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On Littlewood's boundedness problem for relativistic oscillators with singular potentials ☆

Yu Han, Xiong Li *

School of Mathematics Sciences, Beijing Normal University, Beijing 100875, PR China

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ABSTRACT

In this paper, we study the following nonlinear differential equations of motions of relativistic oscillators with singular potentials

$$\frac{d}{dt}\left(\frac{x'}{\sqrt{1-(x')^2}}\right) + V'(x) = p(t),$$

where V is a singular potential and p is a 1-periodic function. We will prove the boundedness of all solutions and the existence of infinitely many quasi-periodic solutions via Moser's twist theorem.

1. Introduction

In the classical Newtonian mechanics, the motion of a particle can be described by Duffing equation

$$x'' + V'(x) = p(t), \tag{1.1}$$

where $x' = \frac{dx}{dt}$. The question of the boundedness of all solutions is the famous Littlewood's problem, which was investigated by many authors since 1960s. For example, several authors considered the question with superlinear potentials, see [7,13,23,30–32] and the references therein. The sublinear case was studied in [12,17,29], and the results about the boundedness or unboundedness of all solutions of the semilinear equation can be found in [16,25,26] and the references therein.

Recently, in particular, attention has been devoted to singular potentials. In this framework, we refer to the papers [4,15,18]. In [4], the authors studied a second order scalar equation (1.1) with a singular potential

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* Corresponding author.
E-mail addresses: yhan@mail.bnu.edu.cn (Y. Han), xli@bnu.edu.cn (X. Li).

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$$V(x) = \frac{1}{2}x_+^2 + \frac{1}{(1-x_-^2)^\gamma} - 1,$$

where γ is a positive integer, $x_+ = \max\{x, 0\}$, and $x_- = \max\{-x, 0\}$, and found some sufficient conditions on V, p such that all solutions are bounded, also obtained the existence of Aubry–Mather set.

On the other hand, in the last years, relativistic oscillator models also have been considered by several authors. First of all, let us introduce this model. In relativistic mechanics ([9]), the quality m of a particle in motion increases as the speed increases, that is,

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}},$$

where m_0 is the quality at rest, c is the light speed and m is the quality at the speed of v. Thus the momentum can be written in the following form

$$p = mv = \frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}},$$

and the motions of relativistic oscillators can be described by the nonlinear differential equation

$$\frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1 - \frac{v^2}{c^2}}}\right) = f(x,t),$$

where $f(x,t)$ is an external field. After normalized by $m_0 = 1$, $c = 1$, the equation can be written in the form

$$\frac{d}{dt}\left(\frac{x'}{\sqrt{1-(x')^2}}\right) = f(x,t),$$

where $x' = v$ is the speed of the relativistic oscillator. Recently, the stability of the equilibrium of relativistic oscillators had been studied in [5], and the existence of periodic solutions also had been investigated by many authors, see [1–3,6,8,20,22,27] and the references therein.

In the relativistic framework, the motions of relativistic oscillators with the potentials are described by the following form

$$\frac{d}{dt}\left(\frac{x'}{\sqrt{1-(x')^2}}\right) + V'(x) = p(t). \tag{1.2}$$

As far as we know, there are not too much results on the Littlewood's boundedness problem for oscillators with relativistic effects.

When taking a harmonic potential, namely, $V(x) = \frac{1}{2}x^2$, (1.2) can be written in the following form

$$\begin{cases} x' = \dfrac{y}{\sqrt{1+y^2}}, \\ y' = -x + p(t), \end{cases}$$

which is equivalent to

$$y'' + \frac{y}{\sqrt{1+y^2}} = p'(t).$$

Wang ([28]) proved the boundedness of all solutions and the existence of quasi-periodic solutions.

The general case was considered in [19]. In [19], the authors investigated the Littlewood's boundedness problem to the relativistic oscillator with anharmonic potentials

$$V(x) = \frac{1}{\alpha+1}|x|^{\alpha+1}, \quad \alpha > 0,$$

and the result also holds provided the 1-periodic function $p \in \mathcal{C}^5$.

In [21], the author was concerned with the dynamics of the differential equation

$$\frac{d}{dt}\left(\frac{x'}{\sqrt{1-(x')^2}}\right) + a\sin x = f(t),$$

and applied KAM theory to prove that all solutions have bounded momentum and obtained the existence of quasi-periodic solutions in a generalized sense.

In this paper, we extend the Littlewood's boundedness problem to the relativistic oscillator with the singular potential same as in [4], that is,

$$V(x) = \frac{1}{2}x_+^2 + \frac{1}{(1-x_-^2)^\gamma} - 1, \tag{1.3}$$

where γ is a positive integer. The main result of this paper is as follows.

Theorem 1.1. *Assume that the singular potential V is given by (1.3) and the external force $p(t) \in \mathcal{C}^5$, is 1-periodic in t, then every solution of (1.2) with the initial value (x_0, x_0'), $x_0 > -1$, $x_0' \in \mathbb{R}$, is bounded. More precisely, if $x = x(t)$ is a solution of (1.2) with the initial value (x_0, x_0'), $x_0 > -1$, $x_0' \in \mathbb{R}$, then it is defined in $(-\infty, +\infty)$, and satisfies*

$$-1 < x(t) < +\infty, \quad \sup_{t\in\mathbb{R}} |x'(t)| < 1.$$

Furthermore, (1.2) has infinitely many quasi-periodic solutions.

The proof of Theorem 1.1 is based on Moser's twist theorem ([10,11,24]), and the process is similar to that in [19]. On the other hand, since the Hamiltonian function in (1.2) not only possesses the quadratic character in $x > 0$, but also has the singularity at $x = -1$, we have to adopt some estimate techniques from [4,13,19]. After some changes of coordinates and canonical transformations, the associated Poincaré mapping possesses the intersection property and satisfies the corresponding smallness estimates, then Moser's twist theorem can be applied, and thus the invariant closed curves exist, which implies the boundedness of all solutions and the existence of infinitely many quasi-periodic solutions.

The rest of this paper is organized as follows. After introducing action and angle variables, we state some important estimates in Section 3. Because of the complexity of these estimates, we shall show the proofs in Sections 5, 6. In Section 4, we firstly exchange the role of time variable and angle variable, then introduce some canonical transformations, thus obtain the corresponding Poincaré mapping and Theorem 1.1 can be proved by Moser's twist theorem.

2. Reduction to the action-angle variables

In this section, we carry out the standard reduction for (1.2) to the action-angle variables.

Firstly, define a function $\phi : \mathbb{R} \to (-1, 1)$ by

$$\phi(y) = \frac{y}{\sqrt{1+y^2}}.$$

The primitive function of ϕ is

$$\Phi(y) = \int_0^y \frac{\xi}{\sqrt{1+\xi^2}} d\xi = \sqrt{1+y^2} - 1,$$

and the inverse function of Φ is

$$\varphi(x) = \sqrt{x^2 + 2x}.$$

Let $y = \dfrac{x'}{\sqrt{1-(x')^2}}$, then (1.2) can be rewritten into

$$\begin{cases} x' = \dfrac{y}{\sqrt{1+y^2}}, \\ y' = -V'(x) + p(t), \end{cases}$$

which is a planar Hamiltonian system

$$\begin{cases} x' = \dfrac{\partial H}{\partial y}(x, y, t), \\ y' = -\dfrac{\partial H}{\partial x}(x, y, t) \end{cases} \tag{2.1}$$

with

$$H(x, y, t) = \Phi(y) + V(x) - x\, p(t).$$

Consider an auxiliary system

$$\begin{cases} x' = \dfrac{y}{\sqrt{1+y^2}}, \\ y' = -V'(x), \end{cases}$$

which is an integrable Hamiltonian system with the Hamiltonian

$$h(x, y) = \Phi(y) + V(x).$$

For every $h > 0$, denote by $I_0(h)$ the area enclosed by the closed curve

$$\Gamma_h : \quad h = \Phi(y) + V(x).$$

Let $-1 < -\alpha_h < 0 < \beta_h$ be such that

$$V(-\alpha_h) = V(\beta_h) = h.$$

That is,

$$\alpha_h = \sqrt{1 - \left(\frac{1}{1+h}\right)^{\frac{1}{\gamma}}}, \quad \beta_h = \sqrt{2h}.$$

Thus

$$I_0(h) = 2\int_{-\alpha_h}^{\beta_h} \varphi(h - V(\xi))d\xi.$$

For every $h > 0$, let

$$I_+(h) = 2\int_0^{\beta_h} \varphi(h - V(\xi))d\xi,$$

and

$$I_-(h) = 2\int_{-\alpha_h}^{0} \varphi(h - V(\xi))d\xi,$$

then

$$I_0(h) = I_+(h) + I_-(h).$$

Moreover, we define

$$T_0(h) = I_0'(h) = T_+(h) + T_-(h),$$

and

$$T_+(h) := I_+'(h) = 2\int_0^{\beta_h} \frac{h - V(\xi) + 1}{\varphi(h - V(\xi))}d\xi,$$

$$T_-(h) := I_-'(h) = 2\int_{-\alpha_h}^{0} \frac{h - V(\xi) + 1}{\varphi(h - V(\xi))}d\xi.$$

Note also that $T_0^{(n)}(h) = I_0^{(n+1)}(h)$ for all $n \geq 1$.

Now let us construct the mapping $\Psi_0 : (x, y) \to (\theta, I)$ by

$$I(x, y) = I_0(h(x, y)), \tag{2.2}$$

and

$$\theta(x,y)=\begin{cases}\dfrac{T_-(h)}{2T_0(h)}+\dfrac{1}{T_0(h)}\displaystyle\int_0^x\frac{h-V(\xi)+1}{\varphi(h-V(\xi))}d\xi, & x>0, y>0,\\ 1-\dfrac{T_-(h)}{2T_0(h)}-\dfrac{1}{T_0(h)}\displaystyle\int_0^x\frac{h-V(\xi)+1}{\varphi(h-V(\xi))}d\xi, & x>0, y<0,\\ \dfrac{1}{2T_-(h)}\displaystyle\int_{-\alpha_h}^x\frac{h-V(\xi)+1}{\varphi(h-V(\xi))}d\xi, & x<0, y>0,\\ 1-\dfrac{1}{2T_-(h)}\displaystyle\int_{-\alpha_h}^x\frac{h-V(\xi)+1}{\varphi(h-V(\xi))}d\xi, & x<0, y<0,\end{cases}\tag{2.3}$$

where

$$h=h(x,y)=\Phi(y)+V(x).$$

Since the mapping Ψ_0 is symplectic, in the new variables (θ, I), system (2.1) is also the Hamiltonian system

$$\begin{cases}\theta'=\dfrac{\partial H}{\partial I},\\ I'=-\dfrac{\partial H}{\partial\theta}\end{cases}\tag{2.4}$$

with

$$H(\theta,I,t)=h(I)+H_1(\theta,I,t),$$

where $h(I)$ is the inverse function of $I=I_0(h)$, $H_1(\theta,I,t)=-x(\theta,I)\,p(t)$, and $x(\theta,I)$ is determined by the mapping Ψ_0.

3. Some estimates on $I_0(h), h(I)$ and $H_1(\theta,I,t)$

In this section, in order to apply Moser's twist theorem to prove the existence of invariant closed curves, we shall give some estimates on the functions $I_0(h)$, $h(I)$ and $H_1(\theta,I,t)$ for h or I large enough. For this purpose, we first list the properties of the singular potential V. Let us define the function $W:(-1,+\infty)\to\mathbb{R}$ by $W(x)=\frac{V(x)}{V'(x)}$. It is easy to see that for $x>0$, $W(x)=\frac{1}{2}x$, and $W'(x)=\frac{1}{2}$, $W^{(k)}(x)=0$, $k\geq 2$. In the following lemma, we give some estimates on $V(x)$ and $W(x)$ for $-1<x<0$. Although this lemma can be found in [4], here we prove it in detail for the reader's convenience. We denote different constants by $0<c<1$ and $C>1$ in different places through the paper.

Lemma 3.1. *For any $h>0$, the inequality*

$$|(\alpha_h+x)V'(x)|\leq|h-V(x)|,\qquad -\alpha_h\leq x\leq 0\tag{3.1}$$

holds, and for $x\in(-1,0)$, the following inequalities are also true:

$$(1+x)^{k-1}\left|V^{(k)}(x)\right|\leq C\left[|V'(x)|+(1+x)^{k-1}\right],\qquad k\geq 1,\tag{3.2}$$

$$|W(x)|\leq C\,(1+x),\qquad -\gamma\leq W'(x)\leq\frac{1}{2},\tag{3.3}$$

$$\left|W^{(k)}(x)\right| \le C, \qquad k \ge 2. \tag{3.4}$$

Proof. Firstly, we prove inequality (3.1). Recall that

$$V(x) = (1-x^2)^{-\gamma} - 1, \qquad -1 < x \le 0.$$

Then

$$V'(x) = 2\gamma x(1-x^2)^{-\gamma-1} < 0, \qquad -1 < x < 0,$$

and

$$V''(x) = 2\gamma(1-x^2)^{-\gamma-1} + 2\gamma(\gamma+1)x^2(1-x^2)^{-\gamma-2} > 0, \qquad -1 < x < 0,$$

which implies that the potential V in the interval $(-1, 0)$ is monotonically decreasing and convex. Therefore inequality (3.1) is equivalent to

$$-(\alpha_h + x)V'(x) \le h - V(x), \qquad -\alpha_h \le x \le 0.$$

Set $g(x) := h - V(x) + (\alpha_h + x)V'(x)$. Since $g'(x) = V''(x)(\alpha_h + x) \ge 0$, then $g(x) \ge g(-\alpha_h) = 0$ for $-\alpha_h \le x \le 0$. This finishes the proof of (3.1).

Next let us consider inequality (3.2). Clearly, (3.2) holds for $k = 1$. We rewrite $V'(x)$ and $V''(x)$ $(-1 < x < 0)$ as

$$\begin{aligned} V'(x) &= 2\gamma x(1-x)^{-\gamma-1}(1+x)^{-\gamma-1}, \\ V''(x) &= 2\gamma(1-x)^{-\gamma-1}(1+x)^{-\gamma-1} + 2\gamma(\gamma+1)x(1-x)^{-\gamma-2}(1+x)^{-\gamma-1} \\ &\quad - 2\gamma(\gamma+1)x(1-x)^{-\gamma-1}(1+x)^{-\gamma-2}. \end{aligned}$$

Thus, we have

$$\begin{aligned} (1+x)V''(x) = 2\gamma(1-x)^{-\gamma-1}(1+x)^{-\gamma} + 2\gamma(\gamma+1)x(1-x)^{-\gamma-2}(1+x)^{-\gamma} \\ - 2\gamma(\gamma+1)x(1-x)^{-\gamma-1}(1+x)^{-\gamma-1}. \end{aligned}$$

The second term and the third term in the expression of $(1+x)V''(x)$ can be bounded by $C\,|V'(x)|$. As for the first term, when $x \in (-1, -\frac{1}{2})$, we have

$$2\gamma(1+x)(1-x)^{-\gamma-1}(1+x)^{-\gamma-1} \le |V'(x)|,$$

and when $x \in (-\frac{1}{2}, 0)$, we get

$$2\gamma(1+x)(1-x)^{-\gamma-1}(1+x)^{-\gamma-1} \le \gamma 2^{\gamma+2}(1+x),$$

which together imply that for $-1 < x < 0$,

$$2\gamma(1-x)^{-\gamma-1}(1+x)^{-\gamma} \le C\left(|V'(x)| + 1 + x\right).$$

Thus we have finished the proof of (3.2) for $k = 2$. By Leibniz formula, differentiating $V'(x)$ with respect to x by $k-1$ times, one can prove (3.2) for $k > 2$.

Thirdly we prove inequality (3.3). For $x \in (-1, 0)$,

$$W(x) = \frac{(1-x^2)[1-(1-x^2)^\gamma]}{2\gamma x}.$$

Since

$$\lim_{x\to 0} \frac{1-(1-x^2)^\gamma}{2\gamma x} = 0,$$

then

$$|W(x)| \le C(1+x), \qquad x \in (-1, 0).$$

Finally we estimate $W'(x)$. Also,

$$W'(x) = 1 - \varphi(x), \qquad -1 < x < 0,$$

where

$$\begin{aligned}\varphi(x) &:= \frac{1}{2\gamma}\frac{[1-(1-x^2)^\gamma][1+(2\gamma+1)x^2]}{x^2}\\ &= \frac{1}{2\gamma}[1+(1-x^2)+\cdots+(1-x^2)^{\gamma-1}][1+(2\gamma+1)x^2].\end{aligned}$$

Therefore

$$\varphi(x) \le \frac{1}{2\gamma}\cdot\gamma\cdot(2\gamma+2) = \gamma+1,$$

which implies that

$$W'(x) \ge -\gamma.$$

Next we prove $W'(x) \le \frac{1}{2}$. If $\gamma = 1$, then $\varphi(x) = \frac{1+3x^2}{2} \ge \frac{1}{2}$, and (3.3) holds in this case. For $\gamma \ge 2$, since $\varphi(-x) = \varphi(x)$, we consider $0 < x < 1$ instead of $-1 < x < 0$. A direct computation yields that

$$2\gamma\varphi'(x) = \frac{2\left[-1+(1-x^2)^\gamma+\gamma x^2(1-x^2)^\gamma+\gamma(2\gamma+1)x^4(1-x^2)^{\gamma-1}\right]}{x^3}.$$

Let

$$\psi(x) = -1+(1-x^2)^\gamma+\gamma x^2(1-x^2)^\gamma+\gamma(2\gamma+1)x^4(1-x^2)^{\gamma-1}.$$

Then

$$\psi'(x) = 2\gamma(\gamma+1)x^3(1-x^2)^{\gamma-2}[3-(2\gamma+1)x^2],$$

which implies that the function $\psi(x)$ increases for $0 < x < \sqrt{\frac{3}{2\gamma+1}}$, decreases for $1 > x > \sqrt{\frac{3}{2\gamma+1}}$. Since $\psi(0) = 0$, $\psi(1) = -1$, then $\varphi(x)$ first increases in the interval $(0, \bar{x})$ and then decreases in the interval $(\bar{x}, 1)$ for some $\bar{x} \in (0, 1)$. By $\varphi(0) = \frac{1}{2}$, $\varphi(1) = \frac{\gamma+1}{\gamma}$, we know that $\varphi(x) \ge \frac{1}{2}$ for $0 < x < 1$. Consequently, $-\gamma \le W'(x) \le \frac{1}{2}$. Thus we have finished the proof of (3.3). The inequality (3.4) can be proved easily by noticing that $W'(x)$ is a polynomial. $\square$

Lemma 3.2. *For all h large enough, the following inequalities are valid:*

$$c\,h^{\frac{3}{2}} \le I_+(h) \le C\,h^{\frac{3}{2}}, \quad c\,h \le I_-(h) \le C\,h, \quad c\,h^{\frac{3}{2}} \le I_0(h) \le C\,h^{\frac{3}{2}},$$
$$c\,h^{-k} I_+(h) \le I_+^{(k)}(h) \le C\,h^{-k} I_+(h), \qquad |I_-^{(k)}(h)| \le C\,h^{-k} I_-(h), \quad k = 1, 2,$$
$$c\,h^{-k} I_0(h) \le I_0^{(k)}(h) \le C\,h^{-k} I_0(h), \quad k = 1, 2,$$
$$|I_+^{(k)}(h)| \le C\,h^{-k} I_+(h), \quad |I_-^{(k)}(h)| \le C\,h^{-k} I_-(h), \quad 3 \le k \le 9,$$
$$|I_0^{(k)}(h)| \le C\,h^{-k} I_0(h), \quad 3 \le k \le 9.$$

The proof of the lemma is very complicate and we postpone it in Section 5. Although the following lemma was obtained in [13], we restate it here for the reader's convenience.

Lemma 3.3. *If a real function $\varphi(x)$ satisfies $|\varphi^{(m)}(x)| \le C\,x^{-m}\varphi(x), 1 \le m \le k$ and $\varphi'(x) \ge x^{-1}\varphi(x)$ for all x large enough, then the inverse function $\psi = \varphi^{-1}$ satisfies $|\psi^{(m)}(y)| \le C\,y^{-m}\psi(y), 1 \le m \le k$. If, moreover, $|\varphi''(x)| \ge c\,x^{-2}\varphi(x)$, then $|\psi''(y)| \ge c\,y^{-2}\psi(y)$.*

Applying Lemma 3.3 to $h(I)$, by Lemma 3.2, it is easy to obtain the properties of $h(I)$, here $h(I)$ is the inverse function of $I_0(h)$.

Lemma 3.4. *For all I large enough, the following inequalities hold:*

$$c\,I^{\frac{2}{3}} \le h(I) \le C\,I^{\frac{2}{3}},$$
$$c\,I^{-k} h(I) \le h^{(k)}(I) \le C\,I^{-k} h(I), \quad k = 1, 2,$$
$$|h^{(k)}(I)| \le C\,I^{-k} h(I), \quad 3 \le k \le 9.$$

In order to obtain the estimates of $H_1(\theta, I, t)$, we first give the estimates on $x(\theta, I)$.

Lemma 3.5. *For all I large enough and $\theta \in \mathbb{S}^1$, we have*

$$|\partial_I^k x(\theta, I)| \le C\,I^{-k} x(\theta, I), \quad x > 0,\ 0 \le k \le 7,$$
$$|\partial_I^k x(\theta, I)| \le C\,I^{-k}[1 + x(\theta, I)], \quad x < 0,\ 0 \le k \le 7.$$

Taking into account the complexity of the proof process, we put it as a separate section. The following lemma gives the estimate on the perturbation term H_1.

Lemma 3.6. *For I large enough and $(\theta, t) \in \mathbb{T}^2$, the perturbation term H_1 possesses the following estimates*

$$|\partial_I^i \partial_t^j \partial_\theta^k H_1(\theta, I, t)| \le C\,I^{-i+\frac{1}{3}}, \quad k = 0, 1,$$
$$|\partial_I^i \partial_t^j \partial_\theta^2 H_1(\theta, I, t)| \le C\,I^{-i+1} \cdot \frac{1}{(h - V(x) + 1)^3}, \qquad x > 0,$$
$$|\partial_I^i \partial_t^j \partial_\theta^2 H_1(\theta, I, t)| \le C\,I^{-i+\frac{2}{3}} \cdot \frac{|V'(x)| + 1 + x}{(h - V(x) + 1)^3}, \qquad x < 0,$$

where $0 \le i \le 7$, $0 \le j \le 5$, and $h = h(I)$, $x = x(\theta, I)$.

Proof. Let us consider $k = 0$ first. $H_1(\theta, I, t) = -x(\theta, I)\,p(t)$ yields that $\partial_I^i \partial_t^j H_1(\theta, I, t) = -\partial_I^i x(\theta, I) p^{(j)}(t)$. On one hand, if $x > 0$, by Lemma 3.4 and Lemma 3.5, we obtain

$$|\partial_I^i \partial_t^j H_1(\theta, I, t)| = |-\partial_I^i x(\theta, I) p^{(j)}(t)| \leq C\, I^{-i+\frac{1}{3}}.$$

On the other hand, if $x < 0$, by Lemma 3.5, we also have

$$|\partial_I^i \partial_t^j H_1(\theta, I, t)| = |-\partial_I^i x(\theta, I) p^{(j)}(t)| \leq C\, I^{-i}(1+x) \leq C I^{-i},$$

which completes the proof of this lemma in the case $k = 0$.

Now we prove this lemma for $k = 1$. First we remark that the closed curve Γ_h is symmetric with respect to x-axis, we only need to prove the conclusion in the cases: $x < 0, y > 0$ and $x > 0, y > 0$, which correspond to $0 < \theta < \frac{T_-(h)}{2T_0(h)}$ and $\frac{T_-(h)}{2T_0(h)} < \theta < \frac{1}{2}$, respectively.

The definition of θ leads to

$$x_\theta = T_0(h) \cdot \frac{\varphi(h - V(x))}{h - V(x) + 1}, \tag{3.5}$$

which implies that

$$\partial_I^i \partial_t^j \partial_\theta H_1(\theta, I, t) = -\sum_{k+l=i} C_{kl}\, \partial_I^k T_0(h) \cdot \partial_I^l \left(\frac{\varphi(h - V(x))}{h - V(x) + 1} \right) \cdot p^{(j)}(t).$$

Here we use Leibniz formula and C_{kl} is a constant depending only on k, l. By Lemmas 3.2 and 3.4, it follows that

$$\begin{aligned}
|\partial_I^k T_0(h)| &= \left| \sum_{k_1+\cdots+k_s=k} C_{k_1\cdots k_s}\, T_0^{(s)}(h) \partial_I^{k_1} h \cdots \partial_I^{k_s} h \right| \\
&\leq C \sum_{k_1+\cdots+k_s=k} h^{-s} |T_0(h)| I^{-k_1} h \cdots I^{-k_s} h \\
&\leq C\, I^{-k} |T_0(h)| \leq C\, I^{-k+\frac{1}{3}},
\end{aligned}$$

here we use Bruno formula and $C_{k_1\cdots k_s}$ is a constant depending only on $k_1, \cdots, k_s$.

Also by the proof process of Lemma 3.5, we have

$$\left| \partial_I^l \left(\frac{\varphi(h - V(x))}{h - V(x) + 1} \right) \right| \leq C\, I^{-l} \left(\frac{\varphi(h - V(x))}{h - V(x) + 1} + 1 + x \right) \leq C\, I^{-l}.$$

Thus,

$$|\partial_I^i \partial_t^j \partial_\theta H_1(\theta, I, t)| \leq C\, I^{-i+\frac{1}{3}},$$

which leads to completeness of the lemma in the case $k = 1$.

Finally we prove this lemma under the case $k = 2$. Similarly, let us first consider $x > 0$, $y > 0$. Differentiating (3.5) by θ on both sides yields that

$$\partial_\theta^2 x(\theta, I) = \frac{-4(T_0(h))^2 \cdot V'(x)}{(h - V(x) + 1)^3},$$

which implies that

$$\partial_t^j \partial_\theta^2 H_1(\theta, I, t) = \frac{-4(T_0(h))^2 \cdot V'(x)}{(h - V(x) + 1)^3} \cdot p^{(j)}(t).$$

From Lemmas 3.2, 3.4 and the proof process of Lemma 3.5, we obtain

$$\begin{aligned}|\partial_I^k (T_0(h))^2| &= \left|\sum_{k_1+k_2=k} C_{k_1k_2}\,\partial_I^{k_1}T_0(h)\cdot\partial_I^{k_2}T_0(h)\right|\\ &\le C\sum_{k_1+k_2=k} I^{-k_1}T_0(h)\cdot I^{-k_2}T_0(h)\\ &\le C\,I^{-k+\frac{2}{3}},\end{aligned}$$

$$|\partial_I^l V'(x)| = |\partial_I^l x| \le C\,I^{-l+\frac{1}{3}},$$

$$\begin{aligned}&\left|\partial_I^p \frac{1}{(h-V(x)+1)^3}\right|\\ &= \left|\sum_{p_1+\cdots+p_s=p} C_{p_1\cdots p_s}\cdot\frac{1}{(h-V(x)+1)^{3+s}}\partial_I^{p_1}(h-V(x))\cdots\partial_I^{p_s}(h-V(x))\right|\\ &\le C\sum_{p_1+\cdots+p_s=p} I^{-p_1}\cdots I^{-p_s}\cdot\frac{1}{(h-V(x)+1)^3}\\ &\le C\,I^{-p}\cdot\frac{1}{(h-V(x)+1)^3}.\end{aligned}$$

Consequently, it follows that

$$\begin{aligned}&|\partial_I^i\partial_t^j\partial_\theta^2 H_1(\theta,I,t)|\\ &= \left|\sum_{k+l+p=i} C_{klp}\,\partial_I^k(T_0(h))^2\cdot\partial_I^l V'(x)\cdot\partial_I^p\frac{1}{(h-V(x)+1)^3}\cdot p^{(j)}(t)\right|\\ &\le C\sum_{k+l+p=i} I^{-k+\frac{2}{3}}\cdot I^{-l+\frac{1}{3}}\cdot I^{-p}\cdot\frac{1}{(h-V(x)+1)^3}\\ &\le C\,I^{-i+1}\cdot\frac{1}{(h-V(x)+1)^3}.\end{aligned}$$

As for the case $x<0, y>0$, similarly we have

$$\partial_t^j\partial_\theta^2 H_1(\theta,I,t) = \frac{-4(T_0(h))^2\cdot V'(x)}{(h-V(x)+1)^3}\cdot p^{(j)}(t).$$

By Lemmas 3.2, 3.4 and the proof process of Lemma 3.5, we also obtain

$$\begin{aligned}&|\partial_I^k (T_0(h))^2| \le C\,I^{-k+\frac{2}{3}},\\ &\left|\partial_I^p\frac{1}{(h-V(x)+1)^3}\right| \le CI^{-p}\cdot\frac{1}{(h-V(x)+1)^3}.\end{aligned}$$

Meanwhile, according to Lemma 3.1 and 3.5, one has

$$\begin{aligned}|\partial_I^l V'(x)| &= \left|\sum_{l_1+\cdots+l_s=l} C_{l_1\cdots l_s}\,V^{(s+1)}(x)\partial_I^{l_1}x\cdots\partial_I^{l_s}x\right|\\ &\le C\sum_{l_1+\cdots+l_s=l} |V^{(s+1)}(x)|I^{-l_1}(1+x)\cdots I^{-l_s}(1+x)\end{aligned}$$

$$\le C \sum_{s\ge 1} I^{-l}(|V'(x)| + (1+x)^s)$$
$$\le C\, I^{-l}(|V'(x)| + 1 + x).$$

Therefore we obtain

$$|\partial_I^i \partial_t^j \partial_\theta^2 H_1(\theta, I, t)| \le C\, I^{-i+\frac{2}{3}} \cdot \frac{|V'(x)| + 1 + x}{(h - V(x) + 1)^3}.$$

Thus we complete the proof of this lemma. □

4. New action-angle variables and proof of Theorem 1.1

We remark that $H(\theta, I, t) = h(I) + H_1(\theta, I, t)$. It is easy to see that for I large enough

$$\partial_I H = h'(I) + \partial_I H_1(\theta, I, t) \ge c\, I^{-\frac{1}{3}} - c\, I^{-\frac{2}{3}} > 0.$$

Then by implicit function theorem, there is a function $I_1(t, H, \theta)$ such that

$$I(t, H, \theta) = I_0(H) + I_1(t, H, \theta). \tag{4.1}$$

From the definition of $I(t, H, \theta)$, we have $H = h(I(H)) + H_1(I(H))$, and

$$\begin{aligned} I_1(H) &= I_0(H - H_1) - I_0(H) \\ &= -I_0'(H)H_1 + \int_0^1 (1-s) I_0''(H - sH_1) H_1^2 ds, \end{aligned} \tag{4.2}$$

where we treat H as the independent variable and θ, I as parameters. Therefore, for simplicity, here and hereafter, we use the symbols $I(H), I_1(H)$ instead of $I(t, H, \theta), I_1(t, H, \theta)$ respectively. Then (2.4) is equivalent to

$$\begin{cases} \dfrac{dt}{d\theta} = \dfrac{\partial I}{\partial H}(t, H, \theta), \\ \dfrac{dH}{d\theta} = -\dfrac{\partial I}{\partial t}(t, H, \theta), \end{cases} \tag{4.3}$$

with the Hamiltonian $I = I(t, H, \theta)$ and the action, angle and time variables are $H,\ t,\ \theta$, respectively.

The following lemmas give the estimates on the new perturbation term I_1. The proof is similar to the proof of Lemma 3.1 in [14].

Lemma 4.1. *For H large enough and $(\theta, t) \in \mathbb{T}^2$, the perturbation term $I_1(t, H, \theta)$ satisfies the following inequalities*

$$|\partial_H^i \partial_t^j \partial_\theta^k I_1(t, H, \theta)| \le C\, H^{-i+1},$$

where $0 \le i \le 7,\ 0 \le j \le 5,\ k = 0, 1.$

Proof. We prove this lemma according to the values of i, j, k.

Case (i) $i = 0,\ j = 0,\ k = 0$. Firstly, we estimate $H(I)$ and $I(H)$. By Lemma 3.4 and Lemma 3.6, we have

$$cI^{\frac{2}{3}} \leq cI^{\frac{2}{3}} - cI^{\frac{1}{3}} \leq H(I) = h(I) + H_1(\theta, I, t) \leq CI^{\frac{2}{3}} + CI^{\frac{1}{3}} \leq CI^{\frac{2}{3}}.$$

Since $I(H)$ is the inverse function of $H(I)$, then we have

$$cH^{\frac{3}{2}} \leq I(H) \leq CH^{\frac{3}{2}}.$$

Now we estimate $I_1(H)$. From Lemma 3.6 and the estimate of $I(H)$, we have

$$|H_1(I(H))| \leq C(I(H))^{\frac{1}{3}} \leq CH^{\frac{1}{2}},$$
$$|I_0'(H)H_1| \leq CH^{-1}I_0(H) \cdot H^{\frac{1}{2}} \leq CH,$$
$$|I_0''(H - sH_1)H_1^2| \leq C(H - sH_1)^{-2}I_0(H - sH_1)H_1^2 \leq CH^{-2}I_0(H)H \leq CH^{\frac{1}{2}}.$$

Therefore, (4.2) leads to

$$|I_1(H)| \leq CH + CH^{\frac{1}{2}} \leq CH.$$

Case (ii) $i \geq 1$, $j = 0$, $k = 0$. Differentiating (4.2) by H, we have

$$\partial_H^i I_1(H) = -\sum_{l+p=i} C_{lp} I_0^{(l+1)}(H)\partial_H^p H_1 + \partial_H^i \int_0^1 (1-s)I_0''(H - sH_1)H_1^2 ds.$$

Since

$$\partial_H^p H_1 = \sum_{p_1 + \cdots p_s = p} C_{p_1 \cdots p_s} \partial_I^s H_1(I(H))\partial_H^{p_1} I(H) \cdots \partial_H^{p_s} I(H),$$

then it suffices to show the following estimate:

$$|\partial_H^p I(H)| \leq CH^{-p} I(H). \tag{4.4}$$

The proof of (4.4). Differentiating $I(H) = I_0(H - H_1(I(H)))$ by H leads to

$$I'(H) = I_0'(H - H_1)[1 - H_1'(I(H)) \cdot I'(H)],$$

which implies that

$$I'(H) = \frac{I_0'(H - H_1)}{1 + I_0'(H - H_1)H_1'(I(H))}.$$

Since

$$\begin{aligned} & I_0'(H - H_1)H_1'(I(H)) \\ &= I_0'(H - H_1) \cdot h'(I_0(H - H_1)) \cdot \frac{H_1'(I(H))}{h'(I_0(H - H_1))} \\ &= \frac{H_1'(I(H))}{h'(I_0(H - H_1))}, \end{aligned}$$

it follows that

$$
\begin{aligned}
|I_0'(H-H_1)H_1'(I(H))| &= \left|\frac{H_1'(I(H))}{h'(I_0(H-H_1))}\right| \le C\,\frac{I^{-\frac{2}{3}}}{(I_0(H-H_1))^{-\frac{1}{3}}} \\
&\le C\,\frac{(I_0(H-H_1))^{\frac{1}{3}}}{I^{\frac{2}{3}}} \le C\,\frac{H^{\frac{1}{2}}}{H} \le C\,H^{-\frac{1}{2}},
\end{aligned}
$$

which implies that $I_0'(H-H_1)H_1'(I(H)) \to 0$ as $H \to +\infty$, thus we have

$$
\begin{aligned}
I'(H) < 2I_0'(H-H_1) &\le C\,(H-H_1)^{-1}I_0(H-H_1) \\
&\le C\,H^{-1}I_0(H) \le C\,H^{-1}\cdot H^{\frac{3}{2}} \le C\,H^{-1}I(H).
\end{aligned}
$$

Suppose that for $p \le k$, (4.4) holds. Then the $k+1$ derivative of $I(H)$ possesses the following form

$$
\partial_H^{k+1}I(H) = \sum_{l+p=k} C_{lp}\,\partial_H^l I_0'(H-H_1)\partial_H^p\triangle,
$$

where $\triangle = [1 + I_0'(H-H_1)H_1'(I(H))]^{-1}$.

Firstly, let us prove $|\partial_H^l I_0'(H-H_1)| \le C\,H^{-l}I_0'(H-H_1)$. By Lemma 3.4 and the above assumption, we deduce that

$$
\begin{aligned}
&|\partial_H^l(H-H_1)| \\
&= |\partial_H^l h(I(H))| \\
&= \left|\sum_{l_1+\cdots+l_s=l} C_{l_1\cdots l_s}\,\partial_I^s H_0(I(H))\partial_H^{l_1}I(H)\cdots\partial_H^{l_s}I(H)\right| \\
&\le C \sum_{l_1+\cdots+l_s=l} (I(H))^{-s}h(I(H))H^{-l_1}I(H)\cdots H^{-l_s}I(H) \\
&\le C\,H^{-l}H_0(I(H)) \\
&\le C\,H^{-l}(H-H_1).
\end{aligned}
$$

Hence, by Lemma 3.2 and the above inequality, we have

$$
\begin{aligned}
&|\partial_H^l I_0'(H-H_1)| \\
&= \left|\sum_{l_1+\cdots+l_s=l} C_{l_1\cdots l_s}\,I_0^{(s+1)}(H-H_1)\partial_H^{l_1}(H-H_1)\cdots\partial_H^{l_s}(H-H_1)\right| \\
&\le C \sum_{l_1+\cdots+l_s=l} (H-H_1)^{-s}I_0'(H-H_1)H^{-l_1}(H-H_1)\cdots H^{-l_s}(H-H_1) \\
&\le C\,H^{-l}I_0'(H-H_1).
\end{aligned}
$$

Next, we prove $|\partial_H^p\triangle| \le C\,H^{-p}$. From Lemma 3.6 and the above assumption, we get

$$
\begin{aligned}
&|\partial_H^n H_1'(I(H))| \\
&= \left|\sum_{n_1+\cdots+n_s=n} C_{n_1\cdots n_s}\,H_1^{(s+1)}(I(H))\partial_H^{n_1}I(H)\cdots\partial_H^{n_s}I(H)\right| \\
&\le C \sum_{n_1+\cdots+n_s=n} (I(H))^{-(s+1)+\frac{1}{3}}\cdot H^{-n_1}I(H)\cdots H^{-n_s}I(H)
\end{aligned}
$$

$$\begin{aligned}&\le C\,H^{-n}(I(H))^{-\frac{2}{3}}\\&\le C\,H^{-n-1},\end{aligned}$$

which implies that

$$\begin{aligned}&|\partial_H^p[1+I_0'(H-H_1)H_1'(I(H))]|\\&=\left|\sum_{m+n=p}C_{mn}\,\partial_H^m I_0'(H-H_1)\partial_H^n H_1'(I(H))\right|\\&\le C\sum_{m+n=p}H^{-m}I_0'(H-H_1)\cdot H^{-n-1}\\&\le C\,H^{-p-1}I_0'(H-H_1)\le C\,H^{-p-1}(H-H_1)^{\frac{1}{2}}\\&\le C\,H^{-p-1}\cdot H^{\frac{1}{2}}\le C\,H^{-p}.\end{aligned}$$

Since $I_0'(H-H_1)H_1'(I(H))\to 0$ as $H\to+\infty$, then we have

$$\begin{aligned}&|\partial_H^p\triangle|\\&=\left|\sum_{p_1+\cdots+p_s=p}C_{p_1\cdot p_s}\,[1+I_0'(H-H_1)H_1'(I(H))]^{-1-s}\right.\\&\qquad\left.\partial_H^{p_1}[1+I_0'(H-H_1)H_1'(I(H))]\right|\\&\le C\sum_{p_1+\cdots+p_s=p}H^{-p_1}\cdots H^{-p_s}\\&\le C\,H^{-p},\end{aligned}$$

which leads to

$$\begin{aligned}|\partial_H^{k+1}I(H)|&\le C\sum_{p+l=k}H^{-l}I_0'(H-H_1)\cdot H^{-p}\le C\,H^{-k}I_0'(H-H_1)\\&\le C\,H^{-k}(H-H_1)^{\frac{1}{2}}\le C\,H^{-k}H^{\frac{1}{2}}\\&\le C\,H^{-(k+1)}H^{\frac{3}{2}}\le C\,H^{-(k+1)}I(H).\end{aligned}$$

The proof of (4.4) is completed.

Now we estimate $I_1(H)$. First we remark that

$$\begin{aligned}|\partial_H^p H_1|&\le C\sum_{p_1+\cdots p_s=p}(I(H))^{-s+\frac{1}{3}}\cdot H^{-p_1}I(H)\cdots H^{-p_s}I(H)\\&\le C\,H^{-p}(I(H))^{\frac{1}{3}}\le C\,H^{-p+\frac{1}{2}},\end{aligned}$$

and

$$\begin{aligned}\left|\sum_{l+p=i}I_0^{(l+1)}(H)\partial_H^p H_1\right|&\le C\sum_{l+p=i}H^{-(l+1)}I_0(H)H^{-p+\frac{1}{2}}\\&\le C\,H^{-i-\frac{1}{2}}I_0(H)\le C\,H^{-i+1}.\end{aligned}$$

Therefore, in order to get the estimate of $\partial_H^i I_1(H)$, according to the expression of $\partial_H^i I_1(H)$, we also need to estimate the following term

$$\partial_H^i \int_0^1 (1-s)I_0''(H-sH_1)H_1^2 ds = \int_0^1 (1-s)\partial_H^i (I_0''(H-sH_1)H_1^2)ds.$$

Since

$$\partial_H^i (I_0''(H-sH_1)H_1^2) = \sum_{l+p=i} C_{lp}\, \partial_H^l I_0''(H-sH_1)\partial_H^p H_1^2,$$

we need to estimate $\partial_H^l I_0''(H-sH_1)$ and $\partial_H^p H_1^2$ respectively. From Lemma 3.2 and the previous proof, we obtain

$$\begin{aligned}
|I_0^{(j+2)}(H-sH_1)| &\le C\,(H-sH_1)^{-(j+2)} I_0(H-sH_1) \\
&\le C\,H^{-(j+2)} I_0(H) \le C\,H^{-j-\frac{1}{2}}, \\
|\partial_H^m (H-sH_1)| &= |\partial_H^m [(H-H_1)+(1-s)H_1]| \\
&\le |\partial_H^m (H-H_1)| + C|\partial_H^m H_1| \\
&\le C\,H^{-m}(H-H_1) + C\,H^{-m+\frac{1}{2}} \\
&\le C\,H^{-m+1} + C\,H^{-m+\frac{1}{2}} \le C\,H^{-m+1},
\end{aligned}$$

which implies that

$$\begin{aligned}
&|\partial_H^l I_0''(H-sH_1)| \\
&= \left| \sum_{l_1+\cdots+l_q=l} C_{l_1\cdots l_q}\, I_0^{(q+2)}(H-sH_1)\partial_H^{l_1}(H-sH_1)\cdots\partial_H^{l_q}(H-sH_1) \right| \\
&\le C \sum_{l_1+\cdots+l_q=l} H^{-q-\frac{1}{2}} \cdot H^{-l_1+1} \cdots H^{-l_q+1} \\
&\le C\,H^{-l-\frac{1}{2}}.
\end{aligned}$$

Since

$$\begin{aligned}
|\partial_H^p H_1^2| &= \left| \sum_{p_1+p_2=p} C_{p_1p_2}\, \partial_H^{p_1} H_1 \partial_H^{p_2} H_1 \right| \\
&\le C \sum_{p_1+p_2=p} H^{-p_1+\frac{1}{2}} \cdot H^{-p_2+\frac{1}{2}} \\
&\le C\,H^{-p+1},
\end{aligned}$$

then it follows that

$$\left| \partial_H^i \int_0^1 (1-s)I_0''(H-sH_1)H_1^2 ds \right| \le C \sum_{l+p=i} H^{-l-\frac{1}{2}} \cdot H^{-p+1} \le C\,H^{-i+\frac{1}{2}}.$$

Therefore, we obtain

$$|\partial_H^i I_1(H)| \le C\, H^{-i+1}.$$

Case (iii) $i \ge 0$, $j \ge 1$, $k = 0$. From $I_1(H) = I_0(H - H_1) - I_0(H)$, we deduce that

$$\partial_t I_1(H) = I_0'(H - H_1)(-\partial_t H_1 - \partial_I H_1 \cdot \partial_t I(H)).$$

Since $I(H) = I_0(H - H_1)$, then $\partial_t I(H) = \partial_t I_1(H)$, which leads to

$$\partial_t I_1(H) = \frac{-\partial_t H_1 \cdot I_0'(H - H_1)}{1 + \partial_I H_1 \cdot I_0'(H - H_1)} = -\partial_t H_1 \cdot \partial_H I(H).$$

By Lemma 3.6, differentiating H_1 with respect to t does not change the power of I, thus so is I_1.

Case (iv) $k = 1$. Since $\partial_\theta I_1(H) = -\partial_\theta H_1 \cdot \partial_H I(H)$, according to Lemma 3.6, one can easily prove this case. □

Lemma 4.2. *For H large enough and $(\theta, t) \in \mathbb{T}^2$, the perturbation term $I_1(t, H, \theta)$ satisfies the following inequalities*

$$|\partial_H^i \partial_t^j \partial_\theta^2 I_1(t, H, \theta)| \le C\, H^{-i} \left(H^{\frac{1}{2}} + \frac{H^2}{(h - V(x) + 1)^3} \right), \qquad x > 0,$$
$$|\partial_H^i \partial_t^j \partial_\theta^2 I_1(t, H, \theta)| \le C\, H^{-i} \left(H^{\frac{1}{2}} + H^{\frac{3}{2}} \frac{|V'(x)| + 1 + x}{(h - V(x) + 1)^3} \right), \qquad x < 0,$$

where $0 \le i \le 7$, $0 \le j \le 5$, and $h = H_0(I(H)), x = x(\theta, I(H))$.

Proof. Let us first consider $x > 0$. Differentiating the equality $I(t, H(\theta, I, t), \theta) \equiv I$ with respect to θ yields that

$$\partial_\theta I_1(H) = -\partial_\theta H_1 \cdot \partial_H I(H),$$

which leads to

$$\begin{aligned}
\partial_\theta^2 I_1(H) &= -\partial_H \partial_\theta I_1(H) \cdot \partial_\theta H_1 - \partial_H I(H) \cdot \partial_\theta^2 H_1 \\
&\quad - \partial_H I(H) \cdot \partial_\theta I_1(H) \cdot \partial_\theta \partial_I H_1, \\
\partial_H^i \partial_\theta^2 I_1(H) &= -\Big[\partial_H^i (\partial_H \partial_\theta I_1(H) \cdot \partial_\theta H_1) + \partial_H^i (\partial_H I(H) \cdot \partial_\theta^2 H_1) \\
&\quad + \partial_H^i (\partial_H I(H) \cdot \partial_\theta I_1(H) \cdot \partial_\theta \partial_I H_1) \Big].
\end{aligned}$$

Firstly, we estimate $\partial_H^i(\partial_H \partial_\theta I_1(H) \cdot \partial_\theta H_1)$. It is easy to see that

$$\begin{aligned}
|\partial_H^p(\partial_\theta H_1)| &= \left| \sum_{p_1 + \cdots + p_s = p} C_{p_1 \cdots p_s} \cdot \partial_I^s \partial_\theta H_1 \partial_H^{p_1} I(H) \cdots \partial_H^{p_s} I(H) \right| \\
&\le C \sum_{p_1 + \cdots + p_s = p} I^{-s + \frac{1}{3}} \cdot H^{-p_1} I(H) \cdots H^{-p_s} I(H) \\
&\le C\, H^{-p} \cdot I^{\frac{1}{3}} \\
&\le C\, H^{-p + \frac{1}{2}}.
\end{aligned}$$

Then we have

$$
\begin{aligned}
|\partial_H^i(\partial_H\partial_\theta I_1(H)\cdot\partial_\theta H_1)| &= \left|\sum_{l+p=i} C_{lp}\cdot\partial_H^{l+1}\partial_\theta I_1(H)\cdot\partial_H^p(\partial_\theta H_1)\right| \\
&\le C\sum_{l+p=i} H^{-(l+1)+1}\cdot H^{-p+\frac{1}{2}} \\
&\le C\,H^{-i+\frac{1}{2}}.
\end{aligned}
$$

Next, we estimate $\partial_H^i(\partial_H I(H)\cdot\partial_\theta^2 H_1)$. By Lemma 3.6 and the proof process of previous lemma, we obtain

$$
\begin{aligned}
|\partial_H^p(\partial_\theta^2 H_1)| &= \left|\sum_{p_1+\cdots+p_s=p} C_{p_1\cdots p_s}\cdot\partial_I^s\partial_\theta^2 H_1\cdot\partial_H^{p_1} I(H)\cdots\partial_H^{p_s} I(H)\right| \\
&\le C\sum_{p_1+\cdots+p_s=p} I^{-s+1}\frac{1}{(h-V(x)+1)^3}H^{-p_1}I(H)\cdots H^{-p_s}I(H) \\
&\le C\,\frac{H^{-p+\frac{3}{2}}}{(h-V(x)+1)^3}.
\end{aligned}
$$

Then it follows that

$$
\begin{aligned}
|\partial_H^i(\partial_H I(H)\cdot\partial_\theta^2 H_1)| &= \left|\sum_{l+p=i} C_{lp}\cdot\partial_H^{l+1} I(H)\cdot\partial_H^p(\partial_\theta^2 H_1)\right| \\
&\le C\sum_{l+p=i} H^{-(l+1)}I(H)\cdot H^{-p+\frac{3}{2}}\cdot\frac{1}{(h-V(x)+1)^3} \\
&\le C\,\frac{H^{-i+2}}{(h-V(x)+1)^3}.
\end{aligned}
$$

Finally, we estimate $\partial_H^i(\partial_H I(H)\cdot\partial_\theta I_1(H)\cdot\partial_\theta\partial_I H_1)$. From Lemma 3.6 and the proof of previous lemma, we know

$$
\begin{aligned}
|\partial_H^q(\partial_\theta\partial_I H_1)| &= \left|\sum_{q_1+\cdots+q_s=q} C_{q_1\cdots q_s}\cdot\partial_I^{s+1}\partial_\theta H_1\cdot\partial_H^{q_1} I(H)\cdots\partial_H^{q_s} I(H)\right| \\
&\le C\sum_{q_1+\cdots+q_s=q} I^{-(s+1)+\frac{1}{3}}\cdot H^{-q_1}I(H)\cdots H^{-q_s}I(H) \\
&\le C\,H^{-q-1}.
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
&|\partial_H^i(\partial_H I(H)\cdot\partial_\theta I_1(H)\cdot\partial_\theta\partial_I H_1)| \\
&= \left|\sum_{l+p+q=i} C_{lpq}\cdot\partial_H^{l+1} I(H)\cdot\partial_H^p\partial_\theta I_1(H)\cdot\partial_H^q(\partial_\theta\partial_I H_1)\right| \\
&\le C\sum_{l+p+q=i} H^{-(l+1)}I(H)\cdot H^{-p+1}\cdot H^{-q-1} \\
&\le C\,H^{-i-1}I(H) \\
&\le C\,H^{-i+\frac{1}{2}}.
\end{aligned}
$$

Therefore, we have

$$|\partial_H^i \partial_\theta^2 I_1(H)| \le C\, H^{-i} \left(H^{\frac{1}{2}} + \frac{H^2}{(h - V(x) + 1)^3} \right).$$

Differentiating $\partial_H^i \partial_\theta^2 I_1(H)$ by t does not affect the above estimate, thus we complete the proof of this lemma in $x > 0$.

As for the case $x < 0$, similarly we have

$$\begin{aligned} \partial_H^i \partial_\theta^2 I_1(H) = -\Big[& \partial_H^i (\partial_H \partial_\theta I_1(H) \cdot \partial_\theta H_1) + \partial_H^i (\partial_H I(H) \cdot \partial_\theta^2 H_1) \\ & + \partial_H^i (\partial_H I(H) \cdot \partial_\theta I_1(H) \cdot \partial_\theta \partial_I H_1) \Big]. \end{aligned}$$

Let us estimate $\partial_H^i (\partial_H \partial_\theta I_1(H) \cdot \partial_\theta H_1)$ first. By Lemma 3.6, we still know

$$|\partial_H^p (\partial_\theta H_1)| \le C\, H^{-p+\frac{1}{2}}.$$

Then we have

$$|\partial_H^i (\partial_H \partial_\theta I_1(H) \cdot \partial_\theta H_1)| \le C\, H^{-i+\frac{1}{2}}.$$

Next, we estimate $\partial_H^i (\partial_H I(H) \cdot \partial_\theta^2 H_1)$. By Lemma 3.6 and the proof process of previous lemma, we obtain

$$\begin{aligned} |\partial_H^p (\partial_\theta^2 H_1)| &= \left| \sum_{p_1 + \cdots + p_s = p} C_{p_1 \cdots p_s} \cdot \partial_I^s \partial_\theta^2 H_1 \cdot \partial_H^{p_1} I(H) \cdots \partial_H^{p_s} I(H) \right| \\ &\le C \sum_{p_1 + \cdots + p_s = p} I^{-s+\frac{2}{3}} \frac{|V'(x)| + 1 + x}{(h - V(x) + 1)^3} H^{-p_1} I(H) \cdots H^{-p_s} I(H) \\ &\le C\, H^{-p+1} \frac{|V'(x)| + 1 + x}{(h - V(x) + 1)^3}. \end{aligned}$$

Then it follows that

$$\begin{aligned} |\partial_H^i (\partial_H I(H) \cdot \partial_\theta^2 H_1)| &= \left| \sum_{l+p=i} C_{lp} \cdot \partial_H^{l+1} I(H) \cdot \partial_H^p (\partial_\theta^2 H_1) \right| \\ &\le C \sum_{l+p=i} H^{-(l+1)} I(H) \cdot H^{-p+1} \frac{|V'(x)| + 1 + x}{(h - V(x) + 1)^3} \\ &\le C\, H^{-i+\frac{3}{2}} \frac{|V'(x)| + 1 + x}{(h - V(x) + 1)^3}. \end{aligned}$$

Finally, we need to estimate $\partial_H^i (\partial_H I(H) \cdot \partial_\theta I_1(H) \cdot \partial_\theta \partial_I H_1)$. From Lemma 3.6 and the proof process of previous lemma, we still have

$$|\partial_H^q (\partial_\theta \partial_I H_1)| \le C\, H^{-q-1}.$$

Then we obtain

$$|\partial_H^i (\partial_H I(H) \cdot \partial_\theta I_1(H) \cdot \partial_\theta \partial_I H_1)| \le C\, H^{-i+\frac{1}{2}}.$$

Therefore, we have

$$|\partial_H^i \partial_\theta^2 I_1(H)| \leq C\, H^{-i} \left(H^{\frac{1}{2}} + H^{\frac{3}{2}} \frac{|V'(x)| + 1 + x}{(h - V(x) + 1)^3} \right).$$

By Lemma 3.6, differentiating $\partial_H^i \partial_\theta^2 I_1(H)$ by t yields that the same estimate holds. Hence we have finished the proof of this lemma. □

We remark that the Poincaré mapping corresponding to the new Hamiltonian system is far from a small perturbation of the standard twist mapping. Hence we cannot use Moser's twist theorem directly and now introduce a canonical transformation.

Lemma 4.3. *For H large enough, there exists a canonical transformation*

$$\Psi_1 : (\hat{\tau}, \rho) \mapsto (t, H),$$

which transforms (4.1) into a new Hamiltonian

$$\mathcal{H}(\hat{\tau}, \rho, \theta) = \mathcal{H}_0(\rho, \theta) + \mathcal{H}_1(\hat{\tau}, \rho, \theta), \tag{4.5}$$

where

$$\mathcal{H}_0(\rho, \theta) = I_0(\rho) + [I_1](\rho, \theta), \quad [I_1](\rho, \theta) = \int_0^1 I_1(s, \rho, \theta) ds,$$

and the new perturbation $\mathcal{H}_1$ satisfies

$$\begin{aligned}
|\partial_\rho^i \partial_{\hat{\tau}}^j \mathcal{H}_1(\hat{\tau}, \rho, \theta)| &\leq C\, \rho^{-i+\frac{1}{2}}, \\
|\partial_\theta \partial_\rho^i \partial_{\hat{\tau}}^j \mathcal{H}_1(\hat{\tau}, \rho, \theta)| &\leq C\, \rho^{-i} \left(\rho^{\frac{1}{2}} + \frac{\rho^{\frac{3}{2}}}{(h - V(x) + 1)^3} \right), \qquad x > 0, \\
|\partial_\theta \partial_\rho^i \partial_{\hat{\tau}}^j \mathcal{H}_1(\hat{\tau}, \rho, \theta)| &\leq C\, \rho^{-i} \left(\rho^{\frac{1}{2}} + \rho \cdot \frac{|V'(x)| + 1 + x}{(h - V(x) + 1)^3} \right), \qquad x < 0,
\end{aligned}$$

where $0 \leq i \leq 6$, $0 \leq j \leq 5$, and $h = h(I(\Psi_1(\hat{\tau}, \rho), \theta))$, $x = x(\theta, I(\Psi_1(\hat{\tau}, \rho), \theta))$ are determined by the mappings Ψ_0, Ψ_1.

Proof. We define a time-dependent canonical transformation

$$\Psi_1 : (\hat{\tau}, \rho) \mapsto (t, H)$$

as

$$H = \rho + \frac{\partial G}{\partial t}(t, \rho, \theta), \quad \hat{\tau} = t + \frac{\partial G}{\partial \rho}(t, \rho, \theta),$$

where G will be determined later. Under this transformation, the Hamiltonian system (4.1) is transformed into a new one

$$\begin{aligned}\mathcal{H}(\hat{\tau},\rho,\theta) &= I_0(\rho+\partial_t G)+I_1(t,\rho+\partial_t G,\theta)+\partial_\theta G\\ &= I_0(\rho)+I_0'(\rho)\partial_t G+\int_0^1(1-s)I_0''(\rho+s\partial_t G)(\partial_t G)^2ds+I_1(t,\rho,\theta)\\ &\quad+\int_0^1\partial_H I_1(t,\rho+s\partial_t G,\theta)\partial_t G ds+\partial_\theta G\\ &:= \mathcal{H}_0(\rho,\theta)+\mathcal{H}_1(\hat{\tau},\rho,\theta)+I_0'(\rho)\partial_t G+I_1(t,\rho,\theta)-[I_1](\rho,\theta),\end{aligned}$$

where

$$\mathcal{H}_0(\rho,\theta)=I_0(\rho)+[I_1](\rho,\theta),\quad [I_1](\rho,\theta)=\int_0^1 I_1(s,\rho,\theta)ds,$$

$$\begin{aligned}\mathcal{H}_1(\hat{\tau},\rho,\theta)&=\int_0^1(1-s)I_0''(\rho+s\partial_t G)(\partial_t G)^2ds\\ &\quad+\int_0^1\partial_H I_1(t,\rho+s\partial_t G,\theta)\partial_t G ds+\partial_\theta G.\end{aligned}$$

Now we need to find $G(t,\rho,\theta)$ such that

$$I_0'(\rho)\partial_t G+I_1(t,\rho,\theta)-[I_1](\rho,\theta)=0.$$

Thus it is easy to get that

$$G(t,\rho,\theta)=\int_0^t\frac{1}{I_0'(\rho)}([I_1](\rho,\theta)-I_1(s,\rho,\theta))ds.$$

By Lemma 4.1, we have

$$|\partial_\rho^i\partial_t^j\partial_\theta^k G(t,\rho,\theta)|\le C\,\rho^{-i+\frac{1}{2}},\quad i+j\le 5,\quad k=0,1. \tag{4.6}$$

In particular, $|\partial_\rho\partial_t G(t,\rho,\theta)|\le C\,\rho^{-\frac{1}{2}}$. Then we can solve $t=t(\hat{\tau},\rho,\theta)$ from $\hat{\tau}=t+\frac{\partial G}{\partial\rho}(t,\rho,\theta)$. Moreover, we obtain

$$|\partial_\rho^i\partial_{\hat{\tau}}^j\partial_\theta^k t|\le C\,\rho^{-i-\frac{1}{2}},\quad i+j\le 5,\quad k=0,1. \tag{4.7}$$

By (4.6) and (4.7), one can easily obtain the estimate of the new perturbation $\mathcal{H}_1$. □

After the canonical transformation, we observe that the new perturbation term $\mathcal{H}_1$ is still not small. Therefore we must perform another transformation in the similar way.

Lemma 4.4. *For ρ large enough, there exists a canonical transformation*

$$\Psi_2:(\tau,\mu)\mapsto(\hat{\tau},\rho),$$

which transforms (4.5) into a new Hamiltonian

$$\mathcal{J}(\tau,\mu,\theta)=\mathcal{J}_0(\mu,\theta)+\mathcal{J}_1(\tau,\mu,\theta), \tag{4.8}$$

where

$$\mathcal{J}_0(\mu,\theta)=I_0(\mu)+[I_1](\mu,\theta)+[\mathcal{H}_1](\mu,\theta),$$

and the new perturbation $\mathcal{J}_1$ satisfies

$$|\partial_\mu^i\partial_\tau^j\mathcal{J}_1(\tau,\mu,\theta)|\le C\,\mu^{-i}\left(1+\frac{\mu}{(h-V(x)+1)^3}\right),\qquad x>0,$$
$$|\partial_\mu^i\partial_\tau^j\mathcal{J}_1(\tau,\mu,\theta)|\le C\,\mu^{-i}\left(1+\mu^{\frac{1}{2}}\cdot\frac{|V'(x)|+1+x}{(h-V(x)+1)^3}\right),\qquad x<0,$$

where $i+j\le 5$, and $h=h(I(\Psi_1\circ\Psi_2(\tau,\mu),\theta)), x=x(\theta,I(\Psi_1\circ\Psi_2(\tau,\mu),\theta))$ are determined by the mappings $\Psi_i(i=0,1,2)$.

Now system (4.1) is changed into the following form

$$\begin{cases}\dfrac{d\tau}{d\theta}=\partial_\mu\mathcal{J}_0(\mu,\theta)+\partial_\mu\mathcal{J}_1(\tau,\mu,\theta),\\ \dfrac{d\mu}{d\theta}=-\partial_\tau\mathcal{J}_1(\tau,\mu,\theta).\end{cases} \tag{4.9}$$

Define the transformation Ψ_3 by

$$\lambda=\int_0^1\partial_\mu\mathcal{J}_0(\mu,s)ds,\ \ \tau=\tau,\ \ \theta=\theta, \tag{4.10}$$

then system (4.9) is of the form

$$\begin{cases}\dfrac{d\lambda}{d\theta}=f_1(\tau,\lambda,\theta),\\ \dfrac{d\tau}{d\theta}=\lambda+f_{21}(\tau,\lambda,\theta)+f_{22}(\tau,\lambda,\theta),\end{cases} \tag{4.11}$$

where

$$f_1(\tau,\lambda,\theta)=-\partial_\tau\mathcal{J}_1(\tau,\mu,\theta)\int_0^1\partial_\mu^2\mathcal{J}_0(\mu,s)ds,$$

$$f_{21}(\tau,\lambda,\theta)=\partial_\mu\mathcal{J}_0(\mu,\theta)-\int_0^1\partial_\mu\mathcal{J}_0(\mu,s)ds,$$

$$f_{22}(\tau,\lambda,\theta)=\partial_\mu\mathcal{J}_1(\tau,\mu,\theta).$$

By Lemma 3.2 and Lemma 4.4, it follows that

$$\begin{aligned}
|\partial_\lambda^i \partial_\tau^j f_1(\tau,\lambda,\theta)| &\le C\,\lambda^{-i}\left(\lambda^{-1}+\frac{\lambda}{(h-V(x)+1)^3}\right), && x>0,\\
|\partial_\lambda^i \partial_\tau^j f_1(\tau,\lambda,\theta)| &\le C\,\lambda^{-i}\left(\lambda^{-1}+\frac{|V'(x)|+1+x}{(h-V(x)+1)^3}\right), && x<0,\\
|\partial_\lambda^i \partial_\tau^j f_{21}(\tau,\lambda,\theta)| &\le C\,\lambda^{-i+1},\\
|\partial_\lambda^i \partial_\tau^j f_{22}(\tau,\lambda,\theta)| &\le C\,\lambda^{-i}\left(\lambda^{-2}+\frac{1}{(h-V(x)+1)^3}\right), && x>0,\\
|\partial_\lambda^i \partial_\tau^j f_{22}(\tau,\lambda,\theta)| &\le C\,\lambda^{-i}\left(\lambda^{-2}+\lambda^{-1}\frac{|V'(x)|+1+x}{(h-V(x)+1)^3}\right), && x<0,
\end{aligned}$$

where

$$\begin{aligned}
x = x(\tau,\lambda,\theta) &:= x(\theta, I(\Psi_1\circ\Psi_2\circ\Psi_3^{-1}(\lambda,\tau),\theta)),\\
h = h(\tau,\lambda,\theta) &:= h(I(\Psi_1\circ\Psi_2\circ\Psi_3^{-1}(\lambda,\tau),\theta)).
\end{aligned}$$

Let $(\lambda(\theta,\tau_0,\lambda_0),\tau(\theta,\tau_0,\lambda_0))$ be the solution of (4.11) with the initial value (τ_0,λ_0) at $\theta=0$. Integrating (4.11) from 0 to θ yields that

$$\begin{aligned}
\lambda(\theta,\tau_0,\lambda_0) &= \lambda_0 + E_1(\theta,\tau_0,\lambda_0),\\
\tau(\theta,\tau_0,\lambda_0) &= \tau_0 + \lambda_0\theta + E_2(\theta,\tau_0,\lambda_0),
\end{aligned}$$

where

$$\begin{aligned}
E_1(\theta,\tau_0,\lambda_0) &= \int_0^\theta f_1(\tau(\theta,\tau_0,\lambda_0),\lambda(\theta,\tau_0,\lambda_0),\theta)d\theta,\\
E_2(\theta,\tau_0,\lambda_0) &= \int_0^\theta \Big[E_1(\theta,\tau_0,\lambda_0) + f_{21}(\tau(\theta,\tau_0,\lambda_0),\lambda(\theta,\tau_0,\lambda_0),\theta)\\
&\qquad + f_{22}(\tau(\theta,\tau_0,\lambda_0),\lambda(\theta,\tau_0,\lambda_0),\theta)\Big]d\theta.
\end{aligned}$$

In order to obtain the desired estimates on $E_1(1,\tau_0,\lambda_0), E_2(1,\tau_0,\lambda_0)$, we first prove the following lemma.

Lemma 4.5. *For λ_0 large enough, the following inequalities*

$$\int\limits_{\left[0,\frac{T_-(h)}{2T_0(h)}\right]\cup\left[1-\frac{T_-(h)}{2T_0(h)},1\right]} \frac{|V'(x)|}{(h-V(x)+1)^3}d\theta \le C\,\lambda_0^{-1},$$

$$\int_0^1 \frac{d\theta}{(h-V(x)+1)^3} \le C\,\lambda_0^{-1-\frac{1}{2}}$$

hold, where C is independent of λ_0, and

$$\begin{aligned}
x = x(\theta,\tau_0,\lambda_0) &:= x(\tau(\theta,\tau_0,\lambda_0),\lambda(\theta,\tau_0,\lambda_0),\theta),\\
h = h(\theta,\tau_0,\lambda_0) &:= h(\tau(\theta,\tau_0,\lambda_0),\lambda(\theta,\tau_0,\lambda_0),\theta).
\end{aligned}$$

Proof. The second inequality had been proved in [19]. Now we are going to show that the first inequality holds. We first observe that

$$H(x,y,t) = \Phi(y) + \frac{1}{(1-x^2)^\gamma} - 1 - xp(t),$$

which implies that

$$\left|\frac{1}{(1-x^2)^\gamma}\right| \le C\,H, \qquad \frac{1}{1-x^2} \le C\,H^{\frac{1}{\gamma}}.$$

We also remark that λ is corresponding to the initial Hamiltonian H, and the definitions of $\Psi_i(i=0,1,2,3)$ implies that $H \sim O(\lambda_0^2)$ as $\lambda_0 \to +\infty$. Hence there exist two constants c and C such that

$$c\,\lambda_0^2 \le \Phi(y) + V(x) \le C\,\lambda_0^2,$$

which leads to

$$|V'(x)| \le C\,H^{1+\frac{1}{\gamma}} \le C\,\lambda_0^{2+\frac{2}{\gamma}}.$$

Let $E = \left\{\theta \in [0, \frac{T_-(h)}{2T_0(h)}] \cup [1-\frac{T_-(h)}{2T_0(h)}, 1] : |y(\theta, \tau_0, \lambda_0)| \le \lambda_0^{1+\delta}\right\}$ with $\frac{2}{3\gamma} < \delta < 1$. Let us first consider $\theta \notin E$. In this case we have

$$\frac{1}{(1+y^2)^{\frac{3}{2}}} \le C\,\lambda_0^{-3(1+\delta)} \le C\,\lambda_0^{-3-3\delta}.$$

Then it follows that

$$\int\limits_{([0,\frac{T_-(h)}{2T_0(h)}]\cup[1-\frac{T_-(h)}{2T_0(h)},1])\setminus E} \frac{|V'(x)|}{(1+y^2)^{\frac{3}{2}}} d\theta \le C\,\lambda_0^{-3-3\delta} \cdot \lambda_0^{2+\frac{2}{\gamma}} \le C\,\lambda_0^{-1-3\delta+\frac{2}{\gamma}} \le C\,\lambda_0^{-1}.$$

Now we suppose $\theta \in E$. In this case, $y \le \lambda_0^{1+\delta}$, which implies that $|V(x)| \ge c\,H$. Since $V(x) \to 0$ as $x \to 0$, then $|x| \ge c$. Thus, the following inequalities

$$\left|\frac{1}{1-x^2}\right| \ge c\,H^{\frac{1}{\gamma}}, \qquad |V'(x)| \ge c\,H^{1+\frac{1}{\gamma}} \ge c\,\lambda_0^{2+\frac{2}{\gamma}}$$

are true. Moreover, we have

$$\left|\frac{dy}{dt}\right| = |V'(x) - p(t)| \ge c\,\lambda_0^{2+\frac{2}{\gamma}}, \qquad \left|\frac{dt}{d\theta}\right| \ge c\,\lambda_0,$$

which leads to

$$\left|\frac{dy}{d\theta}\right| \ge c\,\lambda_0^{3+\frac{2}{\gamma}}.$$

Therefore, we obtain

$$\begin{aligned}\int_E \frac{|V'(x)|}{(1+y^2)^{\frac{3}{2}}}d\theta &\le C\,\lambda_0^{2+\frac{2}{\gamma}} \int_0^{\lambda_0^{1+\delta}} \frac{\frac{d\theta}{dy}}{(1+y^2)^{\frac{3}{2}}}dy \\ &\le C\,\lambda_0^{2+\frac{2}{\gamma}}\cdot\lambda_0^{-3-\frac{2}{\gamma}} \int_0^{\lambda_0^{1+\delta}} \frac{dy}{(1+y^2)^{\frac{3}{2}}} \\ &\le C\,\lambda_0^{-1},\end{aligned}$$

and thus the proof of this lemma is completed. □

Now we are in a position to prove Theorem 1.1.

The proof of Theorem 1.1. For λ_0 large enough, integrating (4.11) from $\theta = 0$ to $\theta = 1$, we obtain that the Poincaré mapping $\mathcal{P}$ is of the form

$$\mathcal{P} : \tau_1 = \tau_0 + \lambda_0 + E_1(1, \tau_0, \lambda_0),\ \ \lambda_1 = \lambda_0 + E_2(1, \tau_0, \lambda_0),$$

where, for $i + j \le 4$ and $\lambda_0 \gg 1$,

$$|\partial^i_{\lambda_0}\partial^j_{\tau_0} E_k| \le C\,\lambda_0^{-i-\frac{1}{2}},\ \ (k = 1, 2).$$

Now the Poincaré mapping $\mathcal{P}$ satisfies all assumptions of Moser's twist theorem, which implies that $\mathcal{P}$ possesses a sequence of invariant closed curves tending to infinity. Then all solutions of (2.1) are bounded and there exists a corresponding sequence of invariant closed curves of the Poincaré mapping, that is,

$$-1 < x(t) < +\infty,\ \ \sup_{t\in\mathbb{R}} |y(t)| < +\infty.$$

We notice that $y(t) = \dfrac{x'(t)}{\sqrt{1-(x'(t))^2}}$, then $\sup_{t\in\mathbb{R}} |x'(t)| < 1$. Moreover, a corresponding sequence of invariant closed curves implies the existence of infinitely many quasiperiodic solutions. □

Remark 4.6. The above results are sufficient for any continuous potential $V : (a, +\infty) \to \mathbb{R}^+, a \in \mathbb{R}$, which satisfies the following assumptions:

(V_1) there exists $b > a$ such that $V(b) = 0 = V'(b)$;
(V_2) $V \in \mathcal{C}^{10}((a, +\infty)\backslash\{b\})$, $\lim_{x\to a^-} V(x) = +\infty$;
(V_3) the function $W(x) = \frac{V(x)}{V'(x)}$, is of class $\mathcal{C}^9((a, +\infty)\backslash\{b\})$, and

$$\lim_{x\to a^-, x\to b^\pm} |W^{(j)}(x)| < +\infty,\quad j = 0, \cdots, 9;$$

(V_4) there exists $C > 0$ such that

$$(x-a)^{k-1}\left|\frac{d^k}{dx^k}V(x)\right| \le C(|V'(x)| + (x-a)^{k-1}),\ \text{for}\ x > a\ \text{and}\ k = 1, \cdots, 9;$$

(V_5) $V(x) = \frac{1}{2}n^2x^2 + r(x)$, where r is of the form

$$r(x) = O(x),\quad r'(x) = c + O\left(\frac{1}{x^2}\right),\quad r^{(k)}(x) = O\left(\frac{1}{x^{1+k}}\right),\quad k \ge 2,\ x \to +\infty;$$

(V_6) for $c\,H \le |V(x)| \le C\,H$, we can get

$$c\,H^B \le |V'(x)| \le C\,H^A, (B \le A),$$

where H is a large enough variable.

It is easy to see that the function $V(x) = \frac{1}{2}x_+^2 + \dfrac{1}{(1-x_-^2)^\gamma} - 1$ satisfies (V_1)–(V_6). Moreover, we remark that, as far as Theorem 1.1 is concerned, the restriction $\gamma \in \mathbb{N}$, can be weakened; indeed, it is sufficient to require $\gamma \in \mathbb{R} \setminus \mathbb{N}$ with $\gamma > 8$, where $\gamma > 8$ is to ensure that $W(x)$ satisfies (V_3).

5. Proof of Lemma 3.2

In this section we shall prove Lemma 3.2. The estimate techniques are similar to that in [19] but there are significant differences. Because of the complexity of higher order derivatives, we employ some results in [4].

Case (i) $k = 0$. We first observe that the positive function $y = y(x,h) > 0$ determined by the closed curve Γ_h : $\Phi(y) + V(x) = h$ is convex, here h is a parameter. In fact, differentiating the above equation with respect to x twice yields that

$$\Phi'(y) \cdot \frac{dy}{dx} + V'(x) = 0,$$
$$\Phi''(y) \cdot \left(\frac{dy}{dx}\right)^2 + \Phi'(y) \cdot \frac{d^2y}{d^2x} + V''(x) = 0,$$

which implies that

$$\frac{d^2y}{d^2x} = -\Phi'(y)^{-1}\left(\Phi''(y) \cdot \left(\frac{dy}{dx}\right)^2 + V''(x)\right) < 0, \quad y > 0,$$

since $\Phi'(y), \Phi''(y), V''(x)$ are positive for $-1 < x < +\infty$, $y > 0$. Hence, for h large enough, we get

$$c\,h^{\frac{3}{2}} \le \beta_h \cdot y(0,h) \le I_+(h) \le 2\beta_h \cdot y(0,h) \le 2 \cdot \sqrt{2h} \cdot \sqrt{h^2+2h} \le C\,h^{\frac{3}{2}}.$$

Also, since $\alpha_h \nearrow 1$ as $h \to +\infty$, then for h large enough, $1 > \alpha_h \ge \frac{1}{2}$, and

$$c\,h \le \alpha_h \cdot y(0,h) \le I_-(h) \le 2\alpha_h \cdot y(0,h) \le 2\sqrt{h^2+2h} \le C\,h.$$

Therefore, for h large enough, we have

$$c\,h^{\frac{3}{2}} \le I_0(h) = I_+(h) + I_-(h) \le C\,h^{\frac{3}{2}}.$$

Case (ii) $k = 1$. We recall that

$$\frac{I_+(h)}{2} = \int_0^{\beta_h} \varphi(h - V(\xi))d\xi. \tag{5.1}$$

Choosing $\dfrac{V(\xi)}{h} = s \in [0,1]$ as the new variable of integration, we have

$$\frac{I_+(h)}{2} = \int_0^1 \varphi(h-sh)\frac{h}{V'(\xi)}ds,$$

where ξ is a function of s and h, that is, $\xi = \sqrt{2sh}$.

Differentiating this equality by h, noticing that $\frac{\partial \xi}{\partial h} = \frac{W(\xi)}{h}$, we get

$$\begin{aligned}\frac{I'_+(h)}{2} &= \frac{1}{h}\int_0^1 \left[\frac{(h-sh+1)(h-sh)}{\varphi^2(h-sh)} + W'(\xi)\right]\varphi(h-sh)\cdot\frac{h}{V'(\xi)}ds \\ &= \frac{1}{h}\int_0^{\beta_h} \left[\frac{(h-V(\xi)+1)(h-V(\xi))}{\varphi^2(h-V(\xi))} + W'(\xi)\right]\varphi(h-V(\xi))d\xi,\end{aligned}$$

where $W(x) = \frac{V(x)}{V'(x)}$. Since $1 \leq \frac{(h-V(x)+1)(h-V(x))}{\varphi^2(h-V(x))} + W'(x) \leq \frac{3}{2}$ for $x > 0$, then

$$c\,h^{-1}I_+(h) \leq I'_+(h) \leq C\,h^{-1}I_+(h).$$

Similarly, for $-1 < x < 0$,

$$\frac{I_-(h)}{2} = \int_{-\alpha_h}^0 \varphi(h-V(\xi))d\xi,$$

and

$$\frac{I'_-(h)}{2} = \frac{1}{h}\int_{-\alpha_h}^0 \left[\frac{(h-V(\xi)+1)(h-V(\xi))}{\varphi^2(h-V(\xi))} + W'(\xi)\right]\varphi(h-V(\xi))d\xi.$$

By Lemma 3.1, we have

$$\left|\frac{(h-V(x)+1)(h-V(x))}{\varphi^2(h-V(x))} + W'(x)\right| \leq \gamma + 1,$$

which implies that

$$|I'_-(h)| \leq C\,h^{-1}I_-(h).$$

Therefore, it follows that

$$\begin{aligned}c\,h^{-1}I_0(h) &\leq c\,h^{\frac{1}{2}} \leq c\,h^{\frac{1}{2}} - c \leq c\,h^{-1}I_+(h) - c\,h^{-1}I_-(h) \leq I'_0(h) \\ &= I'_+(h) + I'_-(h) \leq C\,h^{-1}I_+(h) + C\,h^{-1}I_-(h) \leq C\,h^{-1}I_0(h).\end{aligned}$$

Case (iii) $k = 2$. Similar to case (ii), we first obtain that

$$\frac{I'_+(h)}{2} = \frac{1}{h}\int_0^{\beta_h} \frac{h-V(\xi)+1}{\varphi(h-V(\xi))}d\xi,$$

and

$$\frac{I''_+(h)}{2} = \frac{1}{h}\int_0^{\beta_h}\left[-\frac{1}{(h-V(\xi)+1)(h-V(\xi)+2)} + W'(\xi)\right]\frac{h-V(\xi)+1}{\varphi(h-V(\xi))}d\xi.$$

Since $-\dfrac{1}{(h-V(x)+1)(h-V(x)+2)} + W'(x) \le \frac{1}{2}$ for $x > 0$, then

$$I''_+(h) \le C\,h^{-1}I'_+(h) \le C\,h^{-2}I_+(h).$$

In order to obtain the inverse inequality

$$c\,h^{-2}I_+(h) \le I''_+(h),$$

we need another form of $I''_+(h)$. Now let $Q(y) = \dfrac{\Phi(y)}{\Phi'(y)}$. Since $Q(y) \to +\infty$ for $y \to +\infty$, then there exists a constant d_0 such that $|Q(y)| \ge \frac{5}{8}$ for all $|y| > d_0$. We notice that $\dfrac{I_+(h)}{2}$ can be written in another form

$$\frac{I_+(h)}{2} = \int_0^{\gamma_h}\sqrt{2(h-\Phi(\xi))}d\xi,$$

where $\Phi(\gamma_h) = h$ with $\gamma_h > 0$. Differentiating the above equality by h yields that

$$\frac{I'_+(h)}{2} = \int_0^{\gamma_h}\frac{1}{\sqrt{2(h-\Phi(\xi))}}d\xi.$$

Choosing $\dfrac{\Phi(\xi)}{h} = s \in [0,1]$ as the new variable of integration, we obtain

$$\frac{I'_+(h)}{2} = \int_0^1 \frac{1}{\sqrt{2(h-sh)}}\cdot\frac{h}{\Phi'(\xi)}d\xi.$$

Differentiating the above equality by h again yields that

$$\begin{aligned}
\frac{I''_+(h)}{2} &= \frac{1}{h}\int_0^1\left(Q'(\xi)-\frac{1}{2}\right)\frac{1}{\sqrt{2(h-sh)}}\frac{h}{\Phi'(\xi)}d\xi \\
&= \frac{1}{h}\int_0^{\gamma_h}\left(Q'(\xi)-\frac{1}{2}\right)\frac{1}{\sqrt{2(h-\Phi(\xi))}}d\xi \\
&= \frac{1}{h}\left(\int_0^{d_0}+\int_{d_0}^{\gamma_h}\right)\left(Q'(\xi)-\frac{1}{2}\right)\frac{1}{\sqrt{2(h-\Phi(\xi))}}d\xi \\
&\ge \frac{1}{h}\left(\frac{1}{8}\int_{d_0}^{\gamma_h}\frac{1}{\sqrt{2(h-\Phi(\xi))}}d\xi - \frac{1}{2}\int_0^{d_0}\frac{1}{\sqrt{2(h-\Phi(\xi))}}d\xi\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{h}\left(\frac{1}{8}\int_0^{\gamma_h} \frac{1}{\sqrt{2(h-\Phi(\xi))}}d\xi - \frac{5}{8}\int_0^{d_0} \frac{1}{\sqrt{2(h-\Phi(\xi))}}d\xi\right) \\
&\geq \frac{1}{h}\left(\frac{1}{8}I_+'(h) - \frac{5}{8}\frac{1}{\sqrt{2(h-\Phi(d_0))}}\cdot d_0\right) \\
&\geq ch^{-2}I_+(h),
\end{aligned}
$$

which implies that

$$c\,h^{-2}I_+(h) \leq I_+''(h) \leq C\,h^{-2}I_+(h).$$

On the other hand, we have

$$\frac{I_-'(h)}{2} = \frac{1}{h}\int_{-\alpha_h}^{0} \frac{h-V(\xi)+1}{\varphi(h-V(\xi))}d\xi,$$

and

$$\frac{I_-''(h)}{2} = \frac{1}{h}\int_{-\alpha_h}^{0}\left[-\frac{1}{(h-V(\xi)+1)(h-V(\xi)+2)} + W'(\xi)\right]\frac{h-V(\xi)+1}{\varphi(h-V(\xi))}d\xi.$$

By Lemma 3.1, $\left|-\frac{1}{(h-V(x)+1)(h-V(x)+2)} + W'(x)\right| \leq \gamma + \frac{1}{2}$ for $-1 < x < 0$, which implies that

$$|I_-''(h)| \leq C\,h^{-2}I_-(h).$$

Therefore, we get

$$c\,h^{-2}I(h) \leq I_0''(h) \leq C\,h^{-2}I(h).$$

Case (iv) $3 \leq k \leq 9$. First we state some important claims.

Claim 5.1. *Suppose the function $u = u(\xi, h - V(\xi))$ is smooth in ξ and h, and define an operator $\mathfrak{B}$ by*

$$\mathfrak{B}(u) = \frac{1}{h}\left[u\cdot\frac{(h-V(\xi)+1)(h-V(\xi))}{\varphi^2(h-V(\xi))} + u_h h + (u\cdot W(\xi))_\xi\right],$$

then

$$
\begin{aligned}
\partial_h^k \int_0^{\beta_h} u\cdot\varphi(h-V(\xi))d\xi &= \int_0^{\beta_h} \underbrace{\mathfrak{B}\circ\cdots\circ\mathfrak{B}}_{k}(u)\varphi(h-V(\xi))d\xi, \\
\partial_h^k \int_{-\alpha_h}^{0} u\cdot\varphi(h-V(\xi))d\xi &= \int_{-\alpha_h}^{0} \underbrace{\mathfrak{B}\circ\cdots\circ\mathfrak{B}}_{k}(u)\varphi(h-V(\xi))d\xi.
\end{aligned}
$$

Claim 5.2. *For all integer $k \leq 9$, it follows that*

$$\left|\int_0^{\beta_h} \underbrace{\mathfrak{B} \circ \cdots \circ \mathfrak{B}}_{k}(1)\varphi(h - V(\xi))d\xi\right| \leq C\,h^{-k} \int_0^{\beta_h} \varphi(h - V(\xi))d\xi,$$

$$\left|\int_{-\alpha_h}^{0} \underbrace{\mathfrak{B} \circ \cdots \circ \mathfrak{B}}_{k}(1)\varphi(h - V(\xi))d\xi\right| \leq C\,h^{-k} \int_{-\alpha_h}^{0} \varphi(h - V(\xi))d\xi.$$

From Claim 5.1, we obtain

$$\frac{I_+^{(k)}(h)}{2} = \int_0^{\beta_h} \underbrace{\mathfrak{B} \circ \cdots \circ \mathfrak{B}}_{k}(1)\varphi(h - V(\xi))d\xi,$$

which together with Claim 5.2 implies that

$$|I_+^{(k)}(h)| \leq C\,h^{-k} \int_0^{\beta_h} \varphi(h - V(\xi))d\xi \leq C\,h^{-k} I_+(h).$$

Similarly, one can prove

$$|I_-^{(k)}(h)| \leq C\,h^{-k} I_-(h).$$

Therefore, it follows that

$$|I^{(k)}(h)| \leq C\,h^{-k} I(h). \qquad \square$$

Proof of Claim 5.1. We only prove the case $k = 1$, and the case $k > 1$ can be proved by iteration step by step. Let

$$J(h) = \int_0^{\beta_h} u(\xi, h - V(\xi))\varphi(h - V(\xi))d\xi.$$

Using the same change of the variable of integration, we have

$$J(h) = \int_0^1 u(\xi, h - sh)\varphi(h - sh) \cdot \frac{h}{V'(\xi)} ds.$$

Differentiating the above equality by h, we obtain

$$\begin{aligned} J'(h) &= \frac{1}{h}\int_0^1 \Big[u_1(\xi, h - sh)W(\xi) + u_2(\xi, h - sh)(h - sh) + u(\xi, h - sh) \\ &\quad \frac{(h - sh + 1)(h - sh)}{\varphi^2(h - sh)} + u(\xi, h - sh)W'(\xi)\Big]\varphi(h - sh) \cdot \frac{h}{V'(\xi)} ds \\ &= \frac{1}{h}\int_0^{\beta_h} \Big[u_1\big(\xi, h - V(\xi)\big)W(\xi) + u_2\big(\xi, h - V(\xi)\big)(h - V(\xi)) \end{aligned}$$

$$+ u\big(\xi, h - V(\xi)\big) \cdot \frac{(h - V(\xi) + 1)(h - V(\xi))}{\varphi^2(h - V(\xi))}$$
$$+ u\big(\xi, h - V(\xi)\big)W'(\xi)\Big]\varphi\big(h - V(\xi)\big)d\xi,$$

where $u_1(\xi, h - V(\xi))$ denotes the derivative of $u(\xi, h - V(\xi))$ with respect to the first variable ξ, and $u_2(\xi, h - V(\xi))$ represents the derivative of $u(\xi, h - V(\xi))$ with respect to the second variable $h - V(\xi)$. In this way, we have

$$u_h(\xi, h - V(\xi)) = u_2(\xi, h - V(\xi)),$$
$$u_\xi(\xi, h - V(\xi)) = u_1(\xi, h - V(\xi)) + u_2(\xi, h - V(\xi))(-V'(\xi)),$$

and

$$(u(\xi, h - V(\xi))W(\xi))_\xi$$
$$= u_1(\xi, h - V(\xi))W(\xi) + u_2(\xi, h - V(\xi))(-V(\xi)) + u(\xi, h - V(\xi))W'(\xi).$$

Then we have

$$J'(h)$$
$$= \frac{1}{h}\int_0^{\beta_h} \Big[u \cdot \frac{(h - V(\xi) + 1)(h - V(\xi))}{\varphi^2(h - V(\xi))} + u_h h + (u \cdot W(\xi))_\xi\Big]\varphi(h - V(\xi))d\xi$$
$$= \int_0^{\beta_h} \mathfrak{B}(u)d\xi.$$

Using the same method, one can prove the second inequality. This completes the proof of Claim 5.1. $\square$

To prove Claim 5.2, we need to estimate

$$\underbrace{\mathfrak{B} \circ \cdots \circ \mathfrak{B}}_{k}(1).$$

Because of the complexity of $\mathfrak{B}(u)$, we will estimate $\mathfrak{B}(u)$ by $\mathfrak{B}_1(u)$, where

$$\mathfrak{B}_1(u) = \frac{1}{h}\Big[\frac{1}{2}u + u_h h + (u \cdot W(\xi))_\xi\Big].$$

Now we introduce the properties of $\mathfrak{B}_1(u)$. Let $v(\xi, h) = \dfrac{(h - V(\xi) + 1)(h - V(\xi))}{\varphi^2(h - V(\xi))}$.

Claim 5.3. *The operator $\mathfrak{B}_1$ has the following properties:*

(1) $\mathfrak{B}_1$ *is the linear operator;*

(2) $\mathfrak{B}_1\left(\frac{1}{h}u\left(v - \frac{1}{2}\right)\right) = \frac{1}{h}\left(v - \frac{1}{2}\right)\mathfrak{B}_1(u) - \dfrac{1}{h^2}u\dfrac{(h - V(\xi))^4}{\varphi^4(h - V(\xi))}$;

(3) $\mathfrak{B}_1\left(\dfrac{1}{h^2}u\dfrac{(h - V(\xi))^4}{\varphi^4(h - V(\xi))}\right) = \dfrac{1}{h^2}\dfrac{(h - V(\xi))^4}{\varphi^4(h - V(\xi))}\mathfrak{B}_1(u) - \dfrac{1}{h^3}u\dfrac{2(h - V(\xi))^6}{\varphi^6(h - V(\xi))}$,

$\mathfrak{B}_1\left(\dfrac{1}{h^n}u\dfrac{(h - V(\xi))^{2n}}{\varphi^{2n}(h - V(\xi))}\right) = \dfrac{1}{h^n}\dfrac{(h - V(\xi))^{2n}}{\varphi^{2n}(h - V(\xi))}\mathfrak{B}_1(u) - \dfrac{1}{h^{n+1}}u\dfrac{n(h - V(\xi))^{2n+2}}{\varphi^{2n+2}(h - V(\xi))}$;

(4) $\underbrace{\mathfrak{B}_1 \circ \cdots \circ \mathfrak{B}_1}_{k}(1) = \frac{1}{h^k} P_k(\xi)$, *where*

$$\begin{cases} P_{k+1}(\xi) = \left(\frac{1}{2} - k\right) P_k(\xi) + (WP_k)'(\xi), k \geq 1 \\ P_1(\xi) = \frac{1}{2} + W'(\xi). \end{cases}$$

From this one can know that $P_k(\xi)$ is bounded.

Proof. It is easy to see that properties (1)–(3) hold, and property (4) can be found in [4]. □

Proof of Claim 5.2. For $k = 1$, we have

$$\begin{aligned} \left| \int_0^{\beta_h} \mathfrak{B}(1)\varphi(h - V(\xi))d\xi \right| &\leq \left| \int_0^{\beta_h} (\mathfrak{B}(1) - \mathfrak{B}_1(1))\varphi d\xi \right| + \left| \int_0^{\beta_h} \mathfrak{B}_1(1)\varphi d\xi \right| \\ &\leq \int_0^{\beta_h} \frac{1}{h} \left| v - \frac{1}{2} \right| \varphi d\xi + \int_0^{\beta_h} \frac{1}{h} |P_1(\xi)| \varphi d\xi \\ &\leq C\, h^{-1} \int_0^{\beta_h} \varphi d\xi. \end{aligned}$$

If $k = 2$, on one hand, we have

$$\begin{aligned} &\left| \int_0^{\beta_h} \mathfrak{B} \circ \mathfrak{B}(1)\varphi d\xi \right| \\ &\leq \left| \int_0^{\beta_h} (\mathfrak{B} \circ \mathfrak{B}(1) - \mathfrak{B}_1 \circ \mathfrak{B}(1))\varphi d\xi \right| \\ &\quad + \left| \int_0^{\beta_h} (\mathfrak{B}_1 \circ \mathfrak{B}(1) - \mathfrak{B}_1 \circ \mathfrak{B}_1(1))\varphi d\xi \right| + \left| \int_0^{\beta_h} \mathfrak{B}_1 \circ \mathfrak{B}_1(1)\varphi d\xi \right|. \end{aligned}$$

On the other hand, the first term in the above inequality can be estimated in the following way

$$\begin{aligned} \left| \int_0^{\beta_h} (\mathfrak{B} \circ \mathfrak{B}(1) - \mathfrak{B}_1 \circ \mathfrak{B}(1))\varphi d\xi \right| &= \left| \int_0^{\beta_h} (\mathfrak{B} - \mathfrak{B}_1) \circ \mathfrak{B}(1)\varphi d\xi \right| \\ &= \left| \int_0^{\beta_h} \frac{1}{h} \mathfrak{B}(1) \left(v - \frac{1}{2} \right) \varphi d\xi \right| \\ &\leq C \left| \int_0^{\beta_h} \frac{1}{h} \mathfrak{B}(1)\varphi d\xi \right| \\ &\leq C\, \frac{1}{h^2} \int_0^{\beta_h} \varphi d\xi. \end{aligned}$$

The second term has the following estimate

$$\begin{aligned}
&\left|\int_0^{\beta_h}(\mathfrak{B}_1\circ\mathfrak{B}(1)-\mathfrak{B}_1\circ\mathfrak{B}_1(1))\varphi d\xi\right| \\
&=\left|\int_0^{\beta_h}\mathfrak{B}_1(\mathfrak{B}(1)-\mathfrak{B}_1(1))\varphi d\xi\right| \\
&=\left|\int_0^{\beta_h}\mathfrak{B}_1\left(\frac{1}{h}\left(v-\frac{1}{2}\right)\right)\varphi d\xi\right| \\
&=\left|\int_0^{\beta_h}\frac{1}{h}\left(v-\frac{1}{2}\right)\mathfrak{B}_1(1))\varphi d\xi-\int_0^{\beta_h}\frac{1}{h^2}\frac{(h-V(\xi))^4}{\varphi^4(h-V(\xi))}\varphi d\xi\right| \\
&\le C\,\frac{1}{h^2}\int_0^{\beta_h}\varphi d\xi.
\end{aligned}$$

Finally, we obtain

$$\left|\int_0^{\beta_h}\mathfrak{B}_1\circ\mathfrak{B}_1(1)\varphi d\xi\right|=\left|\int_0^{\beta_h}\frac{1}{h^2}P_2(\xi)\varphi d\xi\right|\le C\,\frac{1}{h^2}\int_0^{\beta_h}\varphi d\xi.$$

Therefore, we have

$$\left|\int_0^{\beta_h}\mathfrak{B}\circ\mathfrak{B}(1)\varphi d\xi\right|\le C\,\frac{1}{h^2}\int_0^{\beta_h}\varphi d\xi.$$

Similarly, one can prove the claim for $3\le k\le 9$. Moreover, we have

$$\left|\int_{-\alpha_h}^{0}\underbrace{\mathfrak{B}\circ\cdots\circ\mathfrak{B}}_{k}(1)\varphi(h-V(\xi))d\xi\right|\le C\,h^{-k}\int_{-\alpha_h}^{0}\varphi(h-V(\xi))d\xi.$$

6. Proof of Lemma 3.5

In this section, we will prove Lemma 3.5. Let us consider the case $x>0, y>0$ first.

Case (i) $k=1$. By the definition of θ, we have

$$\left(\theta-\frac{T_-(h)}{2T_0(h)}\right)\cdot T_0(h)=\int_0^x\frac{h-V(\xi)+1}{\varphi(h-V(\xi))}d\xi.$$

Differentiating the above equality on both sides by I yields that

$$\left(\theta\cdot T_0'(h)-\frac{T_-'(h)}{2}\right)\cdot h_I=\partial_I\left(\int_0^x\frac{h-V(\xi)+1}{\varphi(h-V(\xi))}d\xi\right). \tag{6.1}$$

Now we compute the right hand of the above equality. Let $V(\xi) = sh$, under the new variable s of integration, we get

$$\int_0^x \frac{h - V(\xi) + 1}{\varphi(h - V(\xi))} d\xi = \int_0^\sigma \frac{h - sh + 1}{\varphi(h - sh)} \cdot \frac{h}{V'(\xi)} ds,$$

where $\sigma = \frac{V(x)}{h}$. Then the direct computation leads to

$$\begin{aligned}
&\partial_I \left(\int_0^\sigma \frac{h - sh + 1}{\varphi(h - sh)} \cdot \frac{h}{V'(\xi)} ds \right) \\
&= \frac{h - V(x) + 1}{\varphi(h - V(x))} \cdot \frac{h}{V'(x)} \cdot \sigma_I \\
&\quad + \frac{h_I}{h} \int_0^\sigma \left(W'(\xi) - \frac{1}{(h - sh + 1)(h - sh + 2)} \right) \frac{h - sh + 1}{\varphi(h - sh)} \cdot \frac{h}{V'(\xi)} ds \\
&= \frac{h - V(x) + 1}{\varphi(h - V(x))} \cdot \left(x_I - \frac{h_I}{h} W(x) \right) \\
&\quad + \frac{h_I}{h} \int_0^x \left(W'(\xi) - \frac{1}{(h - V(\xi) + 1)(h - V(\xi) + 2)} \right) \frac{h - V(\xi) + 1}{\varphi(h - V(\xi))} d\xi.
\end{aligned}$$

From (6.1), we obtain

$$\begin{aligned}
x_I = \frac{h_I}{h} W(x) + \frac{h_I}{h} \cdot \frac{\varphi(h - V(x))}{h - V(x) + 1} \Bigg[&\frac{T_-(h) \cdot hT_0'(h)}{2T_0(h)} - \frac{hT_-'(h)}{2} \\
&+ \int_0^x \left(\frac{T_0'(h)h}{T_0(h)} - W'(\xi) + \frac{1}{(h - V(\xi) + 1)(h - V(\xi) + 2)} \right) \frac{h - V(\xi) + 1}{\varphi(h - V(\xi))} d\xi \Bigg].
\end{aligned}$$

Let

$$\begin{aligned}
a(h) &= \frac{T_-(h) \cdot hT_0'(h)}{2T_0(h)} - \frac{hT_-'(h)}{2}, \\
f(\xi, I) &= \frac{T_0'(h)h}{T_0(h)} - W'(\xi) + \frac{1}{(h - V(\xi) + 1)(h - V(\xi) + 2)}, \\
d\breve{\xi} &= \frac{h - V(\xi) + 1}{\varphi(h - V(\xi))} d\xi, \\
F(x, I) &= a(h) + \int_0^x f(\xi, I) d\breve{\xi}.
\end{aligned}$$

Then

$$x_I = \frac{h_I}{h} W(x) + \frac{h_I}{h} \cdot \frac{\varphi(h - V(x))}{h - V(x) + 1} \cdot F(x, I). \tag{6.2}$$

From Lemma 3.2 and Lemma 3.3, it is easy to prove that $|I^k \frac{d^k a(h)}{dI^k}| \le C$. Since $|W(x)| \le Cx \le CI^{\frac{1}{3}}$, $\left|\frac{\varphi(h-V(x))}{h-V(x)+1} \cdot \frac{h-V(\xi)+1}{\varphi(h-V(\xi))}\right| \le 1$ for $0 < \xi < x$, and $|f(\xi, I)| \le C$, then $|x_I| \le C\, I^{-1+\frac{1}{3}}$, which completes the proof of $k = 1$.

Case (ii) $k = 2$. Differentiating (6.2) on both sides with respect to I leads to

$$x_{II} = \partial_I\left(\frac{h_I}{h}\right)W(x) + \frac{h_I}{h}W'(x)\cdot x_I + \partial_I\left(\frac{h_I}{h}\right)\cdot\frac{\varphi(h-V(x))}{h-V(x)+1}\cdot F(x,I)$$
$$+\frac{h_I}{h}\cdot\partial_I\left(\frac{\varphi(h-V(x))}{h-V(x)+1}\right)\cdot F(x,I) + \frac{h_I}{h}\cdot\frac{\varphi(h-V(x))}{h-V(x)+1}\cdot\partial_I F(x,I).$$

Since $\left|\frac{\varphi(h-V(x))}{h-V(x)+1}\cdot F(x,I)\right| \le C\, I^{\frac{1}{3}}$, then we only need to prove that

$$\left|\partial_I\left(\frac{\varphi(h-V(x))}{h-V(x)+1}\right)\cdot F(x,I)\right| \le C\, I^{-1+\frac{1}{3}}, \tag{6.3}$$

$$\left|\frac{\varphi(h-V(x))}{h-V(x)+1}\cdot\partial_I F(x,I)\right| \le C\, I^{-1+\frac{1}{3}}. \tag{6.4}$$

The proof of (6.3). We notice that

$$\partial_I\left(\frac{\varphi(h-V(x))}{h-V(x)+1}\right)$$
$$=\frac{h_I}{h}\left[\frac{\varphi(h-V(x))}{(h-V(x)+1)^2(h-V(x)+2)} - \frac{V'(x)F(x,I)}{(h-V(x)+1)^3}\right],$$

it suffices to prove that

$$\left|\frac{V'(x)}{h-V(x)+1}\cdot F(x,I)\right| \le C\,\frac{\varphi(h-V(x))}{h-V(x)+1}, \tag{6.5}$$

which is equivalent to

$$-C\,\frac{\varphi(h-V(x))}{V'(x)} \le F(x,I) \le C\,\frac{\varphi(h-V(x))}{V'(x)}. \tag{6.6}$$

Now we first prove that $F(\beta_h, I) = 0$, that is,

$$\frac{T_-(h)\cdot hT_0'(h)}{2T_0(h)} - \frac{hT_-'(h)}{2} + \int_0^{\beta_h} f(\xi,I)d\check{\xi} = 0.$$

By direct computation, the above equality is equivalent to

$$\int_0^{\beta_h}\left(\frac{T_+'(h)h}{T_+(h)} - W'(\xi) + \frac{1}{(h-V(\xi)+1)(h-V(\xi)+2)}\right)\frac{h-V(\xi)+1}{\varphi(h-V(\xi))}d\xi = 0.$$

Differentiating the equality

$$\frac{T_+(h)}{2} = \int_0^{\beta_h}\frac{h-V(\xi)+1}{\varphi(h-V(\xi))}d\xi$$

on both sides by I yields that

$$\frac{T_+'(h)}{2} \cdot h_I = \partial_I \left(\int_0^{\beta_h} \frac{h - V(\xi) + 1}{\varphi(h - V(\xi))} d\xi \right).$$

At the same time,

$$\frac{T_+'(h)}{2} \cdot h_I = \frac{h_I}{h} \cdot \frac{T_+'(h) h}{T_+(h)} \int_0^{\beta_h} \frac{h - V(\xi) + 1}{\varphi(h - V(\xi))} d\xi,$$

and

$$\begin{aligned} &\partial_I \left(\int_0^{\beta_h} \frac{h - V(\xi) + 1}{\varphi(h - V(\xi))} d\xi \right) \\ &= \frac{h_I}{h} \int_0^{\beta_h} \left(W'(\xi) - \frac{1}{(h - V(\xi) + 1)(h - V(\xi) + 2)} \right) \frac{h - V(\xi) + 1}{\varphi(h - V(\xi))} d\xi, \end{aligned}$$

hence,

$$\begin{aligned} &\frac{h_I}{h} \cdot \frac{T_+'(h) h}{T_+(h)} \int_0^{\beta_h} \frac{h - V(\xi) + 1}{\varphi(h - V(\xi))} d\xi \\ &= \frac{h_I}{h} \int_0^{\beta_h} \left(W'(\xi) - \frac{1}{(h - V(\xi) + 1)(h - V(\xi) + 2)} \right) \frac{h - V(\xi) + 1}{\varphi(h - V(\xi))} d\xi, \end{aligned}$$

which implies that $F(\beta_h, I) = 0$.

Let

$$G(x, I) = \frac{\varphi(h - V(x))}{V'(x)}.$$

Since $G(\beta_h, I) = F(\beta_h, I) = 0$, in order to show (6.6), it is enough to prove that

$$C\, \partial_x G(x, I) \le \partial_x F(x, I) \le -C\, \partial_x G(x, I). \tag{6.7}$$

Indeed, $\partial_x F(x, I), \partial_x G(x, I)$ have the following expressions:

$$\begin{aligned} &|\partial_x F(x, I)| \\ &= \left| \left(\frac{T_0'(h) h}{T_0(h)} - W'(x) + \frac{1}{(h - V(x) + 1)(h - V(x) + 2)} \right) \frac{h - V(x) + 1}{\varphi(h - V(x))} \right| \\ &\le C\, \frac{h - V(x) + 1}{\varphi(h - V(x))}, \end{aligned}$$

and

$$\partial_x G(x, I) = -\left[1 + \frac{V''(x)}{(V'(x))^2} \cdot \frac{\varphi^2(h - V(x))}{h - V(x) + 1} \right] \frac{h - V(x) + 1}{\varphi(h - V(x))}.$$

Meanwhile, we have

$$\frac{V''(x)}{(V'(x))^2}\cdot\frac{\varphi^2(h-V(x))}{h-V(x)+1}>0,$$

and

$$1+\frac{V''(x)}{(V'(x))^2}\cdot\frac{\varphi^2(h-V(x))}{h-V(x)+1}\geq 1.$$

Thus, (6.7) holds, and so is (6.5).

Moreover, we obtain

$$\left|\partial_I\left(\frac{\varphi(h-V(x))}{h-V(x)+1}\right)\right|\leq C\,I^{-1}\frac{\varphi(h-V(x))}{h-V(x)+1},$$

which implies that (6.3) holds.

We shall prove (6.4) by the following claim.

Claim 6.1. *Let $f(\xi,I)$ be a real function continuously differentiable on ξ and I and define an operator $\mathfrak{L}$ by*

$$\mathfrak{L}(f)=\frac{h_I}{h}\left[(f(\xi,I)\cdot W(\xi))_\xi-f(\xi,I)\frac{1}{(h-V(\xi)+1)(h-V(\xi)+2)}\right]+f_I,$$

then

$$\partial_I\left(\int_0^x f(\xi,I)d\breve{\xi}\right)=\int_0^x\mathfrak{L}\big(f(\xi,I)\big)d\breve{\xi}+\frac{h_I}{h}f(x,I)F(x,I),$$

where $F(x,I)$ is defined previously.

Proof. Let $V(\xi)=sh$, under the new variable s of integration, we have

$$\int_0^x f(\xi,I)d\breve{\xi}=\int_0^\sigma f(\xi(s,h),I)\frac{h-sh+1}{\varphi(h-sh)}\cdot\frac{h}{V'(\xi)}ds,$$

where $\sigma=\frac{V(x)}{h}$. Therefore,

$$\begin{aligned}&\partial_I\left(\int_0^x f(\xi,I)d\breve{\xi}\right)\\&=\partial_I\left(\int_0^\sigma f(\xi(s,h),I)\frac{h-sh+1}{\varphi(h-sh)}\cdot\frac{h}{V'(\xi)}ds\right)\\&=\int_0^\sigma\left[\left(f_\xi\cdot\frac{W(\xi)}{h}\cdot h_I+f_I\right)\frac{h-sh+1}{\varphi(h-sh)}\cdot\frac{h}{V'(\xi)}-f\cdot\frac{(1-s)h_I}{\varphi^3(h-sh)}\cdot\frac{h}{V'(\xi)}\right.\\&\quad\left.+f\cdot\frac{h-sh+1}{\varphi(h-sh)}\cdot\frac{h_I}{h}\cdot\frac{h}{V'(\xi)}\cdot W'(\xi)\right]ds+\sigma_I\cdot f(x,I)\cdot\frac{h-V(x)+1}{\varphi(h-V(x))}\cdot\frac{h}{V'(x)}\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\sigma} \left[\frac{h_I}{h} \left((f(\xi, I)W(\xi))_\xi - f(\xi, I) \cdot \frac{1}{(h - sh + 1)(h - sh + 2)} \right) \right. \\
&\quad \left. + f_I \right] \frac{h - sh + 1}{\varphi(h - sh)} \cdot \frac{h}{V'(\xi)} ds + \left(x_I - \frac{h_I}{h} W(x) \right) \cdot f(x, I) \cdot \frac{h - V(x) + 1}{\varphi(h - V(x))} \\
&= \int_0^{x} \left[\frac{h_I}{h} \left((f(\xi, I)W(\xi))_\xi - f(\xi, I) \cdot \frac{1}{(h - V(\xi) + 1)(h - V(\xi) + 2)} \right) \right. \\
&\quad \left. + f_I \right] \frac{h - V(\xi) + 1}{\varphi(h - V(\xi))} d\xi + \frac{h_I}{h} f(x, I) F(x, I),
\end{aligned}
$$

and thus we complete the proof of Claim 6.1. □

The proof of (6.4). From Claim 6.1, we have

$$
\begin{aligned}
&\frac{\varphi(h - V(x))}{h - V(x) + 1} \cdot \partial_I F(x, I) \\
&= \frac{\varphi(h - V(x))}{h - V(x) + 1} \left(\frac{da(h)}{dI} + \int_0^x \mathfrak{L}(f(\xi, I)) d\breve{\xi} + \frac{h_I}{h} f(x, I) F(x, I) \right).
\end{aligned}
$$

Therefore in order to prove (6.4), we only need to prove that

$$
|\mathfrak{L}(f)| \leq C\, I^{-1}. \tag{6.8}
$$

Since

$$
\begin{aligned}
&\mathfrak{L}(f(\xi, I)) \\
&= \left(\frac{T_0'(h)h}{T_0(h)} \right)_I + \frac{h_I}{h} \left[-W''(\xi)W(\xi) - (W'(\xi))^2 + \frac{T_0'(h)h}{T_0(h)} \cdot W'(\xi) \right. \\
&\quad + \left(\frac{-(h - V(\xi))}{(h - V(\xi) + 1)^2} - \frac{-(h - V(\xi))}{(h - V(\xi) + 2)^2} \right) - \frac{1}{(h - V(\xi) + 1)(h - V(\xi) + 2)} \\
&\quad \left. \left(\frac{T_0'(h)h}{T_0(h)} + \frac{1}{(h - V(\xi) + 1)(h - V(\xi) + 2)} - 2W'(\xi) \right) \right],
\end{aligned}
$$

we rewrite the above equality as the following form

$$
\mathfrak{L}(f) = \frac{h_I}{h} \sum e(\xi) g(I) p(h - V(\xi)) + q(I),
$$

where $e(\xi) = c$, $|g^{(k)}(I)| \leq C\, I^{-k}$, $|p^{(k)}(y)| \leq C\, \frac{1}{y+1}$, $y \geq 0$, and $|q^{(k)}(I)| \leq C\, I^{-(k+1)}$.

By the definition of $\mathfrak{L}(f)$, it is easy to verify that (6.8) is valid, which implied that the proof of (6.4) is completed. Therefore, we have

$$
|x_{II}| \leq C\, I^{-2+\frac{1}{3}}.
$$

Let

$$h_1(I) = \partial_I\left(\frac{h_I}{h}\right), \quad f_1(x,I) = \frac{x}{2} + \frac{\varphi(h-V(x))}{h-V(x)+1}\cdot F(x,I),$$
$$h_2(I) = \frac{h_I}{h}, \quad f_2(x,I) = \frac{x_I}{2} + \frac{\varphi(h-V(x))}{h-V(x)+1}\cdot \partial_I F(x,I),$$
$$h_3(I) = \left(\frac{h_I}{h}\right)^2, \quad f_3(x,I) = \left[\frac{\varphi(h-V(x))}{(h-V(x)+1)^2(h-V(x)+2)} - \frac{V'(x)}{(h-V(x)+1)^3}\cdot F(x,I)\right]F(x,I).$$

Then

$$x_{II} = h_1(I)f_1(x,I) + h_2(I)f_2(x,I) + h_3(I)f_3(x,I), \tag{6.9}$$

and

$$|h_1(I)| \le C\,I^{-2},\ |h_2(I)| \le C\,I^{-1},\ |h_3(I)| \le C\,I^{-2},$$
$$|f_1(x,I)| \le C\,I^{\frac{1}{3}},\ |f_2(x,I)| \le C\,I^{-1+\frac{1}{3}},\ |f_3(x,I)| \le C\,I^{\frac{1}{3}}.$$

Case (iii) $k=3$. Differentiating (6.9) on both sides by I, we have

$$x_{III} = \partial_I(h_1(I))f_1(x,I) + \partial_I(h_2(I))f_2(x,I) + \partial_I(h_3(I))f_3(x,I) + h_1(I)\partial_I f_1(x,I) + h_2(I)\partial_I f_2(x,I) + h_3(I)\partial_I f_3(x,I),$$

which implies that it suffices to prove that

$$|\partial_I f_1(x,I)| \le C\,I^{-1+\frac{1}{3}}, \tag{6.10}$$
$$|\partial_I f_2(x,I)| \le C\,I^{-2+\frac{1}{3}}, \tag{6.11}$$
$$|\partial_I f_3(x,I)| \le C\,I^{-1+\frac{1}{3}}. \tag{6.12}$$

The proof of (6.10). According to

$$\partial_I f_1(x,I) = \frac{1}{2}x_I + \partial_I\left(\frac{\varphi(h-V(x))}{h-V(x)+1}\right)\cdot F(x,I) + \frac{\varphi(h-V(x))}{h-V(x)+1}\cdot\partial_I F(x,I),$$

it is easy to show (6.10).

The proof of (6.11) By

$$\partial_I f_2(x,I) = \frac{1}{2}x_{II} + \partial_I\left(\frac{\varphi(h-V(x))}{h-V(x)+1}\right)\cdot \partial_I F(x,I) + \frac{\varphi(h-V(x))}{h-V(x)+1}\cdot\partial_{II} F(x,I),$$

it is enough to prove that

$$\left|\frac{\varphi(h-V(x))}{h-V(x)+1}\cdot\partial_{II}F(x,I)\right| \le C\,I^{-2+\frac{1}{3}}.$$

By Claim 6.1, we have

$$\partial_{II} F(x,I) = \frac{d^2 a(h)}{dI^2} + \int_0^x \mathfrak{L}^2(f) d\check{\xi} + \frac{h_I}{h}\mathfrak{L}(f) \cdot F(x,I) + \partial_I \left(\frac{h_I}{h} \cdot f(x,I) \cdot F(x,I) \right),$$

which implies that we only need to prove that

$$|\mathfrak{L}^2(f)| \le C\, I^{-2}, \tag{6.13}$$

$$|\partial_I f(x,I)| \le C\, I^{-1}. \tag{6.14}$$

Firstly, the definition of $\mathfrak{L}(f)$ yields that

$$\mathfrak{L}^2(f) = \left[\left(\frac{h_I}{h}\right)^2 + \partial_I \left(\frac{h_I}{h}\right) \right] \sum e(\xi) g(I) p(h - V(\xi)) \\ + \frac{h_I}{h} \sum e(\xi) m(I) p(h - V(\xi)) + q(I),$$

where

$$|e(\xi)| = c, |g^{(k)}(I)| \le C\, I^{-k}, |p^{(k)}(y)| \le C\, \frac{1}{y+1}, y \ge 0, \\ |m^{(k)}(I)| \le C\, I^{-(k+1)}, |q^{(k)}(I)| \le C\, I^{-(k+2)}.$$

Hence, (6.13) is true.

From the definition of $f(x,I)$, we have

$$\begin{aligned} &\partial_I f(x,I) \\ &= \partial_I \left(\frac{T_0'(h)h}{T_0(h)} - W'(x) + \frac{1}{(h - V(x) + 1)(h - V(x) + 2)} \right) \\ &= \partial_I \left(\frac{T_0'(h)h}{T_0(h)} \right) + \left[\frac{-1}{(h - V(x) + 1)^2} - \frac{-1}{(h - V(x) + 2)^2} \right] \partial_I (h - V(x)). \end{aligned}$$

Since

$$\partial_I (h - V(x)) = \frac{h_I}{h} \left[(h - V(x)) - \frac{\varphi(h - V(x))}{h - V(x) + 1} \cdot V'(x) \cdot F(x,I) \right],$$

it is easy to see that

$$\left| \frac{1}{h - V(x) + 1} \partial_I (h - V(x)) \right| \le C\, I^{-1}.$$

Thus, (6.14) holds, which implies that the proof of (6.11) is completed.

The proof of (6.12). Differentiating $f_3(x,I)$ with respect to I yields that

$$\begin{aligned} &\partial_I f_3(x,I) \\ &= \left[\partial_I \left(\frac{\varphi(h - V(x))}{h - V(x) + 1} \right) \cdot \frac{1}{(h - V(x) + 1)(h - V(x) + 2)} \right. \\ &\quad \left. + \frac{\varphi(h - V(x))}{h - V(x) + 1} \left(\frac{-1}{(h - V(x) + 1)^2} - \frac{-1}{(h - V(x) + 2)^2} \right) \partial_I (h - V(x)) \right. \end{aligned}$$

$$- V''(x) \cdot x_I \cdot \frac{F(x,I)}{(h-V(x)+1)^3} - V'(x) \cdot \partial_I \left(\frac{1}{(h-V(x)+1)^3} \right) \cdot F(x,I)$$
$$\left. - \frac{V'(x)}{(h-V(x)+1)^3} \partial_I F(x,I) \right] F(x,I)$$
$$+ \left[\frac{\varphi(h-V(x))}{(h-V(x)+1)^2(h-V(x)+2)} - \frac{V'(x)}{(h-V(x)+1)^3} F(x,I) \right] \partial_I F(x,I),$$

thus (6.12) can be proved easily according to the above expression.

Case (iv) $k = 4$. It suffices to prove the following inequalities

$$|\partial_{II} f_1(x,I)| \le C\, I^{-2+\frac{1}{3}},\ |\partial_{II} f_2(x,I)| \le C\, I^{-3+\frac{1}{3}},\ |\partial_{II} f_3(x,I)| \le C\, I^{-2+\frac{1}{3}},$$

which implies that we only need to prove that

$$\left| \partial_{II} \left(\frac{\varphi(h-V(x))}{h-V(x)+1} \right) \right| \le C\, I^{-2} \frac{\varphi(h-V(x))}{h-V(x)+1}, \tag{6.15}$$
$$\left| \frac{\varphi(h-V(x))}{h-V(x)+1} \cdot \partial_{III} F(x,I) \right| \le C\, I^{-3+\frac{1}{3}}, \tag{6.16}$$
$$\left| \frac{1}{\sqrt{1+y^2}} \partial_{II}(h-V(x)) \right| \le C\, I^{-2}. \tag{6.17}$$

The proof of (6.15). From the expression of $\partial_{II} \left(\frac{\varphi(h-V(x))}{h-V(x)+1} \right)$, we only need to prove that

$$\left| V'(x) \frac{1}{h-V(x)+1} \partial_I F(x,I) \right| \le C\, I^{-1} \frac{\varphi(h-V(x))}{h-V(x)+1}.$$

From Claim 6.1, it follows that

$$\partial_I F(x,I) = \frac{da(h)}{dI} + \int_0^x \mathfrak{L}(f(\xi,I)) d\breve{\xi} + \frac{h_I}{h} f(x,I) F(x,I).$$

From (6.5), it suffices to prove that

$$\left| V'(x) \frac{1}{h-V(x)+1} \left(\frac{da(h)}{dI} + \int_0^x \mathfrak{L}(f(\xi,I)) d\breve{\xi} \right) \right| \le C\, I^{-1} \frac{\varphi(h-V(x))}{h-V(x)+1}, \tag{6.18}$$

which is equivalent to

$$C\, I^{-1} \frac{\varphi(h-V(x))}{V'(x)} \le \frac{da(h)}{dI} + \int_0^x \mathfrak{L}(f(\xi,I)) d\breve{\xi} \le -C\, I^{-1} \frac{\varphi(h-V(x))}{V'(x)}.$$

Since $F(\beta_h, I) = a(h) + \int_0^{\beta_h} f(\xi,I) d\breve{\xi} = 0$, differentiating this equality by I yields that $\frac{da(h)}{dI} + \int_0^{\beta_h} \mathfrak{L}(f(\xi,I)) d\breve{\xi} = 0$, then we need to prove that

$$C\,I^{-1}\partial_x\left(\frac{\varphi(h-V(x))}{V'(x)}\right)\le \mathfrak{L}(f)\cdot\frac{\varphi(h-V(x))}{h-V(x)+1}\le -C\,I^{-1}\partial_x\left(\frac{\varphi(h-V(x))}{V'(x)}\right).$$

By a direct computation, we have

$$\partial_x\left(\frac{\varphi(h-V(x))}{V'(x)}\right)=\left(-1-\frac{\varphi^2(h-V(x))}{h-V(x)+1}\cdot\frac{V''(x)}{(V'(x))^2}\right)\cdot\frac{\varphi(h-V(x))}{h-V(x)+1}.$$

Since $\frac{\varphi^2(h-V(x))}{h-V(x)+1}\cdot\frac{V''(x)}{(V'(x))^2}>0$, then

$$1+\frac{\varphi^2(h-V(x))}{h-V(x)+1}\cdot\frac{V''(x)}{(V'(x))^2}\ge 1.$$

Therefore, $|\mathfrak{L}(f)|\le C\,I^{-1}$ implies that the estimate (6.18) holds, which leads to (6.15).

The proof of (6.16). Differentiating $\partial_{II}F(x,I)$ with respect to I yields that

$$\begin{aligned}\partial_{III}F(x,I)=&\frac{d^3a(h)}{dI^3}+\int_0^x\mathfrak{L}^3(f)d\breve{\xi}+\frac{h_I}{h}\mathfrak{L}^2(f)\cdot F(x,I)\\&+\partial_I\left(\frac{h_I}{h}\mathfrak{L}(f)\cdot F(x,I)\right)+\partial_{II}\left(\frac{h_I}{h}\cdot f(x,I)\cdot F(x,I)\right),\end{aligned}$$

which implies that it is enough to verify that

$$\begin{aligned}&|\mathfrak{L}^3(f)|\le C\,I^{-3},\\&|\partial_I\mathfrak{L}(f)|\le C\,I^{-2},\\&|\partial_{II}f(x,I)|\le C\,I^{-2}.\end{aligned}$$

These inequalities and (6.17) can be proved in a similar way.

Case (v) $5\le k\le 7$. The proof does not contain any new difficulties, and we omit it here.

Next let us prove the inequality in Lemma 3.5 for $x<0$, $y>0$.

Case (i) $k=1$. The definition of θ yields that

$$\theta\cdot T_0(h)=\int_{-\alpha_h}^x\frac{h-V(\xi)+1}{\varphi(h-V(\xi))}d\xi,$$

then it follows that

$$x_I=\frac{h_I}{h}W(x)+\frac{h_I}{h}\cdot\frac{\varphi(h-V(x))}{h-V(x)+1}\cdot F(x,I),\tag{6.19}$$

where

$$\begin{aligned}f(\xi,I)&=\frac{T_0'(h)h}{T_0(h)}-W'(\xi)+\frac{1}{(h-V(\xi)+1)(h-V(\xi)+2)},\\d\breve{\xi}&=\frac{h-V(\xi)+1}{\varphi(h-V(\xi))}d\xi,\\F(x,I)&=\int_{-\alpha_h}^x f(\xi,I)d\breve{\xi}.\end{aligned}$$

Since $|W(x)| \leq C(1+x)$, and $|f(\xi, I)| \leq C$, it suffices to prove that

$$\left|\frac{\varphi(h-V(x))}{h-V(x)+1} \cdot F(x,I)\right| \leq C(\alpha_h + x). \tag{6.20}$$

The proof of (6.20). Let

$$M(x,I) = \frac{(\alpha_h + x)(h-V(x)+1)}{\varphi(h-V(x))},$$

then we need to prove that

$$-C\,M(x,I) \leq F(x,I) \leq C\,M(x,I).$$

Indeed, since $F(-\alpha_h, I) = M(-\alpha_h, I) = 0$, it is enough to prove that

$$-C\,\partial_x M(x,I) \leq \partial_x F(x,I) \leq C\,\partial_x M(x,I). \tag{6.21}$$

By a direct computation, we obtain

$$\begin{aligned}
&|\partial_x F(x,I)| \\
&= \left|\left(\frac{T_0'(h)h}{T_0(h)} - W'(x) + \frac{1}{(h-V(x)+1)(h-V(x)+2)}\right)\frac{h-V(x)+1}{\varphi(h-V(x))}\right| \\
&\leq C\,\frac{h-V(x)+1}{\varphi(h-V(x))},
\end{aligned}$$

and

$$\partial_x M(x,I) = \left[1 + \frac{(\alpha_h+x)V'(x)}{\varphi^2(h-V(x))(h-V(x)+1)}\right]\frac{h-V(x)+1}{\varphi(h-V(x))}.$$

Since $\left|\frac{(\alpha_h+x)V'(x)}{2(h-V(x))}\right| \leq \frac{1}{2}$, $\left|\frac{2(h-V(x))}{\varphi^2(h-V(x))(h-V(x)+1)}\right| \leq 1$, then

$$\left|\frac{(\alpha_h+x)V'(x)}{\varphi^2(h-V(x))(h-V(x)+1)}\right| \leq \frac{1}{2},$$

which implies that

$$\frac{3}{2} \geq 1 + \frac{(\alpha_h+x)V'(x)}{\varphi^2(h-V(x))(h-V(x)+1)} \geq \frac{1}{2}.$$

Therefore (6.21) holds, and as a consequence, (6.20) is valid. Thus, by (6.19) and (6.20), we obtain

$$|x_I| \leq C\,I^{-1}(1+x),$$

which completes the proof of this case.
Case (ii) $k=2$. Differentiating x_I with respect to I yields that

$$\begin{aligned}
x_{II} &= \partial_I(\frac{h_I}{h})W(x) + \frac{h_I}{h}W'(x)\cdot x_I + \partial_I(\frac{h_I}{h})\cdot\frac{\varphi(h-V(x))}{h-V(x)+1}\cdot F(x,I) \\
&\quad + \frac{h_I}{h}\cdot\partial_I(\frac{\varphi(h-V(x))}{h-V(x)+1})\cdot F(x,I) + \frac{h_I}{h}\cdot\frac{\varphi(h-V(x))}{h-V(x)+1}\cdot\partial_I F(x,I),
\end{aligned}$$

which implies that it is enough to prove that

$$\left|\partial_I\left(\frac{\varphi(h-V(x))}{h-V(x)+1}\right)\cdot F(x,I)\right|\le C\,I^{-1}(1+x), \tag{6.22}$$

$$\left|\frac{\varphi(h-V(x))}{h-V(x)+1}\cdot\partial_I F(x,I)\right|\le C\,I^{-1}(1+x). \tag{6.23}$$

The proof of (6.22). Since

$$\partial_I\left(\frac{\varphi(h-V(x))}{h-V(x)+1}\right)=\frac{h_I}{h}\left[\frac{\varphi(h-V(x))}{(h-V(x)+1)^2(h-V(x)+2)}-\frac{V'(x)F(x,I)}{(h-V(x)+1)^3}\right],$$

it is enough to verify that

$$\left|V'(x)\frac{1}{h-V(x)+1}F(x,I)\right|\le C\,\frac{\varphi(h-V(x))}{h-V(x)+1},$$

which can be proved similar to (6.20). Therefore, we obtain

$$\left|\partial_I\left(\frac{\varphi(h-V(x))}{h-V(x)+1}\right)\right|\le C\,I^{-1}\frac{\varphi(h-V(x))}{h-V(x)+1},$$

and (6.22) follows from the above discussions.

The proof of (6.23). Similarly, using the same method as in Claim 6.1, we have

$$\partial_I\left(\int_{-\alpha_h}^{x} f(\xi,I)d\breve{\xi}\right)=\int_{-\alpha_h}^{x}\mathfrak{L}\big(f(\xi,I)\big)d\breve{\xi}+\frac{h_I}{h}f(x,I)F(x,I),$$

where

$$\mathfrak{L}(f)=\frac{h_I}{h}\left[(f(\xi,I)\cdot W(\xi))_\xi-f(\xi,I)\frac{1}{(h-V(\xi)+1)(h-V(\xi)+2)}\right]+f_I,$$

and $F(x,I)$ is defined previously. From the above equality, we have

$$\frac{\varphi(h-V(x))}{h-V(x)+1}\cdot\partial_I F(x,I)=\frac{\varphi(h-V(x))}{h-V(x)+1}\left(\int_{-\alpha_h}^{x}\mathfrak{L}(f(\xi,I))d\breve{\xi}+\frac{h_I}{h}f(x,I)F(x,I)\right).$$

In order to prove (6.23), we need to state an important claim. The proof of this claim is similar to that of (6.20), we omit it here.

Claim 6.2. *Suppose that the function $g(\xi,I)$ is continuous and there is a constant C such that $|g(\xi,I)|\le C\,I^{-k}$, for some $k\in\mathbb{N}$. Then there exists a constant C such that, for $-\alpha_h\le x\le 0$,*

$$\left|\frac{\varphi(h-V(x))}{h-V(x)+1}\int_{-\alpha_h}^{x} g(\xi,I)d\breve{\xi}\right|\le C\,I^{-k}(\alpha_h+x).$$

By Claim 6.2, it is enough to prove that

$$|\mathfrak{L}(f)| \le C\, I^{-1}.$$

A direct computation leads to

$$\mathfrak{L}(f) = \frac{h_I}{h} \sum e(\xi) g(I) p(h - V(\xi)) + q(I),$$

where $|e^{(k)}(\xi)| \le C, |g^{(k)}(I)| \le C\, I^{-k}, |p^{(k)}(y)| \le C\, \frac{1}{y+1}, y \ge 0, |q^{(k)}(I)| \le C\, I^{-(k+1)}$. As a consequence, the above inequality holds, which leads to (6.23). Thus we have

$$|x_{II}| \le C\, I^{-2}(1+x).$$

Let

$$\begin{aligned}
h_1(I) &= \partial_I\left(\frac{h_I}{h}\right), \quad f_1(x,I) = W(x) + \frac{\varphi(h-V(x))}{h-V(x)+1} \cdot F(x,I),\\
h_2(I) &= \frac{h_I}{h}, \quad f_2(x,I) = W'(x)\cdot x_I + \frac{\varphi(h-V(x))}{h-V(x)+1} \cdot \partial_I F(x,I),\\
h_3(I) &= \left(\frac{h_I}{h}\right)^2, \quad f_3(x,I) = \left[\frac{\varphi(h-V(x))}{(h-V(x)+1)^2(h-V(x)+2)} - \frac{V'(x)F(x,I)}{(h-V(x)+1)^3}\right] F(x,I).
\end{aligned}$$

Then

$$x_{II} = h_1(I) f_1(x,I) + h_2(I) f_2(x,I) + h_3(I) f_3(x,I),$$

and

$$\begin{gathered}
|h_1(I)| \le C\, I^{-2},\ |h_2(I)| \le C\, I^{-1},\ |h_3(I)| \le C\, I^{-2},\\
|f_1(x,I)| \le C\,(1+x),\ |f_2(x,I)| \le C\, I^{-1}(1+x),\ |f_3(x,I)| \le C\,(1+x).
\end{gathered}$$

Case (iii) $k = 3$**.** Differentiating x_{II} by I leads to

$$\begin{aligned}
x_{III} = {}& \partial_I(h_1(I)) f_1(x,I) + \partial_I(h_2(I)) f_2(x,I) + \partial_I(h_3(I)) f_3(x,I)\\
&+ h_1(I)\partial_I f_1(x,I) + h_2(I)\partial_I f_2(x,I) + h_3(I)\partial_I f_3(x,I),
\end{aligned}$$

which implies that it suffices to verify that

$$\begin{aligned}
|\partial_I f_1(x,I)| &\le C\, I^{-1}(1+x),\\
|\partial_I f_2(x,I)| &\le C\, I^{-2}(1+x),\\
|\partial_I f_3(x,I)| &\le C\, I^{-1}(1+x).
\end{aligned}$$

In fact, we need to prove the following inequalities

$$\left|\frac{\varphi(h-V(x))}{h-V(x)+1} \cdot \partial_{II} F(x,I)\right| \le C\, I^{-2}(1+x), \tag{6.24}$$

$$\left|\frac{1}{h-V(x)+1}\partial_I(h-V(x))\right| \le C\, I^{-1}, \tag{6.25}$$

$$\left|\frac{V''(x)\cdot x_I}{(h-V(x)+1)^3}\cdot F(x,I)\right| \le C\,I^{-1}\left[\frac{\varphi(h-V(x))}{h-V(x)+1}+1+x\right]. \tag{6.26}$$

The proofs of (6.24) and (6.25) contain no new difficulties, here we only prove (6.26).

The proof of (6.26). By Lemma 3.1, $V(x)$ possesses the following property

$$(1+x)^{k-1}|V^{(k)}(x)| \le C\,\left[|V'(x)|+(1+x)^{k-1}\right],$$

which implies that we need to prove that

$$|F(x,I)| \le C.$$

From the definition of $F(x,I)$, we know

$$|F(x,I)| = \left|\int_{-\alpha_h}^{x} f(\xi,I)d\check{\xi}\right| \le C\left|\int_{-\alpha_h}^{x} d\check{\xi}\right| \le C\left|\int_{-\alpha_h}^{0} d\check{\xi}\right| = C\cdot\frac{T_-(h)}{2} \le C.$$

Therefore, we obtain

$$\begin{aligned}
&\left|\frac{V''(x)\cdot x_I}{(h-V(x)+1)^3}\cdot F(x,I)\right| \\
&\le C\,I^{-1}(1+x)V''(x)\left|\frac{F(x,I)}{(h-V(x)+1)^3}\right| \\
&\le C\,I^{-1}(|V'(x)|+(1+x))\left|\frac{F(x,I)}{(h-V(x)+1)^3}\right| \\
&\le C\,I^{-1}\left(\frac{\varphi(h-V(x))}{h-V(x)+1}+1+x\right),
\end{aligned}$$

which implies that (6.26) holds.

Case (iv) $k=4$. It is enough to verify that

$$\begin{aligned}
|\partial_{II}f_1(x,I)| &\le C\,I^{-2}(1+x),\\
|\partial_{II}f_2(x,I)| &\le C\,I^{-3}(1+x),\\
|\partial_{II}f_3(x,I)| &\le C\,I^{-2}(1+x),
\end{aligned}$$

which are guaranteed by the following inequalities

$$\left|\partial_{II}\left(\frac{\varphi(h-V(x))}{h-V(x)+1}\right)\right| \le C\,I^{-2}\left(\frac{\varphi(h-V(x))}{h-V(x)+1}+1+x\right), \tag{6.27}$$

$$|\partial_I F(x,I)| \le C\,I^{-1}, \tag{6.28}$$

$$\left|\frac{\varphi(h-V(x))}{h-V(x)+1}\cdot\partial_{III}F(x,I)\right| \le C\,I^{-3}(1+x), \tag{6.29}$$

$$\left|\frac{1}{h-V(x)+1}\partial_{II}(h-V(x))\right| \le C\,I^{-2}. \tag{6.30}$$

The proofs of (6.27), (6.29), (6.30) are similar to that for $x>0$, here we verify (6.28) merely.

The proof of (6.28). We remark that

$$\partial_I F(x,I) = \int\limits_{-\alpha_h}^{x} \mathfrak{L}(f(\xi,I))d\breve{\xi} + \frac{h_I}{h} f(x,I)F(x,I),$$

then it suffices to prove that

$$\left| \int\limits_{-\alpha_h}^{x} \mathfrak{L}(f(\xi,I))d\breve{\xi} \right| \leq C\, I^{-1}.$$

Indeed,

$$\begin{aligned}\left| \int\limits_{-\alpha_h}^{x} \mathfrak{L}(f(\xi,I))d\breve{\xi} \right| &\leq C\, I^{-1} \left| \int\limits_{-\alpha_h}^{x} d\breve{\xi} \right| \leq C\, I^{-1} \left| \int\limits_{-\alpha_h}^{0} d\breve{\xi} \right| \\ &= C\, I^{-1} \cdot \frac{T_-(h)}{2} \leq C\, I^{-1},\end{aligned}$$

which implies that (6.28) holds.

Case (v) $5 \leq k \leq 7$. The proof does not contain any new difficulties, and we omit it here.

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