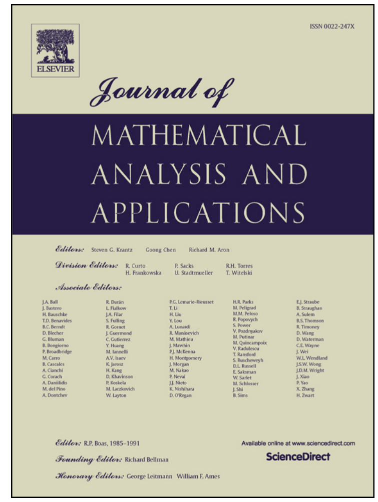

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Regularization of an inverse nonlinear parabolic problem with time-dependent coefficient and locally Lipschitz source term

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Abstract

We consider a backward problem of finding a function u satisfying a nonlinear parabolic equation in the form $u_t + a(t)Au(t) = f(t, u(t))$ subject to the final condition $u(T) = \varphi$. Here A is a positive self-adjoint unbounded operator in a Hilbert space H and f satisfies a locally Lipschitz condition. This problem is ill-posed. Using quasi-reversibility method, we shall construct a regularized solution u_ε from the measured data a_ε and φ_ε. We show that the regularized problems are well-posed and that their solutions converge to the exact solutions. Error estimates of logarithmic type are given and a simple numerical example is presented to illustrate the method as well as verify the error estimates given in the theoretical parts.

Keywords and phrases: Nonlinear parabolic problem, Backward problem, Quasi-reversibility, Ill-posed problem, Contraction principle.
Mathematics subject Classification 2000: 35K05, 35K99, 47J06, 47H10.

1. Introduction

Let $(H, \|\cdot\|)$ be a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$. Let A be a positive self-adjoint operator defined on a dense subspace $D(A) \subset H$ such that $-A$ generates a compact contraction semi-group $S(t)$ on H. Let $f : [0, T] \times H \to H$ satisfy the locally Lipschitz condition: for each $M > 0$, there exists $k(M) > 0$ such that

$$\|f(t, u) - f(t, v)\| \le k(M)\|u - v\| \text{ if } \max\{\|u\|, \|v\|\} \le M. \tag{1}$$

We shall consider a backward problem of finding a function $u : [0, T] \to H$ such that

$$\begin{aligned} u_t + a(t)Au(t) &= f(t, u(t)), \quad 0 < t < T, \\ u(T) &= \varphi, \end{aligned} \tag{2}$$

where $a \in C([0, T])$ is a given real-valued function and $\varphi \in H$ is a prescribed final value.

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This nonlinear nonhomogeneous problem is severely ill-posed. In fact, the problem is extremely sensitive to measurement errors (see, e.g., [2]). The final data is usually the result of discrete experimental measurements and is subject to error. Hence, a solution corresponding to the data does not always exist, and in the case of existence, does not depend continuously on the given data. This, of course, shows that a naturally numerical treatment is impossible. Thus one has to resort to a regularization.

The backward problem (2) has a long history. The linear homogeneous case $f = 0$ has been considered by many authors such as quasi-reversibility method [7, 8, 6, 10, 1], quasi-boundary value method [4, 5]. The problem with constant coefficient and nonlinear source term, i.e.

$$\begin{aligned} u_t + Au(t) &= f(t, u(t)), \quad 0 < t < T, \\ u(T) &= \varphi, \end{aligned} \tag{3}$$

was studied in [3, 13, 14, 15]. However, in these papers, the source function f is assumed to be globally Lipschitz, that is

$$\|f(t, u) - f(t, v)\| \le k\|u - v\|$$

where k is independent of t, u, v. Recently, in [16], a regularization method for locally Lipschitz source term has been established under an extra condition on the source term:

$$\text{There exists a constant } L \ge 0, \text{ such that } \langle f(t, u) - f(t, v), u - v\rangle + L\|u - v\|^2 \ge 0.$$

This condition holds for the source $f(u) = u\,\|u\|_H^2$ (see [16]). However, it is not satisfied in several cases, for example, $f(u) = au - bu^3$ $(b > 0)$ when $u \in H^1(I)$ where I denotes some interval in $\mathbb{R}$ with finite length. Hence, another regularization method which can be applied to any locally Lipschitz source term is of interest. In this paper, we shall assume that the source term f is locally Lipschitz with respect to u (i.e. f satisfies (1)). Our main idea is approximating the function f by a sequence f_ε of Lipschitz functions

$$\|f_\varepsilon(t, u) - f_\varepsilon(t, v)\| \le k_\varepsilon\|u - v\|.$$

Then, we use the results in [13, 15] to approximate problem (3) by the following problem

$$\begin{aligned} \frac{d}{dt}u^\varepsilon(t) + A_\varepsilon u^\varepsilon(t) &= B(\varepsilon, t) f_\varepsilon(t, u^\varepsilon(t)), \qquad t \in [0, T], \\ u^\varepsilon(T) &= \varphi, \end{aligned} \tag{4}$$

where A_ε, $B(\varepsilon, t)$ are defined appropriately.

When the perturbed coefficient a is time-dependent, the problems turns to be more complicated. Indeed, the strategies used for constant coefficient cannot be applied to the time-dependent coefficient case. The problem with time-dependent coefficient has been recently investigated in [9]. However, the methods proposed in [9] can be merely applied either for zero source with perturbed time-dependent coefficient or for globally Lipschitz source with unperturbed time-dependent coefficient. We would like to emphasize that our regularization method for constant coefficient also works for unperturbed time-dependent coefficient.

The paper is organized as follows. In Section 2, we shall investigate a regularization method for the case of constant coefficient $a \equiv 1$. In particular, we shall give precise formulas of A_ε, $B(\varepsilon, t)$ and $f_\varepsilon(t, v)$; show that the regularized problem (4) is well-posed and prove the convergence of u^ε to the exact solution in $C([0, T]; H)$ with explicit error estimates. Section 3 provides a regularization method for perturbed time-dependent coefficient $a(t)$. A simple numerical example to illustrate the method is given in Section 4 and we end this paper by a conclusion in Section 5.

2. Regularization of backward parabolic problem with constant coefficient

2.1. The well-posedness of the regularized problem (4)

We shall first give the precise formula of the compact contraction semigroup $S(t)$ that is generated by $-A$. Assume that A is a positive self-adjoint operator in the separable Hilbert space $(H, \langle\cdot,\cdot\rangle)$ and 0 belongs to its resolvent set. Since A^{-1} is a compact self-adjoint operator, there is an orthonormal eigenbasis $\{\phi_n\}_{n=1}^{\infty}$ of H corresponding to a sequence of its eigenvalues $\{\lambda_n^{-1}\}_{n=1}^{\infty}$ in which

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \lim_{n\to\infty} \lambda_n = \infty.$$

Thus $A^{-1}\phi_n = \lambda_n^{-1}\phi_n$ and $A\phi_n = \lambda_n\phi_n$ for each $n \geq 1$. The compact contraction semi-group $S(t)$ corresponding to A is

$$S(t)v = \sum_{n=1}^{\infty} e^{-t\lambda_n} \langle v, \phi_n\rangle \phi_n,\ v \in H.$$

Problem (3) can be written in the language of semi-groups as follows:

$$u(t) = S^{-1}(T-t)\varphi - \int_t^T S^{-1}(s-t) f(s, u(s))\,\mathrm{d}s. \tag{5}$$

For each $\varepsilon > 0$, we define the bounded operator

$$A_\varepsilon(v) \quad = \quad -\frac{1}{T}\sum_{n=1}^{\infty} \ln(\varepsilon + e^{-T\lambda_n}) \langle v, \phi_n\rangle \phi_n. \tag{6}$$

The compact contraction semi-group $S_\varepsilon(t)$ corresponding to A_ε is

$$S_\varepsilon(t)v = \sum_{n=1}^{\infty} \left(\varepsilon + e^{-T\lambda_n}\right)^{\frac{t}{T}} \langle v, \phi_n\rangle \phi_n,\ v \in H.$$

It is easy to see that

$$S_\varepsilon^{-1}(t)v = \sum_{n=1}^{\infty} \left(\varepsilon + e^{-T\lambda_n}\right)^{\frac{-t}{T}} \langle v, \phi_n\rangle \phi_n,\ v \in H,$$

for all $t \in [0, T]$. Obviously, (4) can be written as

$$u^\varepsilon(t) = S_\varepsilon^{-1}(T-t)\varphi_\varepsilon - \int_t^T S_\varepsilon^{-1}(s-t) B(\varepsilon, s) f_\varepsilon(s, u^\varepsilon(s))\,\mathrm{d}s, \tag{7}$$

For each $t \leq T$, we define by $B(\varepsilon, t)$ the bounded operator

$$B(\varepsilon, t) := S_\varepsilon^{-1}(T-t)S(T-t).$$

The operator $B(\varepsilon, t)$ can be written explicitly as

$$B(\varepsilon, t)(v) = \sum_{n=1}^{\infty} (1 + \varepsilon e^{T\lambda_n})^{\frac{t}{T}-1} \langle v, \phi_n\rangle \phi_n,\ v \in H. \tag{8}$$

In particular,

$$\begin{aligned} B(\varepsilon, t)\phi_n &= S_\varepsilon^{-1}(T-t)S(T-t)\phi_n = S_\varepsilon^{-1}(T-t)\left(e^{-(T-t)\lambda_n}\phi_n\right) \\ &= \left(\varepsilon + e^{-T\lambda_n}\right)^{\frac{t-T}{T}} e^{-(T-t)\lambda_n}\phi_n = (\varepsilon e^{T\lambda_n} + 1)^{\frac{t-T}{T}}\phi_n,\ \forall n \geq 1. \end{aligned}$$

We shall need some upper bounds of the operators $S_\varepsilon(-t)$ and $B(\varepsilon, t)$.

Lemma 1. *Let $0 \le t \le T$. Then $S_\varepsilon^{-1}(t)$ and $B(\varepsilon, t)$ are bounded operators and*

$$\left\|S_\varepsilon^{-1}(t)\right\|_{\mathcal{L}(H)} \le \varepsilon^{-\frac{t}{T}}, \qquad \|B(\varepsilon, t)\|_{\mathcal{L}(H)} \le 1.$$

Moreover,

$$\|[B(\varepsilon, t) - I]\,\phi_n\| \le \varepsilon e^{T\lambda_n}, \forall n \ge 1.$$

Proof. For each $n \ge 1$, one has

$$\begin{aligned}
\left\|S_\varepsilon^{-1}(t)\phi_n\right\| &= \left(\varepsilon + e^{-T\lambda_n}\right)^{-\frac{t}{T}} \le \varepsilon^{-\frac{t}{T}}, \\
\|B(\varepsilon, t)\phi_n\| &= (1 + \varepsilon e^{T\lambda_n})^{\frac{t}{T}-1} \le 1, \\
\|[I - B(\varepsilon, t)]\,\phi_n\| &= 1 - (1 + \varepsilon e^{T\lambda_n})^{\frac{t}{T}-1} \le 1 - (1 + \varepsilon e^{T\lambda_n})^{-1} \le \varepsilon e^{T\lambda_n}.
\end{aligned}$$

The desired result follows. □

Next, we define an approximation f_ε of f. Recall that $f : [0, T] \times H \to H$ satisfies the locally Lipschitz condition (1):

$$\begin{gathered}
\text{For each } M > 0, \text{ there exists } k(M) > 0 \text{ such that} \\
\|f(t, u) - f(t, v)\| \le k(M)\,\|u - v\| \quad \text{if} \quad \max\{\|u\|, \|v\|\} \le M.
\end{gathered}$$

It is obvious that the function $k(\cdot)$ is increasing on $[0, \infty)$. We can choose a set $\{M_\varepsilon > 0\}_{\varepsilon>0}$ satisfying $\lim\limits_{\varepsilon\to 0^+} M_\varepsilon = \infty$ and $k(M_\varepsilon) \le \ln(\ln(\varepsilon^{-1}))/(4T)$. Define

$$f_\varepsilon(t, v) = f\left(t, \min\left\{\frac{M_\varepsilon}{\|v\|}, 1\right\} v\right), \qquad \forall (t, v) \in [0, T] \times H, \tag{9}$$

in particular $f_\varepsilon(t, 0) = f(t, 0)$. With this definition, we claim that f_ε is a Lipschitz function. In fact, we have

Lemma 2. *For $\varepsilon > 0$, $t \in [0, T]$ and $v_1, v_2 \in H$, one has*

$$\|f_\varepsilon(t, v_1) - f_\varepsilon(t, v_2)\| \le k_\varepsilon \|v_1 - v_2\|,$$

where $k_\varepsilon = 2k(M_\varepsilon) \le \ln(\ln(\varepsilon^{-1}))/(2T)$.

Proof. It is sufficient to prove Lemma 2 for non-zero vectors v_1, v_2. Assume that $\|v_1\| \ge \|v_2\| > 0$. Using the locally Lipschitz property of f, one has

$$\begin{aligned}
\|f_\varepsilon(t, v_1) - f_\varepsilon(t, v_2)\| &= \left\| f\left(t, \min\left\{\frac{M_\varepsilon}{\|v_1\|}, 1\right\} v_1\right) - f\left(t, \min\left\{\frac{M_\varepsilon}{\|v_2\|}, 1\right\} v_2\right)\right\| \\
&\le k(M_\varepsilon)\left\|\min\left\{\frac{M_\varepsilon}{\|v_1\|}, 1\right\} v_1 - \min\left\{\frac{M_\varepsilon}{\|v_2\|}, 1\right\} v_2\right\|.
\end{aligned}$$

It remains to show that

$$\left\|\min\left\{\frac{M_\varepsilon}{\|v_1\|}, 1\right\} v_1 - \min\left\{\frac{M_\varepsilon}{\|v_2\|}, 1\right\} v_2\right\| \le 2\,\|v_1 - v_2\|.$$

This inequality is trivial if $M_\varepsilon \geq \|v_1\| \geq \|v_2\|$. When $\|v_1\| \geq \|v_2\| \geq M_\varepsilon$, one has

$$\begin{aligned}
\left\| \frac{M_\varepsilon}{\|v_1\|} v_1 - \frac{M_\varepsilon}{\|v_2\|} v_2 \right\| &= M_\varepsilon \left\| \frac{v_1 - v_2}{\|v_1\|} + \frac{\|v_2\| - \|v_1\|}{\|v_1\| . \|v_2\|} v_2 \right\| \\
&\leq M_\varepsilon \left(\left\| \frac{v_1 - v_2}{\|v_1\|} \right\| + \left\| \frac{\|v_2\| - \|v_1\|}{\|v_1\| . \|v_2\|} v_2 \right\| \right) \\
&= \frac{M_\varepsilon}{\|v_1\|} (\|v_1 - v_2\| + |\|v_2\| - \|v_1\||) \leq 2 \|v_1 - v_2\| .
\end{aligned}$$

Finally, if $\|v_1\| \geq M_\varepsilon \geq \|v_2\|$ then

$$\begin{aligned}
\left\| \frac{M_\varepsilon}{\|v_1\|} v_1 - v_2 \right\| &= \left\| \frac{M_\varepsilon - \|v_1\|}{\|v_1\|} v_1 + v_1 - v_2 \right\| \\
&\leq \left\| \frac{M_\varepsilon - \|v_1\|}{\|v_1\|} v_1 \right\| + \|v_1 - v_2\| \\
&= |M_\varepsilon - \|v_1\|| + \|v_1 - v_2\| \leq 2 \|v_1 - v_2\| .
\end{aligned}$$

Here we have used the inequality $|M_\varepsilon - \|v_1\|| \leq |\|v_2\| - \|v_1\|| \leq \|v_1 - v_2\|$. □

We now study the existence, the uniqueness and the stability of a (weak) solution of problem (4).

Theorem 1. *Let $\varepsilon > 0$. For each $\varphi \in H$, problem (4) has a unique solution $u^\varepsilon \in C([0, T]; H)$. Moreover, the solutions depend continuously on the data in the sense that if u_j^ε is the solution corresponding to φ_j, $j = 1, 2$, then*

$$\|u_1^\varepsilon(t) - u_2^\varepsilon(t)\| \leq \varepsilon^{\frac{t-T}{T}} e^{k_\varepsilon (T-t)} \|\varphi_1 - \varphi_2\|.$$

Proof. **Step 1: Uniqueness**

Fix $\varphi \in H$. For each $w \in C([0, T]; H)$, define by

$$F(w)(t) := S_\varepsilon^{-1}(T - t)_\varepsilon - \int_t^T S_\varepsilon^{-1}(s - t) B(\varepsilon, s) f_\varepsilon(s, w(s))\, \mathrm{d}s.$$

It is sufficient to show that F has a unique fixed point in $C([0, T]; H)$. This fact will be proved by contraction principle.

We claim by induction with respect to $m = 1, 2, ...$ that, for all $w, v \in C([0, T]; H)$,

$$\|F^m(w)(t) - F^m(v)(t)\| \leq \left(\frac{k_\varepsilon}{\varepsilon}\right)^m \frac{(T - t)^m}{m!} |||w - v|||, \tag{10}$$

where $|||.|||$ is the sup norm in $C([0, T]; H)$. For $m = 1$, using lemmas 1 and 2, we have

$$\begin{aligned}
\|F(w)(t) - F(v)(t)\| &= \left\| \int_t^T S_\varepsilon^{-1}(s - t) B(\varepsilon, s) \left[f_\varepsilon(s, w(s)) - f_\varepsilon(s, v(s)) \right] \mathrm{d}s \right\| \\
&\leq \int_t^T \left\| S_\varepsilon^{-1}(s - t) \right\|_{\mathcal{L}(H)} \cdot \|B(\varepsilon, s)\|_{\mathcal{L}(H)} \cdot \|f_\varepsilon(s, w(s)) - f_\varepsilon(s, v(s))\| \,\mathrm{d}s \\
&\leq k_\varepsilon \int_t^T \varepsilon^{\frac{t-s}{T}} \|w - v\| \,\mathrm{d}s \leq \frac{k_\varepsilon}{\varepsilon} \int_t^T \|w - v\| \,\mathrm{d}s \\
&\leq \frac{k_\varepsilon}{\varepsilon} (T - t) |||w - v|||.
\end{aligned}$$

Suppose that (10) holds for $m = j$. We prove that (10) holds for $m = j+1$. Infact, we have

$$\begin{aligned}\left\|F^{j+1}(w)(t) - F^{j+1}(v)(t)\right\| &= \left\|F(F^j(w))(t) - F(F^j(v))(t)\right\| \\ &\le \frac{k_\varepsilon}{\varepsilon}\int_t^T \left\|F^j(w)(s) - F^j(v)(s)\right\| \mathrm{d}s \\ &\le \frac{k_\varepsilon}{\varepsilon}\int_t^T \left(\frac{k_\varepsilon}{\varepsilon}\right)^j \frac{(T-s)^j}{j!}|||w-v|||\,\mathrm{d}s \\ &= \left(\frac{k_\varepsilon}{\varepsilon}\right)^{j+1}\frac{(T-t)^{j+1}}{(j+1)!}|||w-v|||.\end{aligned}$$

Therefore (10) holds for all $m = 1, 2, ...$ by the induction principle. In particular, one has

$$|||F^m(w)(t) - F^m(v)(t)||| \le \left(\frac{k_\varepsilon T}{\varepsilon}\right)^m \frac{1}{m!}|||w-v|||.$$

Since

$$\lim_{m\to\infty}\left(\frac{k_\varepsilon T}{\varepsilon}\right)^m \frac{1}{m!} = 0,$$

there exists a positive integer m_0 such that F^{m_0} is a contraction mapping. It follows that F^{m_0} has a unique fixed point u^ε in $C([0,T];H)$. Since $F^{m_0}(F(u^\varepsilon)) = F(F^{m_0}(u^\varepsilon)) = F(u^\varepsilon)$, we obtain $F(u^\varepsilon) = u^\varepsilon$ due to the uniqueness of the fixed point of F^{m_0}. The uniqueness of the fixed point of F also follows the uniqueness of the fixed point of F^{m_0}. The unique fixed point u^ε of F is the solution of (7) corresponding to final value φ.

Step 2: Continuous dependence on the data

We now let u_1^ε and u_2^ε be two solutions corresponding to final values φ_1 and φ_2, respectively. In the same manner as Step 1, we have for every $w, v \in C([0,T];H)$

$$\|F(w)(t) - F(v)(t)\| \le k_\varepsilon \int_t^T \varepsilon^{\frac{t-s}{T}}\|w(s) - v(s)\|\,\mathrm{d}s.$$

Hence

$$\begin{aligned}\left\|u_1^\varepsilon(t) - u_2^\varepsilon(t)\right\| &= \left\|S_\varepsilon^{-1}(T-t)\,(\varphi_1 - \varphi_2) + F(u_1^\varepsilon)(t) - F(u_2^\varepsilon)(t)\right\| \\ &\le \left\|S_\varepsilon^{-1}(T-t)\right\|_{\mathcal{L}(H)} \cdot \|\varphi_1 - \varphi_2\| + \left\|F(u_1^\varepsilon)(t) - F(u_2^\varepsilon)(t)\right\| \\ &\le \varepsilon^{\frac{t-T}{T}}\|\varphi_1 - \varphi_2\| + k_\varepsilon \int_t^T \varepsilon^{\frac{t-s}{T}}\left\|u_1^\varepsilon(s) - u_2^\varepsilon(s)\right\| ds.\end{aligned}$$

The latter inequality can be written as

$$\varepsilon^{-\frac{t}{T}}\left\|u_1^\varepsilon(t) - u_2^\varepsilon(t)\right\| \le \varepsilon^{-1}\|\varphi_1 - \varphi_2\| + k_\varepsilon \int_t^T \varepsilon^{-\frac{s}{T}}\left\|u_1^\varepsilon(s) - u_2^\varepsilon(s)\right\| ds.$$

It follows from Grönwall's inequality that

$$\varepsilon^{-\frac{t}{T}}\left\|u_1^\varepsilon(t) - u_2^\varepsilon(t)\right\| \le \varepsilon^{-1}e^{k_\varepsilon(T-t)}\|\varphi_1 - \varphi_2\|,\ t \in [0,T].$$

This completes the proof of Theorem 1. □

2.2. Regularization of problem (3)

Our purpose in this section is to construct a regularized solution of the ill-posed problem (3). We mention that the existence of a solution of (3) is not considered here. Instead, we assume that there is an exact solution u corresponding to the exact datum φ, and our aim is to construct, from the given datum φ_ε approximating φ, a regularized solution U_ε which approximates u.

Denote by u^ε the solution of problem (4) corresponding to the final condition φ_ε. We shall show that for each fixed time $t > 0$, the function $u^\varepsilon(t)$ gives a good approximation of $u(t)$. Notice that, it is more difficult to derive an approximation at $t = 0$ than at large t. We therefore need an adjustment in choosing the regularized solution at $t = 0$. The main idea is that we first use the continuity of u to approximate the initial value $u(0)$ by $u(t_\varepsilon)$ for some suitable small time $t_\varepsilon > 0$, and then approximate $u(t_\varepsilon)$ by $u^\varepsilon(t_\varepsilon)$. The parameter t_ε will be choosen as follows.

Lemma 3. *Let $T > 0$ and let $\varepsilon > 0$ small enough. There exists a unique $t_\varepsilon > 0$ such that $\varepsilon^{\frac{t_\varepsilon}{2T}} = t_\varepsilon$. Moreover,*

$$t_\varepsilon \leq \frac{2T\ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})}.$$

Proof. Note that each solution $t > 0$ of $\varepsilon^{\frac{t}{2T}} = t$ is a zero of the function

$$h(t) = \ln(t) + \frac{\ln(\varepsilon^{-1})}{2T}t, \qquad t > 0.$$

We have h is strictly increasing as $h'(t) > 0$. Moreover, $\lim\limits_{t\to 0^+} h(t) = -\infty$ and

$$h\left(\frac{2T\ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})}\right) = \ln\left[2T\ln(\ln(\varepsilon^{-1}))\right] > 0$$

for $\varepsilon > 0$ small enough. Thus the equation $h(t) = 0$ has a unique solution $t_\varepsilon > 0$ such that

$$t_\varepsilon \leq \frac{2T\ln\left(\ln\left(\frac{1}{\varepsilon}\right)\right)}{\ln\left(\frac{1}{\varepsilon}\right)}.$$

□

We have the following regularization result.

Theorem 2. *Let $u \in C^1([0,T];H)$ be a solution of problem (3) corresponding to $\varphi \in H$. Assume that*

$$\sup_{t\in[0,T]}\left(\|u_t(\cdot,t)\| + \varepsilon^{\frac{T-t}{T}}\left[\sum_{n=1}^{\infty} e^{T\lambda_n}|\langle\phi_n, u(t)\rangle|\right]\right) = M < \infty.$$

Let φ_ε be a measured datum satisfying $\|\varphi_\varepsilon - \varphi\| \leq \varepsilon$ with $\varepsilon \in (0, e^{-1})$, and let u^ε be the solution of problem (4) corresponding to φ_ε. Choose $t_\varepsilon > 0$ as in Lemma 3. Define the regularized solution $U^\varepsilon : [0,T] \to H$ by

$$U^\varepsilon(t) = u^\varepsilon(\max\{t, t_\varepsilon\}),\ t \in [0,T].$$

Then one has the error estimate, for $\varepsilon \in (0, e^{-1})$, $t \in [0,T]$,

$$\|U^\varepsilon(t) - u(t)\| \leq 2T^2(3M+2)\frac{\ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})}.$$

Proof. We have in view of (5)

$$u(t) = S^{-1}(T-t)\varphi - \int_t^T S^{-1}(s-t) f(s, u(s))\, \mathrm{d}s.$$

Using $B(\varepsilon, t) = S_\varepsilon(t-T)S(T-t)$, one has

$$B(\varepsilon, t)u(t) = S_\varepsilon^{-1}(T-t)\varphi - \int_t^T S_\varepsilon^{-1}(s-t)B(\varepsilon, s) f(s, u(s))\, \mathrm{d}s.$$

We have in view of (7)

$$u^\varepsilon(t) = S_\varepsilon^{-1}(T-t)\varphi_\varepsilon - \int_t^T S_\varepsilon^{-1}(s-t)B(\varepsilon, s) f_\varepsilon(s, u^\varepsilon(s))\, \mathrm{d}s.$$

Thus

$$\begin{aligned} u^\varepsilon(t) - u(t) &= S_\varepsilon^{-1}(T-t)\,(\varphi_\varepsilon - \varphi) + [B(\varepsilon, t) - I]\, u(t) + \\ &\quad - \int_t^T S_\varepsilon^{-1}(s-t)B(\varepsilon, s)\left[f_\varepsilon(s, u^\varepsilon(s)) - f(s, u(s))\right] \mathrm{d}s. \end{aligned}$$

Since the eigenfunctions $\{\phi_n\}$ forms an orthonormal basis of H, we can write $u(t)$ as

$$u(t) = \sum_{n=1}^{\infty} \langle u(t), \phi_n\rangle\, \phi_n.$$

Hence, using Lemma 1, we have

$$\begin{aligned} \|[B(\varepsilon, t) - I]u(t)\| &= \left\| \sum_{n=1}^{\infty} \langle u(t), \phi_n\rangle\, [B(\varepsilon, t) - I]\phi_n \right\| \le \sum_{n=1}^{\infty} |\langle u(t), \phi_n\rangle|\, \|[B(\varepsilon, t) - I]\phi_n\| \\ &\le \varepsilon \sum_{n=1}^{\infty} |\langle u(t), \phi_n\rangle|\, e^{T\lambda_n}. \end{aligned}$$

Employing Lemma 1 again, and noting that for $\varepsilon > 0$ small enough, $M_\varepsilon \ge \sup_{t\in[0,T]} \|u(t)\|$. This leads to $f(s, u(s)) = f_\varepsilon(s, u(s))$. Thus, taking into account Lemma 2 we get

$$\begin{aligned} \|u^\varepsilon(t) - u(t)\| &\le \left\|S_\varepsilon^{-1}(T-t)\right\| . \|\varphi_\varepsilon - \varphi\| + \|[B(\varepsilon, t) - I]\, u(t)\| + \\ &\quad + \int_t^T \left\|S_\varepsilon^{-1}(s-t)\right\|_{\mathcal{L}(H)} . \|B(\varepsilon, s)\|_{\mathcal{L}(H)} . \|f_\varepsilon(s, u^\varepsilon(s)) - f(s, u(s))\|\, \mathrm{d}s \\ &\le \varepsilon^{\frac{t-T}{T}} \cdot \varepsilon + \varepsilon \sum_{n=1}^{\infty} e^{T\lambda_n}\, |\langle u(t), \phi_n\rangle| + k_\varepsilon \int_t^T \varepsilon^{\frac{t-s}{T}}\, \|u^\varepsilon(s) - u(s)\|\, \mathrm{d}s \\ &\le (M+1)\varepsilon^{\frac{t}{T}} + k_\varepsilon \int_t^T \varepsilon^{\frac{t-s}{T}}\, \|u^\varepsilon(s) - u(s)\|\, \mathrm{d}s. \end{aligned}$$

The latter inequality can be written as

$$\varepsilon^{-\frac{t}{T}} \|u^\varepsilon(t) - u(t)\| \leq (M+1) + k_\varepsilon \int_t^T \varepsilon^{-\frac{s}{T}} \|u^\varepsilon(s) - u(s)\| \, \mathrm{d}s.$$

It follows from Grönwall's inequality that

$$\varepsilon^{-\frac{t}{T}} \|u^\varepsilon(t) - u(t)\| \leq (M+1)e^{k_\varepsilon T}, \ \forall t \in (0, T].$$

Notice that, for $z > 1$ it holds that $e^z \geq z^2$. This implies $\sqrt{z} \geq \ln(z)$. Hence, if we take $z = \ln(\varepsilon^{-1})$, we see that $z > 1$ for $\varepsilon < e^{-1}$. Therefore, we have

$$\sqrt{\ln(\varepsilon^{-1})} \geq \ln(\ln(\varepsilon^{-1})).$$

The above inequality is equivalent to

$$\sqrt{\ln(\varepsilon^{-1})} \leq \frac{\ln(\varepsilon^{-1})}{\ln(\ln(\varepsilon^{-1}))}.$$

Taking into account the definition of k_ε in Lemma 2 and that of t_ε in Lemma 3, we can bound the term $e^{k_\varepsilon T}$ from above as follows

$$e^{k_\varepsilon T} \leq \sqrt{\ln(\varepsilon^{-1})} \leq \frac{\ln(\varepsilon^{-1})}{\ln(\ln(\varepsilon^{-1}))} \leq 2T t_\varepsilon^{-1} = 2T\varepsilon^{-\frac{t_\varepsilon}{2T}}.$$

Therefore, if $t \in [t_\varepsilon, T]$, since $\varepsilon \ll 1$, we have

$$\begin{aligned} \|U^\varepsilon(t) - u(t)\| &= \|u^\varepsilon(t) - u(t)\| \leq (M+1)e^{k_\varepsilon T}\varepsilon^{\frac{t}{T}} \leq 2T(M+1)\varepsilon^{-\frac{t_\varepsilon}{2T}}\varepsilon^{\frac{t}{T}} \leq 2T(M+1)\varepsilon^{\frac{t_\varepsilon}{2T}} \\ &= 2T(M+1)t_\varepsilon \leq \frac{2T^2(2M+2)\ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})}. \end{aligned}$$

Let us now consider $t \in [0, t_\varepsilon]$. One has

$$\|U^\varepsilon(t) - u(t)\| = \|u^\varepsilon(t_\varepsilon) - u(t)\| \leq \|u^\varepsilon(t_\varepsilon) - u(t_\varepsilon)\| + \|u(t_\varepsilon) - u(t)\|.$$

Thanks to Jensen's inequality, we get

$$\|u(\cdot, t_\varepsilon) - u(\cdot, t)\| = \left\| \int_t^{t_\varepsilon} u_t(\cdot, s) ds \right\| \leq (t_\varepsilon - t) \int_t^{t_\varepsilon} \|u_t(\cdot, s)\| \, ds \leq MTt_\varepsilon.$$

Thus, for $t \in [0, t_\varepsilon]$,

$$\begin{aligned} \|U^\varepsilon(t) - u(t)\| &\leq 2T(M+1)\varepsilon^{\frac{t_\varepsilon}{2T}} + MTt_\varepsilon = (3M+2)Tt_\varepsilon \\ &\leq 2T^2(3M+2)\frac{\ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})}. \end{aligned}$$

This completes the proof of Theorem 2. □

3. Regularization of backward parabolic problem with time-dependent coefficient

In this section, we consider the following backward nonlinear parabolic problem with time-dependent coefficient

$$\begin{aligned} u_t + a(t)Au(t) &= f(t, u(t)), \quad 0 < t < T, \\ u(T) &= \varphi, \end{aligned} \tag{11}$$

where $a \in C([0, T])$ is given. The function a is noised by the perturbed data $a_\varepsilon \in C[0, T]$ such that

$$\|a_\varepsilon - a\|_{C([0,T])} \le \varepsilon, \tag{12}$$

where the norm $\|\cdot\|_{C([0,T])}$ is given by the sup norm, i.e., $\|v\|_{C([0,T])} = \sup_{0\le t\le T} |v(t)|$ for every continuous function $v : [0, T] \to \mathbb{R}$. We would like to emphasize that it is impossible to apply the technique in Section 2 to solve problem (11) when the time-dependent coefficient is perturbed by noise. Therefore, we investigate a new regularized problem as follows

$$\begin{cases} \dfrac{d}{dt}v_\varepsilon(t) + a_\varepsilon(t)\widetilde{A}_\varepsilon v_\varepsilon(t) = f_\varepsilon(t, v_\varepsilon(t)), & 0 < t < 1, \\ v_\varepsilon(T) = \varphi_\varepsilon, \end{cases} \tag{13}$$

where $\widetilde{A}_\varepsilon$ is defined by

$$\widetilde{A}_\varepsilon(v) := -\frac{1}{QT}\sum_{n=1}^{\infty} \ln\left(\varepsilon + e^{-QT\lambda_n}\right)\langle v, \phi_n\rangle\phi_n \tag{14}$$

Moreover, we get

$$\|\widetilde{A}_\varepsilon(v)\|^2 = \frac{1}{QT}\sum_{n=1}^{\infty} \ln\frac{1}{(\varepsilon + e^{-QT\lambda_n})^2}\langle v, \phi_n\rangle \le \frac{1}{QT}\ln^2(1/\varepsilon)$$

and $Q := \|a_\varepsilon\|_{C([0,T])}$. The regularization result for time-dependent perturbed coefficient is given in the following theorem.

Theorem 3. *Let $u \in C^1([0, T]; H)$ be a solution of problem (11) corresponding to $\varphi \in H$. Assume that*

$$\sup_{t\in[0,T]}\left[\sum_{n=1}^{\infty} e^{2QT\lambda_n}|(\phi_n, u(t))|^2 + \|u'(t)\|\right] = E_Q < \infty.$$

Let φ_ε and a_ε be measured data satisfying $\|\varphi_\varepsilon - \varphi\| \le \varepsilon$ and $\|a_\varepsilon - a\|_{C([0,T])} \le \varepsilon$ for $\varepsilon > 0$. We denote by v_ε the solution of problem (13) corresponding to φ_ε and a_ε. Choose $t_\varepsilon > 0$ as in Lemma 3. Define the regularized solution $W^\varepsilon : [0, T] \to H$ by

$$W^\varepsilon(t) = v_\varepsilon(\max\{t, t_\varepsilon\}), \ t \in [0, T].$$

Then one has the following error estimate for $\varepsilon > 0$ small enough and $t \in [0, T]$,

$$\|W^\varepsilon(t) - u(t)\| \le 2E_Q\sqrt{2\left(\frac{1}{Q} + 1\right)}e^{2T}\min\left\{\varepsilon^{\frac{t}{2T}}, \frac{2T\ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})}\right\}.$$

Proof. The existence of solutions to problem (11) is similar to that of Theorem 1 and we will not consider it here. It remains to prove the error estimation between W_ε and u. To this end, we first need the error estimation between v_ε and u. The technique we use here is different from Theorem 2. The problem (11) can be written as

$$\begin{cases} u'(t) + a_\varepsilon(t)\widetilde{A}_\varepsilon u(t) &=& a_\varepsilon(t)\widetilde{A}_\varepsilon u(t) - a(t)Au(t) + f(t, u(t)), \\ u(T) &=& \varphi. \end{cases} \tag{15}$$

Recall that v_ε solves the following problem

$$\begin{cases} v'_\varepsilon(t) + a_\varepsilon(t)\widetilde{A}_\varepsilon v_\varepsilon(t) &=& f_\varepsilon(t, v_\varepsilon(t)), \\ v_\varepsilon(T) &=& \varphi_\varepsilon. \end{cases} \tag{16}$$

Subtracting (15) from (16) side-by-side, we obtain

$$\begin{cases} v'_\varepsilon(t) - u'(t) &=& -a_\varepsilon(t)\widetilde{A}_\varepsilon(v_\varepsilon(t) - u(t)) - a_\varepsilon(t)\widetilde{A}_\varepsilon u(t) + a(t)Au(t) \\ & & +f_\varepsilon(t, v_\varepsilon(t)) - f(t, u(t)), \\ v_\varepsilon(T) - u_\varepsilon(T) &=& \varphi_\varepsilon - \varphi. \end{cases} \tag{17}$$

For $\widetilde{b} > 0$, we define by

$$z_\varepsilon(t) := e^{\widetilde{b}(t-T)}\Big(v_\varepsilon(t) - u(t)\Big).$$

By differentiating $z_\varepsilon(t)$ with respect to t and plugging (17) into this result, we get

$$\begin{aligned} z'_\varepsilon(t) &= \widetilde{b}e^{\widetilde{b}(t-T)}\Big(v_\varepsilon(t) - u(t)\Big) + e^{\widetilde{b}(t-T)}(v'_\varepsilon(t) - u'(t)) \\ &= \widetilde{b}z_\varepsilon(t) + e^{\widetilde{b}(t-T)}\Big[- a_\varepsilon(t)\widetilde{A}_\varepsilon\Big(v_\varepsilon(t) - u(t)\Big) + f_\varepsilon(t, v_\varepsilon(t)) - f(t, u(t))\Big] \\ &\quad -e^{\widetilde{b}(t-T)}\Bigg[(a_\varepsilon(t) - a(t))Au(t) + a_\varepsilon(t)\Big(\widetilde{A}_\varepsilon - A\Big)u(t)\Bigg] \\ &= \widetilde{b}z_\varepsilon(t) - a_\varepsilon(t)\widetilde{A}_\varepsilon z_\varepsilon(t) + e^{\widetilde{b}(t-T)}\Big[f_\varepsilon(t, v_\varepsilon(t)) - f(t, u(t))\Big] \\ &\quad -e^{\widetilde{b}(t-T)}(a_\varepsilon(t) - a(t))Au(t) - e^{\widetilde{b}(t-T)}a_\varepsilon(t)(\widetilde{A}_\varepsilon - A)u(t). \end{aligned} \tag{18}$$

By taking the inner product of (18) and $z_\varepsilon(t)$, we get

$$\begin{aligned} \langle z'_\varepsilon(t) + a_\varepsilon(t)\widetilde{A}_\varepsilon z_\varepsilon(t) - \widetilde{b}z_\varepsilon(t), z_\varepsilon(t)\rangle &= \Big\langle e^{\widetilde{b}(t-T)}\Big[f_\varepsilon(t, v_\varepsilon(t)) - f(t, u(t))\Big], z_\varepsilon(t)\Big\rangle \\ &\quad -e^{\widetilde{b}(t-T)}\langle(a_\varepsilon(t) - a(t))Au(t), z_\varepsilon(t)\rangle \\ &\quad -e^{\widetilde{b}(t-T)}\langle a_\varepsilon(t)(\widetilde{A}_\varepsilon - A)u(t), z_\varepsilon(t)\rangle. \end{aligned} \tag{19}$$

A direct computation implies that

$$\begin{aligned} \frac{d}{dt}\Big\|z_\varepsilon(t)\Big\|_H^2 = 2\,\langle z'_\varepsilon(t), z_\varepsilon(t)\rangle &= 2\langle - a_\varepsilon(t)\widetilde{A}_\varepsilon z_\varepsilon(t), z_\varepsilon(t)\rangle + 2\widetilde{b}\langle z_\varepsilon(t), z_\varepsilon(t)\rangle \\ &\quad +2\Big\langle e^{\widetilde{b}(t-T)}\Big[f_\varepsilon(t, v_\varepsilon(t)) - f(t, u(t))\Big], z_\varepsilon(t)\Big\rangle \\ &\quad -2e^{\widetilde{b}(t-T)}\langle(a_\varepsilon(t) - a(t))Au(t), z_\varepsilon(t)\rangle \\ &\quad -2e^{\widetilde{b}(t-T)}\langle a_\varepsilon(t)\Big(\widetilde{A}_\varepsilon - A\Big)u(t), z_\varepsilon(t)\rangle \\ &= 2(\widetilde{I}_1 + \widetilde{I}_2 + \widetilde{I}_3 + \widetilde{I}_4), \end{aligned} \tag{20}$$

where

$$\begin{aligned}
\widetilde{I}_1 &= \langle -a_\varepsilon(t)\widetilde{A}_\varepsilon z_\varepsilon(t), z_\varepsilon(t)\rangle + \widetilde{b}\langle z_\varepsilon(t), z_\varepsilon(t)\rangle, \\
\widetilde{I}_2 &= \Big\langle e^{\widetilde{b}(t-T)}\Big[f_\varepsilon(t, v_\varepsilon(t)) - f(t, u(t))\Big], z_\varepsilon(t)\Big\rangle, \\
\widetilde{I}_3 &= -e^{\widetilde{b}(t-T)}\langle (a_\varepsilon(t) - a(t))Au(t), z_\varepsilon(t)\rangle, \\
\widetilde{I}_4 &= -e^{\widetilde{b}(t-T)}\Big\langle a_\varepsilon(t)(\widetilde{A}_\varepsilon - A\Big)u(t), z_\varepsilon(t)\rangle.
\end{aligned}$$

Since $Q := \sup_{t\in[0,T]} \left|a_\varepsilon(t)\right|$, we have

$$\begin{aligned}
\left|\langle -a_\varepsilon(t)\widetilde{A}_\varepsilon z_\varepsilon(t), z_\varepsilon(t)\rangle\right| &\le \sup_{t\in[0,T]} \left|a_\varepsilon(t)\right|\left\|\widetilde{A}_\varepsilon z_\varepsilon(t)\right\|_H \left\|z_\varepsilon(t)\right\|_H \\
&\le Q\frac{1}{QT}\ln\left(\frac{1}{\varepsilon}\right)\left\|z_\varepsilon(t)\right\|_H^2 \\
&\le \frac{1}{T}\ln\left(\frac{1}{\varepsilon}\right)\left\|z_\varepsilon(t)\right\|_H^2,
\end{aligned}$$

which gives

$$\langle -a_\varepsilon(t)\widetilde{A}_\varepsilon z_\varepsilon(t), z_\varepsilon(t)\rangle \ge -\frac{1}{T}\ln\left(\frac{1}{\varepsilon}\right)\left\|z_\varepsilon(t)\right\|_H^2.$$

Then the term $\widetilde{I}_1$ is estimated by

$$\begin{aligned}
\widetilde{I}_1 &= \langle -a_\varepsilon(t)\widetilde{A}_\varepsilon z_\varepsilon(t), z_\varepsilon(t)\rangle + \widetilde{b}\langle z_\varepsilon(t), z_\varepsilon(t)\rangle \\
&\ge -\frac{1}{T}\ln\left(\frac{1}{\varepsilon}\right)\left\|z_\varepsilon(t)\right\|_H^2 + \widetilde{b}\left\|z_\varepsilon(t)\right\|_H^2. \qquad (21)
\end{aligned}$$

Using Lemma 1 and noting that $f(s, u(s)) = f_\varepsilon(s, u(s))$ for $\varepsilon > 0$ small enough, $M_\varepsilon \ge \sup_{t\in[0,T]} \|u(t)\|$, we have the following estimate

$$\begin{aligned}
\widetilde{I}_2 &= \Big\langle e^{-\widetilde{b}(T-t)}\Big[f_\varepsilon(t, v_\varepsilon(t)) - f(t, u(t))\Big], z_\varepsilon(t)\Big\rangle \\
&= e^{-2\widetilde{b}(T-t)}\langle f_\varepsilon(v_\varepsilon(t), t) - f_\varepsilon(t, u(t)), v_\varepsilon(t) - u(t)\rangle \\
&\ge -k_\varepsilon e^{-2\widetilde{b}(T-t)}\left\|v_\varepsilon(t) - u(t)\right\|_H^2 \\
&= -k_\varepsilon\left\|z_\varepsilon\right\|_H^2. \qquad (22)
\end{aligned}$$

Employing the Hölder inequality, we can bound $\widetilde{I}_3$ as follows

$$\begin{aligned}
\widetilde{I}_3 &= \langle e^{-\widetilde{b}(T-t)}(a_\varepsilon(t) - a(t))Au(t), z_\varepsilon(t)\rangle \\
&\le e^{-2\widetilde{b}(T-t)}\left|a_\varepsilon(t) - a(t)\right|^2\left\|Au(t)\right\|_H^2 + \left\|z_\varepsilon(t)\right\|_H^2 \\
&\le e^{-2\widetilde{b}(T-t)}\left|a_\varepsilon(t) - a(t)\right|^2\left(\sum_{n=1}^{\infty}\lambda_n^2\left|\langle u(t), \phi_n\rangle\right|^2\right) + \left\|z_\varepsilon(t)\right\|_H^2 \\
&\le e^{-2\widetilde{b}(T-t)}\left|a_\varepsilon(t) - a(t)\right|^2\left(\sum_{n=1}^{\infty}\frac{1}{Q^2T^2}\, e^{2QT\lambda_n}\left|\langle u(t), \phi_n\rangle\right|^2\right) + \left\|z_\varepsilon(t)\right\|_H^2 \\
&\le \frac{e^{-2\widetilde{b}(T-t)}\varepsilon^2 E_Q^2}{QT} + \left\|z_\varepsilon(t)\right\|_H^2. \qquad (23)
\end{aligned}$$

Using the Hölder inequality again, $\widetilde{I_4}$ can be bounded as

$$\begin{aligned}
\widetilde{I_4} &= \langle e^{-\widetilde{b}(T-t)} a_\varepsilon(t)(\widetilde{A}_\varepsilon(t) - A(t))u(t), z_\varepsilon(t)\rangle \\
&\leq e^{-2\widetilde{b}(T-t)} \left|a_\varepsilon(t)\right|^2 \left\|(\widetilde{A}_\varepsilon - A)u(t)\right\|_H^2 + \left\|z_\varepsilon(t)\right\|_H^2 \\
&\leq e^{-2\widetilde{b}(T-t)} \left|a_\varepsilon(t)\right|^2 \sum_{n=1}^{\infty} \left| \frac{1}{QT} \ln\left(\frac{1}{\varepsilon + e^{-QT\lambda_n}}\right) - \frac{1}{QT}\ln(e^{QT\lambda_n}) \right|^2 \left|\langle u(t), \phi_n\rangle\right|^2 \\
&\quad + \left\|z_\varepsilon(t)\right\|_H^2 \\
&\leq e^{-2\widetilde{b}(T-t)} \left|a_\varepsilon(t)\right|^2 \frac{1}{Q^2T^2} \sum_{n=1}^{\infty} \left| \ln\left(\frac{1}{\varepsilon e^{QT\lambda_n} + 1}\right)\right|^2 \left|\langle u(t), \phi_n\rangle\right|^2 + \left\|z_\varepsilon(t)\right\|_H^2 \\
&\leq \frac{1}{T^2} e^{-2\widetilde{b}(T-t)} \sum_{n=1}^{\infty} \ln^2\left(\varepsilon e^{QT\lambda_n} + 1\right) \left|\langle u(t), \phi_n\rangle\right|^2 + \left\|z_\varepsilon(t)\right\|_H^2 \\
&\leq \frac{1}{T^2} e^{-2\widetilde{b}(T-t)} \varepsilon^2 \sum_{n=1}^{\infty} e^{2QT\lambda_n} \left|\langle u(t), \phi_n\rangle\right|^2 + \left\|z_\varepsilon(t)\right\|_H^2 \\
&\leq \frac{1}{T^2} e^{-2\widetilde{b}(T-t)} \varepsilon^2 E_Q^2 + \left\|z_\varepsilon(t)\right\|_H^2. \qquad (24)
\end{aligned}$$

Thus, (20), (21), (22), (23) and (24) yields

$$\begin{aligned}
\frac{d}{dt}\left\|z_\varepsilon(t)\right\|_H^2 &\geq \left(-\frac{2}{T}\ln\left(\frac{1}{\varepsilon}\right) + 2\widetilde{b} - 2k_\varepsilon - 4\right)\left\|z_\varepsilon(t)\right\|_H^2 \\
&\quad -2e^{-2\widetilde{b}(T-t)} \varepsilon^2 E_Q^2 \left(\frac{1}{QT} + \frac{1}{T}\right). \qquad (25)
\end{aligned}$$

Since $b = \frac{1}{T}\ln\left(\frac{1}{\varepsilon}\right)$ we obtain

$$\frac{d}{dt}\left\|z_\varepsilon(t)\right\|_H^2 \geq (-2k_\varepsilon - 4)\left\|z_\varepsilon(t)\right\|_H^2 - 2\varepsilon^2 E_Q^2 \left(\frac{1}{QT} + \frac{1}{T}\right).$$

Integrating the above inequality from t to T, we get

$$\begin{aligned}
\left\|z_\varepsilon(T)\right\|_H^2 - \left\|z_\varepsilon(t)\right\|_H^2 &\geq (-2k_\varepsilon - 4)\int_t^T \left\|z_\varepsilon(s)\right\|_H^2 ds \\
&\quad -2E_Q^2\varepsilon^2\left(\frac{1}{QT} + \frac{1}{T}\right)(T-t).
\end{aligned}$$

Since $\left\|z_\varepsilon(T)\right\|_H^2 = \|\varphi_\varepsilon - \varphi\| \leq \varepsilon$, we have

$$\left\|z_\varepsilon(t)\right\|_H^2 \leq (2k_\varepsilon + 4)\int_t^1 \left\|z_\varepsilon(s)\right\|_H^2 ds + 2E_Q^2\varepsilon^2\left(\frac{1}{Q} + 1\right) + \varepsilon^2.$$

This implies that

$$\begin{aligned}
e^{-2\widetilde{b}(T-t)}\left\|v_\varepsilon(t) - u(t)\right\|_H^2 &\leq (2k_\varepsilon + 4)\int_t^T e^{-2\widetilde{b}(T-s)}\left\|v_\varepsilon(s) - u(s)\right\|_H^2 ds \\
&\quad +2E_Q^2\varepsilon^2\left(\frac{1}{Q} + 1\right) + \varepsilon^2.
\end{aligned}$$

Multiplying bothside by $e^{2\widetilde{b}T}$, we obtain

$$\begin{aligned} e^{2\widetilde{b}t}\left\|v_\varepsilon(t)-u(t)\right\|_H^2 &\le (2k_\varepsilon+4)\int_t^T e^{2bs}\left\|v_\varepsilon(s)-u(s)\right\|_H^2 ds \\ &\quad +2E_Q^2\left(\frac{1}{Q}+1\right). \end{aligned}$$

Applying Grönwall's inequality, we get

$$e^{2\widetilde{b}t}\left\|v_\varepsilon(t)-u(t)\right\|_H^2 \le 2E_Q^2\left(\frac{1}{Q}+1\right)e^{\int_t^T(2k_\varepsilon+4)ds},$$

or

$$e^{2\widetilde{b}t}\left\|v_\varepsilon(t)-u(t)\right\|^2 \le 2E_Q^2\left(\frac{1}{Q}+1\right)e^{(2k_\varepsilon+4)(T-t)}.$$

Hence

$$\left\|v_\varepsilon(t)-u(t)\right\|_H^2 \le 2E_Q^2\left(\frac{1}{Q}+1\right)e^{(2k_\varepsilon+4)(T-t)}e^{-\frac{2t}{T}\ln\left(\frac{1}{\varepsilon}\right)}.$$

In particular, if $t\in[t_\varepsilon,T]$ then

$$\begin{aligned} \|W^\varepsilon(t)-u(t)\| = \|v_\varepsilon(t)-u(t)\| &\le E_Q\sqrt{2\left(\frac{1}{Q}+1\right)}e^{2T}e^{k_\varepsilon T}\varepsilon^{\frac{t}{T}} \\ &\le E_Q\sqrt{2\left(\frac{1}{Q}+1\right)}e^{2T}\varepsilon^{\frac{t}{2T}} \\ &\le E_Q\sqrt{2\left(\frac{1}{Q}+1\right)}e^{2T}\frac{2T\ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})}. \end{aligned}$$

Let us now consider $t\in[0,t_\varepsilon]$. One has

$$\|W^\varepsilon(t)-u(t)\| = \|v_\varepsilon(t_\varepsilon)-u(t)\| \le \|v_\varepsilon(t_\varepsilon)-u(t_\varepsilon)\| + \|u(t_\varepsilon)-u(t)\|.$$

Due to the continuity, we get for ε small enough

$$\|u(t_\varepsilon)-u(t)\| = \left\|\int_t^{t_\varepsilon} u_t(s)ds\right\| \le \int_0^{t_\varepsilon}\|u_t(s)\|\,ds \le E_Q t_\varepsilon.$$

Thus, for $t\in[0,t_\varepsilon]$,

$$\begin{aligned} \|W^\varepsilon(t)-u(t)\| &\le E_Q\sqrt{2\left(\frac{1}{Q}+1\right)}e^{2T}\varepsilon^{\frac{t_\varepsilon}{2T}} + E_Q t_\varepsilon \\ &\le 2E_Q\sqrt{2\left(\frac{1}{Q}+1\right)}e^{2T}\min\left\{\varepsilon^{\frac{t}{2T}}, \frac{2T\ln(\ln(\varepsilon^{-1}))}{\ln(\varepsilon^{-1})}\right\}. \end{aligned}$$

This completes the proof of Theorem 3. □

4. Numerical example

We end this paper by considering a simple example to illustrate our numerical strategy and verify the error estimates given in the theoretical parts.

We shall investigate the following backward equation in one-dimension

$$\begin{aligned} &u_t - u_{xx} = \|u\|^3 u + h(x,t), \quad x \in (0,\pi), \quad t \in (0,1), \\ &u(0,t) = u(\pi,t) = 0, \quad t \in (0,1), \\ &u(x,1) = \varphi(x), \quad x \in [0,\pi], \end{aligned}$$

where we postulate that an exact solution u exists and $u(\cdot,t) \in L^2(0,\pi)$ for every $t \in (0,1)$. By $\|u\|$, we mean the $L^2(0,\pi)$-norm of $u(\cdot,t)$ when t is fixed. Indeed, when we choose

$$\begin{aligned} h(x,t) \quad := \quad & 0.6(\sin(x) + \frac{1}{2}\sin(2x)) + 0.6t(\sin(x) + 2\sin(2x)) \\ & -0.6^4 t^4 (\sin(x) + \frac{1}{2}\sin(2x)) \left(\int_0^\pi (\sin(x) + \frac{1}{2}\sin(2x))^2 \, \mathrm{d}x \right)^{3/2}, \end{aligned}$$

and

$$\varphi(x) := 0.6(\sin(x) + \frac{1}{2}\sin(2x)),$$

the above problem admits the exact solution

$$u_{ex} = 0.6t(\sin(x) + \frac{1}{2}\sin(2x)),$$

satisfying that $u_{ex}(\cdot,t) \in L^2(0,\pi)$ for every $t \in (0,1)$.

ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
M_ε	0.3736	0.4570	0.4943	0.5177	0.5345	0.5475
t_ε	0.5382	0.3990	0.3252	0.2780	0.2446	0.2195

Table 1: M_ε and t_ε.

ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}
# iter	11	11	11	12	12	12
$t = 0.6$	0.15491	0.13584	0.12958	0.12903	0.12895	0.12891
$t = 0$	0.08153	0.09460	0.10648	0.11507	0.11580	0.12635

Table 2: Comparing of errors when $N = 18$.

Notice that, a similar problem has been examined in [11] using the truncated Fourier series method. In the present paper, we shall regularize the problem using quasi-reversibility method. We shall follow the notations introduced in the theoretical parts. In this case, the coefficient $a = 1$, the Hilbert space H is chosen to be $L^2(0,\pi)$, the positive operator A defined in a dense subset of

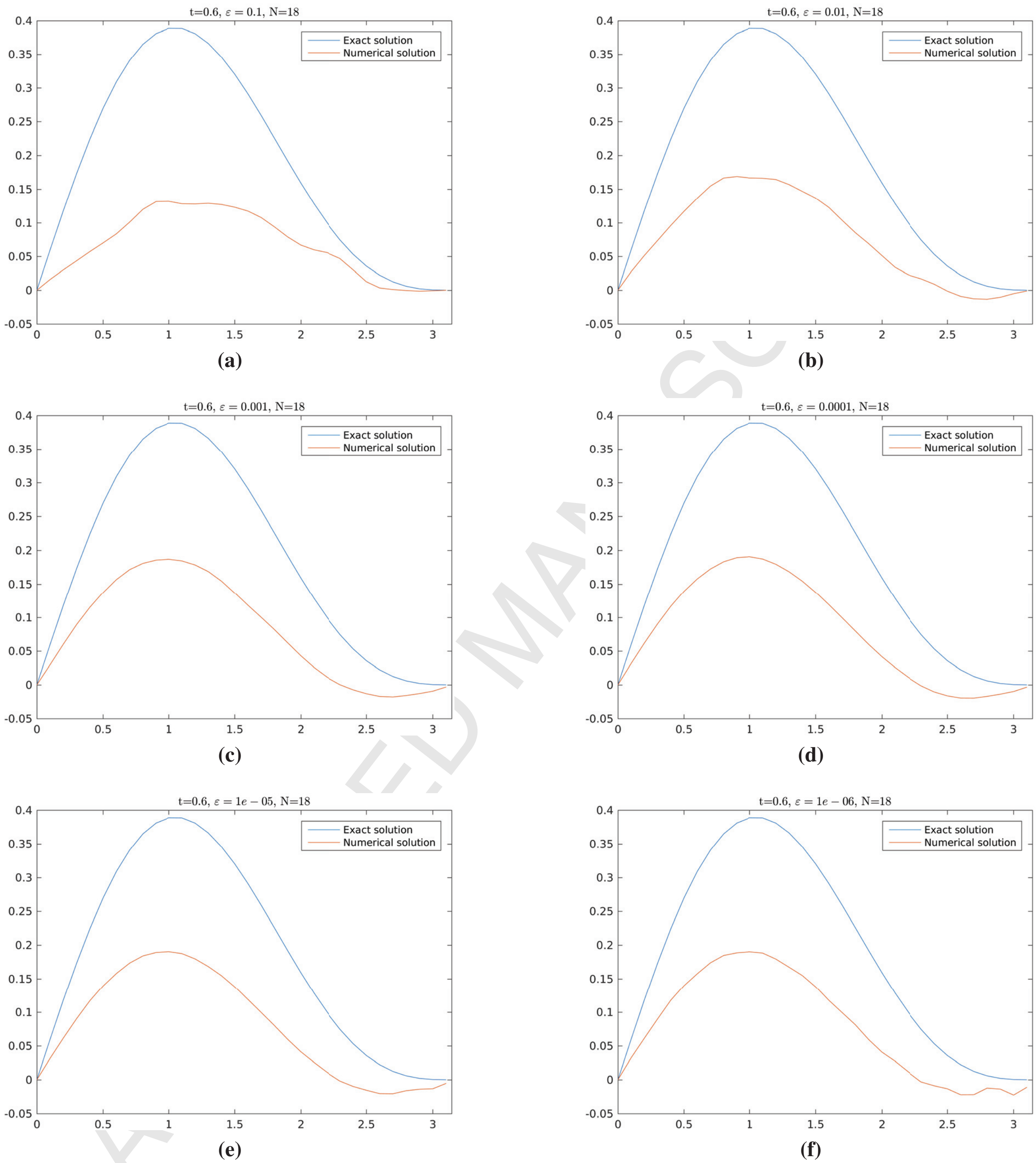

Figure 1: Exact solution and regularized solution at $t = 0.6$ when the noise level $\varepsilon = 10^{-1}, \ldots, 10^{-6}$ and $N = 18$.

$L^2(0, \pi)$ is $-\frac{\partial^2}{\partial x}$. It is easy to check that A has eigenvalues $\lambda_n = n^2$ with corresponding eigenvectors $\phi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$. The sequence of eigenvectors $\{\phi_n\}$ forms an orthonormal basis of $L^2(0, \pi)$. The compact contraction semigroup $S(t)$ that is generated by $-A$ reads

$$S(t)v = \sum_{n=1}^{N} e^{-t\lambda_n}(\phi_n, v)\phi_n \quad \text{for all} \quad v \in H.$$

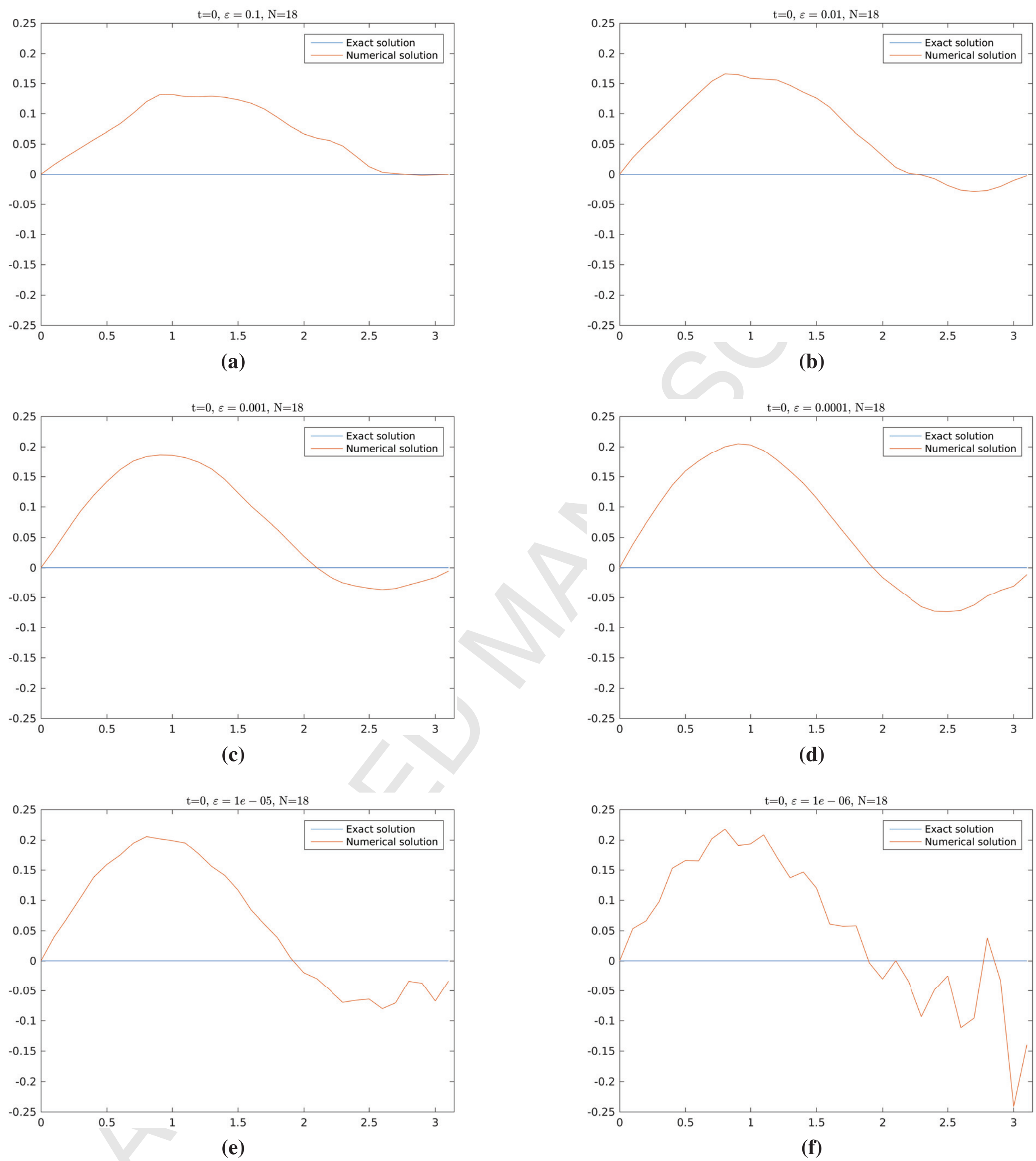

Figure 2: Exact solution and regularized solution at $t = 0$ when the noise level $\varepsilon = 10^{-1}, \ldots, 10^{-6}$ and $N = 1$.

Here, we have truncated the infinite sum of $S(t)$ by a positive integer N. In the following, we shall choose $N = 18$, where $N = 18$ is the largest positive integer so that the condition

$$\sup_{t\in[0,T]} \left(\|u_t(\cdot,t)\| + e^{\frac{T-t}{T}} \left[\sum_{n=1}^{\infty} e^{T\lambda_n} |(\phi_n, u(t))| \right] \right) = M < \infty.$$

in Theorem 2 holds true.

The source function f is given by

$$f(t,x,u) := \|u\|^3 u + h(x,t).$$

Notice that, for any $u \in L^2(0,\pi)$, it may happen that $u^3 \notin L^2(0,\pi)$. For that reason, we consider the source function f of the type $\|u\|^3 u$. By simple computations, one can easily check that f is a locally Lipschitz function w.r.t. $L^2(0,\pi)$-norm, and this function is not globally Lipschitz w.r.t. the same norm. Indeed,

$$\begin{aligned}
\|f(t,\cdot,u) - f(t,\cdot,v)\| &= \|u\|u\|^3 - v\|v\|^3\| \\
&= \|(u-v)\|u\|^3 + v(\|u\|^3 - \|v\|^3)\| \\
&\le \|u\|^3\|u-v\| + \left|\|u\|^3 - \|v\|^3\right| \|v\| \\
&\le \|u\|^3\|u-v\| + |\|u\| - \|v\|| \left(\|u\|^2 + \|u\|\|v\| + \|v\|^2\right) \|v\| \\
&\le \|u-v\|(\|u\|^3 + \|v\|^3 + \|u\|^2\|v\| + \|u\|\|v\|^2),
\end{aligned}$$

where in the last inequality, we have used the fact that $|\|u\| - \|v\|| \le \|u-v\|$. Hence, the source function f w.r.t. the third variable is a locally Lipschitz function with Lipschitz constant $k(M) = \|u\|^3 + \|v\|^3 + \|u\|^2\|v\| + \|u\|\|v\|^2$ for all $\|u\|, \|v\| \le M$. Obviously, k is an increasing function in M.

Let us call $\varepsilon > 0$ the noise level. We define by

$$M_\varepsilon := \sqrt[3]{\frac{1}{16}\ln(\ln(\frac{1}{\varepsilon}))}.$$

Notice that, the condition $\sup_{t\in[0,T]} \|u(t)\| \le M_\varepsilon$ in Theorem 2 fulfills when $\varepsilon \le 10^{-3}$; but our numerical experiments also cover the case $\varepsilon > 10^{-3}$. More precisely, we shall choose the noise level $\varepsilon \in \{10^{-1}, \ldots, 10^{-6}\}$. Obviously, $M_\varepsilon > 0$, $M_\varepsilon \to \infty$ as $\varepsilon \to 0$ and $k(M_\varepsilon) \le \ln(\ln(\varepsilon^{-1}))/4$. See Table 1 for the value of M_ε corresponding to different noise levels.

Now, we define a sequence f_ε that approximates f by replacing the value $f(u)$ when u stays outside the unit ball $B(0, M_\varepsilon)$ by the value of f on the boundary of $B(0, M_\varepsilon)$, i.e.,

$$f_\varepsilon(u) := f\left(\min\left(1, \frac{M_\varepsilon}{\|u\|}\right) u\right).$$

Lemma 2 shows that f_ε is a globally Lipschitz function with Lipschitz constant $k_\varepsilon = 8M_\varepsilon^3$.

To define the measured data φ_ε, we shall perturb the exact data φ with a random noise, which is uniformly-distributed random numbers on the interval $(-1, 1)$

$$\varphi_\varepsilon(x) := \varphi(x) + \frac{\varepsilon}{\sqrt{\pi}}(2 * \mathtt{rand} - 1).$$

It is easy to see that $\|\varphi_\varepsilon - \varphi\|_{L^2(0,\pi)} \le \varepsilon$.

Under the noise level $\varepsilon > 0$, Theorem 1 proves the existence and uniqueness of a solution u^ε, which is implicitly defined by this formula

$$u^\varepsilon(x,t) = S_\varepsilon(t-T)\varphi_\varepsilon(x) - \int_t^T S_\varepsilon(t-s)B(\varepsilon,s) f_\varepsilon(s,x,u^\varepsilon(x,s))\,\mathrm{d}s,$$

where S_ε is the perturbation of $S(t)$, and is defined by

$$S_\varepsilon(t)v := \sum_{n=1}^{N} (\varepsilon + e^{-T\lambda_n})^{\frac{t}{T}} (\phi_n, v)\phi_n, \quad \forall v \in H,$$

and

$$B(\varepsilon, s)(v) := \sum_{n=1}^{N}(1 + \varepsilon e^{T\lambda_n})^{\frac{s}{T}-1}(\phi_n, v)\phi_n, \quad \forall v \in H.$$

Since for all $v \in H$, it holds that

$$S_\varepsilon^{-1}(s-t)B(\varepsilon, s)v = S_\varepsilon^{-1}(T-t)S(T-s)v,$$

this implies

$$u^\varepsilon(x,t) = S_\varepsilon^{-1}(T-t)\varphi_\varepsilon(x) - \int_t^T S_\varepsilon^{-1}(T-t)S(T-s)f_\varepsilon(s, x, u^\varepsilon(x, s))\,\mathrm{d}s, \tag{26}$$

To numerically solve u^ε from (26), we shall discretize the space interval $[0, \pi]$ and the time interval $[0, 1]$ by equi-distant grid points $x_i = (i-1)\Delta x, i = 1, \ldots, N_x + 1, \Delta x = \frac{\pi}{N_x}$ and $t_j = (j-1)\Delta t, j = 1, \ldots, N_t + 1, \Delta t = \frac{1}{N_t}$ respectively, and find an approximation $u_{i,j}$ of u^ε at (x_i, t_j). In this example, we choose $\Delta x = 0.1$ and $\Delta t = 0.1$. From the discrete viewpoint, the $L^2(0, \pi)$-norm will be replaced by the root-mean-square deviation (RMSD)

$$\|u(\cdot, t)\|_{L^2(0,\pi)} \approx \sqrt{\frac{1}{N_x+1}\sum_{i=1}^{N_x+1}|u(x_i, t)|^2}.$$

As a consequence, the inner product of $L^2(0, \pi)$ can also be approximated by

$$(u, v) \approx \frac{1}{N_x+1}\sum_{i=1}^{N_x+1} u(x_i)v(x_i).$$

The integral, on the other side, will be calculated using the extended Simpson formula, that is

$$\int_{t_1}^{t_{N_t+1}} w(t)\,\mathrm{d}t \approx \Delta t\left(\frac{3}{8}w(t_1) + \frac{7}{6}w(t_2) + \frac{23}{24}w(t_3) + \sum_{j=4}^{N_t-2} w(t_j) + \frac{23}{24}w(t_{N_t-1}) + \frac{7}{6}w(t_{N_t}) + \frac{3}{8}w(t_{N_t+1})\right).$$

Now, we are in a position to give the iterative formula to define u^ε

$$u^\varepsilon_{(k+1)}(x_i, t_j) := \sum_{n=1}^{N}(\varepsilon + e^{-T\lambda_n})^{\frac{t_j}{T}-1}\phi_n(x_i)\left((\phi_n, \varphi_\varepsilon) - \int_{t_j}^T e^{-(T-s)\lambda_n}(\phi_n, f_\varepsilon(s, \cdot, u^\varepsilon_{(k)}(\cdot, s)))\,\mathrm{d}s\right)$$

where $i = 1, \ldots, N_x+1$, $j = 1, \ldots, N_t+1$, $k \geq 1$ and the initial guess $u^\varepsilon_{(0)}(x_i, t_j) := \sin(x_i)$. The stopping criterion is when the root-mean-square deviation between $u^\varepsilon_{(k+1)}$ and $u^\varepsilon_{(k)}$ is less than ε^{20}.

The regularized solution U^ε will take the same value as u^ε for large t, and will be regularized as follows when t is near 0.

$$U^\varepsilon(t) := u^\varepsilon(\max(t, t_\varepsilon)), \quad t \in [0, 1],$$

here t_ε is the unique root of the equation $\varepsilon^{\frac{t}{2}} - t = 0$ as in Lemma 3. Notice that, the value of t_ε decreases as the noise tends to 0. See Table 1 for the value of t_ε corresponding to different noise levels. The error estimates between the exact solution and the numerical solution U^ε at different time when $N = 18$ are given in Table 2. Comparing the error estimates in Table 2 at $t = 0.6$ and $t = 0$, we see that the convergence rates become worse and worse when time is close to 0. This reflects the behavior of the rate of convergence in Theorem 2. This phenomenon is also illustrated via the figures: in Figure 1, when $t = 0.6$, the numerical solution slowly converges to the exact solution when the noise tends to 0, whilst in Figure 2, when $t = 0$, the smaller the noise level, the larger the deviation of the numerical solution from the exact solution.

5. Conclusion

In this paper, we have considered the problem of finding a function u satisfying the nonlinear parabolic equation $u_t + a(t)Au(t) = f(t, u(t))$ subject to the final condition $u(T) = \varphi$, where f is a locally Lipschitz function w.r.t. u. This is an ill-posed problem, and we have suggested a regularized scheme using quasi-reversibility method to construct the regularized solution u_ε from the measured data a_ε and φ_ε. Error estimates of logarithm type are given, which explain the slow convergence rates illustrated in figures 1 and 2.

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