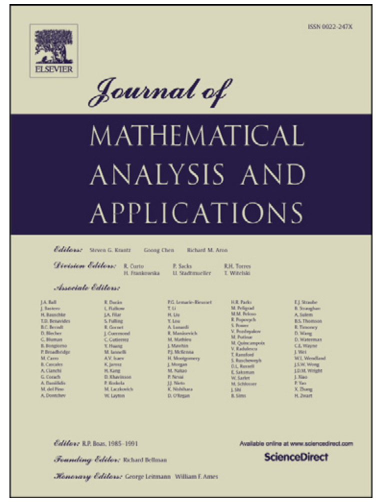


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# A Hilbert-Type Integral Inequality in the Whole Plane Related to the Hypergeometric Function and the Beta Function

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**Abstract**

A new Hilbert-type integral inequality in the whole plane with the non-homogeneous kernel and parameters is given. The constant factor related to the hypergeometric function and the beta function is proved to be the best possible. As applications, equivalent forms, the reverses, some particular examples, two kinds of Hardy-type inequalities, and operator expressions are considered.

**Key words:** Hilbert-type integral inequality; weight function; equivalent form; hypergeometric function; beta function;
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## 1 Introduction

If $f(x), g(y) \geq 0$, satisfy

$$0 < \int_0^\infty f^2(x)dx < \infty$$

and

$$0 < \int_0^\infty g^2(y)dy < \infty,$$

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then we have the following Hilbert's integral inequality (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \pi \left( \int_0^\infty f^2(x)dx \int_0^\infty g^2(y)dy \right)^{\frac{1}{2}}, \tag{1.1}$$

where the constant factor $\pi$ is the best possible. The inequality (1.1) is very important in Mathematical Analysis and its applications (cf. [1], [2]). In recent years, by the use of the method of weight functions, a number of extensions of (1.1) were given by Yang (cf. [3]). Noticing that inequality (1.1) is a homogenous kernel of degree -1, in 2009, a survey of the study of Hilbert-type inequalities with the homogeneous kernels of degree equal to negative numbers and some parameters is given in [4]. Recently, some inequalities with the homogenous kernels of degree 0 and non-homogenous kernels have been proved (cf. [5]-[10]). Other kinds of Hilbert-type inequalities are shown in [11]-[16]. All of the above integral inequalities are constructed in the quarter plane of the first quadrant.

In 2007, Yang [17] presented a new Hilbert-type integral inequality in the whole plane, as follows:

$$\begin{aligned} &\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{f(x)g(y)}{(1+e^{x+y})^\lambda} dxdy \\ < \ & B(\frac{\lambda}{2},\frac{\lambda}{2})(\int_{-\infty}^\infty e^{-\lambda x} f^2(x)dx \int_{-\infty}^\infty e^{-\lambda y} g^2(y)dy)^{\frac{1}{2}}, \end{aligned} \tag{1.2}$$

where the constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})(\lambda > 0)$ is the best possible.
If $0 < \lambda < 1$, $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, Yang [20] derived another new Hilbert-type integral inequality in the whole plane. Namely, he proved that

$$\begin{aligned} &\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{|1+xy|^\lambda} f(x)g(y)dxdy \\ < \ & k_\lambda \left[ \int_{-\infty}^\infty |x|^{p(1-\frac{\lambda}{2})-1} f^p(x)dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^\infty |y|^{q(1-\frac{\lambda}{2})-1} g^q(y)dy \right]^{\frac{1}{q}}, \end{aligned} \tag{1.3}$$

where the constant factor

$$k_\lambda = B(\frac{\lambda}{2},\frac{\lambda}{2}) + 2B(1-\lambda,\frac{\lambda}{2})$$

is still the best possible. Furthermore, Yang et al. [19]-[28] proved as well some new Hilbert-type integral inequalities in the whole plane.

In this paper, using methods from Real Analysis and by estimating the weight functions, a new Hilbert-type integral inequality in the whole plane with the non-homogeneous kernel and multi-parameters is shown, which gives an extension of (1.3). The constant factor related to the hypergeometric function and the beta function is proved to be the best possible. As applications, equivalent forms, the reverses, some particular examples, two kinds of Hardy-type inequalities, and operator expressions are also considered.

## 2 Some Lemmas

Initially, we introduce the following formula of the hypergeometric function $F$ (cf. [29]): If $\mathrm{Re}(\gamma) > \mathrm{Re}(\theta) > 0$, $|\arg(1-z)| < \pi$, $(1-zt)^{-\alpha}|_{z=0} = 1$, then

$$F(\alpha,\theta,\gamma,z) := \frac{\Gamma(\gamma)}{\Gamma(\theta)\Gamma(\gamma-\theta)}\int_0^1 t^{\theta-1}(1-t)^{\gamma-\theta-1}(1-zt)^{-\alpha}dt,$$

where,

$$\Gamma(\eta) = \int_0^\infty x^{\eta-1}e^{-x}dx(\mathrm{Re}(\eta) > 0)$$

is the gamma function. In particular, for $z = -1, \gamma = \theta + 1\ (\theta > 0)$, $\alpha \in \mathbf{R}$, we have

$$\int_0^1 t^{\theta-1}(1+t)^{-\alpha}dt = \frac{1}{\theta}F(\alpha,\theta,1+\theta,-1) \in \mathbf{R}_+. \tag{2.1}$$

**Lemma 2.1.** *If $\beta > -1, \mu, \sigma > -\beta, \mu+\sigma = \lambda < 1-\beta, \delta \in \{-1,1\}$, we define two weight functions $\omega(\sigma,y)$ and $\varpi(\sigma,x)$ as follows:*

$$\omega(\sigma,y) \ : \ = \int_{-\infty}^{\infty} \frac{(\min\{1,|x^\delta y|\})^\beta}{|1+x^\delta y|^{\lambda+\beta}} \frac{|y|^\sigma}{|x|^{1-\delta\sigma}}dx(y \in \mathbf{R}\backslash\{0\}), \tag{2.2}$$

$$\varpi(\sigma,x) \ : \ = \int_{-\infty}^{\infty} \frac{(\min\{1,|x^\delta y|\})^\beta}{|1+x^\delta y|^{\lambda+\beta}} \frac{|x|^{\delta\sigma}}{|y|^{1-\sigma}}dy(x \in \mathbf{R}\backslash\{0\}). \tag{2.3}$$

*Then we have*

$$\begin{aligned}\omega(\sigma,y) \ &= \ \varpi(\sigma,x) = K(\sigma) := \frac{1}{\beta+\sigma}F(\lambda+\beta,\beta+\sigma,1+\beta+\sigma,-1)\\ &+\frac{1}{\beta+\mu}F(\lambda+\beta,\beta+\mu,1+\beta+\mu,-1)\\ &+B(1-\lambda-\beta,\beta+\sigma)+B(1-\lambda-\beta,\beta+\mu) \in \mathbf{R}_+.\end{aligned} \tag{2.4}$$

*Proof.* (i) For $\delta = 1$, by (2.2) it follows that

$$\omega(\sigma,y) = \int_{-\infty}^{\infty} \frac{(\min\{1,|xy|\})^\beta}{|1+xy|^{\lambda+\beta}} \frac{|y|^\sigma}{|x|^{1-\sigma}}dx.$$

(a) If $y < 0$, setting $u = xy$, we obtain

$$\begin{aligned}\omega(\sigma,y) \ &= \ \int_{\infty}^{-\infty} \frac{(\min\{1,|u|\})^\beta}{|1+u|^{\lambda+\beta}} \frac{(-y)^\sigma}{|u/y|^{1-\sigma}} \frac{1}{y}du\\ &= \ \int_{-\infty}^{\infty} \frac{(\min\{1,|u|\})^\beta}{|1+u|^{\lambda+\beta}} \frac{(-y)^\sigma(-y)^{1-\sigma}}{|u|^{1-\sigma}} \frac{1}{(-y)}du\\ &= \ \int_{-\infty}^{\infty} \frac{(\min\{1,|u|\})^\beta|u|^{\sigma-1}}{|1+u|^{\lambda+\beta}}du;\end{aligned}$$

(b) if $y > 0$, setting $u = xy$, it yields

$$\omega(\sigma,y) = \int_{-\infty}^{\infty} \frac{(\min\{1,|u|\})^\beta}{|1+u|^{\lambda+\beta}} \frac{y^\sigma du}{|u/y|^{1-\sigma}y} = \int_{-\infty}^{\infty} \frac{(\min\{1,|u|\})^\beta|u|^{\sigma-1}}{|1+u|^{\lambda+\beta}}du.$$

(ii) For $\delta = -1$, setting $X = x^{-1}$, we obtain

$$\begin{aligned}
\omega(\sigma, y) &= \int_{-\infty}^{0} \frac{(\min\{1, |x^{-1}y|\})^{\beta}}{|1+x^{-1}y|^{\lambda+\beta}} \frac{|y|^{\sigma}}{|x|^{1+\sigma}} dx + \int_{0}^{\infty} \frac{(\min\{1, |x^{-1}y|\})^{\beta}}{|1+x^{-1}y|^{\lambda+\beta}} \frac{|y|^{\sigma}}{|x|^{1+\sigma}} dx \\
&= \int_{0}^{-\infty} \frac{(\min\{1, |Xy|\})^{\beta}}{|1+Xy|^{\lambda+\beta}} \frac{-|y|^{\sigma} dX}{|X^{-1}|^{1+\sigma} X^{2}} + \int_{\infty}^{0} \frac{(\min\{1, |Xy|\})^{\beta}}{|1+Xy|^{\lambda+\beta}} \frac{-|y|^{\sigma} dX}{|X^{-1}|^{1+\sigma} X^{2}} \\
&= \int_{-\infty}^{0} \frac{(\min\{1, |Xy|\})^{\beta}}{|1+Xy|^{\lambda+\beta}} \frac{|y|^{\sigma} dX}{|X|^{1-\sigma}} + \int_{0}^{\infty} \frac{(\min\{1, |Xy|\})^{\beta}}{|1+Xy|^{\lambda+\beta}} \frac{|y|^{\sigma} dX}{|X|^{1-\sigma}} \\
&= \int_{-\infty}^{\infty} \frac{(\min\{1, |xy|\})^{\beta}}{|1+xy|^{\lambda+\beta}} \frac{|y|^{\sigma}}{|x|^{1-\sigma}} dx.
\end{aligned}$$

Hence, for $\delta \in \{-1, 1\}$, we obtain the following expression:

$$\begin{aligned}
\omega(\sigma, y) &= \int_{-\infty}^{\infty} \frac{(\min\{1, |u|\})^{\beta} |u|^{\sigma-1}}{|1+u|^{\lambda+\beta}} du = K_1(\sigma) + K_2(\sigma), \\
K_1(\sigma) &:= \int_{-1}^{1} \frac{(\min\{1, |u|\})^{\beta} |u|^{\sigma-1}}{(1+u)^{\lambda+\beta}} du, \\
K_2(\sigma) &:= \int_{\mathbf{R} \setminus (-1,1)} \frac{(\min\{1, |u|\})^{\beta} |u|^{\sigma-1}}{|1+u|^{\lambda+\beta}} du.
\end{aligned}$$

In view of (2.1), and the following formula of the beta function (cf. [29]):

$$B(p, q) := \int_{0}^{1} \frac{v^{p-1}}{(1-v)^{1-q}} dv \, (p, q > 0),$$

we obtain

$$\begin{aligned}
K_1(\sigma) &= \int_{-1}^{0} \frac{(-u)^{\beta+\sigma-1}}{(1+u)^{\lambda+\beta}} du + \int_{0}^{1} \frac{u^{\beta+\sigma-1}}{(1+u)^{\lambda+\beta}} du \\
&= \int_{0}^{1} \frac{v^{\beta+\sigma-1}}{(1-v)^{\lambda+\beta}} dv + \int_{0}^{1} \frac{u^{\beta+\sigma-1}}{(1+u)^{\lambda+\beta}} du \\
&= B(1-\lambda-\beta, \beta+\sigma) + \frac{1}{\beta+\sigma} F(\lambda+\beta, \beta+\sigma, 1+\beta+\sigma, -1). \qquad (2.5)
\end{aligned}$$

Setting $v = \frac{1}{u}$, it follows that

$$\begin{aligned}
K_2(\sigma) &= \int_{-\infty}^{-1} \frac{(\min\{1, (-u)\})^{\beta} (-u)^{\sigma-1}}{|1+u|^{\lambda+\beta}} du + \int_{1}^{\infty} \frac{(\min\{1, u\})^{\beta} u^{\sigma-1}}{|1+u|^{\lambda+\beta}} du \\
&= -\int_{0}^{-1} \frac{(\min\{1, (-\frac{1}{v})\})^{\beta} (-\frac{1}{v})^{\sigma-1}}{|1+\frac{1}{v}|^{\lambda+\beta} v^{2}} dv - \int_{1}^{0} \frac{(\min\{1, \frac{1}{v}\})^{\beta} (\frac{1}{v})^{\sigma-1}}{|1+\frac{1}{v}|^{\lambda+\beta} v^{2}} dv \\
&= \int_{-1}^{0} \frac{(\min\{1, (-v)\})^{\beta} (-v)^{\lambda-\sigma-1}}{|1+v|^{\lambda+\beta}} dv + \int_{0}^{1} \frac{(\min\{1, v\})^{\beta} v^{\lambda-\sigma-1}}{|1+v|^{\lambda+\beta}} dv \\
&= \int_{-\mathbf{1}}^{1} \frac{(\min\{1, |v|\})^{\beta} |v|^{\mu-1}}{(1+v)^{\lambda+\beta}} dv = K_1(\mu) \\
&= B(1-\lambda-\beta, \beta+\mu) + \frac{1}{\mu} F(\lambda+\beta, \beta+\mu, 1+\beta+\mu, -1). \qquad (2.6)
\end{aligned}$$

Setting $u = x^{\delta} y$ in (2.3), for $x < 0\,(x > 0)$, we also get

$$\varpi(\sigma, x) = \int_{-\infty}^{\infty} \frac{(\min\{1, |u|\})^{\beta} |u|^{\sigma-1}}{|1+u|^{\lambda+\beta}} du = K(\sigma).$$

Hence we have (2.4). □

*Remark* 2.2. By Taylor's formula, we obtain

$$\begin{aligned}
&\frac{1}{\beta+\sigma} F(\lambda+\beta, \beta+\sigma, 1+\beta+\sigma, -1) \\
= &\int_0^1 \frac{u^{\beta+\sigma-1} du}{(1+u)^{\lambda+\beta}} = \int_0^1 \sum_{k=0}^{\infty} \binom{-\lambda-\beta}{k} u^{k+\beta+\sigma-1} du \\
= &\int_0^1 \sum_{k=0}^{\infty} (-1)^k \binom{\lambda+\beta+k-1}{k} u^{k+\beta+\sigma-1} du \\
= &\int_0^1 \sum_{k=0}^{\infty} [\binom{\lambda+\beta+2k-1}{2k} - \binom{\lambda+\beta+2k}{2k+1} u] u^{2k+\beta+\sigma-1} du.
\end{aligned}$$

Since we have

$$\binom{\lambda+\beta+2k-1}{2k} - \binom{\lambda+\beta+2k}{2k+1} u = [1 - \frac{(\lambda+\beta+2k)u}{2k+1}] \binom{\lambda+\beta+2k-1}{2k},$$

there exists a large number $k_0 \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$, such that $\lambda+\beta+2k_0 > 0$, and for any $s \in \mathbf{N}$,

$$\begin{aligned}
&\binom{\lambda+\beta+2(k_0+s)-1}{2(k_0+s)} - \binom{\lambda+\beta+2(k_0+s)+1}{2(k_0+s+1)} u \\
= &[1 - \frac{(\lambda+\beta+2k_0+2s)u}{2(k_0+s)+1}] \binom{\lambda+\beta+2(k_0+s)-1}{2(k_0+s)} \\
= &[1 - \frac{(\lambda+\beta+2k_0+2s)u}{2(k_0+s)+1}] \\
&\times \frac{\lambda+\beta+2k_0+2s-1}{2k_0+2s} \cdots \frac{\lambda+\beta+2k_0}{2k_0+1} \binom{\lambda+\beta+2k_0-1}{2k_0},
\end{aligned}$$

$$\begin{aligned}
&1 - \frac{(\lambda+\beta+2k_0+2s)u}{2(k_0+s)+1} \\
\geq &1 - \frac{\lambda+\beta+2k_0+2s}{2(k_0+s)+1} = \frac{1-\lambda-\beta}{2(k_0+s)+1} > 0 (u \in (0,1]).
\end{aligned}$$

It follows that for any $s \in \mathbf{N}$, we have

$$sgn(\binom{\lambda+\beta+2(k_0+s)-1}{2(k_0+s)} - \binom{\lambda+\beta+2(k_0+s)+1}{2(k_0+s+1)} u) = sgn \binom{\lambda+\beta+2k_0-1}{2k_0}.$$

By Lebesgue's term by term integration theorem (cf. [31]), we have

$$\begin{aligned}
&\frac{1}{\sigma} F(\lambda+\beta, \beta+\sigma, 1+\beta+\sigma, -1) \\
= &\sum_{k=0}^{\infty} \int_0^1 [\binom{\lambda+\beta+2k-1}{2k} - \binom{\lambda+\beta+2k}{2k+1} u] u^{2k+\beta+\sigma-1} du \\
= &\sum_{k=0}^{\infty} \binom{-\lambda-\beta}{k} \int_0^1 u^{k+\beta+\sigma-1} du = \sum_{k=0}^{\infty} \frac{1}{k+\beta+\sigma} \binom{-\lambda-\beta}{k}.
\end{aligned}$$

Similarly, since

$$(-1)^k \binom{-\lambda-\beta}{k} = (-1)^k \frac{(-\lambda-\beta)(-\lambda-\beta-1)\cdots(-\lambda-\beta-k+1)}{k!}$$

$$= (-1)^{2k} \frac{(\lambda+\beta)(\lambda+\beta+1)\cdots(\lambda+\beta+k-1)}{k!} = \binom{\lambda+\beta+k-1}{k},$$

we obtain

$$B(1-\lambda-\beta,\beta+\sigma) = \int_0^1 \frac{u^{\beta+\sigma-1}du}{(1-u)^{\lambda+\beta}} = \int_0^1 \sum_{k=0}^{\infty} (-1)^k \binom{-\lambda-\beta}{k} u^{k+\beta+\sigma-1}du$$

$$= \int_0^1 \sum_{k=0}^{\infty} \binom{\lambda+\beta+k-1}{k} u^{k+\beta+\sigma-1}du.$$

There exists a large number $k_1 \in \mathbf{N}$, such that $\lambda+\beta+k_1 > 0$.
Hence, for any $s \in \mathbf{N}$, we have

$$\binom{\lambda+\beta+k_1+s-1}{k_1+s} = \frac{\lambda+\beta+k_1+s-1}{k_1+s} \cdots \frac{\lambda+\beta+k_1}{k_1+1} \binom{\lambda+\beta+k_1-1}{k_1},$$

and then it follows that

$$sgn\binom{\lambda+\beta+k_1+s-1}{k_1+s} = sgn\binom{\lambda+\beta+k_1-1}{k_1}.$$

Still by Lebesgue's term by term integration theorem, we obtain

$$B(1-\lambda-\beta,\beta+\sigma) = \sum_{k=0}^{\infty} \binom{\lambda+\beta+k-1}{k} \int_0^1 u^{k+\beta+\sigma-1}du = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+\beta+\sigma} \binom{-\lambda-\beta}{k}.$$

Hence, we deduce the following series expressions:

$$K_1(\sigma) = 2\sum_{k=0}^{\infty} \frac{1}{2k+\beta+\sigma} \binom{-\lambda-\beta}{2k}, \tag{2.7}$$

$$K_2(\sigma) = 2\sum_{k=0}^{\infty} \frac{1}{2k+\beta+\mu} \binom{-\lambda-\beta}{2k}, \tag{2.8}$$

$$K(\sigma) = 2\sum_{k=0}^{\infty} \frac{4k+2\beta+\lambda}{(2k+\beta+\sigma)(2k+\beta+\mu)} \binom{-\lambda-\beta}{2k}. \tag{2.9}$$

**Lemma 2.3.** *Suppose that* $p > 1, \frac{1}{p}+\frac{1}{q} = 1, \beta > -1, \mu, \sigma > -\beta, \mu+\sigma = \lambda < 1-\beta$, $\delta \in \{-1,1\}$, $K(\sigma)$ *as indicated by (2.4) (or (2.9)).*
*If* $f(x)$ *is a non-negative measurable function in* $\mathbf{R}$, *then we have*

$$J := \int_{-\infty}^{\infty} |y|^{p\sigma-1} \left[\int_{-\infty}^{\infty} \frac{(\min\{1,|x^{\delta}y|\})^{\beta}}{|1+x^{\delta}y|^{\lambda+\beta}} f(x)dx\right]^p dy$$

$$\leq K^p(\sigma) \int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x)dx. \tag{2.10}$$

*Proof.* By Hölder's inequality (cf. [30]) and (2.2), we derive that

$$\begin{aligned}
&\left[\int_{-\infty}^{\infty}\frac{(\min\{1,|x^{\delta}y|\})^{\beta}}{|1+x^{\delta}y|^{\lambda+\beta}}f(x)dx\right]^{p}\\
&= \left\{\int_{-\infty}^{\infty}\frac{(\min\{1,|x^{\delta}y|\})^{\beta}}{|1+x^{\delta}y|^{\lambda+\beta}}[\frac{|x|^{(1-\delta\sigma)/q}}{|y|^{(1-\sigma)/p}}f(x)][\frac{|y|^{(1-\sigma)/p}}{|x|^{(1-\delta\sigma)/q}}]dx\right\}^{p}\\
&\leq \int_{-\infty}^{\infty}\frac{(\min\{1,|x^{\delta}y|\})^{\beta}}{|1+x^{\delta}y|^{\lambda+\beta}}\frac{|x|^{(1-\delta\sigma)(p-1)}}{|y|^{1-\sigma}}f^{p}(x)dx\\
&\quad\times\left[\int_{-\infty}^{\infty}\frac{(\min\{1,|x^{\delta}y|\})^{\beta}}{|1+x^{\delta}y|^{\lambda+\beta}}\frac{|y|^{(1-\sigma)(q-1)}}{|x|^{1-\delta\sigma}}dx\right]^{p-1} \qquad (2.11)\\
&= \frac{(\omega(\sigma,y))^{p-1}}{|y|^{p\sigma-1}}\int_{-\infty}^{\infty}\frac{(\min\{1,|x^{\delta}y|\})^{\beta}}{|1+x^{\delta}y|^{\lambda+\beta}}\frac{|x|^{(1-\delta\sigma)(p-1)}}{|y|^{1-\sigma}}f^{p}(x)dx.
\end{aligned}$$

Then, by (2.4) and Fubini's theorem (cf. [31]), it follows that

$$\begin{aligned}
J &\leq K^{p-1}(\sigma)\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{(\min\{1,|x^{\delta}y|\})^{\beta}}{|1+x^{\delta}y|^{\lambda+\beta}}\frac{|x|^{(1-\delta\sigma)(p-1)}}{|y|^{1-\sigma}}f^{p}(x)dx\right]dy\\
&= K^{p-1}(\sigma)\int_{-\infty}^{\infty}\varpi(\sigma,x)|x|^{p(1-\delta\sigma)-1}f^{p}(x)dx. \qquad (2.12)
\end{aligned}$$

Hence, by (2.4), inequality (2.10) follows. □

## 3 Main Results and Applications

**Theorem 3.1.** *Let* $p>1,\frac{1}{p}+\frac{1}{q}=1,\beta>-1,\mu,\sigma>-\beta$, $\mu+\sigma=\lambda<1-\beta$, $\delta\in\{-1,1\}$, $K(\sigma)$ *as indicated by (2.4) (or (2.9)).*
*If* $f,g\geq 0$, *satisfy*

$$0<\int_{-\infty}^{\infty}|x|^{p(1-\delta\sigma)-1}f^{p}(x)dx<\infty$$

*and*

$$0<\int_{-\infty}^{\infty}|y|^{q(1-\sigma)-1}g^{q}(y)dy<\infty,$$

*then we have the following equivalent inequalities:*

$$\begin{aligned}
I &:= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(\min\{1,|x^{\delta}y|\})^{\beta}}{|1+x^{\delta}y|^{\lambda+\beta}}f(x)g(y)dxdy\\
&< K(\sigma)\left[\int_{-\infty}^{\infty}|x|^{p(1-\delta\sigma)-1}f^{p}(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q(1-\sigma)-1}g^{q}(y)dy\right]^{\frac{1}{q}}, \qquad (3.1)
\end{aligned}$$

$$\begin{aligned}
J &:= \int_{-\infty}^{\infty}|y|^{p\sigma-1}\left[\int_{-\infty}^{\infty}\frac{(\min\{1,|x^{\delta}y|\})^{\beta}}{|1+x^{\delta}y|^{\lambda+\beta}}f(x)dx\right]^{p}dy\\
&< K^{p}(\sigma)\int_{-\infty}^{\infty}|x|^{p(1-\delta\sigma)-1}f^{p}(x)dx. \qquad (3.2)
\end{aligned}$$

*where, the constant factors $K(\sigma)$ and $K^p(\sigma)$ are the best possible.*

*In particular, for $\delta = 1$, we obtain the following equivalent inequalities:*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(\min\{1,|xy|\})^{\beta}}{|1+xy|^{\lambda+\beta}}f(x)g(y)dxdy < K(\sigma)\left[\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}}, \tag{3.3}$$

$$\int_{-\infty}^{\infty}|y|^{p\sigma-1}\left[\int_{-\infty}^{\infty}\frac{(\min\{1,|xy|\})^{\beta}}{|1+xy|^{\lambda+\beta}}f(x)dx\right]^p dy < K^p(\sigma)\int_{-\infty}^{\infty}|x|^{p(1-\sigma)-1}f^p(x)dx. \tag{3.4}$$

*Proof.* If (2.11) takes the form of equality for some $y \in (-\infty,0)\cup(0,\infty)$, then, there exist constants $A$ and $B$, such that they are not all zero, and

$$A\frac{|x|^{(1-\delta\sigma)(p-1)}}{|y|^{1-\sigma}}f^p(x) = B\frac{|y|^{(1-\sigma)(q-1)}}{|x|^{1-\delta\sigma}}\ a.e.\ in\ \mathbf{R}.$$

Let us suppose that $A \neq 0$ (otherwise $B = A = 0$). Then it follows that

$$|x|^{p(1-\delta\sigma)-1}f^p(x) = |y|^{q(1-\sigma)}\frac{B}{A|x|}\ a.e.\ in\mathbf{R},$$

which contradicts the fact that

$$0 < \int_{-\infty}^{\infty}|x|^{p(1-\delta\sigma)-1}f^p(x)dx < \infty.$$

Hence (2.11) takes the form of strict inequality. So does (2.12), and we obtain (3.2).

By Hölder's inequality (cf. [30]), we have

$$I = \int_{-\infty}^{\infty}\left[|y|^{\sigma-\frac{1}{p}}\int_{-\infty}^{\infty}\frac{(\min\{1,|x^{\delta}y|\})^{\beta}}{|1+x^{\delta}y|^{\lambda+\beta}}f(x)dx\right](|y|^{\frac{1}{p}-\sigma}g(y))dy \leq J^{\frac{1}{p}}\left[\int_{-\infty}^{\infty}|y|^{q(1-\sigma)-1}g^q(y)dy\right]^{\frac{1}{q}}. \tag{3.5}$$

Then by (3.2), we get (3.1). On the other hand, suppose that (3.1) is valid. We then set

$$g(y) := |y|^{p\sigma-1}\left[\int_{-\infty}^{\infty}\frac{(\min\{1,|x^{\delta}y|\})^{\beta}}{|1+x^{\delta}y|^{\lambda+\beta}}f(x)dx\right]^{p-1}\quad(y\neq 0), \tag{3.6}$$

and then

$$J = \int_{-\infty}^{\infty}|y|^{q(1-\sigma)-1}g^q(y)dy.$$

By (2.12), we have $J < \infty$. If $J = 0$, then (3.2) is trivially true; if $0 < J < \infty$, then by (3.1), we obtain

$$\begin{aligned} 0 &< \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)dy = J = I \\ &< K(\sigma)\left[\int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)dy\right]^{\frac{1}{q}} < \infty, \end{aligned} \tag{3.7}$$

$$J^{\frac{1}{p}} = \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)dy\right]^{\frac{1}{p}} < K(\sigma)\left[\int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}. \tag{3.8}$$

Hence, we obtain (3.2), which is equivalent to (3.1).

We set $E_\delta := \{x \in \mathbf{R}; |x|^\delta \geq 1\}$, and $E_\delta^+ := E_\delta \cap \mathbf{R}_+ = \{x \in \mathbf{R}_+; x^\delta \geq 1\}$. For $\varepsilon > 0$, we define two functions $\tilde{f}(x), \tilde{g}(y)$ as follows:

$$\begin{aligned} \tilde{f}(x) &: = \begin{cases} |x|^{\delta(\sigma-\frac{2\varepsilon}{p})-1}, & x \in E_\delta \\ 0, & x \in \mathbf{R}\backslash E_\delta \end{cases}, \\ \tilde{g}(y) &: = \begin{cases} 0, & y \in (-\infty,-1] \cup [1,\infty) \\ |y|^{\sigma+\frac{2\varepsilon}{q}-1}, & y \in (-1,1) \end{cases}. \end{aligned}$$

Then we obtain

$$\begin{aligned} \tilde{L} &: = \left[\int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} \tilde{f}^p(x)dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} \tilde{g}^q(y)dy\right]^{\frac{1}{q}} \\ &= 2\left(\int_{E_\delta^+} x^{-2\delta\varepsilon-1}dx\right)^{\frac{1}{p}} \left(\int_0^1 y^{2\varepsilon-1}dy\right)^{\frac{1}{q}} = \frac{1}{\varepsilon}. \end{aligned}$$

We find

$$\begin{aligned} I(x) &: = \int_{-1}^{1} \frac{(\max\{1,|x^\delta y|\})^\beta}{|1+x^\delta y|^{\lambda+\beta}} |y|^{\sigma+\frac{2\varepsilon}{q}-1} dy \\ &\overset{y=-Y}{=} \int_{-1}^{1} \frac{(\max\{1,|-x^\delta Y|\})^\beta}{|1-x^\delta Y|^{\lambda+\beta}} |-Y|^{\sigma+\frac{2\varepsilon}{q}-1} dY = I(-x), \end{aligned}$$

and then $I(x)$ is an even function. It follows that

$$\begin{aligned} &\tilde{I} := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(\min\{1,|x^\delta y|\})^\beta}{|1+x^\delta y|^{\lambda+\beta}} \tilde{f}(x)\tilde{g}(y)dxdy \\ &= \int_{E_\delta} |x|^{\delta(\sigma-\frac{2\varepsilon}{p})-1} I(x)dx = 2\int_{E_\delta^+} x^{\delta(\sigma-\frac{2\varepsilon}{p})-1} I(x)dx \\ &\overset{u=x^\delta y}{=} 2\int_{E_\delta^+} x^{-2\delta\varepsilon-1}\left[\int_{-x^\delta}^{x^\delta} \frac{(\min\{1,|u|\})^\beta}{|1+u|^{\lambda+\beta}} |u|^{\sigma+\frac{2\varepsilon}{q}-1} du\right] dx. \end{aligned}$$

Setting $v = x^{\delta}$ in the above integral, by Fubini's theorem (cf. [31]), we find

$$\begin{aligned}
\tilde{I} = 2\int_1^{\infty} v^{-2\varepsilon-1}\left[\int_{-v}^{v}\frac{(\min\{1,|u|\})^{\beta}}{|1+u|^{\lambda+\beta}}|u|^{\sigma+\frac{2\varepsilon}{q}-1}du\right]dv \\
= & \ 2\int_1^{\infty} v^{-2\varepsilon-1}\left\{\int_0^{v}[\frac{(\min\{1,u\})^{\beta}}{|1-u|^{\lambda+\beta}}+\frac{(\min\{1,u\})^{\beta}}{(1+u)^{\lambda+\beta}}]u^{\sigma+\frac{2\varepsilon}{q}-1}du\right\}dv \\
= & \ 2\int_1^{\infty} v^{-2\varepsilon-1}\left\{\int_0^{1}[\frac{1}{(1-u)^{\lambda+\beta}}+\frac{1}{(1+u)^{\lambda+\beta}}]u^{\beta+\sigma+\frac{2\varepsilon}{q}-1}du\right\}dv \\
& +2\int_1^{\infty} v^{-2\varepsilon-1}\left\{\int_1^{v}[\frac{1}{(u-1)^{\lambda+\beta}}+\frac{1}{(1+u)^{\lambda+\beta}}]u^{\sigma+\frac{2\varepsilon}{q}-1}du\right\}dv \\
= & \ \frac{1}{\varepsilon}\int_0^{1}[\frac{1}{(1-u)^{\lambda+\beta}}+\frac{1}{(1+u)^{\lambda+\beta}}]u^{\beta+\sigma+\frac{2\varepsilon}{q}-1}du \\
& +2\int_1^{\infty}(\int_u^{\infty} v^{-2\varepsilon-1}dv)[\frac{1}{(u-1)^{\lambda+\beta}}+\frac{1}{(1+u)^{\lambda+\beta}}]u^{\sigma+\frac{2\varepsilon}{q}-1}du \\
= & \ \frac{1}{\varepsilon}\left\{\int_0^{1}[\frac{1}{(1-u)^{\lambda+\beta}}+\frac{1}{(1+u)^{\lambda+\beta}}]u^{\beta+\sigma+\frac{2\varepsilon}{q}-1}du\right. \\
& \left.+\int_1^{\infty}[\frac{1}{(u-1)^{\lambda+\beta}}+\frac{1}{(1+u)^{\lambda+\beta}}]u^{\sigma-\frac{2\varepsilon}{p}-1}du\right\}.
\end{aligned}$$

If the constant factor $K(\sigma)$ in (3.1) is not the best possible, then, there exists a positive number $k$, with $K(\sigma) < k$, such that (3.1) is valid when replacing $K(\sigma)$ by $k$. Then in particular, we have $\varepsilon\tilde{I} < \varepsilon k\tilde{L}$, and

$$\begin{aligned}
& \int_0^{1}[\frac{1}{(1-u)^{\lambda+\beta}}+\frac{1}{(1+u)^{\lambda+\beta}}]u^{\beta+\sigma+\frac{2\varepsilon}{q}-1}du \\
& +\int_1^{\infty}[\frac{1}{(u-1)^{\lambda+\beta}}+\frac{1}{(1+u)^{\lambda+\beta}}]u^{\sigma-\frac{2\varepsilon}{p}-1}du = \varepsilon\tilde{I} < \varepsilon k\tilde{L} = k. \qquad (3.9)
\end{aligned}$$

By (2.5), (2.6) and Fatou's lemma (cf. [31]), we have

$$\begin{aligned}
K(\sigma) \ = & \ \int_0^{1}[\frac{1}{(1-u)^{\lambda+\beta}}+\frac{1}{(1+u)^{\lambda+\beta}}]u^{\beta+\sigma-1}du \\
& +\int_1^{\infty}[\frac{1}{(u-1)^{\lambda+\beta}}+\frac{1}{(1+u)^{\lambda+\beta}}]u^{\sigma-1}du \\
= & \ \int_0^{1}\lim_{\varepsilon\to 0^+}[\frac{1}{(1-u)^{\lambda+\beta}}+\frac{1}{(1+u)^{\lambda+\beta}}]u^{\beta+\sigma+\frac{2\varepsilon}{q}-1}du \\
& +\int_1^{\infty}\lim_{\varepsilon\to 0^+}[\frac{1}{(u-1)^{\lambda+\beta}}+\frac{1}{(1+u)^{\lambda+\beta}}]u^{\sigma-\frac{2\varepsilon}{p}-1}du \\
\leq & \ \underline{\lim}_{\varepsilon\to 0^+}\left\{\int_0^{1}[\frac{1}{(1-u)^{\lambda+\beta}}+\frac{1}{(1+u)^{\lambda+\beta}}]u^{\beta+\sigma+\frac{2\varepsilon}{q}-1}du\right. \\
& \left.+\int_1^{\infty}[\frac{1}{(u-1)^{\lambda+\beta}}+\frac{1}{(1+u)^{\lambda+\beta}}]u^{\sigma-\frac{2\varepsilon}{p}-1}du\right\} \leq k,
\end{aligned}$$

which contradicts the fact that $k < K(\sigma)$. Hence the constant factor $K(\sigma)$ in (3.1) is the best possible.

If the constant factor in (3.2) is not the best possible, then by (3.5) we would reach the contradiction that the constant factor in (3.1) is not the best possible. □

**Theorem 3.2.** *If in the assumptions of Theorem 3.1, we replace $p > 1$ by $0 < p < 1$, we obtain the equivalent reverses of (3.1) and (3.2) with the same best constant factors.*

*Proof.* By Hölder's reverse inequality (cf. [30]), we derive the reverses of (2.11), (2.12), (2.10) and (3.5). It is easy to obtain the reverse of (3.2). In view of the reverses of (3.2) and (3.5), we obtain the reverse of (3.1). On the other hand, if we suppose that the reverse of (3.1) is valid, then if we set $g(y)$ as in (3.6), by the reverse of (2.12), we have $J > 0$. If $J = \infty$, then the reverse of (3.2) is obviously true; if $J < \infty$, then by the reverse of (3.1), we obtain the reverses of (3.7) and (3.8). Hence, we obtain the reverse of (3.2), which is equivalent to the reverse of (3.1).

If the constant factor $K(\sigma)$ in the reverse of (3.1) is not the best possible, then, there exists a positive constant $k$, with $k > K(\sigma)$, such that the reverse of (3.1) is still valid when replacing $K(\sigma)$ by $k$. By the reverse of (3.9), we have

$$\begin{aligned}&\int_0^1 [\frac{1}{(1-u)^{\lambda+\beta}} + \frac{1}{(1+u)^{\lambda+\beta}}]u^{\beta+\sigma+\frac{2\varepsilon}{q}-1}du \\ &+\int_1^\infty [\frac{1}{(u-1)^{\lambda+\beta}} + \frac{1}{(1+u)^{\lambda+\beta}}]u^{\sigma-\frac{2\varepsilon}{p}-1}du > k. \qquad (3.10)\end{aligned}$$

By Levi's theorem (cf. [31]), we find

$$\begin{aligned}&\int_1^\infty [\frac{1}{(u-1)^{\lambda+\beta}} + \frac{1}{(1+u)^{\lambda+\beta}}]u^{\sigma-\frac{2\varepsilon}{p}-1}du \\ \to \ &\int_1^\infty [\frac{1}{(u-1)^{\lambda+\beta}} + \frac{1}{(1+u)^{\lambda+\beta}}]u^{\sigma-1}du(\varepsilon \to 0^+).\end{aligned}$$

There exists a constant $\delta_0 > 0$, such that $\sigma - \frac{1}{2}\delta_0 > -\beta$, and then $K(\sigma - \frac{\delta_0}{2}) \in \mathbf{R}_+$. For $0 < \varepsilon < \frac{\delta_0|q|}{4}$ $(q < 0)$, since $u^{\beta+\sigma+\frac{2\varepsilon}{q}-1} \le u^{\beta+\sigma-\frac{\delta_0}{2}-1}, u \in (0,1]$, and

$$0 < \int_0^1 [\frac{1}{(1-u)^{\lambda+\beta}} + \frac{1}{(1+u)^{\lambda+\beta}}]u^{\beta+\sigma-\frac{\delta_0}{2}-1}du \le K(\sigma - \frac{\delta_0}{2}),$$

then by Lebesgue's control convergence theorem (cf. [31]), we have

$$\begin{aligned}&\int_0^1 [\frac{1}{(1-u)^{\lambda+\beta}} + \frac{1}{(1+u)^{\lambda+\beta}}]u^{\beta+\sigma+\frac{2\varepsilon}{q}-1}du \\ \to \ &\int_0^1 [\frac{1}{(1-u)^{\lambda+\beta}} + \frac{1}{(1+u)^{\lambda+\beta}}]u^{\beta+\sigma-1}du \ (\varepsilon \to 0^+).\end{aligned}$$

By (3.10) and the above results, for $\varepsilon \to 0^+$, we get $K(\sigma) \ge k$, which contradicts the fact that $k > K(\sigma)$. Hence, the constant factor $K(\sigma)$ in the reverse of (3.1) is the best possible.

If the constant factor in the reverse of (3.2) is not the best possible, then, by the reverse of (3.5), we would reach the contradiction that the constant factor in the reverse of (3.1) is not the best possible. □

*Remark* 3.3. (i) For $\delta = -1$ in (3.1) and (3.2), replacing $|x|^{\lambda} f(x)$ by $f(x)$, we obtain $0 < \int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x)dx < \infty$, and the following equivalent inequalities with the homogeneous kernel and the best possible constant factors:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^{\beta}}{|x+y|^{\lambda+\beta}} f(x)g(y)dxdy$$
$$< K(\sigma)\left[\int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)dy\right]^{\frac{1}{q}}, \quad (3.11)$$

$$\int_{-\infty}^{\infty} |y|^{p\sigma-1}\left[\int_{-\infty}^{\infty} \frac{(\max\{|x|,|y|\})^{\beta}}{|x+y|^{\lambda+\beta}} f(x)dx\right]^{p} dy$$
$$< K^p(\sigma)\int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x)dx. \quad (3.12)$$

In particular, for $\lambda = 0 = \mu + \sigma(\mu, \sigma > -\beta)$, $0 < \beta < 1$, we find

$$\begin{aligned} K(\sigma) &= K_0(\sigma) := \frac{1}{\beta+\sigma} F(\beta, \beta+\sigma, 1+\beta+\sigma, -1) \\ &\quad + \frac{1}{\beta+\mu} F(\beta, \beta+\mu, 1+\beta+\mu, -1) \\ &\quad + B(1-\beta, \beta+\sigma) + B(1-\beta, \beta+\mu), \end{aligned} \quad (3.13)$$

and the following equivalent inequalities:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{\min\{|x|,|y|\}}{|x+y|}\right)^{\beta} f(x)g(y)dxdy$$
$$< K_0(\sigma)\left[\int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)dy\right]^{\frac{1}{q}}, \quad (3.14)$$

$$\int_{-\infty}^{\infty} |y|^{p\sigma-1}\left[\int_{-\infty}^{\infty} \left(\frac{\min\{|x|,|y|\}}{|x+y|}\right)^{\beta} f(x)dx\right]^{p} dy$$
$$< K_0^p(\sigma)\int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x)dx. \quad (3.15)$$

(ii) For $\lambda = 0 = \mu + \sigma(\mu, \sigma > -\beta)$, $0 < \beta < 1$ in (3.1) and (3.2), we have the following equivalent inequalities:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{\min\{1,|x^{\delta}y|\}}{|1+x^{\delta}y|}\right)^{\beta} f(x)g(y)dxdy$$
$$< K_0(\sigma)\left[\int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}} \left[\int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)dy\right]^{\frac{1}{q}}, \quad (3.16)$$

$$\int_{-\infty}^{\infty} |y|^{p\sigma-1} \left[ \int_{-\infty}^{\infty} \left( \frac{\min\{1,|x^{\delta}y|\}}{|1+x^{\delta}y|} \right)^{\beta} f(x)dx \right]^{p} dy$$
$$< \quad K_0^p(\sigma) \int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x)dx. \tag{3.17}$$

In particular, for $\delta = 1$,we have the following equivalent inequalities (cf. [25], for $\sigma = \mu = 0$):

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{\min\{1,|xy|\}}{|1+xy|} \right)^{\beta} f(x)g(y)dxdy$$
$$< \quad K_0(\sigma) \left[ \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)dy \right]^{\frac{1}{q}}, \tag{3.18}$$

$$\int_{-\infty}^{\infty} |y|^{p\sigma-1} \left[ \int_{-\infty}^{\infty} \left( \frac{\min\{1,|xy|\}}{|1+xy|} \right)^{\beta} f(x)dx \right]^{p} dy$$
$$< \quad K_0^p(\sigma) \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx. \tag{3.19}$$

(iii) For $\beta = 0 < \lambda < 1$, $\sigma = \mu = \frac{\lambda}{2}$ in (3.3), we obtain

$$K(\frac{\lambda}{2}) = \int_{0}^{\infty} \frac{u^{\frac{\lambda}{2}-1}}{(1+u)^{\lambda}} du + 2\int_{0}^{1} \frac{u^{\frac{\lambda}{2}-1}}{(1-u)^{\lambda}} du = k_{\lambda},$$

and then (1.3) follows. Hence, (3.1)-(3.3) is an extension of (1.3).

# 4 Some Corollaries

In the following two sections, if the constant factors are related to $K_1(\sigma)$, then we call them Hardy-type inequalities (operators) of the first kind; if the constant factors are related to $K_2(\sigma)$, then we call them Hardy-type inequalities (operators) of the second kind.

Setting the kernel

$$H(xy) := \begin{cases} 0, & |xy| > 1 \\ \frac{(\min\{1,|xy|\})^{\beta}}{|1+xy|^{\lambda+\beta}}, & |xy| \leq 1 \end{cases},$$

it follows that

$$H(u) = \begin{cases} 0, & |u| > 1 \\ \frac{(\min\{1,|u|\})^{\beta}}{|1+u|^{\lambda+\beta}}, & |u| \leq 1 \end{cases},$$
$$\int_{-\infty}^{\infty} H(u)|u|^{\sigma-1}du = \int_{-1}^{1} \frac{(\min\{1,|u|\})^{\beta}}{|1+u|^{\lambda+\beta}} |u|^{\sigma-1}du = K_1(\sigma).$$

In view of Theorems 3.1-3.2 (for $\delta = 1$), we obtain the following Hardy-type inequalities of the first kind with the non-homogeneous kernel:

**Corollary 4.1.** *Suppose that* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \beta > -1, \mu, \sigma > -\beta, \mu + \sigma = \lambda < 1 - \beta$, $K_1(\sigma)$ *is indicated by (2.5) (or 2.7). If* $f$, $g \geq 0$, *satisfy*

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx < \infty$$

*and*

$$0 < \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)dy < \infty,$$

*then we have the following equivalent inequalities:*

$$\begin{aligned} &\int_{-\infty}^{\infty} \left[ \int_{-\frac{1}{|y|}}^{\frac{1}{|y|}} \frac{(\min\{1,|xy|\})^{\beta}}{|1+xy|^{\lambda+\beta}} f(x)dx \right] g(y)dy \\ < \quad & K_1(\sigma) \left[ \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)dy \right]^{\frac{1}{q}}, \end{aligned} \tag{4.1}$$

$$\begin{aligned} &\int_{-\infty}^{\infty} |y|^{p\sigma-1} \left[ \int_{-\frac{1}{|y|}}^{\frac{1}{|y|}} \frac{(\min\{1,|xy|\})^{\beta}}{|1+x^{\delta}y|^{\lambda+\beta}} f(x)dx \right]^p dy \\ < \quad & K_1^p(\sigma) \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx. \end{aligned} \tag{4.2}$$

*where, the constant factors* $K_1(\sigma)$ *and* $K_1^p(\sigma)$ *are the best possible. Replacing* $p > 1$ *by* $0 < p < 1$, *we have the equivalent reverses of (4.1) and (4.2) with the same best constant factors.*

If we set $E_y := \{x \in \mathbf{R}; |xy| \geq 1\}$, and

$$H(xy) := \begin{cases} 0, & |xy| < 1 \\ \frac{(\min\{1,|xy|\})^{\beta}}{|1+xy|^{\lambda+\beta}}, & |xy| \geq 1 \end{cases},$$

then it follows that

$$\begin{aligned} H(u) &= \begin{cases} 0, & |u| < 1 \\ \frac{(\min\{1,|u|\})^{\beta}}{|1+u|^{\lambda+\beta}}, & |u| \geq 1 \end{cases}, \\ \int_{-\infty}^{\infty} H(u)|u|^{\sigma-1}du &= \int_{E_1} \frac{(\min\{1,|u|\})^{\beta}}{|1+u|^{\lambda+\beta}} |u|^{\sigma-1}du = K_2(\sigma). \end{aligned}$$

In view of Theorems 3.1-3.2 (for $\delta = 1$), we have the following Hardy-type inequalities of the second kind with the non-homogeneous kernel:

**Corollary 4.2.** *Suppose that* $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \beta > -1, \mu, \sigma > -\beta, \mu + \sigma = \lambda < 1 - \beta$, $K_2(\sigma)$ *is indicated by (2.6) (or 2.8).*
*If* $f$, $g \geq 0$, *satisfy*

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx < \infty$$

*and*

$$0 < \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)dy < \infty,$$

*then we have the following equivalent inequalities:*

$$\int_{-\infty}^{\infty} \left[ \int_{E_y} \frac{(\min\{1,|xy|\})^{\beta}}{|1+xy|^{\lambda+\beta}} f(x)dx \right] g(y)dy$$

$$< K_2(\sigma) \left[ \int_{-\infty}^{\infty} |x|^{p(1-\sigma)-1} f^p(x)dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)dy \right]^{\frac{1}{q}}, \tag{4.3}$$

$$\int_{-\infty}^{\infty} |y|^{p\sigma-1} \left[ \int_{E_y} \frac{(\min\{1,|xy|\})^{\beta}}{|1+xy|^{\lambda+\beta}} f(x)dx \right]^p dy$$

$$< K_2^p(\sigma) \int_{-\infty}^{\infty} |x|^{p(1-\delta\sigma)-1} f^p(x)dx. \tag{4.4}$$

*where, the constant factors $K_2(\sigma)$ and $K_2^p(\sigma)$ are the best possible. Replacing $p > 1$ by $0 < p < 1$, we obtain the equivalent reverses of (4.3) and (4.4) with the same best constant factors.*

If we set $\widetilde{E}_y := \{x \in \mathbf{R}; |\frac{y}{x}| \leq 1\}$ and

$$K_\lambda(x,y) := \begin{cases} 0, & |\frac{y}{x}| > 1 \\ \frac{(\min\{|x|,|y|\})^{\beta}}{|x+y|^{\lambda+\beta}}, & |\frac{y}{x}| \leq 1 \end{cases},$$

then it follows

$$K_\lambda(1,u) = \begin{cases} 0, & |u| > 1 \\ \frac{(\min\{1,|u|\})^{\beta}}{|1+u|^{\lambda+\beta}}, & |u| \leq 1 \end{cases},$$

$$\int_{-\infty}^{\infty} K_\lambda(1,u)|u|^{\sigma-1}du = \int_{-1}^{1} \frac{(\min\{1,|u|\})^{\beta}}{|1+u|^{\lambda+\beta}} |u|^{\sigma-1}du = K_1(\sigma).$$

In view of Remark 3.3 (i), we have the following Hardy-type inequalities of the first kind with the homogeneous kernel:

**Corollary 4.3.** *Suppose that $p > 1, \frac{1}{p}+\frac{1}{q} = 1, \beta > -1, \mu, \sigma > -\beta, \mu+\sigma = \lambda < 1-\beta$, $K_1(\sigma)$ is indicated by (2.5) (or 2.7). If $f$, $g \geq 0$, satisfy*

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x)dx < \infty$$

*and*

$$0 < \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)dy < \infty,$$

*then we have the following equivalent inequalities:*

$$\int_{-\infty}^{\infty} \left[ \int_{\widetilde{E}_y} \frac{(\min\{|x|,|y|\})^{\beta}}{|x+y|^{\lambda+\beta}} f(x)dx \right] g(y)dy$$

$$< K_1(\sigma) \left[ \int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x)dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)dy \right]^{\frac{1}{q}}, \tag{4.5}$$

$$\int_{-\infty}^{\infty} |y|^{p\sigma-1} \left[ \int_{\widetilde{E}_y} \frac{(\min\{|x|,|y|\})^{\beta}}{|x+y|^{\lambda+\beta}} f(x)dx \right]^p dy$$
$$< \quad K_1^p(\sigma) \int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x)dx. \tag{4.6}$$

*where the constant factors $K_1(\sigma)$ and $K_1^p(\sigma)$ are the best possible. Replacing $p > 1$ by $0 < p < 1$, we derive the equivalent reverses of (4.5) and (4.6) with the same best constant factors.*

Setting the kernel

$$K_\lambda(x,y) := \begin{cases} 0, & |\frac{y}{x}| < 1 \\ \frac{(\min\{|x|,|y|\})^{\beta}}{|x+y|^{\lambda+\beta}}, & |\frac{y}{x}| \geq 1 \end{cases},$$

then it follows that

$$K_\lambda(1,u) \quad = \quad \begin{cases} 0, & |u| < 1 \\ \frac{(\min\{1,|u|\})^{\beta}}{|1+u|^{\lambda+\beta}}, & |u| \geq 1 \end{cases},$$
$$\int_{-\infty}^{\infty} K_\lambda(1,u)|u|^{\sigma-1}du \quad = \quad \int_{E_1} \frac{(\min\{1,|u|\})^{\beta}}{|1+u|^{\lambda+\beta}} |u|^{\sigma-1}du = K_2(\sigma).$$

In view of Remark 3.3 (i), we have the following Hardy-type inequalities of the second kind with the homogeneous kernel:

**Corollary 4.4.** *Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \beta < 1, \mu, \sigma > 0, \mu + \sigma = \lambda < 1 - \beta, K_2(\sigma)$ is indicated by (2.6) (or 2.8). If $f$, $g \geq 0$, satisfy*

$$0 < \int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x)dx < \infty$$

*and*

$$0 < \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)dy < \infty,$$

*then we have the following equivalent inequalities:*

$$\int_{-\infty}^{\infty} \left[ \int_{-|y|}^{|y|} \frac{(\min\{|x|,|y|\})^{\beta}}{|x+y|^{\lambda+\beta}} f(x)dx \right] g(y)dy$$
$$< \quad K_2(\sigma) \left[ \int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x)dx \right]^{\frac{1}{p}} \left[ \int_{-\infty}^{\infty} |y|^{q(1-\sigma)-1} g^q(y)dy \right]^{\frac{1}{q}}, \tag{4.7}$$

$$\int_{-\infty}^{\infty} |y|^{p\sigma-1} \left[ \int_{-|y|}^{|y|} \frac{(\min\{|x|,|y|\})^{\beta}}{|x+y|^{\lambda+\beta}} f(x)dx \right]^p dy$$
$$< \quad K_2^p(\sigma) \int_{-\infty}^{\infty} |x|^{p(1-\mu)-1} f^p(x)dx. \tag{4.8}$$

*where the constant factors $K_2(\sigma)$ and $K_2^p(\sigma)$ are the best possible. Replacing $p > 1$ by $0 < p < 1$, we get the equivalent reverses of (4.7) and (4.8) with the same best constant factors.*

## 5 Operator Expressions

Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, \beta > -1, \mu, \sigma > -\beta$, $\mu + \sigma = \lambda < 1 - \beta$. We set the following functions: $\varphi(x) := |x|^{p(1-\sigma)-1}, \psi(y) := |y|^{q(1-\sigma)-1}, \phi(x) := |x|^{p(1-\mu)-1} (x, y \in \mathbf{R})$, wherefrom, $\psi^{1-p}(y) = |y|^{p\sigma-1}$. Define the following real normed linear space:

$$L_{p,\varphi}(\mathbf{R}) := \left\{ f : ||f||_{p,\varphi} := \left( \int_{-\infty}^{\infty} \varphi(x)|f(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\},$$

wherefrom,

$$\begin{aligned} L_{p,\psi^{1-p}}(\mathbf{R}) &= \left\{ h : ||h||_{p,\psi^{1-p}} = \left( \int_{-\infty}^{\infty} \psi^{1-p}(y)|h(y)|^p dy \right)^{\frac{1}{p}} < \infty \right\}, \\ L_{p,\phi}(\mathbf{R}) &= \left\{ g : ||g||_{p,\phi} = \left( \int_{-\infty}^{\infty} \phi(x)|g(x)|^p dx \right)^{\frac{1}{p}} < \infty \right\}. \end{aligned}$$

(a) In view of Theorem 3.1 ($\delta = 1$), for $f \in L_{p,\varphi}(\mathbf{R})$, setting

$$H^{(1)}(y) := \int_{-\infty}^{\infty} \frac{(\min\{1, |xy|\})^\beta}{|1+xy|^{\lambda+\beta}} |f(x)| dx (y \in \mathbf{R}),$$

by (3.4), we have

$$||H^{(1)}||_{p,\psi^{1-p}} = \left[ \int_{-\infty}^{\infty} \psi^{1-p}(y)(H^{(1)}(y))^p dy \right]^{\frac{1}{p}} < K(\sigma)||f||_{p,\varphi} < \infty. \tag{5.1}$$

**Definition 5.1.** Define the Hilbert-type integral operator with the non-homogeneous kernel in the whole plane $T^{(1)} : L_{p,\varphi}(\mathbf{R}) \to L_{p,\psi^{1-p}}(\mathbf{R})$ as follows: For any $f \in L_{p,\varphi}(\mathbf{R})$, there exists a unique representation $T^{(1)}f = H^{(1)} \in L_{p,\psi^{1-p}}(\mathbf{R})$, satisfying

$$T^{(1)}f(y) = H^{(1)}(y),$$

for any $y \in \mathbf{R}$.

In view of (5.1), it follows that $||T^{(1)}f||_{p,\psi^{1-p}} = ||H^{(1)}||_{p,\psi^{1-p}} \leq K(\sigma)||f||_{p,\varphi}$. Then, the operator $T^{(1)}$ is bounded satisfying

$$||T^{(1)}|| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R}_+)} \frac{||T^{(1)}f||_{p,\psi^{1-p}}}{||f||_{p,\varphi}} \leq K(\sigma).$$

Since the constant factor $K(\sigma)$ in (5.1) is the best possible, we have $||T^{(1)}|| = K(\sigma)$.

If we define the formal inner product of $T^{(1)}f$ and $g$ as follows:

$$\begin{aligned} (T^{(1)}f, g) \quad : &= \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \frac{(\min\{1, |xy|\})^\beta}{|1+xy|^{\lambda+\beta}} f(x) dx \right] g(y) dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(\min\{1, |xy|\})^\beta}{|1+xy|^{\lambda+\beta}} f(x)g(y) dx dy, \end{aligned}$$

then we can rewrite (3.3) and (3.4) in the form:

$$(T^{(1)}f,g) < ||T^{(1)}|| \cdot ||f||_{p,\varphi}||g||_{q,\psi},\ ||T^{(1)}f||_{p,\psi^{1-p}} < ||T^{(1)}|| \cdot ||f||_{p,\varphi}. \tag{5.2}$$

(b) In view of Corollary 4.1, for $f \in L_{p,\varphi}(\mathbf{R})$, setting

$$H_1^{(1)}(y) := \int_{-\frac{1}{|y|}}^{\frac{1}{|y|}} \frac{(\min\{1,|xy|\})^{\beta}}{|1+xy|^{\lambda+\beta}}|f(x)|dx (y \in \mathbf{R}),$$

by (3.14), we obtain

$$||H_1^{(1)}||_{p,\psi^{1-p}} = \left[\int_{-\frac{1}{|y|}}^{\frac{1}{|y|}} \psi^{1-p}(y)(H_1^{(1)}(y))^p dy\right]^{\frac{1}{p}} < K_1(\sigma)||f||_{p,\varphi} < \infty. \tag{5.3}$$

**Definition 5.2.** Define the Hilbert-type integral operator of the first kind with the non-homogeneous kernel in the whole plane $T_1^{(1)} : L_{p,\varphi}(\mathbf{R}) \to L_{p,\psi^{1-p}}(\mathbf{R})$ as follows: For any $f \in L_{p,\varphi}(\mathbf{R})$, there exists a unique representation $T_1^{(1)}f = H_1^{(1)} \in L_{p,\psi^{1-p}}(\mathbf{R})$, satisfying

$$T_1^{(1)}f(y) = H_1^{(1)}(y),$$

for any $y \in \mathbf{R}$.

In view of (5.3), it follows that $||T_1^{(1)}f||_{p,\psi^{1-p}} = ||H_1^{(1)}||_{p,\psi^{1-p}} \leq K_1(\sigma)||f||_{p,\varphi}$, and then the operator $T_1^{(1)}$ is bounded satisfying

$$||T_1^{(1)}|| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R}_+)} \frac{||T_1^{(1)}f||_{p,\psi^{1-p}}}{||f||_{p,\varphi}} \leq K_1(\sigma).$$

Since the constant factor $K_1(\sigma)$ in (5.3) is the best possible, we have $||T_1^{(1)}|| = K_1(\sigma)$.

If we define the formal inner product of $T_1^{(1)}f$ and $g$ as follows:

$$(T_1^{(1)}f,g) := \int_{-\infty}^{\infty}\left[\int_{-\frac{1}{|y|}}^{\frac{1}{|y|}} \frac{(\min\{1,|xy|\})^{\beta}}{|1+xy|^{\lambda+\beta}}f(x)dx\right]g(y)dy,$$

then we can rewrite (3.13) and (3.14) in the following way:

$$(T_1^{(1)}f,g) < ||T_1^{(1)}|| \cdot ||f||_{p,\varphi}||g||_{q,\psi}, ||T_1^{(1)}f||_{p,\psi^{1-p}} < ||T_1^{(1)}|| \cdot ||f||_{p,\varphi}. \tag{5.4}$$

(c) In view of Corollary 4.2, for $f \in L_{p,\varphi}(\mathbf{R})$, setting

$$H_2^{(1)}(y) := \int_{E_y} \frac{(\min\{1,|xy|\})^{\beta}}{|1+xy|^{\lambda+\beta}}|f(x)|dx (y \in \mathbf{R}),$$

by (3.16), we have

$$||H_2^{(1)}||_{p,\psi^{1-p}} = \left[\int_{E_y} \psi^{1-p}(y)(H_2^{(1)}(y))^p dy\right]^{\frac{1}{p}} < K_2(\sigma)||f||_{p,\varphi} < \infty. \tag{5.5}$$

**Definition 5.3.** Define the Hilbert-type integral operator of the second kind with the non-homogeneous kernel in the whole plane $T_2^{(1)} : L_{p,\varphi}(\mathbf{R}) \to L_{p,\psi^{1-p}}(\mathbf{R})$ as follows: For any $f \in L_{p,\varphi}(\mathbf{R})$, there exists a unique representation $T_2^{(1)} f = H_2^{(1)} \in L_{p,\psi^{1-p}}(\mathbf{R})$, satisfying

$$T_2^{(1)} f(y) = H_2^{(1)}(y),$$

for any $y \in \mathbf{R}$.

In view of (5.5), it follows that $||T_2^{(1)} f||_{p,\psi^{1-p}} = ||H_2^{(1)}||_{p,\psi^{1-p}} \leq K_2(\sigma)||f||_{p,\varphi}$, and then the operator $T_2^{(1)}$ is bounded satisfying

$$||T_2^{(1)}|| = \sup_{f(\neq\theta)\in L_{p,\varphi}(\mathbf{R}_+)} \frac{||T_2^{(1)} f||_{p,\psi^{1-p}}}{||f||_{p,\varphi}} \leq K_2(\sigma).$$

Since the constant factor $K_2(\sigma)$ in (5.5) is the best possible, we have $||T_2^{(1)}|| = K_2(\sigma)$.

If we define the formal inner product of $T_2^{(1)} f$ and $g$ as follows:

$$(T_2^{(1)} f, g) := \int_{-\infty}^{\infty} \left[ \int_{E_y} \frac{(\min\{1,|xy|\})^\beta}{|1+xy|^{\lambda+\beta}} f(x) dx \right] g(y) dy,$$

then we can rewrite (3.15) and (3.16) as shown below:

$$(T_2^{(1)} f, g) < ||T_2^{(1)}|| \cdot ||f||_{p,\varphi} ||g||_{q,\psi}, ||T_2^{(1)} f||_{p,\psi^{1-p}} < ||T_2^{(1)}|| \cdot ||f||_{p,\varphi}. \tag{5.6}$$

(d) In view of Remark 3.3 (i), for $f \in L_{p,\phi}(\mathbf{R})$, setting

$$H^{(2)}(y) := \int_{-\infty}^{\infty} \frac{(\min\{|x|,|y|\})^\beta}{|x+y|^{\lambda+\beta}} |f(x)| dx (y \in \mathbf{R}),$$

by (3.12), we have

$$||H^{(2)}||_{p,\psi^{1-p}} = \left[ \int_{-\infty}^{\infty} \psi^{1-p}(y) (H^{(2)}(y))^p dy \right]^{\frac{1}{p}} < K(\sigma)||f||_{p,\phi} < \infty. \tag{5.7}$$

**Definition 5.4.** Define the Hilbert-type integral operator with the homogeneous kernel in the whole plane $T^{(2)} : L_{p,\phi}(\mathbf{R}) \to L_{p,\psi^{1-p}}(\mathbf{R})$ as follows: For any $f \in L_{p,\phi}(\mathbf{R})$, there exists a unique representation $T^{(2)} f = H^{(2)} \in L_{p,\psi^{1-p}}(\mathbf{R})$, satisfying

$$T^{(2)} f(y) = H^{(2)}(y),$$

for any $y \in \mathbf{R}$.

In view of (5.7), it follows that $||T^{(2)} f||_{p,\psi^{1-p}} = ||H^{(2)}||_{p,\psi^{1-p}} \leq K(\sigma)||f||_{p,\phi}$, and then the operator $T^{(2)}$ is bounded satisfying

$$||T^{(2)}|| = \sup_{f(\neq\theta)\in L_{p,\phi}(\mathbf{R}_+)} \frac{||T^{(2)} f||_{p,\psi^{1-p}}}{||f||_{p,\phi}} \leq K(\sigma).$$

Since the constant factor $K(\sigma)$ in (5.7) is the best possible, we have $||T^{(2)}|| = K(\sigma)$.

If we define the formal inner product of $T^{(2)}f$ and $g$ as follows:

$$\begin{aligned}(T^{(2)}f,g) \quad &: \quad = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{(\min\{|x|,|y|\})^{\beta}}{|x+y|^{\lambda+\beta}}f(x)dx\right]g(y)dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(\min\{|x|,|y|\})^{\beta}}{|x+y|^{\lambda+\beta}}f(x)g(y)dxdy,\end{aligned}$$

then we can rewrite (3.11) and (3.12) as follows:

$$(T^{(2)}f,g) < ||T^{(2)}||\cdot||f||_{p,\phi}||g||_{q,\psi}, ||T^{(2)}f||_{p,\psi^{1-p}} < ||T^{(2)}||\cdot||f||_{p,\phi}. \quad (5.8)$$

(e) In view of Corollary 4.3, for $f \in L_{p,\phi}(\mathbf{R})$, setting

$$H_1^{(2)}(y) := \int_{\widetilde{E}_y}\frac{(\min\{|x|,|y|\})^{\beta}}{|x+y|^{\lambda+\beta}}|f(x)|dx(y \in \mathbf{R}),$$

by (3.18), we have

$$||H_1^{(2)}||_{p,\psi^{1-p}} = \left[\int_{\widetilde{E}_y}\psi^{1-p}(y)(H_1^{(2)}(y))^p dy\right]^{\frac{1}{p}} < K_1(\sigma)||f||_{p,\phi} < \infty. \quad (5.9)$$

**Definition 5.5.** Define the Hilbert-type integral operator of the fist kind with the homogeneous kernel in the whole plane $T_1^{(2)} : L_{p,\phi}(\mathbf{R}) \to L_{p,\psi^{1-p}}(\mathbf{R})$ as follows: For any $f \in L_{p,\phi}(\mathbf{R})$, there exists a unique representation $T_1^{(2)}f = H_1^{(2)} \in L_{p,\psi^{1-p}}(\mathbf{R})$, satisfying

$$T_1^{(2)}f(y) = H_1^{(2)}(y),$$

for any $y \in \mathbf{R}$.

In view of (5.9), it follows that $||T_1^{(2)}f||_{p,\psi^{1-p}} = ||H_1^{(2)}||_{p,\psi^{1-p}} \leq K_1(\sigma)||f||_{p,\phi}$, and then the operator $T_1^{(2)}$ is bounded satisfying

$$||T_1^{(2)}|| = \sup_{f(\neq\theta)\in L_{p,\phi}(\mathbf{R}_+)}\frac{||T_1^{(2)}f||_{p,\psi^{1-p}}}{||f||_{p,\phi}} \leq K_1(\sigma).$$

Since the constant factor $K_1(\sigma)$ in (5.9) is the best possible, we have $||T_1^{(2)}|| = K_1(\sigma)$.

If we define the formal inner product of $T_1^{(2)}f$ and $g$ as follows:

$$(T_1^{(1)}f,g) := \int_{-\infty}^{\infty}\left[\int_{\widetilde{E}_y}\frac{(\min\{|x|,|y|\})^{\beta}}{|x+y|^{\lambda+\beta}}f(x)dx\right]g(y)dy,$$

then we can rewrite (3.17) and (3.18) as follows:

$$(T_1^{(2)}f,g) < ||T_1^{(2)}||\cdot||f||_{p,\phi}||g||_{q,\psi}, ||T_1^{(2)}f||_{p,\psi^{1-p}} < ||T_1^{(2)}||\cdot||f||_{p,\phi}. \quad (5.10)$$

(f) In view of Corollary 4.4, for $f \in L_{p,\phi}(\mathbf{R})$, setting

$$H_2^{(2)}(y) := \int_{-|y|}^{|y|} \frac{(\min\{|x|,|y|\})^{\beta}}{|x+y|^{\lambda+\beta}} |f(x)| dx (y \in \mathbf{R}),$$

by (4.1), we have

$$||H_2^{(2)}||_{p,\psi^{1-p}} = \left[ \int_{-|y|}^{|y|} \psi^{1-p}(y) (H_2^{(2)}(y))^p dy \right]^{\frac{1}{p}} < K_2(\sigma) ||f||_{p,\phi} < \infty. \quad (5.11)$$

**Definition 5.6.** Define the Hilbert-type integral operator of the second kind with the homogeneous kernel in the whole plane $T_2^{(2)} : L_{p,\phi}(\mathbf{R}) \to L_{p,\psi^{1-p}}(\mathbf{R})$ as follows: For any $f \in L_{p,\phi}(\mathbf{R})$, there exists a unique representation $T_2^{(2)} f = H_2^{(2)} \in L_{p,\psi^{1-p}}(\mathbf{R})$, satisfying

$$T_2^{(2)} f(y) = H_2^{(2)}(y),$$

for any $y \in \mathbf{R}$.

In view of (5.11), it follows that $||T_2^{(2)} f||_{p,\psi^{1-p}} = ||H_2^{(2)}||_{p,\psi^{1-p}} \leq K_2(\sigma) ||f||_{p,\phi}$, and thus the operator $T_2^{(2)}$ is bounded satisfying

$$||T_2^{(2)}|| = \sup_{f(\neq\theta) \in L_{p,\phi}(\mathbf{R}_+)} \frac{||T_2^{(2)} f||_{p,\psi^{1-p}}}{||f||_{p,\phi}} \leq K_2(\sigma).$$

Since the constant factor $K_2(\sigma)$ in (5.11) is the best possible, we have $||T_2^{(2)}|| = K_2(\sigma)$.

If we define the formal inner product of $T_2^{(2)} f$ and $g$ as

$$(T_2^{(2)} f, g) := \int_{-\infty}^{\infty} \left[ \int_{-|y|}^{|y|} \frac{(\min\{|x|,|y|\})^{\beta}}{|x+y|^{\lambda+\beta}} f(x) dx \right] g(y) dy,$$

then we can rewrite (3.19) and (4.1) as follows:

$$(T_2^{(2)} f, g) < ||T_2^{(2)}|| \cdot ||f||_{p,\phi} ||g||_{q,\psi}, ||T_2^{(2)} f||_{p,\psi^{1-p}} < ||T_2^{(2)}|| \cdot ||f||_{p,\phi}. \quad (5.12)$$

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