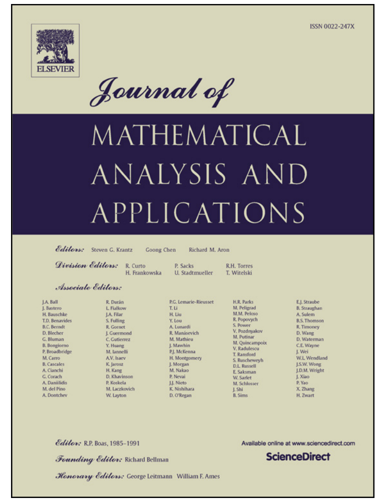

A discrete generalized coupled dispersionless integrable system and its multisoliton solutions

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Highlights

- An integrable discretization of the generalized coupled dispersionless integrable system (dGCD) system via Lax pair is presented.
- A Darboux transformation is proposed for dGCD system.
- Darboux matrix is defined in terms of quasideterminant.
- Quasideterminant multisoliton solutions have been computed.
- Explicit expressions of one and two soliton solutions have been computed.

A discrete generalized coupled dispersionless integrable system and its multisoliton solutions

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Abstract

An integrable discretization of generalized coupled dispersionless (dGCD) integrable system via Lax pair is presented. A Lax pair for the dGCD system is defined. A Darboux transformation is used on the Lax pair to obtain multi-soliton solutions of the dGCD system. The solutions are expressed in terms of quasideterminants. Explicit expressions of discrete one- and two-soliton solutions are obtained for the $SU(2)$ case by using properties of quasideterminants. We also study continuous analogue of the dGCD system by applying continuum limit.

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Keywords: Discrete Integrable systems, solitons, Darboux transformation

1 Introduction

Dispersionless integrable systems have attracted a great deal of interest due to their emergence in various areas of theoretical physics [1]-[7]. Many of these integrable systems arise as semi-classical limits of ordinary integrable systems with a dispersion term. Coupled dispersionless integrable system is an important example of integrable systems which have various applications in diverse areas of physics and mathematics (see e.g. [8]-[19]). This system is referred to as coupled dispersionless integrable system for being not containing the dispersion term rather resulting as a semi-classical limit in the sense mentioned above.

In this paper, we discretize the generalized coupled dispersionless system based on a general nonabelian Lie group by writing down its Lax pair representation. A Darboux transformation is defined on the solutions to the dGCD system. The solutions are then

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expressed in terms of quasideterminants. For the case of $SU(2)$, we obtain explicit expressions of one- and two-soliton solutions by using properties of quasideterminants. The results in this paper are generalizations of results obtained in [19].

2 Lax pair of dGCD system

The Lax pair of dGCD system is written as the following set of difference-difference linear system of equations

$$\Phi_{n+1,\,m} \;=\; \left(I+\lambda^{-1}[S_{n+1,\,m}-S_{n,\,m}]\right)\Phi_{n,\,m}\equiv \mathcal{A}_{n,\,m}\Phi_{n,\,m}, \tag{2.1}$$

$$\Phi_{n,\,m+1} \;=\; \left(I+b[S_{n,\,m+1}G-GS_{n,\,m}+\lambda G]\right)\Phi_{n,\,m}\equiv \mathcal{B}_{n,\,m}\Phi_{n,\,m}, \tag{2.2}$$

where $S_{n,\,m}$ is an $N\times N$ matrix and G is an $N\times N$ constant matrix taking values in some non-abelian Lie algebra $\mathfrak{g}$ of Lie group $\mathcal{G}$ and $\Phi_{n,\,m}$ is also an $N\times N$ spectral parameter dependent matrix which takes value in Lie group $\mathcal{G}$. The subscripts $n,\ m$ of the matrices $\Phi_{n,\,m}$ and $S_{n,\,m}$ represent variables defined on a square lattice. The Lax pair (2.1)-(2.2) of dGCD system satisfying the compatibility condition $\mathcal{A}_{n,\,m+1}\mathcal{B}_{n,\,m}=\mathcal{B}_{n+1,\,m}\mathcal{A}_{n,\,m}$ gives the nonlinear dGCD system

$$\begin{aligned}&\left(S_{n+1,\,m+1}-S_{n+1,\,m}-S_{n,\,m+1}+S_{n,\,m}\right)+b\left(S_{n+1,\,m+1}-S_{n,\,m+1}\right)\left(S_{n,\,m+1}G-GS_{n,\,m}\right)\\&=b\left(S_{n+1,\,m+1}G-GS_{n+1,\,m}\right)\left(S_{n+1,\,m}-S_{n,\,m}\right).\end{aligned} \tag{2.3}$$

From equation (2.3), we can derive the semi-discrete GCD (sdGCD) system, continuous in t-direction by applying a continuum limit. For this, let us define the continuum limit i.e., $\lim_{b\to 0}\frac{f_{n,\,m+1}-f_{n,\,m}}{b}=\frac{d}{dt}f_n$, then accordingly we have sdGCD system

$$\frac{d}{dt}\left(S_{n+1}-S_n\right)+\left(S_{n+1}-S_n\right)\left(S_nG-GS_n\right)=\left(S_{n+1}G-GS_{n+1}\right)\left(S_{n+1}-S_n\right). \tag{2.4}$$

Similarly, by sending a (lattice parameter along x-direction) to zero, we have $\lim_{a\to 0}\frac{f_{n+1,\,m}-f_{n,\,m}}{a}=\frac{d}{dx}f_m$, so that the sdGCD system continuous in x-direction is given by

$$\frac{d}{dx}\left(S_{m+1}-S_m\right)+b\frac{d}{dx}S_{m+1}\left(S_{m+1}G-GS_m\right)=b\left(S_{m+1}G-GS_m\right)\frac{d}{dx}S_m. \tag{2.5}$$

The continuous GCD system, continuous in both x- and t-direction is obtained when both lattice parameters approach to zero i.e. $a,\ b\to 0$, we get

$$\partial_t\partial_x S+[[G,\ S],\ \partial_x S]=0. \tag{2.6}$$

3 Discrete Darboux transformation

Darboux transformation (DT) is widely used to obtain soliton solutions of integrable systems [20]-[23]. Let us define a Darboux transformation on the matrix solutions $\Phi_{n,\,m}$ of Lax pair (2.1)-(2.2) as

$$\Phi_{n,\,m}[1] = D_{n,\,m}(\lambda)\Phi_{n,\,m}, \tag{3.1}$$

where $D_{n,\,m}(\lambda)$ is a discrete Darboux matrix. In our case, we take the Darboux matrix to be

$$D_{n,\,m}(\lambda) = \lambda I - \Theta_{n,\,m}, \tag{3.2}$$

where $\Theta_{n,\,m}$ is an $N \times N$ matrix to be determined and I is the $N \times N$ identity matrix. The discrete Darboux matrix (3.2) gives the transformation on the matrix solutions of dGCD system (2.3)

$$\begin{aligned} &G[1] = G \quad \text{(constant matrix)}, \\ &S_{n+1,\,m}[1] - S_{n,\,m}[1] = S_{n+1,\,m} - S_{n,\,m} - \left(\Theta_{n+1,\,m} - \Theta_{n,\,m}\right). \end{aligned} \tag{3.3}$$

Equation (3.3) can also be written as

$$S_{n,\,m}[1] = S_{n,\,m} - \Theta_{n,\,m}. \tag{3.4}$$

The Lax pair (2.1)-(2.2) is covariant under the discrete DT if the following conditions on the matrix $\Theta_{n,\,m}$ are satisfied at every point on the lattice, i.e.

$$\Theta_{n+1,\,m} - \Theta_{n,\,m} = S_{n+1,\,m} - S_{n,\,m} - \Theta_{n+1,\,m}\left(S_{n+1,\,m} - S_{n,\,m}\right)\Theta^{-1}_{n,\,m}, \tag{3.5}$$

$$\begin{aligned} \Theta_{n,\,m+1} - \Theta_{n,\,m} &= b\left(S_{n,\,m+1}G - GS_{n,\,m}\right)\Theta_{n,\,m} - b\Theta_{n,\,m+1}\left(S_{n,\,m+1}G - GS_{n,\,m}\right) \\ &- b\left(\Theta_{n,\,m+1}G - G\Theta_{n,\,m}\right)\Theta_{n,\,m}. \end{aligned} \tag{3.6}$$

Now for $b \to 0$, the Lax pair of the sdGCD system continuous in t-direction remains unchanged under the DT for the following conditions on the matrix Θ_n,

$$\Theta_{n+1} - \Theta_n = S_{n+1} - S_n - \Theta_{n+1}\left(S_{n+1} - S_n\right)\Theta_n^{-1}, \tag{3.7}$$

$$\frac{d}{dt}\Theta_n = \left[S_nG - GS_n,\ \Theta_n\right] - \left(\Theta_nG - G\Theta_n\right)\Theta_n. \tag{3.8}$$

The transformed matrix solution S_n of the sdGCD system continuous in t-direction is

$$S_n[1] = S_n - \Theta_n. \tag{3.9}$$

Similarly for $a \to 0$, we have a sdGCD system continuous in x-direction

$$\frac{d}{dx}\Theta_m = \left(\frac{d}{dx}S_m\Theta_m - \Theta_m\frac{d}{dx}S_m\right)\Theta_m^{-1}, \tag{3.10}$$

$$\begin{aligned}\Theta_{m+1} - \Theta_m &= b\left(S_{m+1}G - GS_m\right)\Theta_m - b\Theta_{m+1}\left(S_{m+1}G - GS_m\right) \\ &- b\left(\Theta_{m+1}G - G\Theta_m\right)\Theta_m, \end{aligned} \tag{3.11}$$

with DT on S_m matrix

$$S_m[1] = S_m - \Theta_m. \tag{3.12}$$

We now construct the matrix $\Theta_{n,\,m}$ for the dGCD system. Let us define N constant parameters (real or complex) $\lambda_1,\ \lambda_2,\ ...,\ \lambda_N$. For each value of the parameter, there exists a unique column vector solution $|m_i\rangle_{n,\,m}$ to the Lax pair (2.1)-(2.2), i.e.

$$|m_i\rangle_{n+1,\,m} = \left[1 + \lambda_i^{-1}\left(S_{n+1,\,m} - S_{n,\,m}\right)\right]|m_i\rangle_{n,\,m}\,, \tag{3.13}$$

$$|m_i\rangle_{n,\,m+1} = \left[1 + b\left(S_{n,\,m+1}G - GS_{n,\,m} + \lambda_i G\right)\right]|m_i\rangle_{n,\,m}\,. \tag{3.14}$$

Let us now define an $N \times N$ diagonal matrix as $\Lambda = \mathrm{diag}\left(\lambda_1,\ \lambda_2,\ ...,\ \lambda_N\right)$, so that a particular matrix solution $M_{n,\,m}$ with a particular eigenvalue matrix Λ satisfies the following matrix Lax pair (2.1)-(2.2)

$$M_{n+1,\,m} = M_{n,\,m} + \left(S_{n+1,\,m} - S_{n,\,m}\right)M_{n,\,m}\Lambda^{-1}, \tag{3.15}$$

$$M_{n,\,m+1} = M_{n,\,m} + b\left(S_{n,\,m+1}G - GS_{n,\,m}\right)M_{n,\,m} + bGM_{n,\,m}\Lambda. \tag{3.16}$$

If $\det M_{n,\,m} \neq 0$, then we can define a matrix $\Theta_{n,\,m}$ in terms of particular matrix solutions as

$$\Theta_{n,\,m} = M_{n,\,m}\Lambda M_{n,\,m}^{-1}, \tag{3.17}$$

Now we check that the Darboux matrix $\Theta_{n,\,m}$ defined in equation (3.17), satisfies the conditions on the matrix $\Theta_{n,\,m}$ resulting from the Darboux covarience, i.e.

$$
\begin{aligned}
\left(\Theta_{n+1,\,m}-\Theta_{n,\,m}\right)\Theta_{n,\,m} &= M_{n+1,\,m}\Lambda M^{-1}_{n+1,\,m}M_{n,\,m}\Lambda M^{-1}_{n,\,m}-M_{n,\,m}\Lambda M^{-1}_{n,\,m}M_{n,\,m}\Lambda M^{-1}_{n,\,m},\\
&= M_{n+1,\,m}\Lambda M^{-1}_{n+1,\,m}M_{n,\,m}\Lambda M^{-1}_{n,\,m}-M_{n,\,m}\Lambda M^{-1}_{n,\,m}M_{n,\,m}\Lambda M^{-1}_{n,\,m}\\
&+ M_{n+1,\,m}\Lambda M^{-1}_{n,\,m}M_{n,\,m}\Lambda M^{-1}_{n,\,m}-M_{n+1,\,m}\Lambda M^{-1}_{n+1,\,m}M_{n+1,\,m}\Lambda M^{-1}_{n,\,m},\\
&= \left(M_{n+1,\,m}\Lambda M^{-1}_{n,\,m}-M_{n,\,m}\Lambda M^{-1}_{n,\,m}\right)M_{n,\,m}\Lambda M^{-1}_{n,\,m}\\
&- M_{n+1,\,m}\Lambda M^{-1}_{n+1,\,m}\left(M_{n+1,\,m}\Lambda M^{-1}_{n,\,m}-M_{n,\,m}\Lambda M^{-1}_{n,\,m}\right),\\
&= \left(S_{n+1,\,m}-S_{n,\,m}\right)\Theta_{n,\,m}-\Theta_{n+1,\,m}\left(S_{n+1,\,m}-S_{n,\,m}\right), \qquad (3.18)\\
\Theta_{n,\,m+1}-\Theta_{n,\,m} &= M_{n,\,m+1}\Lambda M^{-1}_{n,\,m+1}-M_{n,\,m}\Lambda M^{-1}_{n,\,m},\\
&= M_{n,\,m+1}\Lambda M^{-1}_{n,\,m+1}-M_{n,\,m}\Lambda M^{-1}_{n,\,m}+M_{n,\,m+1}M^{-1}_{n,\,m}M_{n,\,m}\Lambda M^{-1}_{n,\,m}\\
&- M_{n,\,m+1}\Lambda M^{-1}_{n,\,m+1}M_{n,\,m+1}M^{-1}_{n,\,m},\\
&= \left(M_{n,\,m+1}M^{-1}_{n,\,m}-I\right)M_{n,\,m}\Lambda M^{-1}_{n,\,m}\\
&- M_{n,\,m+1}\Lambda M^{-1}_{n,\,m+1}\left(M_{n,\,m+1}M^{-1}_{n,\,m}-I\right),\\
&= b\left(S_{n,\,m+1}G-GS_{n,\,m}\right)\Theta_{n,\,m}-b\Theta_{n,\,m+1}\left(S_{n,\,m+1}G-GS_{n,\,m}\right)\\
&- b\left(\Theta_{n,\,m+1}G-G\Theta_{n,\,m}\right)\Theta_{n,\,m}. \qquad (3.19)
\end{aligned}
$$

So the equations (3.5) and (3.6) are satisfied for the choice (3.17).

For the underlying Lie group $SU(N)$, we require the new solutions $S_{n,\,m}[1]$, $G[1]$ to take values in respective Lie algebra. For anti-Hermitian generators of the Lie algebra of $SU(N)$, we require $S_{n,\,m}[1]$, $G[1]$ be anti-Hermitian and traceless. The uniqueness of the column solution $|m_i\rangle_{n,\,m}$ at a particular value of $\lambda=\lambda_i$ (where $\lambda_i\neq\lambda_j$) preserves the orthonormality condition at each lattice point, i.e.

$$
\begin{aligned}
{}_{n,\,m}\langle m_i|m_j\rangle_{n,\,m} &= \delta_{i,\,j},\\
&= 0 \quad \text{for} \quad i\neq j,\\
&= 1 \quad \text{for} \quad i=j. \qquad (3.20)
\end{aligned}
$$

From the orthonormality condition (3.20) of the column solutions, we can say that the matrix Θ_n must be anti-Hermitian (in case when the generators of the Lie algebra are anti-Hermitian) i.e.

$$
\begin{aligned}
{}_{n,\,m}\langle m_i|\,\Theta^{\dagger}_{n,\,m}+\Theta_{n,\,m}\,|m_j\rangle_{n,\,m} &= \left(\bar{\lambda}_i+\lambda_j\right)\,{}_{n,\,m}\langle m_i|m_j\rangle_{n,\,m},\\
\Theta^{\dagger}_{n,\,m}+\Theta_{n,\,m} &= 0 \quad \text{for} \quad i\neq j. \qquad (3.21)
\end{aligned}
$$

For $\lambda_i = \lambda_j = \lambda_1$

$$_{n,\,m}\langle m_i|\,\Theta^{\dagger}_{n,\,m} + \Theta_{n,\,m}\,|m_j\rangle_{n,\,m} = _{n,\,m}\langle m_i|\,\bar{\lambda}_1 + \lambda_1\,|m_j\rangle_{n,\,m}\,, \tag{3.22}$$

since the column solutions $|m_i\rangle_{n,\,m}$'s are all orthonormal (linearly independent), so equation (3.22) implies

$$\Theta^{\dagger}_{n,\,m} + \Theta_{n,\,m} = \left(\bar{\lambda}_1 + \lambda_1\right) I. \tag{3.23}$$

Further, we also have

$$\Theta^{\dagger}_{n,\,m}\Theta_{n,\,m} = \bar{\lambda}_1\lambda_1. \tag{3.24}$$

Now for a particular solution $|m_i\rangle_{n,\,m}$ at $\lambda = \lambda_i$ and $\lambda_i \neq \lambda_j$ $\left(\bar{\lambda}_i = \lambda_j\right)$, we have

$$\begin{aligned} _{n+1,\,m}\langle m_i|m_j\rangle_{n,\,m} + _{n,\,m}\langle m_i|m_j\rangle_{n+1,\,m} &= \; _{n,\,m}\langle m_i|\left(I + \bar{\lambda}_i\left(S^{\dagger}_{n+1,\,m} - S^{\dagger}_{n,\,m}\right)\right)|m_j\rangle_{n,\,m} \\ &+ \; _{n,\,m}\langle m_i|\left(I + \lambda_j\left(S_{n+1,\,m} - S_{n,\,m}\right)\right)|m_j\rangle_{n,\,m}\,, \\ &= 0, \end{aligned} \tag{3.25}$$

which implies that $S^{\dagger}_{n,\,m} = -S_{n,\,m}$.

For the invertible matrix $M_{n,\,m}$, we write the discrete Darboux matrix in terms of quasideterminants[3]. We write the quasideterminant expression of the Darboux transformed matrix $\Phi_{n,\,m}[1]$ as a solution to the Lax pair (2.1)-(2.2) as

$$\begin{aligned} \Phi_{n,\,m}[1] &\equiv D_{n,\,m}(\lambda)\Phi_{n,\,m} = \lambda\Phi_{n,\,m} - M_{n,\,m}\Lambda M^{-1}_{n,\,m}\Phi_{n,\,m}, \\ &= \lambda\Phi_{n,\,m} + \begin{vmatrix} M_{n,\,m} & \Phi_{n,\,m} \\ M_{n,\,m}\Lambda & \boxed{O} \end{vmatrix} = \begin{vmatrix} M_{n,\,m} & \Phi_{n,\,m} \\ M_{n,\,m}\Lambda & \boxed{\lambda\Phi_{n,\,m}} \end{vmatrix}, \end{aligned} \tag{3.27}$$

where I is an $N \times N$ identity matrix and Λ is an $N \times N$ invertible diagonal matrix. Similarly the expression of one-fold matrix solution $S_{n,\,m}[1]$ in terms of quasideterminant about an $N \times N$ null matrix O can be written as

$$\begin{aligned} S_{n,\,m}[1] &= S_{n,\,m} - M_{n,\,m}\Lambda M^{-1}_{n,\,m}, \\ &= S_{n,\,m} + \begin{vmatrix} M_{n,\,m} & I \\ M_{n,\,m}\Lambda & \boxed{O} \end{vmatrix}. \end{aligned} \tag{3.28}$$

[3]We will use the notion of quasideterminants. In this paper we will consider only quasideterminants that are expanded about an $m \times m$ matrix. The quasideterminant of $J \times J$ matrix expanded about the $m \times m$ matrix D is defined as

$$\begin{vmatrix} A & B \\ C & \boxed{D} \end{vmatrix} = D - CA^{-1}B. \tag{3.26}$$

For further (see e.g. [24]-[26])

In order to express the matrix $S_{n,\,m}[2]$ in terms of quasideterminants, we define particular matrix solutions $M_{n,\,m,\,1}$ at Λ_1 and $M_{n,\,m,\,2}$ at Λ_2 to get

$$S_{n,\,m}[2] = S_{n,\,m}[1] - M_{n,\,m}[1]\Lambda_2 M^{-1}_{n,\,m}[1], \tag{3.29}$$

where

$$\begin{aligned} M_{n,\,m}[1] &= M_{n,\,m,\,2}\Lambda_2 - M_{n,\,m,\,1}\Lambda_1 M^{-1}_{n,\,m,\,1} M_{n,\,m,\,2}, \\ &= \begin{vmatrix} M_{n,\,m,\,1} & M_{n,\,m,\,2} \\ M_{n,\,m,\,1}\Lambda_1 & \boxed{M_{n,\,m,\,2}\Lambda_2} \end{vmatrix}. \end{aligned} \tag{3.30}$$

By substituting equation (3.30) in equation (3.29), rearranging the terms and using non-commutative Jacobi identity and homological relation [24]-[26], we get

$$S_{n,\,m}[2] = S_{n,\,m} + \begin{vmatrix} M_{n,\,m,\,1} & M_{n,\,m,\,2} & O \\ M_{n,\,m,\,1}\Lambda_1 & M_{n,\,m,\,2}\Lambda_2 & I \\ M_{n,\,m,\,1}\Lambda_1^2 & M_{n,\,m,\,2}\Lambda_2^2 & \boxed{O} \end{vmatrix}. \tag{3.31}$$

The results in equations (3.31) can be extended to K-fold Darboux transformation on the solutions. For this, let us define an invertible matrix solutions $M_{n,\,m,\,k}$ at $\Lambda = \Lambda_k$ ($k = 1,\ 2,\ ...,\ K$) to the Lax pair (2.1)-(2.2). The matrix $S_{n,\,m}[K]$ can now be written as

$$S_{n,\,m}[K] = S_{n,\,m} + \begin{vmatrix} M_{n,\,m,\,1} & M_{n,\,m,\,2} & \cdots & M_{n,\,m,\,K} & O \\ M_{n,\,m,\,1}\Lambda_1 & M_{n,\,m,\,2}\Lambda_2 & \cdots & M_{n,\,m,\,K}\Lambda_K & O \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{n,\,m,\,1}\Lambda_1^{K-1} & M_{n,\,m,\,2}\Lambda_2^{K-1} & \cdots & M_{n,\,m,\,K}\Lambda_K^{K-1} & I \\ M_{n,\,m,\,1}\Lambda_1^{K} & M_{n,\,m,\,2}\Lambda_2^{K} & \cdots & M_{n,\,m,\,K}\Lambda_K^{K} & \boxed{O} \end{vmatrix}. \tag{3.32}$$

Similarly the expression for $\Phi_{n,\,m}[K]$ is

$$\Phi_{n,\,m}[K] \;=\; \begin{vmatrix} M_{n,\,m,\,1} & M_{n,\,m,\,2} & \cdots & M_{n,\,m,\,K} & \Phi_{n,\,m} \\ M_{n,\,m,\,1}\Lambda_1 & M_{n,\,m,\,2}\Lambda_2 & \cdots & M_{n,\,m,\,K}\Lambda_K & \lambda\Phi_{n,\,m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ M_{n,\,m,\,1}\Lambda_1^{K} & M_{n,\,m,\,2}\Lambda_2^{K} & \cdots & M_{n,\,m,\,K}\Lambda_K^{K} & \boxed{\lambda^K\Phi_{n,\,m}} \end{vmatrix}. \tag{3.33}$$

The results (3.32) and (3.33) can be proved by mathematical induction (see e.g. [15]). The equation (3.32) and (3.33) can also be written in a more convenient form as

$$S_{n,\,m}[K] = S_{n,\,m} + \Theta^{(K)}_{n,\,m}, \tag{3.34}$$

$$\Phi_{n,\,m}[K] = \Omega^{(K)}_{n,\,m}\Phi_{n,\,m}, \tag{3.35}$$

where $N \times N$ matrices $\Theta_{n,\,m},\ \Omega_{n,\,m}$ is a quasiedeterminant given by

$$\Theta^{(K)}_{n,\,m} = \begin{vmatrix} M_{n,\,m} & E^{(K)} \\ \widehat{M}_{n,\,m} & \boxed{O_N} \end{vmatrix}, \quad \Omega^{(K)}_{n,\,m} = \begin{vmatrix} M_{n,\,m} & \widehat{E}^{(K)} \\ \widehat{M}_{n,\,m} & \boxed{\lambda^K I_N} \end{vmatrix}, \tag{3.36}$$

where $E^{(K)},\ \widehat{E}^{(K)}$ are $NK \times N$ and $\widehat{M}_{n,\,m},\ M_{n,\,m}$ are the $N \times NK,\ NK \times NK$ matrices respectively, i.e.

$$\begin{aligned} E^{(K)} &= \begin{pmatrix} O_N & O_N & \cdots & I_N \end{pmatrix}^T,\ \widehat{E}^{(K)} = \begin{pmatrix} I_N & \lambda I_N & \cdots & \lambda^{K-1} I_N \end{pmatrix}^T, \\ \widehat{M}_{n,\,m} &= \begin{pmatrix} M_{n,\,m,\,1}\Lambda_1^K & M_{n,\,m,\,2}\Lambda_2^K & \cdots & M_{n,\,m,\,K}\Lambda_K^K \end{pmatrix}, \\ M_{n,\,m} &= \begin{pmatrix} M_{n,\,m,\,1} & M_{n,\,m,\,2} & \cdots & M_{n,\,m,\,K} \\ M_{n,\,m,\,1}\Lambda_1 & M_{n,\,m,\,2}\Lambda_2 & \cdots & M_{n,\,m,\,K}\Lambda_K \\ \vdots & \vdots & \ddots & \vdots \\ M_{n,\,m,\,1}\Lambda_1^{K-1} & M_{n,\,m,\,2}\Lambda_2^{K-1} & \cdots & M_{n,\,m,\,K}\Lambda_K^{K-1} \end{pmatrix}. \end{aligned} \tag{3.37}$$

The matrix elements of the matrices $\Theta^{(K)}_{n,\,m}$ and $\Omega^{(K)}_{n,\,m}$ are computed as

$$\begin{aligned} \left(\Theta^{(K)}_{n,\,m}\right)_{ij} &= \left(\begin{vmatrix} M_{n,\,m} & E^{(K)} \\ \widehat{M}_{n,\,m} & \boxed{O} \end{vmatrix}\right)_{ij} = \begin{vmatrix} M_{n,\,m} & E^{(K)}_j \\ (\widehat{M}_{n,\,m})_i & \boxed{0} \end{vmatrix}, \quad i \neq j, \\ \left(\Theta^{(K)}_{n,\,m}\right)_{ij} &= \left(\begin{vmatrix} M_{n,\,m} & E^{(K)} \\ \widehat{M}_{n,\,m} & \boxed{O} \end{vmatrix}\right)_{ii} = \begin{vmatrix} M_{n,\,m} & E^{(K)}_i \\ (\widehat{M}_{n,\,m})_i & \boxed{0} \end{vmatrix}, \quad i = j. \end{aligned} \tag{3.38}$$

$$\begin{aligned} \left(\Omega^{(K)}_{n,\,m}\right)_{ij} &= \left(\begin{vmatrix} M_{n,\,m} & E^{(K)} \\ \widehat{M}_{n,\,m} & \boxed{\lambda^K I} \end{vmatrix}\right)_{ij} = \begin{vmatrix} M_{n,\,m} & \widehat{E}^{(K)}_j \\ (\widehat{M}_{n,\,m})_i & \boxed{0} \end{vmatrix}, \quad i \neq j, \\ \left(\Theta^{(K)}_{n,\,m}\right)_{ij} &= \left(\begin{vmatrix} M_{n,\,m} & E^{(K)} \\ \widehat{M}_{n,\,m} & \boxed{\lambda^K I} \end{vmatrix}\right)_{ii} = \begin{vmatrix} M_{n,\,m} & E^{(K)}_i \\ (\widehat{M}_{n,\,m})_i & \boxed{\lambda^K} \end{vmatrix}, \quad i = j. \end{aligned} \tag{3.39}$$

where $(\widehat{M}_{n,\,m})_i$ represents i-th row of $M_{n,\,m}$ and $(E^{(K)})_j,\ (\widehat{E}^{(K)})_j$ represent j-th column of $E^{(K)},\ \widehat{E}^{(K)}$ respectively. The quasideterminant expressions in the equations (3.38) and (3.39) are used to compute explicit expressions of the Darboux transformations on the scalar solutions to the Lax pair (2.1)-(2.2) and the scalar solutions of the dGCD system. The K-fold solutions (3.32) and (3.33) reduce to those of continuous system when we take a continuum limit on the lattice parameters as $a,\ b \to 0$ [15].

4 Examples of simple models

In order to calculate scalar solutions for simple models, first of all we start with a more general model of 2×2 matrix functions and will carry out reduction to $SU(2)$ model. We start with a 2×2 matrix $S_{n,\,m}$ in terms of a scalar functions $q_{n,\,m},\ r_{n,\,m}$ and $s_{n,\,m}$ given by

$$S_{n,\,m} = i\begin{pmatrix} q_{n,\,m} & r_{n,\,m} \\ s_{n,\,m} & -q_{n,\,m} \end{pmatrix}, \tag{4.1}$$

and for $G = -\frac{i}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, the Lax pair (2.1)-(2.2) of dGCD system takes the form

$$\Phi_{n+1,\,m} = \begin{pmatrix} 1 + i\lambda^{-1}(q_{n+1,\,m} - q_{n,\,m}) & i\lambda^{-1}(r_{n+1,\,m} - r_{n,\,m}) \\ i\lambda^{-1}(s_{n+1,\,m} - s_{n,\,m}) & 1 - i\lambda^{-1}(q_{n+1,\,m} - q_{n,\,m}) \end{pmatrix}\Phi_{n,\,m}, \tag{4.2}$$

$$\Phi_{n,\,m+1} = \begin{pmatrix} 1 + \frac{b}{2}(q_{n,\,m+1} - q_{n,\,m} - i\lambda) & -\frac{b}{2}(r_{n,\,m+1} + r_{n,\,m}) \\ \frac{b}{2}(s_{n,\,m+1} + s_{n,\,m}) & 1 + \frac{b}{2}(q_{n,\,m+1} - q_{n,\,m} + i\lambda) \end{pmatrix}\Phi_{n,\,m}. \tag{4.3}$$

And the compatibility condition of the Lax pair (4.2)-(4.3) gives the dCD system as follows

$$\begin{aligned}
&(q_{n+1,\,m+1} - q_{n+1,\,m} - q_{n,\,m+1} + q_{n,\,m}) + \frac{b}{4}[(r_{n+1,\,m+1} - r_{n,\,m})(s_{n,\,m+1} + s_{n+1,\,m}) \\
&+ (s_{n+1,\,m+1} - s_{n,\,m})(r_{n,\,m+1} + r_{n+1,\,m}) + 2(r_{n+1,\,m}s_{n+1,\,m} - r_{n,\,m+1}s_{n,\,m+1})] \\
&= \frac{b}{2}[(q_{n+1,\,m+1} - q_{n+1,\,m})(q_{n+1,\,m} - q_{n,\,m}) - (q_{n,\,m+1} - q_{n,\,m})(q_{n+1,\,m+1} - q_{n,\,m+1})],
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
&(r_{n+1,\,m+1} - r_{n+1,\,m} - r_{n,\,m+1} + r_{n,\,m}) - \frac{b}{2}[(q_{n+1,\,m+1} - q_{n,\,m})(r_{n,\,m+1} + r_{n+1,\,m}) \\
&- (q_{n,\,m+1} - q_{n+1,\,m})(r_{n+1,\,m+1} + r_{n,\,m})] = 0,
\end{aligned} \tag{4.5}$$

$$\begin{aligned}
&(s_{n+1,\,m+1} - s_{n+1,\,m} - s_{n,\,m+1} + s_{n,\,m}) - \frac{b}{2}[(q_{n+1,\,m+1} - q_{n,\,m})(s_{n,\,m+1} + s_{n+1,\,m}) \\
&- (q_{n,\,m+1} - q_{n+1,\,m})(s_{n+1,\,m+1} + s_{n,\,m})] = 0.
\end{aligned} \tag{4.6}$$

For $r_{n,\,m} = s_{n,\,m}$, we get the dCD system for $SU(2)$ symmetry (in this case, we also need the reduction $SO(2)$ see e.g., [27])

$$\begin{aligned}
&(q_{n+1,\,m+1} - q_{n+1,\,m} - q_{n,\,m+1} + q_{n,\,m}) + \frac{b}{2}[(r_{n+1,\,m+1} - r_{n,\,m+1} + r_{n+1,\,m} - r_{n,\,m}) \\
&\times (r_{n,\,m+1} + r_{n+1,\,m})] - \frac{b}{2}[(q_{n+1,\,m+1} - q_{n+1,\,m})(q_{n+1,\,m} - q_{n,\,m}) \\
&- (q_{n,\,m+1} - q_{n,\,m})(q_{n+1,\,m+1} - q_{n,\,m+1})] = 0,
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
&(r_{n+1,\,m+1} - r_{n+1,\,m} - r_{n,\,m+1} + r_{n,\,m}) - \frac{b}{2}[(q_{n+1,\,m+1} - q_{n,\,m})(r_{n,\,m+1} + r_{n+1,\,m}) \\
&- (q_{n,\,m+1} - q_{n+1,\,m})(r_{n+1,\,m+1} + r_{n,\,m})] = 0.
\end{aligned} \tag{4.8}$$

which is the fully discrete version of CD system. With $r_{n,\,m} = s_{n,\,m}$, the Lax pair (4.2)-(4.3) is written as

$$\Phi_{n+1,\,m} = \begin{pmatrix} 1 + i\lambda^{-1}(q_{n+1,\,m} - q_{n,\,m}) & i\lambda^{-1}(r_{n+1,\,m} - r_{n,\,m}) \\ i\lambda^{-1}(r_{n+1,\,m} - r_{n,\,m}) & 1 - i\lambda^{-1}(q_{n+1,\,m} - q_{n,\,m}) \end{pmatrix}\Phi_{n,\,m}, \tag{4.9}$$

$$\Phi_{n,\,m+1} = \begin{pmatrix} 1 + \frac{b}{2}(q_{n,\,m+1} - q_{n,\,m} - i\lambda) & -\frac{b}{2}(r_{n,\,m+1} + r_{n,\,m}) \\ \frac{b}{2}(r_{n,\,m+1} + r_{n,\,m}) & 1 + \frac{b}{2}(q_{n,\,m+1} - q_{n,\,m} + i\lambda) \end{pmatrix}\Phi_{n,\,m}. \tag{4.10}$$

For further convenience, we apply a gauge transformation to get an equivalent representation of the dCD system. Under the gauge transformation $\mathcal{A}_{n,\,m} \to \widetilde{\mathcal{A}}_{n,\,m} = F^{-1}\mathcal{A}_n F$, where $F = \frac{1}{\sqrt{2}}\left(\begin{smallmatrix} 1 & 1 \\ -i & i \end{smallmatrix}\right)$, the Lax pair (4.9)-(4.10) is written as

$$\widetilde{\Phi}_{n+1,\,m} = \begin{pmatrix} 1 & i\lambda^{-1}[(q_{n+1,\,m}-q_{n,\,m})+i(r_{n+1,\,m}-r_{n,\,m})] \\ i\lambda^{-1}[(q_{n+1,\,m}-q_{n,\,m})+i(r_{n+1,\,m}-r_{n,\,m})] & 1 \end{pmatrix}\widetilde{\Phi}_{n,\,m}, \tag{4.11}$$

$$\widetilde{\Phi}_{n,\,m+1} = \begin{pmatrix} 1+\frac{b}{2}[(q_{n,\,m+1}-q_{n,\,m})+i(r_{n,\,m+1}+r_{n,\,m})] & -\frac{ib}{2}\lambda \\ -\frac{ib}{2}\lambda & 1+\frac{b}{2}[(q_{n,\,m+1}-q_{n,\,m})-i(r_{n,\,m+1}+r_{n,\,m})] \end{pmatrix}\widetilde{\Phi}_{n,\,m}. \tag{4.12}$$

Similarly for $s_{n,\,m} = r^*_{n,\,m}$, we get the complex coupled dispersionless integrable system. The Lax pair (4.11)-(4.12) can be used to get simpler expressions of Darboux transformation on the scalar solutions of the Lax pair and the scalar functions $q_{n,\,m}$ and $r_{n,\,m}$ of the dCD system.

4.1 sdCD system continuous in t-direction

The Lax pair of general sdCD system continuous in t-direction is the difference-differential linear system of equations given by

$$\Phi_{n+1} = \begin{pmatrix} 1+i\lambda^{-1}(q_{n+1}-q_n) & i\lambda^{-1}(r_{n+1}-r_n) \\ i\lambda^{-1}(s_{n+1}-s_n) & 1-i\lambda^{-1}(q_{n+1}-q_n) \end{pmatrix}\Phi_n \equiv \mathcal{A}_n\Phi_n, \tag{4.13}$$

$$\frac{d}{dt}\Phi_n = \begin{pmatrix} -\frac{i\lambda}{2} & -r_n \\ s_n & \frac{i\lambda}{2} \end{pmatrix}\Phi_n \equiv \mathcal{B}_n\Phi_n. \tag{4.14}$$

The compatibility condition $\frac{d}{dt}\mathcal{A}_n + \mathcal{A}_n\mathcal{B}_n - \mathcal{B}_{n+1}\mathcal{A}_n = 0$ is equivalent to the following nonlinear set of equations

$$\frac{d}{dt}(q_{n+1}-q_n) + (r_{n+1}s_{n+1} - r_n s_n) = 0, \tag{4.15}$$

$$\frac{d}{dt}(r_{n+1}-r_n) + (q_{n+1}-q_n)(r_{n+1}+r_n) = 0, \tag{4.16}$$

$$\frac{d}{dt}(s_{n+1}-s_n) + (q_{n+1}-q_n)(s_{n+1}+s_n) = 0. \tag{4.17}$$

The set of equations (4.15)-(4.17) is the sdCD system continuous in t-direction and has been obtained in [19].

4.2 sdCD system continuous in x-direction

The Lax pair of general sdCD system continuous in x-direction is the differential-difference linear system of equations given by

$$\begin{aligned}
\frac{d}{dx}\Phi_m &= i\lambda^{-1}\begin{pmatrix} \frac{d}{dx}q_m & \frac{d}{dx}r_m \\ \frac{d}{dx}s_m & -\frac{d}{dx}q_m \end{pmatrix}\Phi_m \equiv \mathcal{A}_m\Phi_m, && (4.18)\\
\Phi_{m+1} &= \begin{pmatrix} 1+\frac{b}{2}\left(q_{m+1}-q_m-i\lambda\right) & -\frac{b}{2}\left(r_{m+1}+r_m\right) \\ \frac{b}{2}\left(s_{m+1}+s_m\right) & 1+\frac{b}{2}\left(q_{m+1}-q_m+i\lambda\right) \end{pmatrix}\Phi_{n,\,m}, \\
&\equiv \mathcal{B}_m\Phi_m && (4.19)
\end{aligned}$$

The compatibility condition $\frac{d}{dx}\mathcal{B}_m + \mathcal{B}_m\mathcal{A}_m - \mathcal{A}_{m+1}\mathcal{B}_m = 0$ is equivalent to the following nonlinear sdCD system continuous in x-direction

$$\begin{aligned}
\frac{d}{dx}(q_{m+1}-q_m)[1+\frac{b}{2}(q_{m+1}-q_m)] + \frac{b}{4}\frac{d}{dx}[(r_{m+1}+r_m)(s_{m+1}+s_m)] &= 0, && (4.20)\\
\frac{d}{dx}(r_{m+1}-r_m)[1+\frac{b}{2}(q_{m+1}-q_m)] - \frac{b}{2}\frac{d}{dx}(q_{m+1}+q_m)[r_{m+1}+r_m] &= 0, && (4.21)\\
\frac{d}{dx}(s_{m+1}-s_m)[1+\frac{b}{2}(q_{m+1}-q_m)] - \frac{b}{2}\frac{d}{dx}(q_{m+1}+q_m)[s_{m+1}+s_m] &= 0. && (4.22)
\end{aligned}$$

Apparently the sdCD system (4.20)-(4.22) seems to be different as obtained in [16]. This is due to the fact that an additional t-derivative appears in the bilinearization process. The sdCD system (4.20)-(4.22) can be related with that of [16] by an appropriate transformation on the scalar functions q_m, r_m and s_m. For $SU(2)$ i.e., $r_m = s_m$, the set of nonlinear equations for the sdCD system continuous in x-direction (4.20)-(4.22) is

$$\begin{aligned}
\frac{d}{dx}(q_{m+1}-q_m)[1+\frac{b}{2}(q_{m+1}-q_m)] + \frac{b}{4}\frac{d}{dx}(r_{m+1}+r_m)^2 &= 0, && (4.23)\\
\frac{d}{dx}(r_{m+1}-r_m)[1+\frac{b}{2}(q_{m+1}-q_m)] - \frac{b}{2}\frac{d}{dx}(q_{m+1}+q_m)[r_{m+1}+r_m] &= 0. && (4.24)
\end{aligned}$$

In the continuum limit ($b \to 0,\ f_{m+1}-f_m \to 0$), we have $\frac{f_{m+1}-f_m}{b} = f_t,\ f_{m+1}+f_m = 2f$ so that the sdCD systems (4.20)-(4.24) reduces to their counterpart continuous CD systems.

5 Explicit soliton solutions

In what follows, we compute explicit expressions of one- and two-soliton solutions of the dGCD system for $r_{n,\,m} = s_{n,\,m}$. To get one-soliton solution, let us take a seed solution i.e. $q_{n+1,\,m} - q_{n,\,m} = p \neq 0,\ q_{n,\,m+1} - q_{n,\,m} = 0,\ r_{n,\,m} = 0$, where p is a real constant, so the solutions $X_{n,\,m},\ Y_{n,\,m}$ of the Lax pair (4.11) and (4.12) are computed as

$$X_{n,\,m} = \left(1+i\lambda^{-1}p\right)^n\left(1-b\frac{i\lambda}{2}\right)^m + i\left(1-i\lambda^{-1}p\right)^n\left(1+b\frac{i\lambda}{2}\right)^m, \tag{5.1}$$

$$Y_{n,\,m} = \left(1+i\lambda^{-1}p\right)^n\left(1-b\frac{i\lambda}{2}\right)^m - i\left(1-i\lambda^{-1}p\right)^n\left(1+b\frac{i\lambda}{2}\right)^m. \tag{5.2}$$

For one soliton $K=1$, the choice of matrices $\widetilde{E}^{(1)},\ E^{(1)}, H_{j,\,1}$ and Λ_1 are taken to be as follows

$$\widetilde{E}^{(1)} = E^{(1)} = I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_{n,\,m,\,1} = \begin{pmatrix} X_{n,\,m,\,1} & X_{n,\,m,\,1} \\ Y_{n,\,m,\,1} & -Y_{n,\,m,\,1} \end{pmatrix},$$
$$\Lambda_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix}, \qquad M_{n,\,m,\,1}\Lambda_1 = \begin{pmatrix} \lambda_1 X_{n,\,m,\,1} & -\lambda_1 X_{n,\,m,\,1} \\ \lambda_1 Y_{n,\,m,\,1} & \lambda_1 Y_{n,\,m,\,1} \end{pmatrix}. \tag{5.3}$$

By using equation (5.3) in the equation (3.34) with (3.38), the one-fold DT on the scalar solutions $q_{n,\,m},\ r_{n,\,m}$ of the dCD system are given by

$$q_{n,\,m}[1] = q_{n,\,m} - \frac{i}{2}\left(\Theta^{(1)}_{n,\,m,\,12} + \Theta^{(1)}_{n,\,m,\,21}\right), \tag{5.4}$$
$$= q_{n,\,m} - \frac{i}{2}\left(\begin{vmatrix} X_{n,\,m,\,1} & X_{n,\,m,\,1} & 0 \\ Y_{n,\,m,\,1} & -Y_{n,\,m,\,1} & 1 \\ \lambda_1 X_{n,\,m,\,1} & -\lambda_1 X_{n,\,m,\,1} & \boxed{0} \end{vmatrix} + \begin{vmatrix} X_{n,\,m,\,1} & X_{n,\,m,\,1} & 1 \\ Y_{n,\,m,\,1} & -Y_{n,\,m,\,1} & 0 \\ \lambda_1 Y_{n,\,m,\,1} & \lambda_1 Y_{n,\,m,\,1} & \boxed{0} \end{vmatrix}\right),$$

$$r_{n,\,m}[1] = r_{n,\,m} - \frac{1}{2}\left(\Theta^{(1)}_{n,\,m,\,12} - \Theta^{(1)}_{n,\,m,\,21}\right), \tag{5.5}$$
$$= r_{n,\,m} - \frac{1}{2}\left(\begin{vmatrix} X_{n,\,m,\,1} & X_{n,\,m,\,1} & 0 \\ Y_{n,\,m,\,1} & -Y_{n,\,m,\,1} & 1 \\ \lambda_1 X_{n,\,m,\,1} & -\lambda_1 X_{n,\,m,\,1} & \boxed{0} \end{vmatrix} - \begin{vmatrix} X_{n,\,m,\,1} & X_{n,\,m,\,1} & 1 \\ Y_{n,\,m,\,1} & -Y_{n,\,m,\,1} & 0 \\ \lambda_1 Y_{n,\,m,\,1} & \lambda_1 Y_{n,\,m,\,1} & \boxed{0} \end{vmatrix}\right).$$

After simplification, the expressions (5.4) and (5.5) reduce to

$$q_{n,\,m}[1] = q_{n,\,m} + \frac{i}{2}\left(\lambda_1\frac{X_{n,\,m,\,1}}{Y_{n,\,m,\,1}} + \lambda_1\frac{Y_{n,\,m,\,1}}{X_{n,\,m,\,1}}\right), \tag{5.6}$$

$$r_{n,\,m}[1] = r_{n,\,m} + \frac{1}{2}\left(\lambda_1\frac{X_{n,\,m,\,1}}{Y_{n,\,m,\,1}} - \lambda_1\frac{Y_{n,\,m,\,1}}{X_{n,\,m,\,1}}\right). \tag{5.7}$$

The results in equations (5.6) and (5.7) have also been obtained in [19] for the sdCD system. By substituting the set of equations (5.1)-(5.2) in (5.6) and (5.7), explicit calculations for one-soliton with ($\lambda = i\mu$) yield

$$q_{n,\,m}[1] = q_{n,\,m} - \mu_1\frac{\mathcal{C}_{n,\,m} - \mathcal{D}_{n,\,m}}{\mathcal{C}_{n,\,m} + \mathcal{D}_{n,\,m}}, \tag{5.8}$$

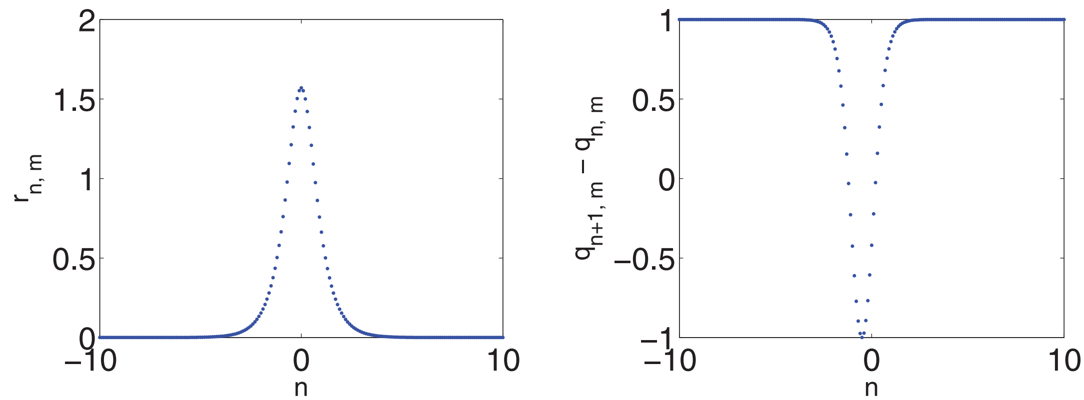
$$r_{n,\,m}[1] = -2\mu_1\frac{\mathcal{E}_{n,\,m}}{\mathcal{C}_{n,\,m} + \mathcal{D}_{n,\,m}}, \tag{5.9}$$

$$q_{n+1,\,m}[1] - q_{n,\,m}[1] = p - 2\mu_1\frac{\mathcal{C}_{n+1,\,m}\mathcal{D}_{n,\,m} - \mathcal{C}_{n,\,m}\mathcal{D}_{n+1,\,m}}{\left(\mathcal{C}_{n+1,\,m} + \mathcal{D}_{n+1,\,m}\right)\left(\mathcal{C}_{n,\,m} + \mathcal{D}_{n,\,m}\right)}, \tag{5.10}$$

with

$$\mathcal{C}_{n,\,m} = \left(1+\mu_1^{-1}p\right)^{2n}\left(1+\frac{\mu_1}{2}b\right)^{2m}, \quad \mathcal{D}_{n,\,m} = \left(1-\mu_1^{-1}p\right)^{2n}\left(1-\frac{\mu_1}{2}b\right)^{2m},$$
$$\mathcal{E}_{n,\,m} = \left(1+\mu_1^{-1}p\right)^{n}\left(1-\mu_1^{-1}p\right)^{n}\left(1+\frac{\mu_1}{2}b\right)^{m}\left(1-\frac{\mu_1}{2}b\right)^{m}. \tag{5.11}$$

The plots of equations (5.9), (5.10) have been sketched in Figure 1



(a) plot of solution (5.9)

(b) plot of solution (5.10)

Figure 1

In the continuum limit, i.e., when both the lattice parameters approach to zero, the solutions obtained in (5.8)-(5.10), can be reduced to those obtained in [8], [13] given by

$$q[1] \;=\; q-\mu_1\tanh z, \quad r[1] = -\mu_1 \mathrm{sech} z, \tag{5.12}$$
$$\partial_x q[1] \;=\; p\left(1-2\mathrm{sech}^2 z\right), \quad z=\frac{2p}{\mu_1}x+\mu_1 t. \tag{5.13}$$

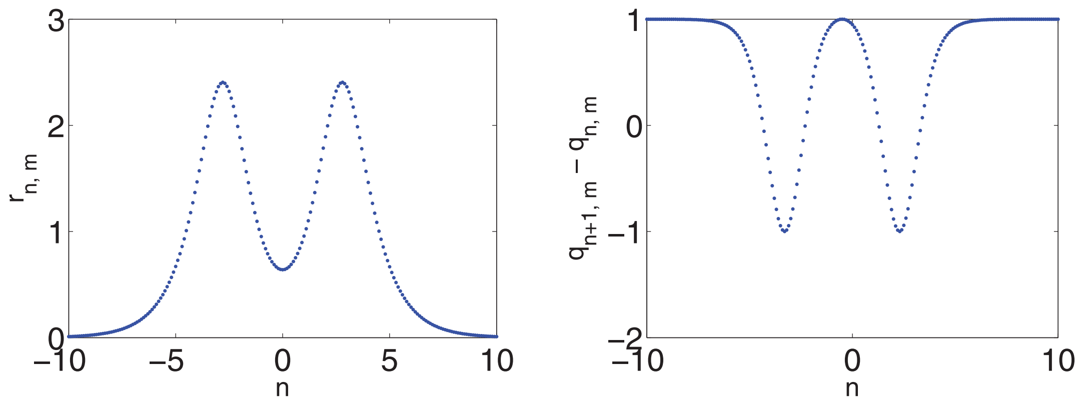
Similarly, after a tedious calculation, explicit expressions of two-soliton solutions of the dGCD system are given by

$$q_{n,\,m}[2] \;=\; q_{n,\,m} - \left(\mu_2^2-\mu_1^2\right)\frac{\xi_{n,\,m}}{\chi_{n,\,m}}, \tag{5.14}$$
$$r_{n,\,m}[2] \;=\; -2\left(\mu_2^2-\mu_1^2\right)\frac{\zeta_{n,\,m}}{\chi_{n,\,m}}, \tag{5.15}$$
$$q_{n+1,\,m}[2]-q_{n,\,m}[2] \;=\; p-\left(\mu_2^2-\mu_1^2\right)\frac{\xi_{n+1,\,m}\chi_{n,\,m}-\xi_{n,\,m}\chi_{n+1,\,m}}{\chi_{n+1,\,m}\chi_{n,\,m}}, \tag{5.16}$$

where

$$\xi_{n,\,m} = \mu_2\left(\mathcal{C}_{n,\,m,\,2}-\mathcal{D}_{n,\,m,\,2}\right)\left(\mathcal{C}_{n,\,m,\,1}+\mathcal{D}_{n,\,m,\,1}\right)-\mu_1\left(\mathcal{C}_{n,\,m,\,1}-\mathcal{D}_{n,\,m,\,1}\right)\left(\mathcal{C}_{n,\,m,\,2}+\mathcal{D}_{n,\,m,\,2}\right),$$
$$\zeta_{n,\,m} = \mu_2\left(\mathcal{C}_{n,\,m,\,1}+\mathcal{D}_{n,\,m,\,1}\right)\mathcal{E}_{n,\,m,\,2}-\mu_1\left(\mathcal{C}_{n,\,m,\,2}+\mathcal{D}_{n,\,m,\,2}\right)\mathcal{E}_{n,\,m,\,1},$$
$$\chi_{n,\,m} = \left(\mu_1^2+\mu_2^2\right)\left(\mathcal{C}_{n,\,m,\,1}+\mathcal{D}_{n,\,m,\,1}\right)\left(\mathcal{C}_{n,\,m,\,2}+\mathcal{D}_{n,\,m,\,2}\right)$$
$$-2\mu_1\mu_2\left[\left(\mathcal{C}_{n,\,m,\,1}-\mathcal{D}_{n,\,m,\,1}\right)\left(\mathcal{C}_{n,\,m,\,2}-\mathcal{D}_{n,\,m,\,2}\right)+4\mathcal{E}_{n,\,m,\,1}\mathcal{E}_{n,\,m,\,2}\right].$$

where $\mathcal{C}_{n,\,m,\,1}$, $\mathcal{D}_{n,\,m,\,1}$ and $\mathcal{C}_{n,\,m,\,2}$, $\mathcal{D}_{n,\,m,\,2}$ are the solutions at μ_1 and μ_2 respectively. Similarly by calculating explicit expressions of matrix elements from the matrix $S_{n,\,m}$, we get the multi-soliton solutions of dCD system.



(a) plot of solution (5.15)

(b) plot of solution (5.16)

Figure 2

By performing a continuous limit along the x-direction ($a \to 0$), we get the one- and two-soliton solutions of the sdCD continuous in x-direction i.e.

$$q_m[1] = q_m - \mu_1 \frac{\mathcal{C}_m - \mathcal{D}_m}{\mathcal{C}_m + \mathcal{D}_m}, \tag{5.17}$$

$$r_m[1] = -2\mu_1 \frac{\mathcal{E}_m}{\mathcal{C}_m + \mathcal{D}_m}, \tag{5.18}$$

$$\frac{d}{dx} q_m[1] = p - 8 \frac{\mathcal{C}_m \mathcal{D}_m}{(\mathcal{C}_m + \mathcal{D}_m)^2}, \tag{5.19}$$

and

$$q_m[2] = q_m - \left(\mu_2^2 - \mu_1^2\right) \frac{\xi_m}{\chi_m}, \tag{5.20}$$

$$r_m[2] = -2\left(\mu_2^2 - \mu_1^2\right) \frac{\zeta_m}{\chi_m}, \tag{5.21}$$

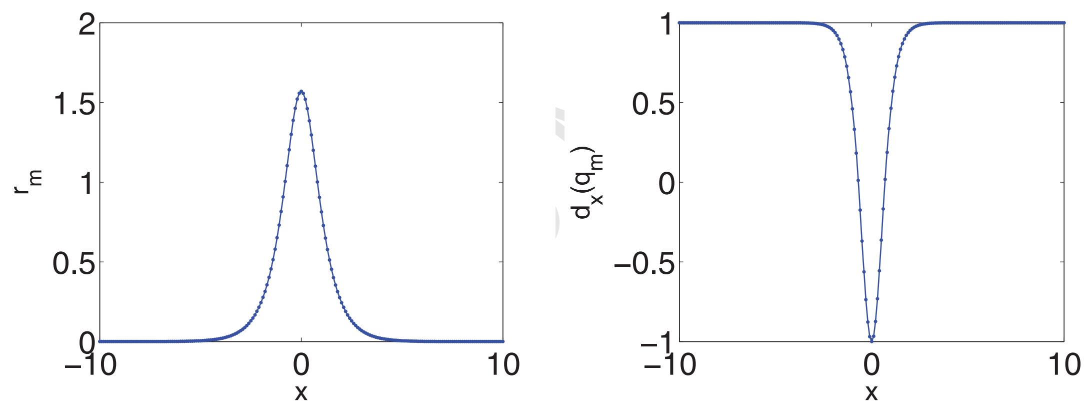
$$\frac{d}{dx} q_m[2] = p - 32\left(\mu_2^2 - \mu_1^2\right) \frac{\varphi_m}{\chi_m^2}, \tag{5.22}$$

where

$$
\begin{aligned}
\xi_m &= \mu_2\left(\mathcal{C}_{m,\,2} - \mathcal{D}_{m,\,2}\right)\left(\mathcal{C}_{m,\,1} + \mathcal{D}_{m,\,1}\right) - \mu_1\left(\mathcal{C}_{m,\,1} - \mathcal{D}_{m,\,1}\right)\left(\mathcal{C}_{m,\,2} + \mathcal{D}_{m,\,2}\right),\\
\zeta_m &= \mu_2\left(\mathcal{C}_{m,\,1} + \mathcal{D}_{m,\,1}\right)\mathcal{E}_{m,\,2} - \mu_1\left(\mathcal{C}_{m,\,2} + \mathcal{D}_{m,\,2}\right)\mathcal{E}_{m,\,1},\\
\varphi_m &= [(\mathcal{C}_{m,\,1} - \mathcal{D}_{m,\,1})^2\mathcal{C}_{m,\,2}\mathcal{D}_{m,\,2} + (\mathcal{C}_{m,\,2} - \mathcal{D}_{m,\,2})^2\mathcal{C}_{m,\,1}\mathcal{D}_{m,\,1}\\
&\qquad - 2(\mathcal{C}_{m,\,1} - \mathcal{D}_{m,\,1})(\mathcal{C}_{m,\,2} - \mathcal{D}_{m,\,2})\mathcal{E}_{m,\,1}\mathcal{E}_{m,\,2}],\\
\chi_m &= \left(\mu_1^2 + \mu_2^2\right)\left(\mathcal{C}_{m,\,1} + \mathcal{D}_{m,\,1}\right)\left(\mathcal{C}_{m,\,2} + \mathcal{D}_{m,\,2}\right)\\
&\qquad - 2\mu_1\mu_2\left[\left(\mathcal{C}_{m,\,1} - \mathcal{D}_{m,\,1}\right)\left(\mathcal{C}_{m,\,2} - \mathcal{D}_{m,\,2}\right) + 4\mathcal{E}_{m,\,1}\mathcal{E}_{m,\,2}\right],
\end{aligned}
$$

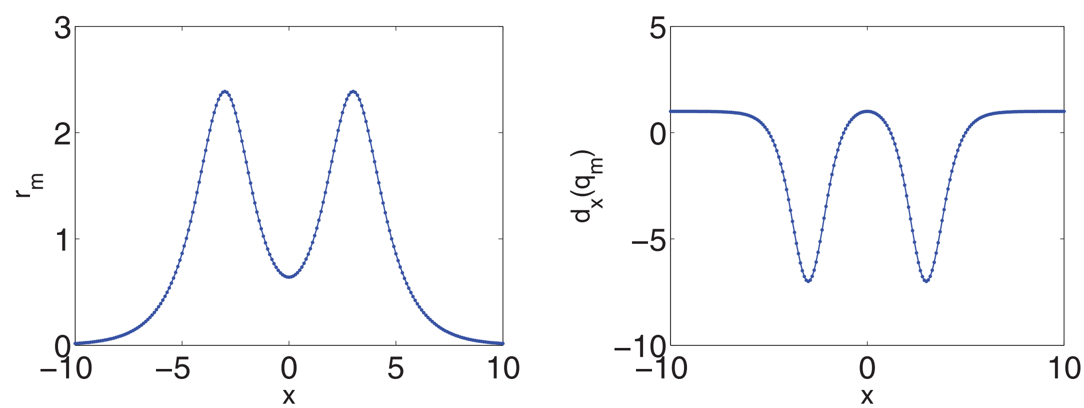
and

$$
\begin{aligned}
\mathcal{C}_{m,\,1,\,2} &= \left(1 + \frac{\mu_{1,\,2}}{2}b\right)^{2m} e^{2\mu_{1,\,2}^{-1}px}, \quad \mathcal{D}_{m,\,1,\,2} = \left(1 - \frac{\mu_{1,\,2}}{2}b\right)^{2m} e^{-2\mu_{1,\,2}^{-1}px},\\
\mathcal{E}_{m,\,1,\,2} &= \left(1 + \frac{\mu_{1,\,2}}{2}b\right)^{m}\left(1 - \frac{\mu_{1,\,2}}{2}b\right)^{m}.
\end{aligned}
\tag{5.23}
$$



(a) plot of solution (5.18) (b) plot of solution (5.19)

Figure 3



(a) plot of solution (5.21) (b) plot of solution (5.22)

Figure 4

Similarly, explicit expressions of one- and two-soliton solutions of sdCD system continuous in t-direction are obtained by performing the continuum limit as $b \to 0$. When we apply the continuum limit along x- and t-direction simultaneously, we obtain soliton solutions of a continuous CD system.

Equations (5.9) and (5.10) represent respectively, the discrete bright- and dark-type one soliton solution of the dGCD system for the $SU(2)$. Similarly bright- and dark-type two soliton solution are also obtained and have been plotted in Figure 2. In the same way, we can also get the multi-soliton of the dGCD system.

6 Concluding remarks

In this paper, we have used Lax pair representation to get an integrable discretization of the generalized coupled dispersionless (dGCD) system. From the fully dGCD, two semi-discrete versions of GCD along x-direction and t-direction, respectively, are obtained. We have also obtained the continuous GCD system from discrete GCD system by reducing discrete variables n and m to continuous variables x and t by taking a continuum limit on the lattice parameters. Multi-soliton solutions are obtained by the action of discrete DT and the solutions are expressed in terms of quasideterminants. Explicit expressions of one- and two-soliton solutions are derived by using properties of quasideterminants.

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