



SOME QUALITATIVE QUESTIONS ON THE EQUATION

$$-div(a(x, u, \nabla u)) = f(x, u)$$

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ABSTRACT. In this article, we establish several applications of Picone's identity for the operator of the form

$$-div(a(x, u, \nabla u)),$$

such as Hardy type inequality, Sturmian comparison theorem, monotonicity property of the first eigenvalue, nonexistence of positive supersolutions and Caccioppoli inequality under certain conditions on a .

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1. INTRODUCTION

The classical Picone's identity says that for differentiable functions $v > 0$ and $u \geq 0$,

$$(1.1) \quad |\nabla u|^2 + \frac{u^2}{v^2} |\nabla v|^2 - 2 \frac{u}{v} \nabla u \nabla v = |\nabla u|^2 - \nabla \left(\frac{u^2}{v} \right) \nabla v \geq 0.$$

(1.1) has enormous applications to second-order elliptic equations and systems, see for instance [11, 12, 13, 38] and the references therein. For a nonlinear version of (1.1), we refer to [45]. In order to apply (1.1) to equations involving p-Laplace operator and other general divergence type operators, (1.1) has been extended in several directions, see [14, 23, 32, 33, 35] and the references cited therein. W. Allegretto and Y.X.Huang [14] proved some qualitative results using the Picone's identity. J. Jaroš established the Picone's identity for Finsler p-Laplacian [33]

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and A-harmonic operator [32]. He also proved various qualitative results such as Caccioppoli type estimates, nonexistence of positive supersolutions, the uniqueness and simplicity of principal eigenvalues for problems involving, domain monotonicity property of first eigenvalue, Barta-type inequality etc. B. Abdellaoui and I. Peral [5] used classical Picone's identity for p-Laplace operator to establish the Picone's inequality in integral form for $W^{1,p}(\Omega)$ functions. They used Picone's inequality to prove several results, see for instance [2, 3, 4, 6, 7, 15, 39, 40]. Picone's identity for the operator

$$-div(a(x, \nabla u)),$$

where

$$a : \Omega \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$

satisfies certain conditions, is established by Kawohl et. al. [35]. They proved that for differentiable functions $v > 0$ and $u \geq 0$, the following equality holds:

(1.2)

$$\begin{aligned} \langle a(x, \nabla u), \nabla u \rangle - \left\langle a(x, \nabla v), \nabla \left(\frac{u^p}{v^{p-1}} \right) \right\rangle &= \langle a(x, \nabla u), \nabla u \rangle - p \left\langle a \left(x, \frac{u}{v} \nabla v \right), \nabla u \right\rangle \\ &\quad + (p-1) \left\langle a \left(x, \frac{u}{v} \nabla v \right), \frac{u}{v} \nabla v \right\rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in \mathbb{R}^n .

In the case when

(1.3)

$$a : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$

and satisfies hypotheses given in Section 2, (1.2) can be obtained in the following form:

$$\begin{aligned} (1.4) \quad \langle a(x, u, \nabla u), \nabla u \rangle - \left\langle a(x, v, \nabla v), \nabla \left(\frac{u^p}{v^{p-1}} \right) \right\rangle &= \langle a(x, u, \nabla u), \nabla u \rangle \\ &\quad - p \left\langle a \left(x, v, \frac{u}{v} \nabla v \right), \nabla u \right\rangle + (p-1) \left\langle a \left(x, v, \frac{u}{v} \nabla v \right), \frac{u}{v} \nabla v \right\rangle, \end{aligned}$$

The proof of (1.4) is on the similar lines as the proof of (1.2). The aim of this paper is to establish several applications of (1.4) and for this purpose, let us consider the model problem

$$(1.5) \quad \begin{cases} -div(a(x, u, \nabla u)) = \lambda f(x, u) + g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Several authors have established the existence of solution to (1.5), with f satisfying standard growth conditions and the source term g in Sobolev and Lebesgue space [1, 10, 17, 18, 20, 21, 34], Orlicz space [9, 29] and Lorentz space [30]. For the problem

$$(1.6) \quad \begin{cases} -div(a(x, \nabla u)) = \lambda f(x, u) + \mu g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

we refer to the work of H. Brezis and L. Oswald [24], where they established the existence and uniqueness of solution with $\mu = 0$ and $f(x, u)$ as $f(u)$ by using minimization techniques and maximum principle. The existence of solutions to (1.6) in case when $\mu = 0$ has been widely studied, see, for instance [19, 22, 25, 27, 28, 36, 37, 42]. The existence of solutions in case when $\lambda > 0$ and $\mu > 0$ has been discussed, for instance in [26, 43].

In this article, we assume the existence of solutions to the problem of type (1.5) and establish several applications such as Hardy type inequality, Sturmian comparison theorem, monotonicity property of the first eigenvalue, nonexistence of positive supersolutions and Caccioppoli inequality for the operator $-div(a(x, u, \nabla u))$.

The organization of this paper is as follows. In Section 2, we list the hypotheses which have been used in this paper, construct some examples and state Picone's identity for the operator under consideration. Section 3 deals with several applications.

2. PICONE'S IDENTITY

Let $\Omega \subseteq \mathbb{R}^n$ be a smooth and bounded domain. Let c, α, β and p be positive constants satisfying $0 < \alpha < \beta$ and $1 < p < \infty$. Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product in \mathbb{R}^n . We consider nonlinear monotone operator

$$A : W_0^{1,p}(\Omega) \rightarrow W^{-1,q}(\Omega)$$

of the form

$$-div(a(x, u, \nabla u)),$$

whose coefficients

$$a : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$

belong to the class of functions satisfying the following hypotheses.

- (H1) $a : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuous map and its restriction to $\Omega \times \mathbb{R} \times (\mathbb{R}^n \setminus \{0\})$ is of class C^1 and $a(x, s, 0) = 0$, $\forall (x, s) \in \Omega \times \mathbb{R}$.
- (H2) (Monotonicity) $0 \leq \langle a(x, s, \xi_1) - a(x, s, \xi_2), (\xi_1 - \xi_2) \rangle$, $\forall \xi_1, \xi_2 \in \mathbb{R}^n$, $\forall s \in \mathbb{R}$ and $\forall x \in \Omega$.
- (H3) (Uniform Ellipticity) $\alpha|\xi|^p \leq \langle a(x, s, \xi), \xi \rangle$, $\forall \xi \in \mathbb{R}^n$, $\forall s \in \mathbb{R}$ and $\forall x \in \Omega$.
- (H4) (Growth) $|a(x, s, \xi)| \leq \beta|\xi|^{p-1}$, $\forall \xi \in \mathbb{R}^n$, $\forall s \in \mathbb{R}$ and $\forall x \in \Omega$.
- (H5) (Positive Homogeneity) $a(x, s, t\xi) = t^{p-1}a(x, s, \xi)$, $\forall \xi \in \mathbb{R}^n$, $\forall t > 0$, $\forall s \in \mathbb{R}$ and $\forall x \in \Omega$.
- (H6) (Oddness) $a(x, s, -\xi) = -a(x, s, \xi)$, $\forall \xi \in \mathbb{R}^n$, $\forall x \in \Omega$, $\forall s \in \mathbb{R}$.
- (H7) (Cyclical Monotonicity) The function $a(\cdot, \cdot, \cdot)$ is said to be cyclically monotone if

$$(2.1) \quad \sum_{i=1}^k \langle a(x, s, \xi_i), \xi_{i+1} - \xi_i \rangle \leq 0,$$

for all $\xi_1, \dots, \xi_{k+1} \in \mathbb{R}^n$, $\xi_{k+1} = \xi_1$, $\forall x \in \Omega$, $\forall s \in \mathbb{R}$.

Let us define $\Phi(x, s, \xi_1, \xi_2) := \langle a(x, s, \xi_1), \xi_1 \rangle + \langle a(x, s, \xi_2), \xi_2 \rangle$ for all $s \in \mathbb{R}$, $\forall x \in \Omega$ and $\forall \xi_1, \xi_2 \in \mathbb{R}^n$.

(H8) (Equi-continuity) Let $\delta = \min(\frac{p}{2}, p-1)$,

$$(2.2) \quad |a(x, s, \xi_1) - a(x, s, \xi_2)| \leq c\Phi(x, s, \xi_1, \xi_2)^{(p-1-\delta)/p} \langle a(x, s, \xi_1) - a(x, s, \xi_2), (\xi_1 - \xi_2) \rangle^{\delta/p},$$

for all $s \in \mathbb{R}$, $\forall x \in \Omega$ and $\forall \xi_1, \xi_2 \in \mathbb{R}^n$.

(H9) (Strong Monotonicity) Let $\gamma = \max(p, 2)$ and let Φ be as above. We say that a is strongly monotone if it satisfies

$$(2.3) \quad \alpha|\xi_1 - \xi_2|^\gamma \Phi(x, s, \xi_1, \xi_2)^{1-\frac{\gamma}{p}} \leq \langle a(x, s, \xi_1) - a(x, s, \xi_2), \xi_1 - \xi_2 \rangle,$$

$\forall \xi_1, \xi_2 \in \mathbb{R}^n, \forall x \in \Omega$ and $s \in \mathbb{R}$.

For the detailed discussion on implications of Hypotheses (H1)–(H9), we refer to Section 3.4 [16].

Example 2.1. *The prototype for such functions is*

$$a(x, s, \xi) := |A(x)\xi \cdot \xi|^{(p-2)/2} A(x)\xi,$$

where $A(\cdot)$ is a continuous function with values in the set of $n \times n$ symmetric matrices which satisfies

$$(2.4) \quad \alpha' |\xi|^2 \leq A(x)\xi \cdot \xi, \quad |A(x)\xi| \leq \beta' |\xi|, \quad \forall \xi \in \mathbb{R}^n, \forall x \in \Omega,$$

for some constants α' and β' .

Example 2.2. *If we take A as identity matrix in Example 2.1, then*

$$a(x, s, \xi) = |\xi|^{p-2} \xi,$$

which corresponds to the p -Laplace operator.

Example 2.3. $a(x, s, \xi) = |\xi|^{p-2} (1 + |\xi|^p)^{\frac{q-p}{p}} \xi$, $1 < q \leq p < \infty$, which reduces to p -Laplace operator when $q = p$.

In next the lemma, we state Picone's identity for the operator $-\operatorname{div}(a(x, u, \nabla u))$ and the proof follows from [23, 35] with some minor changes. So we skip the proof.

Lemma 2.4. [23, 35] *Let $a : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfies (H1)–(H9). Assume that $u \geq 0$ and $v > 0$ are differentiable functions in Ω and denote*

$$(2.5) \quad L(u, v) = \langle a(x, u, \nabla u), \nabla u \rangle - \left\langle a(x, v, \nabla v), \nabla \left(\frac{u^p}{v^{p-1}} \right) \right\rangle.$$

$$(2.6) \quad R(u, v) = \langle a(x, u, \nabla u), \nabla u \rangle - p \left\langle a \left(x, v, \frac{u}{v} \nabla v \right), \nabla u \right\rangle + (p-1) \left\langle a \left(x, v, \frac{u}{v} \nabla v \right), \frac{u}{v} \nabla v \right\rangle.$$

Then (i) $L(u, v) = R(u, v)$ (ii) $L(u, v) \geq 0$ and (iii) $L(u, v) = 0$ in Ω if and only if $u = cv$ for some $c \in \mathbb{R}$.

3. APPLICATIONS

This section deals with the applications of Lemma 2.4.

3.1. Hardy type inequality. In next theorem, we obtain a Hardy type inequality [31].

Theorem 3.1. *Assume that $v \in C^1(\Omega)$ satisfying*

$$(3.1) \quad -\operatorname{div}(a(x, v, \nabla v)) \geq \lambda g |v|^{p-1}, \quad v > 0, \quad \text{in } \Omega$$

for some $\lambda > 0$ and $0 \leq g \in L^\infty(\Omega)$. Then for any $0 \leq u \in C_0^\infty(\Omega)$, the following holds

$$(3.2) \quad \beta \int_{\Omega} |\nabla u|^p dx \geq \lambda \int_{\Omega} g |u|^p dx.$$

Proof. Take $0 < \phi \in C_0^\infty(\Omega)$, then

$$\begin{aligned} 0 &\leq \int_{\Omega} L(\phi, v) dx = \int_{\Omega} R(\phi, v) dx \\ &= \int_{\Omega} \langle a(x, \phi, \nabla \phi), \nabla \phi \rangle dx - \int_{\Omega} \langle \nabla \left(\frac{\phi^p}{v^{p-1}} \right), a(x, v, \nabla v) \rangle dx \\ &\leq \beta \int_{\Omega} |\nabla \phi| |\nabla \phi|^{p-1} dx + \int_{\Omega} \frac{\phi^p}{v^{p-1}} \operatorname{div}(a(x, v, \nabla v)) dx \\ &\leq \beta \int_{\Omega} |\nabla \phi|^p dx - \lambda \int_{\Omega} \phi^p g dx \quad (\text{by (3.1)}) \end{aligned}$$

Letting $\phi \rightarrow u$, we get

$$\beta \int_{\Omega} |\nabla u|^p dx \geq \lambda \int_{\Omega} g |u|^p dx.$$

□

3.2. Sturmian comparison theorem. Sturmian comparison theorem plays an important role in the qualitative theory of elliptic partial differential equations. In the next theorem, we establish a comparison theorem for the operator under consideration.

Consider the problem

$$(3.3) \quad \begin{aligned} -\operatorname{div}(a(x, u, \nabla u)) &= g(x, u) \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where $a(\cdot, \cdot, \cdot)$ satisfies (H1)–(H9) and $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that

$$(3.4) \quad \left. \begin{aligned} g(x, s)s &\geq 0, \quad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega \\ |g(x, s)| &\leq |s|^{p-1}, \quad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega. \end{aligned} \right\}$$

We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of (3.3) if

$$(3.5) \quad \int_{\Omega} \langle a(x, u, \nabla u), \nabla \phi \rangle dx = \int_{\Omega} g(x, u) \phi dx,$$

for all $\phi \in W_0^{1,p}(\Omega)$. Bensoussan et. al. [17] proved the existence of a weak solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ to the Problem (3.3). By using regularity result (Theorem 8.10 [41]) and maximum principle (Theorem 3.2.1 [44] and Corollary 8.17 [41]), we conclude that u is positive in Ω and $u \in C^{1,\gamma}(\bar{\Omega})$ for some $0 < \gamma < 1$. Now, we prove a Sturm type comparison theorem.

Theorem 3.2. *Let u be a positive solution to the problem*

$$(3.6) \quad \begin{cases} -\operatorname{div}(a(x, u, \nabla u)) = f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f(x, s)$ is a Carathéodory function such that

$$(3.7) \quad \begin{cases} f(x, s)s \geq 0, \quad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega \\ f(x, s) = h_1(x)g_1(s), \end{cases}$$

where $h_1 \in L^1(\Omega)$, $0 \leq h_1(x) \leq 1$ a.e. in Ω and g_1 is a continuous and increasing function with finite value on \mathbb{R}^+ with growth condition $|g_1(s)| \leq |s|^{p-1}$, $\forall s \in \mathbb{R}$.

Consider

$$(3.8) \quad \begin{cases} -\operatorname{div}(a(x, v, \nabla v)) = h_2(x)|v|^{p-2}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $h_2 \in L^1(\Omega)$. If $h_1(x) < h_2(x)$ a.e in Ω , then any nontrivial solution of (3.8) changes sign in Ω .

Proof. We shall prove this theorem by the method of contradiction. Let us assume that v does not change sign in Ω . Then either $v > 0$ in Ω or $v < 0$ in Ω . We assume that $v > 0$ in Ω , the case $v < 0$ in Ω can be dealt similarly. Then by Lemma 2.4, we have

$$(3.9) \quad \begin{aligned} 0 &\leq \int_{\Omega} L(u, v) dx = \int_{\Omega} R(u, v) dx \\ &= \int_{\Omega} \langle a(x, u, \nabla u), \nabla u \rangle dx - \int_{\Omega} \left\langle \nabla \left(\frac{u^p}{v^{p-1}} \right), a(x, v, \nabla v) \right\rangle dx. \end{aligned}$$

On using $\phi = u$ and $\phi = \frac{u^p}{v^{p-1}}$ as the test function in the weak formulation of (3.6) and (3.8) respectively, we get

$$(3.10) \quad \int_{\Omega} \langle a(x, u, \nabla u), \nabla u \rangle dx = \int_{\Omega} f(x, u) u dx$$

and

$$(3.11) \quad \int_{\Omega} \left\langle a(x, v, \nabla v), \nabla \left(\frac{u^p}{v^{p-1}} \right) \right\rangle dx = \int_{\Omega} h_2(x) v^{p-1} \frac{u^p}{v^{p-1}} dx = \int_{\Omega} h_2(x) u^p dx.$$

On using (3.10) and (3.11) in (3.9), we get

$$\begin{aligned} 0 &\leq \int_{\Omega} f(x, u) u dx - \int_{\Omega} h_2(x) u^p dx \\ &= \int_{\Omega} h_1(x) g_1(u) u dx - \int_{\Omega} h_2(x) u^p dx \\ &\leq \int_{\Omega} h_1(x) u^p dx - \int_{\Omega} h_2(x) u^p dx \quad (\text{Since } |g_1(s)| \leq |s|^{p-1}) \\ &= \int_{\Omega} (h_1(x) - h_2(x)) u^p dx < 0, \end{aligned}$$

which is a contradiction. Thus v must change sign. \square

3.3. A nonlinear system with singular nonlinearity. In this subsection, we consider a system with singular nonlinearity. We establish a linear relationship between u and v , using Lemma 2.4.

Theorem 3.3. Consider

$$(3.12) \quad \begin{cases} -\operatorname{div}(a(x, u, \nabla u)) = |v|^{p-2}v & \text{in } \Omega, \\ -\operatorname{div}(a(x, v, \nabla v)) = \frac{(|v|^{p-2}v)^2}{u^{p-1}} & \text{in } \Omega, \\ u > 0, v > 0 & \text{in } \Omega, \\ u = 0 = v & \text{on } \partial\Omega. \end{cases}$$

If (u, v) is a weak solution of (3.12). Then $u = cv$, where c is a constant.

Proof. Let (u, v) be a weak solution of (3.12). Now for any $\phi_1, \phi_2 \in W_0^{1,p}(\Omega)$, we have

$$(3.13) \quad \int_{\Omega} \langle a(x, u, \nabla u), \nabla \phi_1 \rangle dx = \int_{\Omega} v^{p-1} \phi_1 dx.$$

$$(3.14) \quad \int_{\Omega} \langle a(x, v, \nabla v), \nabla \phi_2 \rangle dx = \int_{\Omega} \frac{v^{2p-2}}{u^{p-1}} \phi_2 dx.$$

By choosing $\phi_1 = u$ and $\phi_2 = \frac{u^p}{v^{p-1}}$ as a test function in (3.13) and (3.14) respectively, we get

$$(3.15) \quad \int_{\Omega} \langle a(x, u, \nabla u), \nabla u \rangle dx = \int_{\Omega} uv^{p-1} dx.$$

$$(3.16) \quad \int_{\Omega} \left\langle a(x, v, \nabla v), \nabla \left(\frac{u^p}{v^{p-1}} \right) \right\rangle dx = \int_{\Omega} uv^{p-1} dx.$$

Using (3.15) and (3.16), we get

$$\int_{\Omega} R(u, v) dx = \int_{\Omega} \langle a(x, u, \nabla u), \nabla u \rangle dx - \int_{\Omega} \left\langle a(x, v, \nabla v), \nabla \left(\frac{u^p}{v^{p-1}} \right) \right\rangle dx = 0.$$

By the nonnegativity of $R(u, v)$, we conclude that $R(u, v) = 0$, which implies $u = cv$, for some $c \in \mathbb{R}$, by Lemma 2.4. \square

3.4. Monotonicity property of first eigenvalue. Consider the eigenvalue problem

$$(3.17) \quad \begin{aligned} -\operatorname{div}(a(x, u, \nabla u)) &= \lambda |u|^{p-2} u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

A real number λ such that (3.17) admits a nontrivial solution u is called an eigenvalue of the operator $-\operatorname{div}(a(x, u, \nabla u))$ and u is called the corresponding eigenfunction. The first eigenvalue of (3.17) is denoted by λ_1 and is defined as:

$$(3.18) \quad \lambda_1 = \inf_{0 \neq u \in W_0^{1,p}(\Omega)} \frac{\int_{\Omega} \langle a(x, u, \nabla u), \nabla u \rangle dx}{\int_{\Omega} |u|^p dx}.$$

Following the arguments of Lemma 3.6 [16], it is easy to see that, infimum in (3.18) is attained. Using the arguments in the spirit of Bonder et. al. [23] and Kawohl et. al. [35], we can show that λ_1 is simple and corresponding eigenfunction is of fixed sign. Next, we prove the monotonicity property of λ_1 .

Theorem 3.4. *Suppose $\Omega_1 \subset \Omega_2$ and $\Omega_1 \neq \Omega_2$. Then $\lambda_1(\Omega_1) > \lambda_1(\Omega_2)$, if both exist.*

Proof. Let u_i be positive eigenfunction associated with λ_i , $i = 1, 2$. Then

$$(3.19) \quad \begin{cases} -\operatorname{div}(a(x, u_1, \nabla u_1)) = \lambda_1(\Omega_1) |u_1|^{p-2} u_1 & \text{in } \Omega_1, \\ u_1 > 0 & \text{in } \Omega_1, \\ u_1 = 0 & \text{on } \partial\Omega_1, \end{cases}$$

and

$$(3.20) \quad \begin{cases} -\operatorname{div}(a(x, u_2, \nabla u_2)) = \lambda_1(\Omega_2) |u_2|^{p-2} u_2 & \text{in } \Omega_2, \\ u_2 > 0 & \text{in } \Omega_2, \\ u_2 = 0 & \text{on } \partial\Omega_2. \end{cases}$$

For $0 \leq \phi \in C_0^\infty(\Omega_1)$, By Lemma 2.4 we have,

$$\begin{aligned} 0 &\leq \int_{\Omega_1} L(\phi, u_2) dx = \int_{\Omega_1} R(\phi, u_2) dx \\ &= \int_{\Omega_1} \langle a(x, \phi, \nabla \phi), \nabla \phi \rangle dx - \int_{\Omega_1} \left\langle a(x, u_2, \nabla u_2), \nabla \left(\frac{\phi^p}{u_2^{p-1}} \right) \right\rangle dx. \end{aligned} \quad (3.21)$$

By multiplying (3.19) and (3.20) with test functions $\eta \in C_0^\infty(\Omega_1)$ and $\psi \in C_0^\infty(\Omega_2)$, respectively and then by integrating, we obtain

$$(3.22) \quad \int_{\Omega_1} \langle a(x, u_1, \nabla u_1), \nabla \eta \rangle dx = \lambda_1(\Omega_1) \int_{\Omega_1} |u_1|^{p-1} u_1 \eta dx$$

and

$$(3.23) \quad \int_{\Omega_2} \langle a(x, u_2, \nabla u_2), \nabla \psi \rangle dx = \lambda_1(\Omega_2) \int_{\Omega_2} |u_2|^{p-2} u_2 \psi dx.$$

Since $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, therefore, (3.22) holds for each $\eta \in W_0^{1,p}(\Omega_1)$ and (3.23) holds for each $\psi \in W_0^{1,p}(\Omega_2)$. By choosing $\psi = \frac{\phi^p}{u_2^{p-1}}$ as test function in (3.23), we get

$$(3.24) \quad \int_{\Omega_2} \left\langle a(x, u_2, \nabla u_2), \nabla \left(\frac{\phi^p}{u_2^{p-1}} \right) \right\rangle dx = \lambda_1(\Omega_2) \int_{\Omega_2} \frac{\phi^p}{u_2^{p-1}} u_2^{p-1} dx.$$

Since ϕ vanishes outside Ω_1 , therefore, we have

$$(3.25) \quad \int_{\Omega_1} \left\langle a(x, u_2, \nabla u_2), \nabla \left(\frac{\phi^p}{u_2^{p-1}} \right) \right\rangle dx = \lambda_1(\Omega_2) \int_{\Omega_1} \phi^p dx.$$

By using (3.25) in (3.21), we obtain,

$$(3.26) \quad 0 \leq \int_{\Omega_1} L(\phi, u_2) dx = \int_{\Omega_1} \langle a(x, \phi, \nabla \phi), \nabla \phi \rangle dx - \lambda_1(\Omega_2) \int_{\Omega_1} \phi^p dx.$$

Letting $\phi = u_1$ in (3.26), we get

$$\begin{aligned} 0 &\leq \int_{\Omega_1} L(u_1, u_2) dx = \int_{\Omega_1} \langle a(x, u_1, \nabla u_1), \nabla u_1 \rangle dx - \lambda_1(\Omega_2) \int_{\Omega_1} u_1^p dx \\ &= \lambda_1(\Omega_1) \int_{\Omega_1} u_1^p dx - \lambda_1(\Omega_2) \int_{\Omega_1} u_1^p dx, \text{ (By choosing } \eta = u_1 \text{ in (3.22))} \\ (3.27) \quad &= (\lambda_1(\Omega_1) - \lambda_1(\Omega_2)) \int_{\Omega_1} u_1^p dx. \end{aligned}$$

(3.27) leads $\lambda_1(\Omega_1) - \lambda_1(\Omega_2) \geq 0$. Now, if $\lambda_1(\Omega_1) - \lambda_1(\Omega_2) = 0$, then $L(u_1, u_2) = 0$ and an application of Lemma 2.4 implies that $u_1 = cu_2$; which is not possible as $\Omega_1 \subset \Omega_2$ and $\Omega_1 \neq \Omega_2$. This completes the proof. \square

3.5. Nonexistence of positive supersolutions. Consider the equation

$$(3.28) \quad -\operatorname{div}(a(x, u, \nabla u)) - g(x)|u|^{p-2}u = f(x),$$

where $p > 1$, $0 \leq g \in L^\infty(\Omega)$, $0 \leq f \in L^q(\Omega)$, $q = \frac{p-1}{p}$.

We call $u \in W^{1,p}(\Omega)$ a weak supersolution of Equation (3.28) in $\Omega \subseteq \mathbb{R}^n$ if it satisfies

$$(3.29) \quad \int_{\Omega} \langle a(x, u, \nabla u), \nabla \phi \rangle dx - \int_{\Omega} g(x)|u|^{p-2}u\phi dx \geq \int_{\Omega} f(x)\phi dx, \quad \forall 0 \leq \phi \in W_0^{1,p}(\Omega).$$

A weak subsolution $u \in W^{1,p}(\Omega)$ of (3.28) is defined analogously with the inequality (3.29) reversed.

Let us consider a functional J associated with (3.28), defined by

$$(3.30) \quad J(u) := \int_{\Omega} \langle a(x, u, \nabla u), \nabla u \rangle dx - \int_{\Omega} g(x)|u|^p dx.$$

Theorem 3.5. Assume that (3.28) has positive supersolution $v \in W^{1,p}(\Omega)$. Then for any $0 \leq u \in W_0^{1,p}(\Omega)$, the inequality

$$(3.31) \quad J(u) \geq \int_{\Omega} R(u, v) dx + \int_{\Omega} f(x) \frac{u^p}{v^{p-1}} dx$$

holds.

Proof. Let $v > 0$ be a supersolution of (3.28) and $0 \leq u \in W_0^{1,p}(\Omega)$. By choosing

$$\phi = \frac{u^p}{v^{p-1}}$$

as a test function in (3.29), we obtain

$$(3.32) \quad \int_{\Omega} \left\langle a(x, v, \nabla v), \nabla \left(\frac{u^p}{v^{p-1}} \right) \right\rangle dx - \int_{\Omega} g(x)|v|^{p-2}v \frac{u^p}{v^{p-1}} dx \geq \int_{\Omega} f(x) \frac{u^p}{v^{p-1}} dx.$$

On rearrangement of terms in (3.32), we obtain

$$\begin{aligned} \int_{\Omega} g(x)u^p dx &\leq \int_{\Omega} \left\langle a(x, v, \nabla v), \nabla \left(\frac{u^p}{v^{p-1}} \right) \right\rangle dx - \int_{\Omega} f(x) \frac{u^p}{v^{p-1}} dx \\ &= \int_{\Omega} \langle a(x, u, \nabla u), \nabla u \rangle dx - \int_{\Omega} \left[\langle a(x, u, \nabla u), \nabla u \rangle - \left\langle a(x, v, \nabla v), \nabla \left(\frac{u^p}{v^{p-1}} \right) \right\rangle \right] dx \\ &\quad - \int_{\Omega} f(x) \frac{u^p}{v^{p-1}} dx \\ &= \int_{\Omega} \langle a(x, u, \nabla u), \nabla u \rangle dx - \int_{\Omega} L(u, v) dx - \int_{\Omega} f(x) \frac{u^p}{v^{p-1}} dx \\ (3.33) \quad &= \int_{\Omega} \langle a(x, u, \nabla u), \nabla u \rangle dx - \int_{\Omega} R(u, v) dx - \int_{\Omega} f(x) \frac{u^p}{v^{p-1}} dx. \end{aligned}$$

By rearranging terms in (3.33), we get

$$J(u) = \int_{\Omega} \langle a(x, u, \nabla u), \nabla u \rangle dx - \int_{\Omega} g(x)|u|^p dx \geq \int_{\Omega} R(u, v) dx + \int_{\Omega} f(x) \frac{u^p}{v^{p-1}} dx,$$

which proves (3.31). \square

Remark 3.6. If, in particular, $v \in W^{1,p}(\Omega)$ is a positive weak solution of (3.28), then (3.31) becomes equality.

Remark 3.7. Since $R(u, v) \geq 0$ in Ω , so if v is a positive supersolution of (3.28) and $0 \leq u \in W_0^{1,p}(\Omega)$, then

$$(3.34) \quad J(u) \geq \int_{\Omega} f(x) \frac{|u|^p}{v^{p-1}} dx \geq 0.$$

Corollary 3.8. Assume that there exists $0 \leq u \in W_0^{1,p}(\Omega)$ such that $J(u) < 0$. Then (3.28) has no positive supersolution in Ω .

Proof. The proof is done by the method of contradiction. Suppose (3.28) has a positive supersolution in Ω . Then by Remark 3.7, (3.34) holds, which is a contradiction to $J(u) < 0$. This completes the proof. \square

In the next subsection, we obtain a Caccioppoli inequality using Picone's identity.

3.6. Caccioppoli Inequality. Let us consider the following equation

$$(3.35) \quad -\operatorname{div}(a(x, v, \nabla v)) = g(x)v,$$

where $0 \leq g \in L^{\infty}(\Omega)$. We say that $v \in H^1(\Omega) \cap C(\bar{\Omega})$ is a continuous weak subsolution to (3.35) if

$$(3.36) \quad \int_{\Omega} \langle a(x, v, \nabla v), \nabla \phi \rangle dx \leq \int_{\Omega} g(x)v\phi dx, \quad \forall 0 \leq \phi \in H^1(\Omega) \cap C(\bar{\Omega}).$$

Next, we establish a Caccioppoli type inequality for the subsolutions of (3.35).

Theorem 3.9. Let $0 < v \in H^1(\Omega) \cap C(\bar{\Omega})$ be a continuous weak subsolution of (3.35) and $0 \leq \eta \in C_0^{\infty}(\Omega)$, then

$$(3.37) \quad \int_{\Omega} \langle a(x, v, \eta \nabla v), \eta \nabla v \rangle dx \leq \int_{\Omega} g(x)\eta^p v^2 dx - p \int_{\Omega} \langle a(x, v, \eta \nabla v), v \nabla \eta \rangle.$$

Proof. Let us choose $u = \eta v$ in (2.5) and (2.6), where $v \in H^1(\Omega) \cap C(\bar{\Omega})$ is a positive subsolution of (3.36) and $0 \leq \eta \in C_0^{\infty}(\Omega)$, we get

$$(3.38) \quad -\frac{1}{p} \int_{\Omega} \langle a(x, v, \nabla v), \nabla(\eta^p v) \rangle dx = - \int_{\Omega} \langle a(x, v, \eta \nabla v), \nabla(\eta v) \rangle dx \\ + \frac{p-1}{p} \int_{\Omega} \langle a(x, v, \eta \nabla v), \eta \nabla v \rangle dx.$$

By the regularity theory, $v \in C^{1,\gamma}$, $\gamma < 1$, and therefore, we can compute

$$\nabla(\eta v) = \eta \nabla v + v \nabla \eta.$$

Inserting this expression in (3.38), we obtain

$$-\frac{1}{p} \int_{\Omega} \langle a(x, v, \nabla v), \nabla(\eta^p v) \rangle dx = - \int_{\Omega} \langle a(x, v, \eta \nabla v), \eta \nabla v + v \nabla \eta \rangle dx \\ + \frac{p-1}{p} \int_{\Omega} \langle a(x, v, \eta \nabla v), \eta \nabla v \rangle dx \\ = - \int_{\Omega} (\langle a(x, v, \eta \nabla v), \eta \nabla v \rangle + \langle a(x, v, \eta \nabla v), v \nabla \eta \rangle) dx \\ + \frac{p-1}{p} \int_{\Omega} \langle a(x, v, \eta \nabla v), \eta \nabla v \rangle dx$$

$$= \left(\frac{p-1}{p} - 1 \right) \int_{\Omega} \langle a(x, v, \eta \nabla v), \eta \nabla v \rangle dx - \int_{\Omega} \langle a(x, v, \eta \nabla v), v \nabla \eta \rangle.$$

This implies that

$$(3.39) \quad - \int_{\Omega} \langle a(x, v, \nabla v), \nabla(\eta^p v) \rangle dx = - \int_{\Omega} \langle a(x, v, \eta \nabla v), \eta \nabla v \rangle dx - p \int_{\Omega} \langle a(x, v, \eta \nabla v), v \nabla \eta \rangle.$$

Let us choose $\phi = \eta^p v$ as a test function in (3.36), where $v \in H^1(\Omega) \cap C(\bar{\Omega})$, we obtain

$$(3.40) \quad \int_{\Omega} \langle a(x, v, \nabla v), \nabla(\eta^p v) \rangle dx \leq \int_{\Omega} g(x) \eta^p v^2 dx, \quad \forall 0 \leq \phi \in H^1(\Omega) \cap C(\bar{\Omega}).$$

Using (3.40) in (3.39), we obtain

$$\int_{\Omega} \langle a(x, v, \eta \nabla v), \eta \nabla v \rangle dx \leq \int_{\Omega} g(x) \eta^p v^2 dx - p \int_{\Omega} \langle a(x, v, \eta \nabla v), v \nabla \eta \rangle.$$

This completes the proof. \square

Corollary 3.10. *Let $g(x) \equiv 0$ in (3.35). Let $0 < v \in H^1(\Omega) \cap C(\bar{\Omega})$ be a weak solution of (3.35) and $0 \leq \eta \in C_0^\infty(\Omega)$, then*

$$(3.41) \quad \int_{\Omega} |\eta \nabla v|^p dx \leq \left(\frac{p\beta}{\alpha} \right)^p \int_{\Omega} |v \nabla \eta|^p.$$

Proof. By putting $g(x) \equiv 0$ in (3.40) and using it in (3.39), we obtain

$$0 \leq - \int_{\Omega} \langle a(x, v, \eta \nabla v), \eta \nabla v \rangle dx - p \int_{\Omega} \langle a(x, v, \eta \nabla v), v \nabla \eta \rangle, \text{ i.e.,}$$

$$\int_{\Omega} \langle a(x, v, \eta \nabla v), \eta \nabla v \rangle dx \leq p \int_{\Omega} |a(x, v, \eta \nabla v)| |v \nabla \eta| dx.$$

This implies that

$$\begin{aligned} \alpha \int_{\Omega} |\eta \nabla v|^p dx &\leq p\beta \int_{\Omega} |\eta \nabla v|^{p-1} |v \nabla \eta| dx, \text{ (by (H3) and (H4))} \\ &\leq p\beta \left(\int_{\Omega} |\eta \nabla v|^{(p-1)q} dx \right)^{\frac{1}{q}} \left(\int_{\Omega} |v \nabla \eta|^p dx \right)^{\frac{1}{p}} \\ &= p\beta \left(\int_{\Omega} |\eta \nabla v|^p dx \right)^{\frac{1}{q}} \left(\int_{\Omega} |v \nabla \eta|^p dx \right)^{\frac{1}{p}} \\ \int_{\Omega} |\eta \nabla v|^p dx &\leq \frac{p\beta}{\alpha} \left(\int_{\Omega} |\eta \nabla v|^p dx \right)^{\frac{1}{q}} \left(\int_{\Omega} |v \nabla \eta|^p dx \right)^{\frac{1}{p}} \\ \int_{\Omega} |\eta \nabla v|^p dx &\leq \left(\frac{p\beta}{\alpha} \right)^p \int_{\Omega} |v \nabla \eta|^p. \end{aligned}$$

This completes the proof. \square

Corollary 3.11. *Let $g(x) \equiv 0$ in (3.35). Let $0 < v \in H^1(\Omega) \cap C(\bar{\Omega})$ be a weak solution of (3.35) and $B_{2r}(x_0) \subseteq \Omega$, then*

$$\int_{B_{2r}(x_0)} |\nabla v|^p dx \leq \left(\frac{p\beta}{\alpha r} \right)^p \int_{B_{2r}(x_0) \setminus B_r(x_0)} |v|^p dx.$$

Proof. Let us choose $\eta \in C_0(B_{2r}(x_0))$ such that

$$\eta \equiv 1 \text{ in } B_r(x_0), \quad |\nabla \eta| \leq \frac{c}{r}, \quad \text{in } B_{2r}(x_0) \setminus B_r(x_0),$$

where $c > 0$. Then from Corollary 3.10, we have the estimates (3.41) and therefore

$$\int_{B_{2r}(x_0)} |\nabla v|^p dx \leq \left(\frac{p\beta}{\alpha r} \right)^p \int_{B_{2r}(x_0) \setminus B_r(x_0)} |v|^p dx.$$

□

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