



Tingley's problem on finite von Neumann algebras



Ryotaro Tanaka

Faculty of Mathematics, Kyushu University, Fukuoka, 819-0395, Japan

ARTICLE INFO

Article history:

Received 20 September 2016

Available online 15 February 2017

Submitted by M. Mathieu

Keywords:

Isometry

Face

Unitary group

Jordan *-isomorphism

ABSTRACT

In this paper, we give an affirmative answer to Tingley's problem between finite von Neumann algebras. To do this, we see that every norm closed proper face of the unit ball of a unital C^* -algebra can be represented as the intersection of a family of maximal proper faces.

© 2017 Elsevier Inc. All rights reserved.

1. Introduction

The study of preserver problems concerning linear isometries on function spaces or operator algebras has a long and rich history, and it has been provided many important theories. The Banach–Stone theorem is a typical and fundamental result in that research area which describes the structure of surjective linear isometries between the spaces of continuous functions on compact Hausdorff spaces. In 1951, Kadison [17] gave a non-commutative version of this theorem. It turned out that every surjective linear isometry between C^* -algebras is in fact a Jordan *-isomorphism followed by multiplication by a fixed unitary element. In other words, linear isomorphisms between C^* -algebras “preserving” the metric structure also preserves the Jordan structure. This reveals a strong connection among certain structures of C^* -algebras such as linear, or metric, or Jordan structures. There are some other settings for linear preserver problems; see, for example, [29].

Recently, the main stream of preserver problems has been changed into another direction, so-called non-linear preserver problems. In those settings, the linearity of mappings are no longer assumed. Nevertheless, we have many useful results such as the celebrated Mazur–Ulam theorem which states that each surjective isometry between two Banach spaces is always affine. Since affine operators are just translations of real-linear operators, as a consequence, the real-linear structure of Banach spaces is controlled by the metric structure of those spaces. In 1972, Mankiewicz [21] gave a localization of the Mazur–Ulam theorem by showing that every surjective isometry between open connected subsets of normed spaces uniquely extends to an

E-mail address: r-tanaka@math.kyushu-u.ac.jp.

affine isometry between the whole spaces. In this direction, so far, various settings for non-linear preserver problems have been considered [1,10,12–14,23,24].

As a consequence of Mankiewicz's theorem, every surjective isometry between the unit balls of two Banach spaces extends to a real-linear isometric isomorphism. Motivated by this fact, Tingley [33] proposed the following problem concerning a further localization of the Mazur–Ulam theorem. For a Banach space X , let S_X be the unit sphere of X .

Problem 1.1 (*Tingley's problem*). Let X, Y be Banach spaces, and let $T : S_X \rightarrow S_Y$ be a surjective isometry. Does there exist a real-linear isometric isomorphism $\hat{T} : X \rightarrow Y$ such that $\hat{T}|_{S_X} = T$?

This problem is also known as the isometric extension problem between spheres. Many papers were devoted to studying Tingley's problem; and the problem was solved positively for some classical real Banach spaces such as $\ell_p(\Gamma)$ [3,5,6], L_p [25–27] and $C(X)$ [4,28]; see also a survey of Ding [7]. However, Tingley's problem is still open even in the simplest case of $X = Y$ and $\dim X = 2$. The difficulties come from the lack of interior points and appropriate geometric or algebraic structures. To overcome these, recently, some interesting ideas were provided by several authors [8,29]; and Tingley's problem was solved affirmatively for the cases of finite dimensional polyhedral Banach spaces [16] and a special class of Banach spaces called generalized lush spaces [28].

On the other hand, the study of Tingley's problem on operator algebras was started by [31]. Usually, main tools for Tingley's problem come from Banach space geometry which provides many fundamental notions such as support functionals, slices and various extremal structures of convex subsets. However, in the case of operator algebras, we can also make use of C^* - or operator algebraic techniques such as polar decompositions, functional calculus for self-adjoint operators and so on. Tingley's problem on operator algebras provides an interesting element of the intersection of Banach space geometry and the theory of operator algebras. As in [32], Tingley's problem has an affirmative answer if both X and Y are finite dimensional C^* -algebras.

A key result for the methods described in the two previous papers [31,32] is a characterization of the form of isometries between the unitary groups of two von Neumann algebras given by Hatori and Molnár [15]. This can be viewed as a Mazur–Ulam type result; and since unitary groups of C^* -algebras are included in the sets of extreme points of unit balls, it can be also considered as a further localization of Tingley's problem. What is interesting is that, although the above mentioned result of Hatori and Molnár is heavily based on operator algebraic techniques (such as Kadison's result [18] and Stone's theorem), some suitable Banach space geometric arguments allow us to use it to solve Tingley's problem.

The purpose of this paper is to give an affirmative answer to Tingley's problem on finite von Neumann algebras. This generalizes the main results in [31,32] to a much wider class of operator algebras. For this, we first remark that every norm closed proper face of the unit ball of a unital C^* -algebra can be represented as the intersection of a family of maximal proper faces. Applying this result, we present a solution to Tingley's problem on finite von Neumann algebras.

2. Notation and preliminaries

Let X be a Banach space. Then B_X and S_X are the unit ball and unit sphere, respectively. The (norm) dual space of X is denoted by $X^\#$. An element $f \in S_{X^\#}$ is called a support functional for B_X if $f(x) = 1$ for some $x \in S_X$, in which case, f supports B_X at x . Now let C be a nonempty closed convex subset of a Banach space. A nonempty closed convex subset D of C is a *face* of (or *extremal* in) C if it has the following property: If $x, y \in C$ and $tx + (1 - t)y \in D$ for some $t \in (0, 1)$, then $x, y \in D$. If a singleton $\{x\}$ is a face of C , then x is called an *extreme point* of C . Another typical example of a face is an exposed face, that is, a face $f^{-1}(1) \cap B_X$, where f is a support functional for B_X . If an exposed face of B_X is a singleton $\{x\}$, then x is called an *exposed point* of B_X , and denoted by $x \in \exp(B_X)$.

In [31], as an invariant for spherical isometries, the notion of *proper extreme point* was introduced. (In fact, they are called proper “exposed” points in [31]). An element $x \in B_X$ is called a *proper extreme point* of B_X , in symbols $x \in \text{pext}(B_X)$, if

$$\{x\} = \bigcap \{C : \text{maximal convex subset of } S_X \text{ containing } x\}.$$

This notion plays an important role to studying Tingley’s problem.

Throughout this paper, the term “ C^* -algebra” means a unital C^* -algebra. Suppose that $\mathfrak{A}$ is a C^* -algebra. Then an element $\rho \in \mathfrak{A}^\#$ is said to be *positive* if $\rho(A) \geq 0$ for each $A \in \mathfrak{A}$ with $A \geq 0$. A *state* of $\mathfrak{A}$ is a positive linear functional ρ satisfying $\rho(I) = \|\rho\| = 1$. It is now clear that the set $\mathcal{S}(\mathfrak{A})$ of all states of $\mathfrak{A}$ is weak* compact subset of $B_{\mathfrak{A}^\#}$. The Krein–Milman theorem assures that the set $\mathcal{P}(\mathfrak{A}) := \text{ext}(\mathcal{S}(\mathfrak{A}))$ is nonempty. An element of $\mathcal{P}(\mathfrak{A})$ is called a *pure state*.

A *finite* von Neumann algebra is a von Neumann algebra that admits the center valued trace. Suppose that $\mathcal{R}$ is a finite von Neumann algebra. Then, for each $A \in \mathcal{R}$, there exists a unitary element $U \in \mathcal{R}$ such that $A = U|A|$. Consequently, if $A \in B_{\mathcal{R}}$ then $A = (U_1 + U_2)/2$ for some pair of unitaries $U_1, U_2 \in \mathcal{R}$. From this and the well-known characterization of extreme points of the unit ball of a C^* -algebra, in particular, $\text{ext}(B_{\mathcal{R}}) = \mathcal{U}$, where $\mathcal{U}$ is the unitary group of $\mathcal{R}$.

3. Facial structure

We begin the study of maximal faces with the following lemma which gives a simple but important fact. The proof is based on an easy application of Eidelheit’s separation theorem [22, Theorem 2.2.26]; see, for example, [31, Lemma 3.3].

Lemma 3.1. *Let X be a Banach space. Suppose that C is a maximal convex subset of the unit sphere S_X of X . Then C is an exposed face of B_X .*

As a consequence of the preceding lemma, we easily have the following.

Lemma 3.2. *Let X be a Banach space. Suppose that C is a convex subset of S_X . Then C is maximal as a convex subset of S_X if and only if it is maximal as a proper face of B_X .*

Proof. If C is maximal as a convex subset, then it is maximal as a proper face since each proper face of B_X is a convex subset of S_X . Conversely, suppose that C is maximal as a proper face of B_X . Since C is a convex subset of S_X , by a routine argument on Zorn’s lemma, there exists a maximal convex subset C_0 of S_X that contains C . However, by Lemma 3.1, C_0 is in fact a proper face of B_X ; and hence $C = C_0$ by the maximality of C as a proper face of B_X . $\square$

Thus a subset of B_X is a maximal convex subset of S_X if and only if it is a maximal proper face of B_X . In what follows, we use the term “maximal proper face” rather than “maximal convex subset”.

The facial structure of C^* -algebras was studied in detail by Akemann and Pedersen [2]. In particular, we make use of the duality between norm closed faces of the unit ball of a C^* -algebra and weak* closed faces of the unit ball of the dual of that algebra. For this, we introduce the notions of *facear* and *pre-facear*. Let $\mathfrak{A}$ be a C^* -algebra. For each $F \subset B_{\mathfrak{A}}$ and each $G \subset B_{\mathfrak{A}^\#}$, define their facear F^f and pre-facear G_f by

$$F^f = \{f \in B_{\mathfrak{A}^\#} : f(A) = 1 \text{ for each } A \in F\}$$

$$G_f = \{A \in B_{\mathfrak{A}} : f(A) = 1 \text{ for each } f \in G\},$$

respectively. Then it is clear that F^f and G_f are weak* and norm closed faces. Moreover, we have the following result of Akemann and Pedersen [2, Theorem 5.3].

Theorem 3.3 (Akemann and Pedersen [2]). *Let $\mathfrak{A}$ be a C^* -algebra. Then the mapping $F \rightarrow F^f$ is an order-reversing isomorphism from the set of norm closed faces of $B_{\mathfrak{A}}$ onto that of weak* closed faces of $B_{\mathfrak{A}^\#}$. Moreover, the inverse is given by $G \rightarrow G_f$.*

Using this duality theorem, we can show our main purpose of this section.

Corollary 3.4. *Let $\mathfrak{A}$ be a C^* -algebra, and let F be a norm closed proper face of $B_{\mathfrak{A}}$. Then F is the intersection of a family of maximal proper faces of $B_{\mathfrak{A}}$.*

Proof. We first note that, by the preceding theorem, if ρ is an extreme point (and hence is a minimal weak* closed face) of $B_{\mathfrak{A}^\#}$ then its exposed face $F(\rho) = \{\rho\}_f$ is a maximal proper face of $B_{\mathfrak{A}}$. Moreover, using the Krein–Milman theorem, we have

$$F = (F^f)_f = \bigcap_{\rho \in F^f} F(\rho) = \bigcap_{\rho \in \text{ext}(F^f)} F(\rho).$$

Since F^f is a face, it follows that $\text{ext}(F^f) \subset \text{ext}(B_{\mathfrak{A}^\#})$. This proves the corollary. $\square$

4. Application to Tingley's problem

We first recall the following result that plays a central role in the study of Tingley's problem; the proof can be found in [30, Lemma 3.5].

Lemma 4.1. *Let X, Y be Banach spaces, and let $T : S_X \rightarrow S_Y$ be a surjective isometry. Then C is a maximal proper face of S_X if and only if $T(C)$ is a maximal proper face of S_Y .*

Combining Lemma 4.1 with [31, Lemma 3.8] yields the following lemma.

Lemma 4.2. *Let X, Y be Banach spaces, and let $T : S_X \rightarrow S_Y$ be a surjective isometry. Then $T(\text{pext}(B_X)) = \text{pext}(B_Y)$.*

The following result provides a simple inclusion relation between $\text{pext}(B_X)$ and $\text{ext}(B_X)$ in a Banach space X . The proof can be found in [31, Proposition 3.4].

Proposition 4.3 ([31]). *Let X be Banach spaces. Then $\text{pext}(B_X) \subset \text{ext}(B_X)$.*

As an application of Corollary 3.4, we have the following result.

Proposition 4.4. *Let $\mathfrak{A}$ be a C^* -algebra. Then $\text{pext}(B_{\mathfrak{A}}) = \text{ext}(B_{\mathfrak{A}})$.*

Proof. By the preceding proposition, it is enough to show that $\text{pext}(B_{\mathfrak{A}}) \supset \text{ext}(B_{\mathfrak{A}})$. Let $A \in \text{ext}(B_{\mathfrak{A}})$. Then $\{A\}$ is a norm closed proper face of $B_{\mathfrak{A}}$; and hence, by Corollary 3.4, it is the intersection of a family of maximal proper faces. This means that $A \in \text{pext}(B_{\mathfrak{A}})$. The proof is complete. $\square$

Corollary 4.5. *Let $\mathfrak{A}$ be an abelian C^* -algebra. Then $\text{pext}(B_{\mathfrak{A}}) = \mathcal{U}$, where $\mathcal{U}$ is the unitary group of $\mathfrak{A}$.*

Corollary 4.6. *Let $\mathcal{R}$ be a finite von Neumann algebra. Then $\text{pext}(B_{\mathcal{R}}) = \mathcal{U}$, where $\mathcal{U}$ is the unitary group of $\mathcal{R}$.*

Remark 4.7. In [31], it was shown that $\text{pext}(B_X) \subset \exp(B_X)$ for separable Banach spaces X ; and a counterexample was given in the non-separable case. In this context, Theorem 3.4 provides a more natural example of a non-separable Banach space X with $\text{pext}(B_X) \not\subset \exp(B_X)$. Indeed, Grzaślewicz [11] showed that if $\mathcal{H}$ is not separable then $\exp(B_{\mathcal{B}(\mathcal{H})}) = \emptyset$, while our Proposition 4.4 guarantees that $\text{pext}(B_{\mathcal{B}(\mathcal{H})}) = \text{ext}(B_{\mathcal{B}(\mathcal{H})}) \neq \emptyset$.

Remark 4.8. By Proposition 4.4, we have $\text{pext}(B_{\mathcal{B}(\mathcal{H})}) = \mathcal{U}$ when $\dim \mathcal{H} < \infty$, where $\mathcal{U}$ is the unitary group of $\mathcal{B}(\mathcal{H})$; see also [31, Theorem 4.3]. However, if $\dim \mathcal{H} = \infty$, we cannot have the same conclusion since $\mathcal{U} \subsetneq \text{ext}(B_{\mathcal{B}(\mathcal{H})})$. (For example, consider isometries onto proper subspaces such as shift operators.)

As in the two previous papers [31,32], a key ingredient for our argument is the following theorem by Hatori and Molnár [15, Corollary 3].

Theorem 4.9 (Hatori and Molnár [15]). *Let $\mathcal{R}_j$ be a von Neumann algebra with the unitary group $\mathcal{U}_j$, $j = 1, 2$. Then the mapping $T : \mathcal{U}_1 \rightarrow \mathcal{U}_2$ is a surjective isometry if and only if there exist a central projection P in $\mathcal{R}_2$ and a Jordan $*$ -isomorphism $J : \mathcal{R}_1 \rightarrow \mathcal{R}_2$ such that*

$$T(U) = T(I)(PJ(U) + (I - P)J(U)^*)$$

for each $U \in \mathcal{U}_1$.

Recall that Jordan $*$ -isomorphisms have some useful properties. If J is a Jordan $*$ -isomorphism between two C^* -algebras, then

- (i) $J(I) = I$;
- (ii) $J(AB) = J(A)J(B)$ whenever $AB = BA$; and
- (iii) $J(E)$ is a projection if and only if E is a projection.

See, for example, [19, Exercise 10.5.22] and its solution in [20]. In particular, J preserves the centers and unitary groups. On the other hand, it is well-known that each Jordan $*$ -isomorphism is an isometry. Consequently, if J is a Jordan $*$ -isomorphism, it easily follows that the formula $T(A) = T(I)(PJ(A) + (I - P)J(A)^*)$ defines a real-linear isometric isomorphism.

The following result is a special case of the result of Mankiewicz [21].

Lemma 4.10 (Mankiewicz [21]). *Let X, Y be Banach spaces, and let $T : B_X \rightarrow B_Y$ be a surjective isometry. Then there exists a real-linear isometric isomorphism $\hat{T} : X \rightarrow Y$ such that $\hat{T}|_{B_X} = T$.*

We also make use of the facial structure of von Neumann algebras given by Edwards and Rüttimann [9]; see also [2].

Theorem 4.11 (Edwards and Rüttimann [9]). *Let $\mathcal{R}$ be a von Neumann algebra and F be a weak-operator closed proper face of $B_{\mathcal{R}}$. Then there exists a partial isometry V in $\mathcal{R}$ such that*

$$F = V + (I - VV^*)B_{\mathcal{R}}(I - V^*V).$$

We are now in a position to attack Tingley's problem on finite von Neumann algebras. The following theorem answers affirmatively to it.

Theorem 4.12. Let $\mathcal{R}_j$ be a finite von Neumann algebra, $j = 1, 2$. If $T : S_{\mathcal{R}_1} \rightarrow S_{\mathcal{R}_2}$ is a surjective isometry, then there exist a central projection P in $\mathcal{R}_2$ and a Jordan $*$ -isomorphism $J : \mathcal{R}_1 \rightarrow \mathcal{R}_2$ such that

$$T(A) = T(I)(PJ(A) + (I - P)J(A)^*)$$

for each $A \in S_{\mathcal{R}_1}$.

Proof. Let $\mathcal{U}_j$ denote the unitary group of $\mathcal{R}_j$, $j = 1, 2$. Since $\text{pext}(B_{\mathcal{R}_j}) = \mathcal{U}_j$ by Proposition 4.4, we have $T(\mathcal{U}_1) = T(\text{pext}(B_{\mathcal{R}_1}) = \text{pext}(B_{\mathcal{R}_2}) = \mathcal{U}_2$ by Corollary 4.6. Therefore, by Theorem 4.9, there exist a central projection P in $\mathcal{R}_2$ and a Jordan $*$ -automorphism $J : \mathcal{R}_1 \rightarrow \mathcal{R}_2$ such that

$$T(U) = T(I)(PJ(U) + (I - P)J(U)^*)$$

for each $U \in \mathcal{U}_1$. Put $\widehat{T}(A) = T(I)(PJ(A) + (I - P)J(A)^*)$ for each $A \in \mathcal{R}_1$. What only remains is to show that $T(A) = \widehat{T}(A)$ holds for all $A \in S_{\mathcal{R}_1}$.

We first show that the above equality holds for each element in $S_{\mathcal{R}_1}$ that lies in some weak-operator closed proper face F of $B_{\mathcal{R}_1}$. By Corollary 3.4, F can be written as the intersection of a family (F_λ) of maximal proper faces of $B_{\mathcal{R}_1}$. Fix a λ . We shall show that $T(F_\lambda) = \widehat{T}(F_\lambda)$. Let $B \in T(F_\lambda)$. Since $\mathcal{R}_2$ is finite, $B = (V_1 + V_2)/2$ for some $V_1, V_2 \in \mathcal{U}_2$. Moreover, $T(F_\lambda)$ is a face of $B_{\mathcal{R}_2}$ by Lemma 4.1, which implies that $V_1, V_2 \in T(F_\lambda)$. It follows from $T|_{\mathcal{U}_1} = \widehat{T}|_{\mathcal{U}_1}$ that $V_1, V_2 \in \widehat{T}(F_\lambda)$. Therefore one has that $B \in \widehat{T}(F_\lambda)$. This shows that $T(F_\lambda) \subset \widehat{T}(F_\lambda)$ while $T(F_\lambda)$ is a maximal proper face of $B_{\mathcal{R}_2}$. Thus we obtain $T(F_\lambda) = \widehat{T}(F_\lambda)$, as desired. Now it follows that

$$T(F) = T\left(\bigcap_{\lambda} F_{\lambda}\right) = \bigcap_{\lambda} T(F_{\lambda}) = \bigcap_{\lambda} \widehat{T}(F_{\lambda}) = \widehat{T}\left(\bigcap_{\lambda} F_{\lambda}\right) = \widehat{T}(F).$$

On the other hand, the Jordan $*$ -isomorphism J is ultraweakly continuous on $\mathcal{R}_1$ (see [19, Exercise 10.6.26] and its solution in [20]), which together with the ultraweak continuity of the involution operation implies that $\widehat{T}$ is ultraweakly continuous on $\mathcal{R}_1$. Moreover, a subset of $B_{\mathcal{R}_1}$ is weak-operator closed if and only if it is ultraweakly compact. Thus $T(F) (= \widehat{T}(F))$ is a weak-operator closed proper face of $B_{\mathcal{R}_2}$.

In the preceding paragraph, we showed that T preserves weak-operator closed proper faces of $B_{\mathcal{R}_1}$. Again, let F be such a set. Since F and $T(F)$ are weak-operator closed proper faces of $B_{\mathcal{R}_1}$ and $B_{\mathcal{R}_2}$, respectively, by Theorem 4.11, there exist partial isometries $V \in \mathcal{R}_1$ and $W \in \mathcal{R}_2$ such that

$$F = V + (I - VV^*)B_{\mathcal{R}_1}(I - V^*V)$$

and

$$T(F) = W + (I - WW^*)B_{\mathcal{R}_2}(I - W^*W).$$

Let $T_1 : A \rightarrow V + A$ for each $A \in \mathcal{R}_1$ and $T_2 : B \rightarrow -W + B$ for each $B \in \mathcal{R}_2$. Then T_1 and T_2 are surjective affine isometry on $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. Moreover, $T_0 = T_2 T T_1$ is an isometry from $(I - VV^*)B_{\mathcal{R}_1}(I - V^*V)$ onto $(I - WW^*)B_{\mathcal{R}_2}(I - W^*W)$; and these sets are the unit balls of the Banach spaces $(I - VV^*)\mathcal{R}_1(I - V^*V)$ and $(I - WW^*)\mathcal{R}_2(I - W^*W)$. Hence T_0 extends to a real-linear isometric isomorphism between those spaces by Lemma 4.10. In particular, $T|_F = T_2^{-1}T_0T_1^{-1}|_F$ is affine. Now let $A \in F$. Then, as in the preceding paragraph, $A = (U_1 + U_2)/2$ for some $U_1, U_2 \in \mathcal{U}_1 \cap F$. It follows that

$$T(A) = \frac{T(U_1) + T(U_2)}{2} = \frac{\widehat{T}(U_1) + \widehat{T}(U_2)}{2} = \widehat{T}(A).$$

Finally, since $\mathcal{R}_1$ is a von Neumann algebra, it can be viewed as the dual of the space $(\mathcal{R}_1)_\#$ of all ultraweakly continuous linear functionals on $\mathcal{R}_1$; see, for example, [19, Theorem 7.4.2]. The action of $A \in \mathcal{R}_1$ on $(\mathcal{R}_1)_\#$ is given by $A(\varphi) = \varphi(A)$ for each $\varphi \in (\mathcal{R}_1)_\#$. Now take an arbitrary $A \in S_{\mathcal{R}_1}$. Then, by the Bishop–Phelps theorem (see, for example, [22, Theorem 2.11.13]), for each n , there exists a support functional $B_n \in S_{\mathcal{R}_1}$ for $B_{(\mathcal{R}_1)_\#}$ such that $\|A - B_n\| < 1/n$. Let $\varphi_n \in S_{(\mathcal{R}_1)_\#}$ be an element satisfying $\varphi_n(B_n) = 1$. The exposed face $\varphi_n^{-1}(1) \cap B_{\mathcal{R}_1}$ with respect to φ_n is weak-operator closed and contains B_n . Therefore, from the preceding paragraph, we have $T(B_n) = \widehat{T}(B_n)$. It now follows from the norm continuity of T and $\widehat{T}$ that $T(A) = \lim_n T(B_n) = \lim_n \widehat{T}(B_n) = \widehat{T}(A)$. This completes the proof. $\square$

As a final remark, we note that the core of the proof of Theorem 4.12 is strongly based on the following properties possessed by finite von Neumann algebras $\mathcal{R}$:

- $\text{pext}(B_{\mathcal{R}}) = \mathcal{U}$, the unitary group of $\mathcal{R}$.
- Each $A \in B_{\mathcal{R}}$ is the midpoint of two unitary elements $\in \mathcal{R}$.

For this reason, our argument cannot apply even to infinite type I factors. To study Tingley’s problem on general (infinite) von Neumann algebras, we need to find other routes that do not depend on the above special properties.

Acknowledgment

The author was supported in part by Grants-in-Aid for Scientific Research Grant Number 16J01162, Japan Society for the Promotion of Science.

References

- [1] T. Abe, S. Akiyama, O. Hatori, Isometries of the special orthogonal group, *Linear Algebra Appl.* 439 (2013) 174–188.
- [2] C.A. Akemann, G.K. Pedersen, Facial structure in operator algebra theory, *Proc. Lond. Math. Soc.* (3) 64 (1992) 418–448.
- [3] G.G. Ding, The isometric extension problem in the spheres of $l^p(\Gamma)$ ($p > 1$) type spaces, *Sci. China Ser. A* 46 (2003) 333–338.
- [4] G.G. Ding, On the extension of isometries between unit spheres of E and $C(\Omega)$, *Acta Math. Sin. (Engl. Ser.)* 19 (2003) 793–800.
- [5] G.G. Ding, The representation theorem of onto isometric mappings between two unit spheres of l^∞ -type spaces and the application on isometric extension problem, *Sci. China Ser. A* 47 (2004) 722–729.
- [6] G.G. Ding, The representation theorem of onto isometric mappings between two unit spheres of $l^1(\Gamma)$ type spaces and the application to the isometric extension problem, *Acta Math. Sin. (Engl. Ser.)* 20 (2004) 1089–1094.
- [7] G.G. Ding, On isometric extension problem between two unit spheres, *Sci. China Ser. A* 52 (2009) 2069–2083.
- [8] G.G. Ding, The isometric extension problem between unit spheres of two separable Banach spaces, *Acta Math. Sin. (Engl. Ser.)* 31 (2015) 1872–1878.
- [9] C.M. Edwards, G.T. Rüttimann, On the facial structure of the unit balls in a JBW^* -triple and its predual, *J. Lond. Math. Soc.* (2) 38 (1988) 317–332.
- [10] G.P. Gehér, P. Šemrl, Isometries of Grassmann spaces, *J. Funct. Anal.* 270 (2016) 1585–1601.
- [11] R. Grzaślewicz, Exposed points of the unit ball of $\mathcal{L}(H)$, *Math. Z.* 193 (1986) 595–596.
- [12] O. Hatori, Algebraic properties of isometries between groups of invertible elements in Banach algebras, *J. Math. Anal. Appl.* 376 (2011) 84–93.
- [13] O. Hatori, Maps on the sphere of the algebras of matrices, *Nihonkai Math. J.* 26 (2015) 121–125.
- [14] O. Hatori, L. Molnár, Isometries of the unitary group, *Proc. Amer. Math. Soc.* 140 (2012) 2127–2140.
- [15] O. Hatori, L. Molnár, Isometries of the unitary groups and Thompson isometries of the spaces of invertible positive elements in C^* -algebras, *J. Math. Anal. Appl.* 409 (2014) 158–167.
- [16] V. Kadets, M. Martín, Extension of isometries between unit spheres of finite-dimensional polyhedral Banach spaces, *J. Math. Anal. Appl.* 396 (2012) 441–447.
- [17] R.V. Kadison, Isometries of operator algebras, *Ann. of Math.* (2) 54 (1951) 325–338.
- [18] R.V. Kadison, A generalized Schwarz inequality and algebraic invariants for operator algebras, *Ann. of Math.* (2) 56 (1952) 494–503.
- [19] R.V. Kadison, J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras. Vol. II, Advanced Theory*, Pure and Applied Mathematics, vol. 100, Academic Press, Inc., Orlando, FL, 1986.

- [20] R.V. Kadison, J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras. Vol. IV, Special Topics. Advanced Theory—an Exercise Approach*, Birkhäuser Boston, Inc., Boston, MA, 1992.
- [21] P. Mankiewicz, On extension of isometries in normed linear spaces, *Bull. Acad. Pol. Sci., Sér. Sci. Math. Astron. Phys.* 20 (1972) 367–371.
- [22] R.E. Megginson, *An Introduction to Banach Space Theory*, Graduate Texts in Mathematics, vol. 83, Springer-Verlag, New York, 1998.
- [23] L. Molnár, Thompson isometries of the space of invertible positive operators, *Proc. Amer. Math. Soc.* 137 (2009) 3849–3859.
- [24] L. Molnár, P. Šemrl, Transformations of the unitary group on a Hilbert space, *J. Math. Anal. Appl.* 388 (2012) 1205–1217.
- [25] D. Tan, Extension of isometries on unit sphere of L^∞ , *Taiwanese J. Math.* 15 (2011) 819–827.
- [26] D. Tan, On extension of isometries on the unit spheres of L^p -spaces for $0 < p \leq 1$, *Nonlinear Anal.* 74 (2011) 6981–6987.
- [27] D. Tan, Extension of isometries on the unit sphere of L^p spaces, *Acta Math. Sin. (Engl. Ser.)* 28 (2012) 1197–1208.
- [28] D. Tan, X. Huang, R. Liu, Generalized-lush spaces and the Mazur–Ulam property, *Studia Math.* 219 (2013) 139–153.
- [29] D. Tan, R. Liu, A note on the Mazur–Ulam property of almost-CL-spaces, *J. Math. Anal. Appl.* 405 (2013) 336–341.
- [30] R. Tanaka, A further property of spherical isometries, *Bull. Aust. Math. Soc.* 90 (2014) 304–310.
- [31] R. Tanaka, The solution of Tingley’s problem for the operator norm unit sphere of complex $n \times n$ matrices, *Linear Algebra Appl.* 494 (2016) 274–285.
- [32] R. Tanaka, Spherical isometries of finite dimensional C^* -algebras, *J. Math. Anal. Appl.* 445 (1) (2017) 337–341.
- [33] D. Tingley, Isometries of the unit sphere, *Geom. Dedicata* 22 (1987) 371–378.