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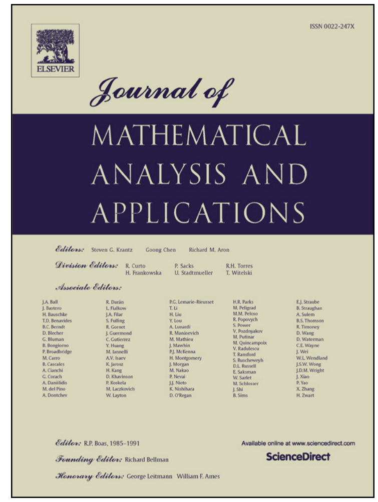

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Peter Hornung, Matthäus Pawelczyk, Igor Velčić

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Peter Hornung
FB Mathematik, TU Dresden
01062 Dresden (Germany)

Matthäus Pawelczyk
FB Mathematik, TU Dresden
01062 Dresden (Germany)

Igor Velčić
University of Zagreb,
Faculty of Electrical Engineering and Computing
Unska 3, Zagreb (Croatia)

Abstract

We use the notion of stochastic two-scale convergence introduced in [ZP06] to solve the problem of stochastic homogenization of the elastic plate in the bending regime.

Keywords: elasticity, dimension reduction, stochastic homogenization, stochastic two-scale convergence.

1 Introduction

The problem of rigorously deriving a two-dimensional model approximating a three-dimensional (nonlinear) elastic plate with very small thickness was long outstanding. It was finally solved in [FJM02] in terms of Γ-convergence after establishing the geometric rigidity estimate. With this estimate they further managed in [FJM06] to derive a multitude of related models. Their results have since been generalized in various directions, e.g., different dimensions involved (e.g., [MM04]), convergence of equilibria instead of convergence of minimizer as Γ-convergence yields (e.g., [MP08, MM04]), or an inhomogeneous plate (e.g., [HNV14, NO15, Vel15]). This paper falls into the last category.

We consider a thin plate with a fine microstructure on the midplane, extended constantly in normal direction. A very similar problem was studied in [HNV14], where the microstructure was assumed to be periodic, while we consider more general random materials and recover their main results as a special case. Another interesting generalization of the periodic case was given in [BDF15], where the microstructure was allowed to oscillate on two different scales $\varepsilon_1(h)$ and $\varepsilon_2(h)$, where the 'coarser' structure dominates the homogenization effect.

Already in the periodic case it was seen that the homogenization and the dimension reduction interact non-trivially with each other. To be more precise let $h > 0$ denote the thickness of the plate, and $\varepsilon(h)$ the 'fineness' of the microstructure, e.g., in the periodic case the length of a periodic cell, at thickness h with $\varepsilon(h) \to 0$ if $h \to 0$ and assume $\gamma = \lim_{h\to 0} h\varepsilon^{-1}(h) \in [0, \infty]$ exists. One might imagine the case $\gamma = \infty$ corresponds to the situation, where we apply purely dimension reduction to an already homogeneous plate, while one could expect $\gamma = 0$ to be the case where a 2D plate is homogenized; the latter, however, is wrong at least in the plate scaling as comparing the results obtained in [Vel15] and [NO15] shows. This intuition, however, holds true for the von Kármán plate [NV13]. The intermediate case $0 < \gamma < \infty$ corresponds to the case, where both effects strongly interact; in some sense thus the most interesting case.

With a periodic microstructure in [HNV14] the range $\gamma \in (0, \infty]$ was covered, excluding the 0 entirely. The methods developed in [Vel15] allows the treatment, at least partially, of the case $\gamma = 0$. Only partially, since we have to assume the microstructure is still sufficiently strong 'homogenizing', i.e. $h \gg (\varepsilon(h))^2$. In [CC15] the authors use a different approach (smoothening and unfolding operator) to deal with the homogenization of plate. Firstly, they recover the result of [Vel15] in the simpler case, when the energy density does not additionally depend on x_3 variable and then they conclude that in the regime when $h \ll (\varepsilon(h))^2$ the limit model is the same as the one obtained in [NO15]. The regime $h \sim (\varepsilon(h))^2$ remains uncovered.

The stochastic homogenization incorporates periodic setting, almost periodic setting, but also some completely non-periodic examples (see the Example 4.2 below). Since it is possible to have the situation where periodicity is completely destroyed and since we are not able to treat all cases of the periodic homogenization, we find that it is important to establish the result on the

stochastic homogenization of the bending plate.
As in [HNV14] we make heavy use of two-scale convergence. The first generalization to the stochastic setting of two-scale convergence was done in [BMW94], which is too crude to recover the information on the limit material. An alternative was introduced by [ZP06], which is more flexible, and which we will use. Recently the notion of an unfolding operator for the stochastic homogenization was introduced in [Neu17]. However, since this notion also averages over the probability space and thus is an analogue of the notion of stochastic two-scale convergence introduced in [BMW94], it is also too crude to recover the results obtained here.
Compared to the periodic setting the identification of the two-scale limits is more involved and subtle. In fact we are not able to recover all the limits derived in [HNV14]. This is due to the lack of the notion of oscillatory convergence in the stochastic setting. Namely, the notion of oscillatory convergence, introduced in [HNV14] and later developed in [BDF15], for the multiscale homogenization, has useful consequences only in the periodic setting. In the stochastic setting one has to completely rely on duality arguments, which can be used by stochastic two-scale convergence.
To cope with this, we use methods developed in [Vel15] and make use of further cancellation effects (see Lemma 3.9 and Lemma 3.10, see also Remark 3.11). Furthermore, the precise relationship between solenoids and potential fields were not proven, in the case where the differential operators div and ∇ were not either purely classical, or purely stochastic derivatives, but mixtures between them. For this we introduce in section A.4.2 the correct notion of mixed potentials and solenoids. In contrast to the purely stochastic case we have to take into account the boundary condition in the physical space. On one hand they have to be chosen restrictive enough to allow the orthogonality property to hold, proved in Lemma A.5, on the other hand they have to complement each other to L^2 in the sense, that Theorem A.4 (i) holds. In the appendix we first recall previous results for the purely stochastic case, and then prove the Helmholtz-decomposition for the mixed one. This decomposition allows us to reveal a gradient structure in the two-scale limits by testing with solenoids, a subclass of functions used in the oscillatory convergence, introduced in [HNV14].
For simplicity, we will state and prove the case $\gamma \in (0, \infty)$, but the other cases covered in [HNV14, Vel15] can be proved analogously. This includes the case $\gamma = \infty$ as well as $\gamma = 0$, under the additional assumption that $\varepsilon(h)^2 \ll h \ll \varepsilon(h)$. The regime $h \sim (\varepsilon(h))^2$ and $h \ll (\varepsilon(h))^2$ remains uncovered.
Without the notion of stochastic two-scale convergence we are not able to solve the problem; the usual approach for stochastic homogenization in the

context of calculus of variation however does not necessarily need stochastic two-scale convergence, cf. [DMM86a, DMM86b] for the convex case and [DG16] for the non-convex case. The main claims of the paper are given in Theorem 3.6 and Theorem 3.8.

Notation

The inner product of the Hilbert space H for the vectors $a, b \in H$ is denoted by $\langle a, b \rangle_H$. If $H = \mathbf{R}^n$ it is also denoted by $a \cdot b$. By $\iota : \mathbf{R}^{2\times 2} \to \mathbf{R}^{3\times 3}$ we denote the natural inclusion

$$\iota(G) = \sum_{\alpha,\beta=1,2} G_{\alpha\beta} e_\alpha \otimes e_\beta.$$

For a matrix $M \in \mathbf{R}^{n\times n}$ we denote its transpose by M^T and by $\operatorname{cof} M$ its cofactor matrix. By $I_{n\times n}$ we denote the identity matrix in $\mathbf{R}^{n\times n}$. By $\nabla' = (\partial_1, \partial_2)$ we denote the gradient with respect to the first two variables and similarly $x' = (x_1, x_2) \in \mathbf{R}^2$ for $x \in \mathbf{R}^3$. For $h > 0$ we furthermore denote by $\nabla_h = (\nabla', \frac{1}{h}\partial_3)$ the scaled gradient. By $\mathbf{R}^{n\times n}_{\text{sym}}$ we denote the space of symmetric matrices, while by $\mathbf{R}^{n\times n}_{\text{skw}}$ we denote the space of antisymmetric matrices. For a normed space X and $S \subset X$ we denote by $\text{adh}_X S$ the closure of the set S in the norm defined on the space X. By $C_0^\infty(S)$ we denote the smooth functions with compact support on S.

2 Stochastic two-scale convergence

2.1 Probability framework

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. We will assume that $\mathcal{F}$ is countably generated which implies that the spaces $L^p(\Omega)$, for $p \in [1, \infty)$, are separable. By S we will denote the domain in $\mathbf{R}^n$. With I we denote the interval $I = [-\frac{1}{2}, \frac{1}{2}]$.

Definition 2.1. *A family $(T_x)_{x\in\mathbf{R}^n}$ of measurable bijective mappings $T_x : \Omega \to \Omega$ on a probability space $(\Omega, \mathcal{F}, P)$ is called a dynamical system on Ω with respect to P if*

a. $T_x \circ T_y = T_{x+y}$;

b. $P(T_x F) = P(F)$, $\forall x \in \mathbf{R}^n$, $F \in \mathcal{F}$;

c. $\mathcal{T} : \Omega \times \mathbf{R}^n \to \Omega$, $(\omega, x) \to T_x(\omega)$ *is measurable (for the standard σ-algebra on the product space, where on $\mathbf{R}^n$ we take the Lebesgue σ-algebra).*

We define the notion of ergodicity for the dynamical system.

Definition 2.2. *A dynamical system is called ergodic if one of the following equivalent conditions is fulfilled*

a. f *measurable,* $f(\omega) = f(T_x\omega)$, $\forall x \in \mathbf{R}^n$, *a.e.* $\omega \in \Omega \implies f(\omega) =$ *const. for* P*-a.e.* $\omega \in \Omega$.

b. $\Big[\forall x \in \mathbf{R}^n, P((T_xB \cup B)\backslash(T_xB \cap B)) = 0\Big] \implies P(B) \in \{0,1\}$.

Remark 2.3. *Note that for the condition b the implication* $P(B) \in \{0,1\}$ *has to hold, if the symmetric difference between* T_xB *and* B *is a null set. It can be shown (e.g., [CFS82]), that ergodicity is also equivalent if a priori only the weaker implication*

$$\Big[\forall x \in \mathbf{R}^N, \quad T_xB = B\Big] \implies P(B) \in \{0,1\}$$

holds. This formulation will however only be used in the appendix to show that the product of an ergodic system with a periodic one is once more ergodic.

On $L^2(\Omega)$ we can define the unitary action

$$U(x)f = f \circ T_x, \quad \forall f \in L^2(\Omega).$$

It can be shown that a, b, c of Definition 2.1 imply that this is a strongly continuous group (see [ZKO94]). We define the operator D_i as the infinitesimal generator of the unitary group U_{x_i}. This means that

$$D_i f(\omega) = \lim_{x_i \to 0} \frac{f(T_{x_i}\omega) - f(\omega)}{x_i},$$

where the limit is taken in L^2 sense. Also we have that $iD_1, \ldots, iD_n$ are commuting, self-adjoint, closed, and densely defined linear operators on the separable Hilbert space $L^2(\Omega)$. The domain $\mathcal{D}_i(\Omega)$ of such an operator is given by the set of L^2 functions for which the limit exists. We denote by $W^{1,2}(\Omega)$ the set

$$W^{1,2}(\Omega) := \mathcal{D}_1(\Omega) \cap \cdots \cap \mathcal{D}_n(\Omega)$$

and similarly

$$W^{k,2}(\Omega) = \{f \in L^2(\Omega) : D_1^{\alpha_1} \ldots D_n^{\alpha_n} f \in L^2(\Omega),\ \alpha_1 + \cdots + \alpha_n = k\};$$
$$W^{\infty,2}(\Omega) = \bigcap_{k \in \mathbf{N}} W^{k,2}(\Omega).$$

On $W^{k,2}$ we define the norm in the usual way. By the standard semigroup property it can be shown that $W^{\infty,2}(\Omega)$ is dense in $L^2(\Omega)$. We also define the space

$$\mathcal{C}^\infty(\Omega) = \left\{ f \in W^{\infty,2}(\Omega) : \forall (\alpha_1, \dots, \alpha_n) \in \mathbf{N}_0^n, \quad D_1^{\alpha_1} \dots D_n^{\alpha_n} f \in L^\infty(\Omega) \right\}.$$

By the smoothening procedure explained below it can be shown that $\mathcal{C}^\infty(\Omega)$ is dense in $L^p(\Omega)$ for any $p \in [1, \infty)$, as well as in $W^{k,2}(\Omega)$ for any k. Notice that $D_i f$, due to the closedness property of the infinitesimal generator, can be equivalently defined as the function that satisfies the property

$$\int_\Omega D_i f g = -\int_\Omega f D_i g, \quad \forall g \in \mathcal{C}^\infty(\Omega).$$

We can identify $f : \Omega \to \mathbf{R}$ with $f : \Omega \times \mathbf{R}^n \to \mathbf{R}$, its realization, given by $f(\omega, x) := f(T_x \omega)$. After identifying $f \in W^{1,2}(\Omega)$ with its realization, one can show that

$$\begin{aligned} W^{1,2}(\Omega) &= \{ f \in W^{1,2}_{\text{loc}}(\mathbf{R}^n, L^2(\Omega)) : f(x+y, \omega) = f(x, T_y \omega), \quad \forall x, y, \text{for a.e. } \omega \} \\ &= \{ f \in C^1(\mathbf{R}^n, L^2(\Omega)) : f(x+y, \omega) = f(x, T_y \omega), \quad \forall x, y, \text{for a.e. } \omega \}. \end{aligned} \tag{1}$$

A proof of this fact can be found in [DG16][Lemma A.7].
As in [ZKO94] we define a stochastic mollifier. For $\varphi \in L^\infty(\Omega)$ and $K \in C_0^\infty(\mathbf{R}^n)$ even, i.e., $K(x) = K(-x)$ for all $x \in \mathbf{R}^n$, we set

$$(\varphi * K)(\omega) := \int_{\mathbf{R}^n} \varphi(T_x \omega) K(x) \mathrm{d}x, \quad \omega \in \Omega.$$

It is easily seen that $\varphi \mapsto \varphi * K$ is well defined and continuous from $L^2(\Omega)$ to $L^2(\Omega)$. By using this mollifier one can show that there exists a countable dense subset of $L^2(\Omega)$ and $W^{1,2}(\Omega)$ (see [BMW94]). Following [SW11] we denote by $\|\cdot\|_{\#,k,2}$ the seminorm on $\mathcal{C}^\infty(\Omega)$ given by

$$\|u\|_{\#,k,2}^2 = \sum_{\alpha \in \mathbb{N}^n, |\alpha| = k} \|D^\alpha u\|_{L^2}^2.$$

By $\mathcal{W}^{k,2}(\Omega)$ we denote the completion of $\mathcal{C}^\infty(\Omega)$ with respect to the seminorm $\|\cdot\|_{\#,k,2}$. The gradient operator $\nabla_\omega = (D_1, \dots, D_n)$ and $\operatorname{div}_\omega = \nabla_\omega \cdot$ operator extend by continuity uniquely to mappings from $\mathcal{W}^{1,2}(\Omega)$ to $L^2(\Omega, \mathbf{R}^n)$, respectively $\mathcal{W}^{1,2}(\Omega, \mathbf{R}^n)$ to $L^2(\Omega)$. By the density argument it is easily seen that $\mathcal{W}^{1,2}(\Omega)$ is also the completion of $W^{1,2}(\Omega)$ in $\|\cdot\|_{\#,1,2}$ seminorm. We also define $\mathcal{W}^{1,2}_{\text{sym}}(\Omega, \mathbf{R}^n)$ as the completion of $\mathcal{C}^\infty(\Omega, \mathbf{R}^n)$ with respect to the seminorm $\|\cdot\|_{\#,\text{sym},2,n}$ defined by

$$\|b\|_{\#,\text{sym},2,n} = \|\operatorname{sym} \nabla b\|_{L^2}, \forall b \in \mathcal{C}^\infty(\Omega, \mathbf{R}^n).$$

2.2 Definition and basic properties

The key property of ergodic systems is the ergodic theorem, due to Birkhoff:

Theorem 2.4 (Ergodic theorem)**.** *Let $(\Omega, \mathcal{F}, P)$ be a probability space with an ergodic dynamical system $(T_x)_{x\in\mathbf{R}^n}$ on Ω. Let $f \in L^1(\Omega)$ be a function and $A \subset \mathbf{R}^n$ be a bounded open set. Then for P-a.e. $\widetilde{\omega} \in \Omega$ we have*

$$\lim_{\varepsilon\to 0} \int_A f(T_{\varepsilon^{-1}x}\widetilde{\omega})\mathrm{d}x = |A| \int_\Omega f(\omega)\mathrm{d}P(\omega). \tag{2}$$

Furthermore, for every $f \in L^p(\Omega)$, $1 \le p \le \infty$, and a.e. $\widetilde{\omega} \in \Omega$, the function $f(\widetilde{\omega}, x) = f(T_x\widetilde{\omega})$ satisfies $f(\widetilde{\omega}, \cdot) \in L^p_{loc}(\mathbf{R}^n)$. For $p < \infty$ we have $f(\widetilde{\omega}, \cdot/\varepsilon) = f(T_{\cdot/\varepsilon}\widetilde{\omega}) \rightharpoonup \int_\Omega f\mathrm{d}P$ weakly in $L^p_{loc}(\mathbf{R}^n)$ as $\varepsilon \to 0$.

Note that the exception set, where (2) doesn't hold, will in general depend on f. The elements $\widetilde{\omega}$ such that (2) holds for every $f \in L^1(\Omega)$ are called typical elements, the corresponding trajectories $(T_x\widetilde{\omega})_{x\in\mathbf{R}^N}$ are called typical trajectories. Note that the separability of $L^1(\Omega)$ implies that almost every $\widetilde{\omega} \in \Omega$ is typical.

In the following we denote by $S \subset \mathbf{R}^n$ a bounded domain, if not otherwise stated. For vector spaces V_1, V_2 we denote by $V_1 \otimes V_2$ the usual tensor product of the spaces V_1, V_2. We define the following notion of stochastic two-scale convergence, a slight variation of the definition given in [ZP06]. We will stay in the L^2 setting, since it suffices for our analysis.

Definition 2.5. *Let $(T_x\widetilde{\omega})_{x\in\mathbf{R}^n}$ be a typical trajectory and (v^ε) a bounded sequence in $L^2(S)$. We say that (v^ε) stochastically weakly two-scale converges to $v \in L^2(\Omega \times S)$ w.r.t. $\widetilde{\omega}$ and we write $v^\varepsilon \xrightharpoonup{2} v$ if*

$$\lim_{\varepsilon\downarrow 0} \int_S v^\varepsilon(x)g(T_{\varepsilon^{-1}x}\widetilde{\omega}, x)\mathrm{d}x = \int_\Omega\int_S v(\omega, x)g(\omega, x)\mathrm{d}x\,\mathrm{d}P(\omega)$$

for all $g \in \mathcal{C}^\infty(\Omega) \otimes C_0^\infty(S)$.
If additionally

$$\|v^\varepsilon\|_{L^2(S)} \to \|v\|_{L^2(\Omega\times S)}$$

holds, then we say (v^ε) strongly two-scale converges to v and write $v^\varepsilon \xrightarrow{2} v$.

For vector-valued functions we define the convergence componentwise.

Remark 2.6. *The convergence of the sequence (v^ε) is defined along a typical trajectory and thus the limit can also be $\widetilde{\omega}$-dependent. We don't write this dependence, since we will always look at the problem on a typical trajectory (which can be imagined to be fixed).*

Remark 2.7. *Note that the two-scale limit is defined on the whole space $\Omega \times S$. Furthermore by density we can extend the space of test functions g to $L^\infty(\Omega) \otimes L^2(S)$.*

Remark 2.8. *Since we will assume that the material oscillates only in the in-plane direction on the domain $S \times I$ we will often use the notion of in-plane two-scale convergence. A uniformly bounded sequence (v^ε) in $L^2(S\times I)$ stochastically weakly two-scale converges to $v \in L^2(\Omega\times S\times I)$ w.r.t. $\widetilde{\omega}$, denoted by $v^\varepsilon \xrightarrow{2} v$, if*

$$\lim_{\varepsilon\downarrow 0} \int_{S\times I} v^\varepsilon(x) g(T_{\varepsilon^{-1}x'}\widetilde{\omega}, x)\mathrm{d}x = \int_\Omega \int_{S\times I} v(\omega, x) g(\omega, x)\mathrm{d}x\, \mathrm{d}P(\omega)$$

for all $g \in \mathcal{C}^\infty(\Omega) \otimes C_0^\infty(S \times I)$. All the properties of the previous stochastic two-scale convergence remain valid for this variation as well.

Sometimes we will make the decomposition for the two-scale limit v

$$v(\widetilde{\omega}, x) = \int_\Omega v(\omega, x)\mathrm{d}P(\omega) + \left(v(\widetilde{\omega}, x) - \int_\Omega v(\omega, x)\mathrm{d}P(\omega) \right),$$

separating the weak limit from the oscillatory part. We will then write

$$v^\varepsilon \xrightarrow{2c} v - \int_\Omega v(\omega, \cdot)\mathrm{d}P(\omega).$$

Proposition 2.9 (Compactness)**.** *Let (v^ε) be a bounded sequence in $L^2(S)$. Then there exists a subsequence (not relabeled) and $v \in L^2(\Omega \times S)$ such that $v^\varepsilon \xrightarrow{2} v$.*

A proof can be found in [ZP06][Lemma 5.1].
The following proposition states the compatibility of strongly convergent sequences with weakly two-scale convergent sequences.

Proposition 2.10.
a. If $(u^\varepsilon) \subset L^2(S)$ is a bounded sequence with $u^\varepsilon \to u$ in $L^2(S)$ for some $u \in L^2(S)$, then, after extending u trivially to $\Omega\times S$, it holds $u^\varepsilon \xrightarrow{2} u$.

b. If $(v^\varepsilon) \subset L^\infty(S)$ is uniformly bounded by a constant and $v^\varepsilon \to v$ strongly in $L^1(S)$ for some $v \in L^\infty(S)$, and if (u^ε) is bounded in $L^2(S)$ with $u^\varepsilon \xrightarrow{2} u$ for some $u \in L^2(\Omega \times S)$, then we have that $v^\varepsilon u^\varepsilon \xrightarrow{2} vu$.

The proof is straightforward. The next lemma is useful to prove the following Lemma 2.12, which gives us the form of stochastic two-scale limits of gradients.

Lemma 2.11. *Let $f \in (L^2(\Omega))^n$ be such that*

$$\int_\Omega f \cdot g = 0, \quad \forall g \in \mathcal{C}^\infty(\Omega, \mathbf{R}^n) \textit{ satisfying } \operatorname{div}_\omega g = 0.$$

Then there exists $\psi \in \mathcal{W}^{1,2}(\Omega)$ such that $f = \nabla_\omega \psi$.

Proof. It is an immediate consequence of Theorem A.1. □

Lemma 2.12. *Let (u^ε) be a bounded sequence in $W^{1,2}(S)$. Then there exist $u^0 \in W^{1,2}(S)$ and $u^1 \in L^2(S, \mathcal{W}^{1,2}(\Omega))$, such that on a subsequence we have*

$$u^\varepsilon \rightharpoonup u^0 \textit{ in } W^{1,2}(S) \quad \textit{and} \quad \nabla u^\varepsilon \stackrel{2}{\rightharpoonup} \nabla u^0 + \nabla_\omega u^1.$$

Proof. The statement follows immediately from the previous lemma in the same way as in the periodic case (see [All02]). □

Similar results hold for second gradients. We will prove it for the case $n = 2$ in the next two lemmas (for the proof of the slightly more general claim in the periodic setting by the duality arguments, see [Vel13, Lemma 3.8]).

Lemma 2.13. *Let $f \in L^2(\Omega, \mathbf{R}^{2\times 2}_{\mathrm{sym}})$ be such that*

$$\int_\Omega f \cdot \operatorname{cof} \nabla_\omega g = 0, \quad \forall g \in \mathcal{C}^\infty(\Omega, \mathbf{R}^2).$$

Then there exists $\psi \in \mathcal{W}^{2,2}(\Omega)$ such that $f = \nabla^2_\omega \psi$.

Proof. It is an immediate consequence of Theorem A.2. □

Lemma 2.14. *Let $S \subset \mathbf{R}^2$ be a bounded domain and let (u^ε) be a bounded sequence in $W^{2,2}(S)$. Then there exists $u^0 \in W^{2,2}(S)$ and $u^1 \in L^2(S, \mathcal{W}^{2,2}(\Omega))$ such that on a subsequence we have*

$$u^\varepsilon \rightharpoonup u^0 \quad \textit{in } W^{2,2}(S) \quad \textit{and} \quad \nabla^2 u^\varepsilon \stackrel{2}{\rightharpoonup} \nabla^2 u^0 + \nabla^2_\omega u^1 .$$

Proof. The existence of such an u^0 follows by classical compactness. By Prop. 2.9 there exists $f \in L^2(\Omega \times S, \mathbf{R}^{2\times 2})$ and a subsequence with

$$\nabla^2 u^\varepsilon \stackrel{2}{\rightharpoonup} \nabla^2 u^0 + f.$$

Since $\nabla^2 u^\varepsilon - \nabla^2 u^0 \in \mathbf{R}^{2\times 2}_{\mathrm{sym}}$ almost everywhere on S, we get $f \in \mathbf{R}^{2\times 2}_{\mathrm{sym}}$ almost everywhere on $\Omega \times S$. Thus by Lemma 2.13 it suffices to show that for almost every $x \in S$ we have

$$\int_\Omega f(x,\omega) \cdot \operatorname{cof} \nabla_\omega g(\omega) = 0, \quad \forall g \in \mathcal{C}^\infty(\Omega, \mathbf{R}^2).$$

For this fix some $g \in \mathcal{C}^\infty(\Omega, \mathbf{R}^2)$ and $\varphi \in C_0^\infty(S)$. Then by definition of two-scale convergence we have

$$\begin{aligned}
&\int_S \int_\Omega f(\omega, x) \cdot \operatorname{cof} \nabla_\omega g(\omega) \varphi(x) \mathrm{d}P(\omega) \mathrm{d}x \\
&= \lim_{\varepsilon \downarrow 0} \int_S \big(\nabla^2 u^\varepsilon(x) - \nabla^2 u^0(x)\big) \cdot (\operatorname{cof} \nabla_\omega g)(T_{\varepsilon^{-1}x}\widetilde{\omega})\varphi(x)\mathrm{d}x \\
&= \lim_{\varepsilon \downarrow 0} \int_S \operatorname{cof}\big(\nabla^2 u^\varepsilon(x) - \nabla^2 u^0(x)\big) \cdot \varepsilon \nabla \Big(g(T_{\varepsilon^{-1}x}\widetilde{\omega})\varphi(x)\Big)\mathrm{d}x \\
&\quad - \lim_{\varepsilon \downarrow 0} \varepsilon \int_S \operatorname{cof}\big(\nabla^2 u^\varepsilon(x) - \nabla^2 u^0(x)\big) \cdot \Big[g(T_{\varepsilon^{-1}x}\widetilde{\omega}) \otimes \nabla\varphi(x)\Big]\mathrm{d}x.
\end{aligned}$$

The first term vanishes identically, since div cof $\nabla v = 0$ distributionally for all $v \in W^{1,2}(S, \mathbf{R}^2)$, while the second one vanishes by the uniform bound on the integral. Since φ and g were arbitrary, the claim follows. □

In the periodic case the purely oscillatory two-scale convergence turns out to be a good concept (see e.g., [HNV14]). The test functions considered there were fast oscillating periodic functions with vanishing mean value. Since in the periodic case this implies a predictable rate of convergence, strong results have been obtained. We have to rely on Birkhoff's Ergodic Theorem (Theorem 2.4), which cannot provide such information. Instead we focus on derivatives of test functions, which naturally have vanishing mean value. The following lemma states that we do not lose information by restricting ourselves to this smaller class of functions.

Lemma 2.15. *The set* $\{\operatorname{div}_\omega v : \ v \in \mathcal{C}^\infty(\Omega, \mathbf{R}^n)\}$ *is dense in*

$$\Big\{ b \in L^2(\Omega) \Big| \int_\Omega b(\omega)\mathrm{d}P(\omega) = 0 \Big\},$$

with respect to the strong $L^2(\Omega)$ *topology.*

Proof. See [ZP06][Lemma 2.5]. □

The following lemma is needed for proving Lemma 3.9, which in turn is essential for proving Theorem 3.6.

Lemma 2.16. *Let* $(f^\varepsilon) \subset W^{1,2}(S)$, $(g^\varepsilon) \subset W^{1,2}(S)$ *uniformly bounded in these spaces, and converging weakly in* $W^{1,2}(S)$ *to* f^0, *respectively* g^0. *Assume further that*

$$\limsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \|f^\varepsilon g^\varepsilon\|_{L^1} < \infty,$$

and that there exist $\phi^f, \phi^g \in L^2(S, \mathcal{W}^{1,2}(\Omega))$ *with*

$$\nabla' f^\varepsilon \xrightharpoonup{2} \nabla' f^0 + \nabla_\omega \phi^f, \qquad \nabla' g^\varepsilon \xrightharpoonup{2} \nabla' g^0 + \nabla_\omega \phi^g.$$

Then for every $v \in \mathcal{C}^\infty(\Omega, \mathbf{R}^n)$ *and* $\varphi \in C_0^\infty(S)$ *we have*

$$\int_S \frac{f^\varepsilon g^\varepsilon(x)}{\varepsilon} (\operatorname{div}_\omega v)(T_{\varepsilon^{-1}x}\widetilde{\omega})\varphi(x)\mathrm{d}x \to \int_{\Omega\times S} \Big(\phi^f(\omega, x)\cdot g^0(x) + f^0(x)\cdot\phi^g(\omega,x)\Big)\operatorname{div}_\omega v(\omega)\varphi(x)\mathrm{d}x\mathrm{d}P(\omega). \tag{3}$$

Proof. The proof consists in an integration by parts:

$$\begin{aligned}
&\int_S \frac{f^\varepsilon g^\varepsilon(x)}{\varepsilon} (\operatorname{div}_\omega v)(T_{\varepsilon^{-1}x}\widetilde{\omega})\varphi(x)\mathrm{d}x \\
&= -\int_S \nabla'(f^\varepsilon g^\varepsilon)(x)\cdot v(T_{\varepsilon^{-1}x}\widetilde{\omega})\varphi(x)\mathrm{d}x - \int_S (f^\varepsilon g^\varepsilon)(x)\nabla'\varphi(x)\cdot v(T_{\varepsilon^{-1}x}\widetilde{\omega})\mathrm{d}x \\
&= -\int_S \Big(\nabla' f^\varepsilon(x)\cdot g^\varepsilon(x) + f^\varepsilon(x)\cdot\nabla' g^\varepsilon(x)\Big)\cdot v(T_{\varepsilon^{-1}x}\widetilde{\omega})\varphi(x)\mathrm{d}x \\
&\quad - \varepsilon\int_S \frac{(f^\varepsilon g^\varepsilon)}{\varepsilon}(x)\nabla'\varphi(x)\cdot v(T_{\varepsilon^{-1}x}\widetilde{\omega})\mathrm{d}x \\
&\quad \to -\int_S\int_\Omega \Big(\nabla_\omega\phi^f(\omega,x)\cdot g(x) + f(x)\cdot\nabla_\omega\phi^g(\omega,x)\Big)\cdot v(\omega)\varphi(x)\mathrm{d}x\mathrm{d}P(\omega).
\end{aligned}$$

The claim now follows after integrating by parts once more, this time in ω. □

Remark 2.17. *The right-hand side in* (3) *actually makes sense only via an integration by parts since we do not have that* $\phi^f(x,\cdot)g(x) + f(x)\phi^g(x,\cdot) \in L^2(\Omega)$*, for a.e.* $x \in S$*. However, if we knew that there exists* $h \in L^2(\Omega\times S)$ *such that for all* $v \in \mathcal{C}^\infty(\Omega, \mathbf{R}^n)$ *and* $\varphi \in C_0^\infty(S)$ *we have*

$$-\int_{\Omega\times S}\big(\nabla_\omega\phi^f\cdot g + f\cdot\nabla_\omega\phi^g\big)\cdot v\varphi\,\mathrm{d}x\,\mathrm{d}P(\omega) = \int_{\Omega\times S} h(\operatorname{div}_\omega v)\varphi\,\mathrm{d}x\,\mathrm{d}P(\omega),$$

then we would be able to conclude, by the closedness property of the operator ∇_ω*, that* $\phi^f\cdot g + f\cdot\phi^g \in L^2(\Omega\times S)$*. This will be used in the proof of Lemma 3.9.*

We now introduce the 'mixed' spaces. The integral of $L^2(\Omega)$-valued functions will be in the sense of Bochner. For $A \subset \mathbf{R}^n$ measurable we can define the space $W^{1,2}(A, L^2(\Omega))$ in the usual way. Notice that, since $L^2(\Omega)$ is a separable

Hilbert space, the analysis has many analogies with the analysis in $\mathbf{R}^n$ (see, e.g., [Kre15] when the target space is a general Banach space).
For the main part of the paper we only need $A = I$, the one-dimensional interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. In the appendix we will however make use of this more general notion.
In the case $A = I$ we denote by D_{x_3} the derivative of $f : \Omega \times I \to \mathbf{R}$ in the I-component, i.e., the differential operator mapping $W^{1,2}(I, L^2(\Omega))$ to $L^2(\Omega \times I)$. We define the space $W^{1,2}(\Omega \times I)$ as the space

$$W^{1,2}(\Omega \times I) = W^{1,2}(I, L^2(\Omega)) \cap L^2(I, W^{1,2}(\Omega)). \tag{4}$$

On the space $W^{1,2}(\Omega \times I)$ we again define the seminorm $\|\cdot\|_{\#,2}$ in the following way

$$\|u\|_{\#,2}^2 = \|D_1 u\|_{L^2(\Omega\times I)}^2 + \|D_2 u\|_{L^2(\Omega\times I)}^2 + \|D_{x_3} u\|_{L^2(\Omega\times I)}^2.$$

By $\mathcal{W}^{1,2}(\Omega \times I)$ we denote the completion of the space $W^{1,2}(\Omega \times I)$ with respect to the seminorm $\|\cdot\|_{\#,2}$. By a density argument it can also be seen as the completion of the space $\mathcal{C}^\infty(\Omega) \otimes C^\infty(I)$ with the same norm. We can also naturally define the operators ∇ and div on $\mathcal{W}^{1,2}(\Omega \times I)$ resp. $\mathcal{W}^{1,2}(\Omega \times I, \mathbf{R}^3)$. For $\gamma > 0$ we also define $\mathcal{W}^{1,2}_{\mathrm{sym},\gamma}(\Omega \times I, \mathbf{R}^3)$ as the completion of the space $(\mathcal{C}^\infty(\Omega) \otimes C^\infty(I))^3$ with respect to the seminorm $\|\cdot\|_{\#,\mathrm{sym},\gamma,2}$ given by

$$\|b\|_{\#,\mathrm{sym},\gamma,2} = \|\operatorname{sym}(D_1 b, D_2 b, \tfrac{1}{\gamma} D_{x_3} b)\|_{L^2}, \quad \forall b \in (\mathcal{C}^\infty(\Omega) \otimes C^\infty(I))^3.$$

The following lemma is useful for proving Lemma 2.19.

Lemma 2.18. *Let $\gamma > 0$ and $f \in L^2(\Omega \times I, \mathbf{R}^3)$ be such that*

$$\int_{\Omega\times I} f \cdot g = 0, \quad \forall g \in (\mathcal{C}^\infty(\Omega) \otimes C_0^\infty(I))^3 \text{ that satisfy}$$
$$D_1 g_1 + D_2 g_2 + \tfrac{1}{\gamma} D_{x_3} g_3 = 0.$$

Then there exists $\psi \in \mathcal{W}^{1,2}(\Omega \times I)$ such that

$$f = (D_1\psi, D_2\psi, \tfrac{1}{\gamma} D_{x_3}\psi).$$

Proof. This follows immediately from the decomposition and density result in Theorem A.4. □

We will now assume $\varepsilon = \varepsilon(h)$ depends additionally on $h > 0$ and satisfies $\varepsilon(h) \downarrow 0$ if $h \downarrow 0$. The definition of two-scale convergence extends naturally to sequences $(v^h)_{h>0}$. We assume further that

$$\gamma := \lim_{h\downarrow 0} \frac{h}{\varepsilon(h)} \in (0, \infty) \tag{5}$$

is well-defined. In the sequel we will often suppress the dependence of $\varepsilon(h)$ on h.
Similar to Lemma 2.11 implying Lemma 2.12, we can prove the following lemma, using Lemma 2.18.

Lemma 2.19. *Let $\gamma > 0$ be given by (5) and let $S \subset \mathbf{R}^2$ a bounded domain. Let (u^h) be a bounded sequence in $L^2(S \times I)$, such that the sequence of scaled gradients $(\nabla_h u^h)$ is bounded in $L^2(S \times I, \mathbf{R}^3)$. Assume further there exists $u^0 \in W^{1,2}(S \times I)$ such that $u^h \to u^0$ strongly in $L^2(S \times I)$. Then there exists a subsequence $h_k \to 0$, and $u^1 \in L^2(S, \mathcal{W}^{1,2}(\Omega \times I))$ such that*

$$\nabla_{h_k} u^{h_k} \xrightarrow{2} (\nabla' u^0, 0) + \big(D_1 u^1, D_2 u^1, \tfrac{1}{\gamma} D_{x_3} u^1\big).$$

Proof. The proof relies on the previous lemma and works in the same way as in periodic case (see [Neu10] for details). □

The following Lemma 2.21 shows that convex functionals are compatible with the concept of stochastic two-scale convergence. In the stochastic setting we cannot rely on the unfolding operator (see e.g., [Vis07] for the periodic case) and thus we require more to obtain the continuity of integral functionals with respect to strong stochastic two-scale convergence (see Remark 2.22). Before stating and proving the lemma we give the following definition:

Definition 2.20. *Consider a measurable map $Q : \Omega \times \mathbf{R}^n \times \mathbf{R}^m \to [0, +\infty]$. We say that Q is T-stationary if for a.e. $(\omega, x, y, v) \in \Omega \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^m$ we have*

$$Q(T_y\omega, x, v) = Q(\omega, x + y, v).$$

By $Q^0 : \Omega \times \mathbf{R}^m \to [0, \infty)$ we denote the mapping $Q^0(\omega, v) = Q(\omega, 0, v)$. Without loss of generality we can assume that for a.e. $x \in \mathbf{R}^n$, for all $v \in \mathbf{R}^m$ we have $Q(\omega, x, v) = Q^0(T_x\omega, v)$.

Lemma 2.21. *Let (u^ε) be a bounded sequence in $L^2(S, \mathbf{R}^m)$, such that $u^\varepsilon \xrightarrow{2} u^0 \in L^2(\Omega \times S, \mathbf{R}^m)$. Let $Q^0 : \Omega \times \mathbf{R}^n \times \mathbf{R}^m \to [0, \infty)$ be a T-stationary map such that $Q^0(\omega, x, \cdot)$ is a convex function for a.e. $(\omega, x) \in \Omega \times \mathbf{R}^n$. Assume additionally that there exists a constant $C > 0$ such that $Q^0(\omega, x, v) \leq C(1 + |v|^2)$, for a.e. $(\omega, x) \in \Omega \times \mathbf{R}^n$, for all $v \in \mathbf{R}^m$. Then for a.e. $\widetilde{\omega} \in \Omega$ we have*

$$\begin{aligned} \liminf_{\varepsilon \downarrow 0} \int_S Q^0(\widetilde{\omega}, x/\varepsilon, u^\varepsilon(x))\mathrm{d}x &= \liminf_{\varepsilon \downarrow 0} \int_S Q^0(T_{\varepsilon^{-1}x}\widetilde{\omega}, u^\varepsilon(x))\mathrm{d}x \\ &\geq \int_S \int_\Omega Q^0\big(\omega, u^0(\omega, x)\big)\mathrm{d}P(\omega)\mathrm{d}x. \end{aligned}$$

If additionally $u^\varepsilon \xrightarrow{2} u^0$*, then*

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\int_S Q^0(\widetilde{\omega}, x/\varepsilon, u^\varepsilon(x))\mathrm{d}x &= \lim_{\varepsilon\downarrow 0}\int_S Q^0(T_{\varepsilon^{-1}x}\widetilde{\omega}, u^\varepsilon(x))\mathrm{d}x \\ &= \int_S\int_\Omega Q^0\big(\omega, u^0(\omega,x)\big)\mathrm{d}P(\omega)\mathrm{d}x\end{aligned}$$

holds for almost every $\widetilde{\omega}\in\Omega$.

Proof. We start with the lower semicontinuity: Let $(u^\varepsilon)\subset L^2(S,\mathbf{R}^m)$ be uniformly bounded with $u^\varepsilon \xrightharpoonup{2} u^0$, where $u^0\in L^2(\Omega\times S,\mathbf{R}^m)$. Then take a subsequence such that

$$\liminf_{\varepsilon\downarrow 0}\int_S Q^0(T_{\varepsilon^{-1}x}\widetilde{\omega}, u^\varepsilon(x))\mathrm{d}x = \lim_{k\to\infty}\int_S Q^0\Big(T_{\varepsilon_k^{-1}x}\widetilde{\omega}, u^{\varepsilon_k}(x)\Big)\mathrm{d}x.$$

Denote these limits by $M\in[0,\infty]$. If $M=\infty$, then there is nothing to show. Else we have

$$\Big(Q^0\big(T_{\varepsilon_k^{-1}x}\widetilde{\omega}, u^{\varepsilon_k}(x)\big)\Big)_{k\in\mathbb{N}}\subset L^1(S),$$

with a uniform bound. Thus we may extract another subsequence (not relabeled) such that the sequence converges weakly-$*$ in measure to some μ. By the lower semicontinuity we have $\mu(S)\le M$. We will show that

$$\int_\Omega Q^0\big(\omega, u^0(\omega,x)\big)\mathrm{d}P(\omega)\le\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^n}(x),$$

for almost every $x\in S$, where the right-hand side represents the Radon-Nikodym derivative in x, i.e.,

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mathcal{L}^n}(x) := \lim_{r\downarrow 0}\frac{\mu(B_r(x))}{\mathcal{L}^n(B_r(x))}.$$

Let x^* be a Lebesgue point of $x\mapsto\int_\Omega u^0(\omega,x)\mathrm{d}P(\omega)$ such that the limit

$$h(x^*) = \lim_{r\downarrow 0}\frac{\mu(B_r(x^*))}{\mathcal{L}^n(B_r(x^*))}$$

exists and such that $u^0(\cdot,x^*)\in L^2(\Omega,\mathbf{R}^m)$. Let $a\colon\Omega\to\mathbf{R}^m, b\colon\Omega\to\mathbf{R}$ be measurable, bounded functions with

$$a(\omega)\cdot v + b(\omega)\le Q^0(\omega,v),\qquad\text{for all } v\in\mathbf{R}^m \text{ and almost every } \omega\in\Omega. \tag{6}$$

For a.e. $r > 0$ we have $\mu(\partial(B_r(x^*))) = 0$ and for these r we get

$$\begin{aligned}\mu(B_r(x^*)) &= \lim_{k\to\infty} \int_{B_r(x^*)} Q^0(T_{\varepsilon_k^{-1}x}\widetilde{\omega}, u^{\varepsilon_k}(x))\mathrm{d}x \\ &\geq \lim_{k\to\infty} \int_{B_r(x^*)} \Big(a(T_{\varepsilon_k^{-1}x}\widetilde{\omega}) \cdot u^{\varepsilon_k}(x) + b(T_{\varepsilon_k^{-1}x}\widetilde{\omega})\Big)\,\mathrm{d}x \\ &= \int_{B_r(x^*)} \int_\Omega \big(a(\omega)\cdot u^0(\omega, x) + b(\omega)\big)\,\mathrm{d}P(\omega)\mathrm{d}x,\end{aligned}$$

where we used that $u^\varepsilon \xrightharpoonup{2} u^0$. Therefore for a suitable sequence $r \downarrow 0$ we have

$$\begin{aligned}h(x^*) &\geq \lim_{r\downarrow 0} \frac{1}{|B_r(x^*)|} \int_{B_r(x^*)} \int_\Omega \big(a(\omega)\cdot u^0(\omega, x) + b(\omega)\big)\,\mathrm{d}P(\omega)\mathrm{d}x \\ &= \int_\Omega \big(a(\omega)\cdot u^0(\omega, x^*) + b(\omega)\big)\,\mathrm{d}P(\omega).\end{aligned}$$

By taking the supremum over the functions a, b satisfying (6) we obtain

$$h(x^*) \geq \int_\Omega Q^0\big(\omega, u^0(\omega, x^*)\big)\mathrm{d}P(\omega).$$

Integrating both sides w.r.t. x^* yields the first claim.
For the continuity assume that $u^\varepsilon \xrightarrow{2} u^0$ and assume that $u^0 \in (L^2(\Omega) \otimes L^2(S))^m$. Then from the strong two-scale convergence follows

$$\|u^\varepsilon(x) - u^0(T_{\varepsilon^{-1}x}\widetilde{\omega}, x)\|_{L^2(S)} \to 0.$$

From the convexity and the uniform bound of Q follows the existence of a constant $C > 0$ such that for almost every $\omega \in \Omega$ it holds

$$|Q^0(\omega, v_1) - Q^0(\omega, v_2)| \leq C(1 + |v_1| + |v_2|)|v_1 - v_2|, \forall v_1, v_2 \in \mathbf{R}^n. \tag{7}$$

Using (7) and the Ergodic theorem we conclude

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} \int_S Q^0(T_{\varepsilon^{-1}x}\widetilde{\omega}, u^\varepsilon(x))\mathrm{d}x &= \lim_{\varepsilon\downarrow 0} \int_S Q^0(T_{\varepsilon^{-1}x}\widetilde{\omega}, u^0(T_{\varepsilon^{-1}x'}\widetilde{\omega}, x))\mathrm{d}x \\ &= \int_\Omega \int_S Q^0(\omega, u^0(\omega, x))\mathrm{d}x\mathrm{d}P(\omega).\end{aligned}$$

For general $u^0 \in L^2(\Omega \times S, \mathbf{R}^m)$ the claim follows by approximation and using (7). $\square$

Remark 2.22. *Notice that in the proof of the second claim we only use the relation (7), but not the convexity. Notice also that for $u^0 \in L^2(\Omega \times S, \mathbf{R}^m)$ the function $x \mapsto u^0(T_{\varepsilon^{-1}x}\widetilde{\omega}, x)$ does not need to be measurable (see [BMW94] for details). This is why we proved it first for $u^0 \in (L^2(\Omega) \otimes L^2(S))^m$ and then argued by density.*

Remark 2.23. *Lemma 2.21 also holds for bounded sequences in $L^2(S \times I, \mathbf{R}^m)$ which stochastically two-scale converge in the sense of Remark 2.8.*

3 Homogenization of the plate model

3.1 General framework and main result

In this chapter $S \subset \mathbf{R}^2$ is a bounded domain and the interval $I = [-\frac{1}{2}, \frac{1}{2}]$. Let γ be as in (5). The main results are Theorem 3.6 (lower bound) and Theorem 3.8 (upper bound). To prove the Γ-limit result we will need some additional assumption on the domain S. We will assume that the domain S is piecewise C^1. This assumption is necessary only for the proof of upper bound, and can be weakened (see Theorem 3.8 for a precise definition). For the lower bound we only require S to be a Lipschitz domain.
Consider a measurable map $W : \Omega \times \mathbf{R}^2 \times \mathbf{R}^{3\times3} \to [0, +\infty]$, representing the stored energy density function, satisfying the following:

Assumption 3.1. *We assume that W is T-stationary as in Definition 2.20 and that $W(\omega, x', \cdot)$ is continuous on $\mathbf{R}^{3\times3}$ for a.e. $(\omega, x') \in \Omega \times \mathbf{R}^2$. This will ensure the measurability of all composition mappings that appear (see, e.g., the expression (10)) We also assume that the following properties are satisfied:*

a. Objectivity property

$$W(\omega, x', RF) = W(\omega, x', F) \quad \text{for a.e. } (\omega, x') \in \Omega \times \mathbf{R}^2, \text{ for all } F \in \mathbf{R}^{3\times3},\ R \in \mathrm{SO}(3).$$

b. There exist constants $c_1, c_2, \rho > 0$ such that

$$\begin{aligned} &W(\cdot,\cdot,F) \geq c_1 \operatorname{dist}^2(F, \mathrm{SO}(3)), \ \text{a.e. on } \Omega \times \mathbf{R}^2 \text{ and for all } F \in \mathbf{R}^{3\times3} \\ &W(\cdot,\cdot,F) \leq c_2 \operatorname{dist}^2(F, \mathrm{SO}(3)), \ \text{a.e. on } \Omega \times \mathbf{R}^2 \text{ and for all } F \in \mathbf{R}^{3\times3} \\ &\qquad\qquad \text{with } \operatorname{dist}^2(F, \mathrm{SO}(3)) \leq \rho. \end{aligned} \tag{8}$$

c. There exists a monotone function $r : [0,\infty) \to [0,\infty]$ *with* $r(t) \downarrow 0$ *as* $t \downarrow 0$ *such that, for a.e.* $(\omega, x') \in \Omega \times \mathbf{R}^2$*, there exists a quadratic form* $Q(\omega, x', \cdot)$ *on* $\mathbf{R}^{3\times 3}$ *with*

$$|W(\omega, x', I_{3\times 3} + G) - Q(\omega, x', G)| \leq r(|G|)|G|^2 \text{ for all } G \in \mathbf{R}^{33}. \quad (9)$$

For $\omega \in \Omega$ we define the energy functionals $I^h : W^{1,2}(S \times I, \mathbf{R}^3) \to [0,\infty]$ by

$$I^h(u) = \frac{1}{h^2} \int_{S\times I} W\big(\omega, x'/\varepsilon, \nabla_h u(x', x_3)\big) \mathrm{d}x' \mathrm{d}x_3. \quad (10)$$

As a consequence of relations (8)-(9) we have the following lemma.

Lemma 3.2. *Let* W *be as in Assumption 3.1 and let* Q *be the quadratic form associated with* W *via* (9)*. Then*

(Q1) Q *is* T*-stationary,*

(Q2) for a.e. $(\omega, x') \in \Omega \times \mathbf{R}^2$ *we have that*

$$c_1 |\mathrm{sym}\, G|^2 \leq Q(\omega, x', G) = Q(\omega, x', \mathrm{sym}\, G) \leq c_2 |\mathrm{sym}\, G|^2, \ \forall G \in \mathbf{R}^{3\times 3}.$$

As before by $Q^0 : \Omega \times \mathbf{R}^{3\times 3} \to [0,\infty)$ we denote the mapping $Q^0(\omega, G) = Q(\omega, 0, G)$. Again without loss of generality we can assume that for a.e. $x' \in \mathbf{R}^2$, for all $G \in \mathbf{R}^{3\times 3}$ we have $Q(\omega, x', G) = Q^0(T_{x'}\omega, G)$.

Definition 3.3 (The relaxation formula)**.** *Let* $\gamma > 0$ *and define the map* $\mathcal{Q}^\gamma : \mathbf{R}^{2\times 2} \to [0,\infty)$ *as follows*

$$\mathcal{Q}^\gamma(G) = \inf_{\phi, B} \int_{\Omega\times I} Q^0\big(\omega, \iota(B + x_3 G) + \mathrm{sym}(D_1\phi, D_2\phi, \tfrac{1}{\gamma} D_{x_3}\phi)\big) \mathrm{d}P(\omega) \mathrm{d}x_3, \quad (11)$$

where the infimum is taken over $B \in \mathbf{R}^{2\times 2}$ *and* $\phi \in \mathcal{W}^{1,2}_{\mathrm{sym},\gamma}(\Omega \times I, \mathbf{R}^3)$.

It can be shown that $\mathcal{Q}^\gamma$ is a quadratic form which is coercive on symmetric matrices. Namely the expression on the right-hand side of (11) can be viewed as the projection of $x_3 G$ onto the closed subspace of $L^2(\Omega \times I, \mathbf{R}^{3\times 3}_{\mathrm{sym}})$ defined by $\iota(\mathbf{R}^{2\times 2}_{\mathrm{sym}}) \oplus D\mathcal{W}^{1,2}_{\mathrm{sym},\gamma}(\Omega \times I, \mathbf{R}^3)$ (the orthogonal decomposition) in the norm induced by the quadratic form Q^0, where

$$D\mathcal{W}^{1,2}_{\mathrm{sym},\gamma}(\Omega \times I, \mathbf{R}^3) = \{\mathrm{sym}(D_1\phi, D_2\phi, \tfrac{1}{\gamma} D_{x_3}\phi) : \phi \in \mathcal{W}^{1,2}_{\mathrm{sym},\gamma}(\Omega \times I, \mathbf{R}^3)\}.$$

The coercivity property follows easily from the coercivity property of Q^0.

In the bending regime we assume that the sequence of minimizers (u^h) satisfies

$$\limsup_{h\downarrow 0} I^h(u^h) < \infty.$$

By the compactness result (see Lemma 3.4), it can be concluded that the limit deformations are Sobolev isometries. By $W^{2,2}_{\mathrm{iso}}(S)$ we denote the set

$$W^{2,2}_{\mathrm{iso}}(S) = \{u \in W^{2,2}(S, \mathbf{R}^3) : \partial_\alpha u \cdot \partial_\beta u = \delta_{\alpha\beta} \text{ for } \alpha, \beta = 1, 2\},$$

where δ denotes the Kronecker delta symbol. For $u \in W^{2,2}_{\mathrm{iso}}(S)$ we define its normal $n^u \in W^{1,2}(S, \mathbf{R}^3)$ as $n^u = \partial_1 u \wedge \partial_2 u$ and the second fundamental form II^u as

$$\mathrm{II}^u_{\alpha\beta} = \partial_\alpha u \cdot \partial_\beta n^u = -\partial_{\alpha\beta} u \cdot n^u, \quad \alpha, \beta = 1, 2.$$

We define the limit functional $I^0 : W^{2,2}_{\mathrm{iso}}(S) \to [0, \infty)$ in the following way

$$I^0_\gamma(u) = \int_S \mathcal{Q}^\gamma(\mathrm{II}^u(x'))\mathrm{d}x'.$$

The following compactness result is the consequence of the compactness result given in [FJM02] and is explained in [Vel15][Lemma 3.3, Remark 4, proof of Proposition 3.2].

Lemma 3.4. *There exists a constant $C > 0$, depending only on S, such that for every $u \in W^{1,2}(S \times I, \mathbf{R}^3)$ there exists: a map $R : S \to \mathrm{SO}(3)$, which is piecewise constants on squares $x' + h[0,1)^2$, $x' \in h\mathbf{Z}^2$, as well as $\widetilde{R} \in W^{1,2}(S, \mathbf{R}^{3\times 3})$ such that for every $\xi \in \mathbf{R}^2$ with $|\xi|_\infty = \max\{|\xi \cdot e_1|, |\xi \cdot e_2|\} \le h$ and for each $S' \subset S$ with $\mathrm{dist}(S', \partial S) \ge h$ w.r.t. the $|\cdot|_\infty$ norm, we have*

$$\begin{aligned}\|\nabla_h u - R\|^2_{L^2(S'\times I)} &+ \|R - \widetilde{R}\|^2_{L^2(S')} + h^2\|R - \widetilde{R}\|^2_{L^\infty(S')} \\ &+ h^2\|\nabla'\widetilde{R}\|^2_{L^2(S')} + \|R(\cdot + \xi) - R\|^2_{L^2(S')} \le C\|\mathrm{dist}(\nabla_h u, \mathrm{SO}(3))\|^2_{L^2(S\times I)}.\end{aligned}$$

If additionally S' is open with $\partial S'$ of class $C^{1,1}$, then there exists $\widetilde{u} \in W^{2,2}(S')$ such that

$$\begin{aligned} h^2\|\widetilde{u}\|^2_{W^{2,2}(S')} + \|\nabla'\widetilde{u} - (\widetilde{R}e_1, \widetilde{R}e_2)\|^2_{L^2(S')} + \|\nabla'\widetilde{u} - \nabla'\overline{u}\|^2_{L^2(S')} \\ \le C\|\mathrm{dist}(\nabla_h u, \mathrm{SO}(3))\|^2_{L^2(S\times I)},\end{aligned}$$

where $\overline{u} = \int_I u(x_3)\mathrm{d}x_3$.

Remark 3.5. *The existence of the function R follows from the geometric rigidity, proved in [FJM02], while $\widetilde{R}$ is the mollification of R on scale h. The function $\widetilde{u}$ is the projection of $\widetilde{R}$ onto gradient fields.*

The following two theorems are the main result of this paper. They correspond to the statement of lower and upper bound for the Γ-limit.

Theorem 3.6. *Let $S \subset \mathbf{R}^2$ be a bounded domain with Lipschitz boundary. Let $(u^h) \subset W^{1,2}(S \times I, \mathbf{R}^3)$ be a family with finite elastic energy, i.e.*

$$\limsup_{h \downarrow 0} I^h(u^h) < \infty.$$

a. There exists $u \in W^{2,2}_{iso}(S)$ such that (up to a subsequence) we have

$$u^h - \int_S u^h \to u \qquad \text{strongly in } W^{1,2}(S \times I, \mathbf{R}^3), \tag{12}$$

$$\nabla_h u^h \to (\nabla' u, n^u) \qquad \text{strongly in } L^2(S \times I, \mathbf{R}^{3\times 3}). \tag{13}$$

b. For a.e. $\omega \in \Omega$ and any sequence (u^h) satisfying (12), (13) for some $u \in W^{2,2}_{\mathrm{iso}}(S)$ we have that

$$\liminf_{h \to 0} I^h(u^h) \geq I^0_\gamma(u).$$

Remark 3.7. *The claim a is the standard compactness result for the bending regime, whose proof can be found in [FJM02].*

Theorem 3.8. *Let $S \subset \mathbf{R}^2$ be a bounded domain with Lipschitz boundary, such that its normal is continuous away from a subset of ∂S with length zero (e.g., the boundary is piecewise C^1). Let $u \in W^{2,2}_{\mathrm{iso}}(S)$. Then for a.e. every $\omega \in \Omega$ there exists a sequence $(u^h) \subset W^{1,2}(S \times I, \mathbf{R}^3)$ such that we have*

a. $u^h \to u$ strongly in $W^{1,2}(S \times I, \mathbf{R}^3)$;

b. $I^h(u^h) \to I^0_\gamma(u)$.

3.2 Identifications of two-scale limits and proof of Theorem 3.6

3.2.1 Two-scale limits of the most important terms

In this section we explicitly compute the two-scale limits, which will be needed to prove the lower bound stated by Theorem 3.6.

Lemma 3.9. *Let $S' \subset \mathbf{R}^2$ be a bounded domain. Let $(\widetilde{u}^h) \subset W^{2,2}(S')$, $(R^h) \subset L^\infty(S', \mathrm{SO}(3))$ and $(\widetilde{R}^h) \subset W^{1,2}(S', \mathbf{R}^{3\times 3})$ with*

$$h^2 \|\widetilde{u}^h\|^2_{W^{2,2}(S')} + \|\nabla' \widetilde{u}^h - (\widetilde{R}^h e_1, \widetilde{R}^h e_2)\|^2_{L^2(S')} + \|R^h - \widetilde{R}^h\|^2_{L^2(S')} + h^2 \|R^h - \widetilde{R}^h\|^2_{L^\infty(S')} + h^2 \|\nabla' \widetilde{R}^h\|^2_{L^2(S')} \leq Ch^2.$$

Then there exist a (not relabeled) subsequence and functions $w_\alpha^0 \in L^2(S')$, $\phi^{\widetilde{u}} \in L^2(S', \mathcal{W}^{2,2}(\Omega, \mathbf{R}^3))$, $\phi^{\widetilde{R}} \in L^2(S', \mathcal{W}^{1,2}(\Omega, \mathbf{R}^3))$ such that for $\alpha = 1, 2$ we have

$$\begin{aligned} \frac{\langle R^h e_\alpha, \widetilde{R}^h e_3\rangle + \langle R^h e_3, \partial_\alpha \widetilde{u}^h\rangle}{h} &\xrightharpoonup{2} \frac{1}{\gamma} w_\alpha^0 + \frac{1}{\gamma}\big\langle Re_3, D_\alpha \phi^{\widetilde{u}}\big\rangle + \frac{1}{\gamma}\langle \phi^{\widetilde{R}e_3}, Re_\alpha\rangle, \\ \nabla^2 \widetilde{u}^h &\xrightharpoonup{2c} \nabla^2_\omega \phi^{\widetilde{u}}, \\ \nabla(\widetilde{R}^h e_3) &\xrightharpoonup{2c} \nabla_\omega \phi^{\widetilde{R}e_3}, \end{aligned}$$

and $\langle Re_3, D_\alpha \phi^{\widetilde{u}}\rangle + \langle \phi^{\widetilde{R}e_3}, Re_\alpha\rangle \in L^2(S' \times \Omega, \mathbf{R}^3)$.

Proof. Notice that

$$\langle R^h e_\alpha, \widetilde{R}^h e_3\rangle + \langle R^h e_3, \partial_\alpha \widetilde{u}^h\rangle = \langle R^h e_3 - \widetilde{R}^h e_3, \partial_\alpha \widetilde{u}^h - R^h e_\alpha\rangle + \langle \widetilde{R}^h e_3, \partial_\alpha \widetilde{u}^h\rangle.$$

The left-hand side is of order h, while the first term on the right-hand side is of order h^2. Thus the second term on the right-hand side is of order h. After dividing by h the first term on the right-hand side converges strongly to 0 as $h \to 0$ and thus does not contribute to the two-scale limit. We define

$$f^h := \widetilde{R}^h e_3, \quad g_\alpha^h := \partial_\alpha \widetilde{u}^h \quad \alpha = 1, 2,$$

and notice that, after extracting a subsequence, the components $f_i^h, (g_\alpha^h)_i$, $i = 1, 2, 3$ satisfy the assumptions of Lemma 2.16 (see also Remark 2.17). Thus

$$\frac{1}{h}\big\langle f^h, g_\alpha^h\big\rangle = \frac{\varepsilon}{h}\bigg(\frac{1}{\varepsilon}\big\langle f^h, g_\alpha^h\big\rangle\bigg) \xrightharpoonup{2} \frac{1}{\gamma} w_\alpha^0 + \frac{1}{\gamma}\langle f, \phi^{g_\alpha}\rangle + \frac{1}{\gamma}\langle \phi^f, g_\alpha\rangle,$$

for some $w_\alpha^0 \in L^2(S')$ and $\phi^f, \phi^{g_\alpha} \in L^2(S', \mathcal{W}^{1,2}(\Omega, \mathbf{R}^3))$ such that $\langle f, \phi^{g_\alpha}\rangle + \langle \phi^f, g_\alpha\rangle \in L^2(S' \times \Omega, \mathbf{R}^3)$. From Lemma 2.14 we additionally deduce that there exists $\phi^g \in L^2(S', \mathcal{W}^{2,2}(\Omega, \mathbf{R}^3))$ with $D_\alpha \phi^g = \phi^{g_\alpha}$ for $\alpha = 1, 2$. This yields

$$\frac{\langle R^h e_\alpha, \widetilde{R}^h e_3\rangle + \langle R^h e_3, \partial_\alpha \widetilde{u}^h\rangle}{h} \xrightharpoonup{2} \frac{1}{\gamma} w_\alpha^0 + \frac{1}{\gamma}\big\langle Re_3, D_\alpha \phi^{\widetilde{u}}\big\rangle + \frac{1}{\gamma}\langle \phi^{\widetilde{R}e_3}, Re_\alpha\rangle$$

for some $w_\alpha^0 \in L^2(S')$, $\phi^{\widetilde{R}e_3} \in L^2(\Omega \times S', \mathbf{R}^3)$ and $\phi^{\widetilde{u}} \in L^2(S', \mathcal{W}^{2,2}(\Omega, \mathbf{R}^3))$. □

The following lemma identifies the most sensitive term in our analysis.

Lemma 3.10. *Let $S' \subset \mathbf{R}^2$ be a bounded domain. Let $(\widetilde{R}^h) \subset W^{1,2}(S', \mathbf{R}^{3\times 3})$, and let $(R^h) \subset L^\infty(S', \mathrm{SO}(3))$ be such that for each $h > 0$ the map R^h is piecewise constant on each square $x' + h[0,1)^2$ with $x' \in h\mathbf{Z}^2$. Assume further that for each $\xi \in \mathbf{R}^2$ with $|\xi|_\infty \leq h$ we have*

$$\|R^h - \widetilde{R}^h\|^2_{L^2(S')} + h^2\|R^h - \widetilde{R}^h\|^2_{L^\infty(S')} + h^2\|\nabla'\widetilde{R}^h\|^2_{L^2(S')} + \|R^h(\cdot + \xi) - R^h\|^2_{L^2(S^h)} \leq Ch^2$$

for each sequence of subdomains $S^h \subset S'$ which satisfy $\mathrm{dist}(S^h, \partial S') \geq h$ w.r.t. the $|\cdot|_\infty$ norm.
Finally assume that $\widetilde{R}^h$ is the mollification of R^h on scale h.
Then there exist $R \in W^{1,2}(S', \mathrm{SO}(3))$, $w_3^0 \in L^2(S')$ and $\phi^{\widetilde{R}e_3} \in L^2(S', \mathcal{W}^{1,2}(\Omega, \mathbf{R}^3))$ such that on a subsequence we have $R^h \to R$ in $L^2(S', \mathbf{R}^{3\times 3})$ and

$$\begin{aligned} \frac{\langle R^h e_3, \widetilde{R}^h e_3\rangle - 1}{h} &\xrightharpoonup{2} \frac{1}{\gamma} w_3^0 + \frac{1}{\gamma}\langle Re_3, \phi^{\widetilde{R}e_3}\rangle, \\ \nabla(\widetilde{R}^h e_3) &\xrightharpoonup{2c} \nabla_\omega \phi^{\widetilde{R}e_3}, \end{aligned}$$

with $\langle Re_3, \phi^{\widetilde{R}e_3}\rangle \in L^2(S' \times \Omega)$.

Proof. From

$$f^h := \frac{\langle R^h e_3, \widetilde{R}^h e_3\rangle - 1}{h} = \frac{1}{h}\langle R^h e_3, \widetilde{R}^h e_3 - R^h e_3\rangle$$

we easily see that (f^h) is uniformly bounded in $L^2(S')$. Thus up to a subsequence we have

$$f^h \rightharpoonup \frac{1}{\gamma} w_3^0 \quad \text{and} \quad f^h \xrightharpoonup{2} \frac{1}{\gamma} w_3^0 + \phi$$

for some $w_3^0 \in L^2(S')$ and $\phi \in L^2(\Omega \times S')$. To further identify ϕ we test the sequence against derivatives. For this fix some $b \in \mathcal{C}^\infty(\Omega)$ and $\varphi \in C_0^\infty(S')$. Let $h > 0$ be small enough and such that there is a subdomain $S^h \subset S'$ with $\mathrm{dist}(S^h, \partial S') \geq h$ and the compact support K of φ is contained in S^h.
First note that

$$\begin{aligned} \int_K f^h(x')(D_\alpha b)(T_{\varepsilon^{-1}x'}\widetilde{\omega})\varphi(x')\mathrm{d}x' &= \varepsilon \int_K f^h(x')\partial_\alpha[b(T_{\varepsilon^{-1}x'}\widetilde{\omega})\varphi(x')]\mathrm{d}x' \\ &\quad - \varepsilon \int_K f^h(x')b(T_{\varepsilon^{-1}x'}\widetilde{\omega})\partial_\alpha\varphi(x')\mathrm{d}x'. \end{aligned}$$

The last term converges to 0, and so we focus on the first. For this we define $q_z := (z + h[0,1)^2) \cap K$ and compute

$$
\begin{aligned}
\varepsilon \int_K f^h \partial_\alpha [b(T_{\varepsilon^{-1}x'}\widetilde{\omega})\varphi] \mathrm{d}x' &= \frac{\varepsilon}{h} \int_K \langle R^h e_3, \widetilde{R}^h e_3 \rangle \partial_\alpha [b(T_{\varepsilon^{-1}x'}\widetilde{\omega})\varphi] \mathrm{d}x' \\
&= \frac{\varepsilon}{h} \sum_{z \in h\mathbf{Z}^2} \int_{q_z} \langle R^h(z) e_3, \widetilde{R}^h e_3 \rangle \partial_\alpha [b(T_{\varepsilon^{-1}x'}\widetilde{\omega})\varphi] \mathrm{d}x' \\
&= \frac{\varepsilon}{h} \sum_{z \in h\mathbf{Z}^2} \int_{q_z} \partial_\alpha \Big[\langle R^h(z) e_3, \widetilde{R}^h e_3 \rangle b(T_{\varepsilon^{-1}x'}\widetilde{\omega})\varphi \Big] \mathrm{d}x' \\
&\quad - \frac{\varepsilon}{h} \sum_{z \in h\mathbf{Z}^2} \int_{q_z} \Big[\langle R^h(z) e_3, \partial_\alpha \widetilde{R}^h e_3 \rangle b(T_{\varepsilon^{-1}x'}\widetilde{\omega})\varphi \Big] \mathrm{d}x'.
\end{aligned}
$$

For the last term we use Lemma 2.12 to conclude there exists $\phi^{\widetilde{R}e_3} \in L^2(S', \mathcal{W}^{1,2}(\Omega, \mathbf{R}^3))$ with

$$
\nabla' \widetilde{R}^h e_3 \xrightarrow{2} \nabla' Re_3 + \nabla_\omega \phi^{\widetilde{R}e_3}.
$$

Together with $R^h \to R$ strongly in $L^2(S')$, we obtain

$$
\begin{aligned}
&\lim_{h \downarrow 0} \left(-\frac{\varepsilon}{h} \sum_{z \in h\mathbf{Z}^2} \int_{q_z} \Big[\langle R^h(z) e_3, \partial_\alpha \widetilde{R}^h e_3 \rangle b(T_{\varepsilon^{-1}x'}\widetilde{\omega})\varphi \Big] \mathrm{d}x' \right) \\
&= \lim_{h \downarrow 0} \left(-\frac{\varepsilon}{h} \int_K \Big[\langle R^h e_3, \partial_\alpha \widetilde{R}^h e_3 \rangle b(T_{\varepsilon^{-1}x'}\widetilde{\omega})\varphi \Big] \mathrm{d}x' \right) \\
&= -\frac{1}{\gamma} \int_\Omega \int_K \Big[\langle Re_3, \partial_\alpha Re_3 + D_\alpha \phi^{\widetilde{R}e_3} \rangle b(\omega)\varphi \Big] \mathrm{d}x' \mathrm{d}P(\omega) \\
&= -\frac{1}{\gamma} \int_\Omega \int_K \Big[\langle Re_3, D_\alpha \phi^{\widetilde{R}e_3} \rangle b(\omega)\varphi \Big] \mathrm{d}x' \mathrm{d}P(\omega)
\end{aligned}
$$

Assume that

$$
\frac{\varepsilon}{h} \sum_{z \in h\mathbf{Z}^2} \int_{q_z} \partial_\alpha \Big[\langle R^h(z) e_3, \widetilde{R}^h e_3 \rangle b(T_{\varepsilon^{-1}x'}\widetilde{\omega})\varphi \Big] \mathrm{d}x' \xrightarrow{h \to 0} 0. \tag{14}
$$

holds. Then

$$
\begin{aligned}
&\lim_{h \downarrow 0} \int_K f^h(x') (D_\alpha b)(T_{\varepsilon^{-1}x'}\widetilde{\omega}) \varphi(x') \mathrm{d}x' \\
&= -\frac{1}{\gamma} \int_\Omega \int_K \Big[\langle Re_3, D_\alpha \phi^{\widetilde{R}e_3} \rangle b(\omega)\varphi \Big] \mathrm{d}x' \mathrm{d}P(\omega)
\end{aligned}
$$

By compactness we obtained $f^h \xrightarrow{2c} \phi \in L^2(\Omega \times S')$. By Remark 2.17 we obtain $\langle Re_3, \phi^{\widetilde{R}e_3} \rangle \in L^2(\Omega \times S')$. After integrating by parts, this implies

$f^h \xrightharpoonup{2c} \phi = \frac{1}{\gamma}\langle Re_3, \phi^{\widetilde{R}e_3}\rangle - \frac{1}{\gamma}\int_\Omega \langle Re_3, \phi^{\widetilde{R}e_3}\rangle$ and thus, after absorbing the last term into w_0^3, also the claim. It remains to prove (14). If $q_z \neq \emptyset$ then let $\Gamma_z^{\rm pos}, \Gamma_z^{\rm neg}$ be the boundary of q_z perpendicular to e_α with normals e_α resp. $-e_\alpha$, else $\Gamma_z^{\rm pos}, \Gamma_z^{\rm neg} := \emptyset$. The Gauss's theorem yields

$$\begin{aligned}
&\frac{\varepsilon}{h}\sum_{z\in h\mathbf{Z}^2}\int_{q_z}\partial_\alpha\Big[\langle R^h(z)e_3, \widetilde{R}^h e_3\rangle b(T_{\varepsilon^{-1}x'}\widetilde{\omega})\varphi\Big]\mathrm{d}x' \\
&\quad= \frac{\varepsilon}{h}\sum_{z\in h\mathbf{Z}^2}\int_{\Gamma_z^{\rm pos}}\langle R^h(z)e_3, \widetilde{R}^h e_3\rangle b(T_{\varepsilon^{-1}x'}\widetilde{\omega})\varphi\mathrm{d}x' \\
&\qquad- \frac{\varepsilon}{h}\sum_{z\in h\mathbf{Z}^2}\int_{\Gamma_z^{\rm neg}}\langle R^h(z)e_3, \widetilde{R}^h e_3\rangle b(T_{\varepsilon^{-1}x'}\widetilde{\omega})\varphi\mathrm{d}x',
\end{aligned}$$

where the integral is taken in the sense of traces. We rearrange the sum and obtain

$$\begin{aligned}
&\frac{\varepsilon}{h}\sum_{z\in h\mathbf{Z}^2}\int_{\Gamma_z^{\rm pos}}\langle R^h(z)e_3, \widetilde{R}^h e_3\rangle b(T_{\varepsilon^{-1}x'}\widetilde{\omega})\varphi\mathrm{d}x' \\
&\quad- \frac{\varepsilon}{h}\sum_{z\in h\mathbf{Z}^2}\int_{\Gamma_z^{\rm neg}}\langle R^h(z)e_3, \widetilde{R}^h e_3\rangle b(T_{\varepsilon^{-1}x'}\widetilde{\omega})\varphi\mathrm{d}x' \\
&= \frac{\varepsilon}{h}\sum_{z\in h\mathbf{Z}^2}\Big\langle R^h(z)e_3 - R^h(z+he_\alpha)e_3, \int_{\Gamma_z^{\rm pos}}\widetilde{R}^h e_3 b(T_{\varepsilon^{-1}x'}\widetilde{\omega})\varphi\mathrm{d}x'\Big\rangle.
\end{aligned}$$

By assumption

$$\frac{R^h(z) - R^h(z+he_\alpha)}{h}$$

is uniformly bounded in z and h, which implies

$$\limsup_{h\downarrow 0}\sum_{z\in h\mathbf{Z}^2}\big|R^h(z) - R^h(z+he_\alpha)\big|^2 < \infty.$$

Denote by $\mathcal{Z} \subset h\mathbf{Z}^2$ the z-values such that $\Gamma_z^{\rm pos}$ has positive $\mathcal{H}^2$-measure $|\Gamma_z^{\rm pos}|$. Using the trace inequality and Poincaré's inequality afterwards, we get for $z \in \mathcal{Z}$ that

$$\int_{\Gamma_z^{\rm pos}}\Big|\widetilde{R}^h - \frac{1}{|\Gamma_z^{\rm pos}|}\int_{\Gamma_z^{\rm pos}}\widetilde{R}^h\Big|^2 \le Ch\int_{q_z}|\nabla\widetilde{R}^h|^2\mathrm{d}x'.$$

Combining both previous statements we see that

$$\begin{aligned}
&\lim_{h\to 0}\frac{\varepsilon}{h}\sum_{z\in h\mathbf{Z}^2}\Big\langle R^h(z)e_3 - R^h(z+he_\alpha)e_3, \int_{\Gamma_z^{\rm pos}}\widetilde{R}^h e_3 b(T_{\varepsilon^{-1}x'}\widetilde{\omega})\varphi\mathrm{d}x'\Big\rangle \\
&= \frac{1}{\gamma}\lim_{h\to 0}\sum_{z\in\mathcal{Z}}\Big\langle R^h(z)e_3 - R^h(z+he_\alpha)e_3, \frac{1}{|\Gamma_z^{\rm pos}|}\int_{\Gamma_z^{\rm pos}}\widetilde{R}^h e_3\Big\rangle\int_{\Gamma_z^{\rm pos}} b(T_{\varepsilon^{-1}x'}\widetilde{\omega})\varphi\mathrm{d}x'.
\end{aligned}$$

Noticing the uniform bound

$$\left|\int_{\Gamma_z^{\rm pos}} b(T_{\varepsilon^{-1}x'}\widetilde{\omega})\varphi \mathrm{d}x'\right| \leq h\|b\|_{L^\infty(\Omega)}\|\varphi\|_{L^\infty(S')}$$

we only need to show that

$$\limsup_{h\downarrow 0}\sum_{z\in\mathcal{Z}}\left|\left\langle R^h(z)e_3 - R^h(z+he_\alpha)e_3, \frac{1}{|\Gamma_z^{\rm pos}|}\int_{\Gamma_z^{\rm pos}}\widetilde{R}^h e_3\right\rangle\right| < \infty$$

to conclude the vanishing of the product. For this bound note that $\widetilde{R}^h$ is the mollification of R^h on scale h. Therefore there exist z-independent constants $0 \leq \eta_1, \eta_2, \eta_3 \leq 1$ with

$$\begin{aligned}\frac{1}{|\Gamma_z^{\rm pos}|}\int_{\Gamma_z^{\rm pos}}\widetilde{R}^h =& \eta_1\Big(R^h(z) + R^h(z+he_\alpha)\Big)\\ &+ \eta_2\Big(R^h(z+he_\alpha^\perp) + R^h(z+h(e_\alpha+e_\alpha^\perp))\Big)\\ &+ \eta_3\Big(R^h(z-he_\alpha^\perp) + R^h(z+h(e_\alpha-e_\alpha^\perp))\Big).\end{aligned}$$

We compute

$$\begin{aligned}\mathrm{I} :=& \eta_1\sum_{z\in\mathcal{Z}}\big|\big\langle R^h(z)e_3 - R^h(z+he_\alpha)e_3, R^h(z)e_3 + R^h(z+he_\alpha)e_3\big\rangle\big|\\ =& \eta_1\sum_{z\in\mathcal{Z}}\Big[|R^h(z)e_3|^2 - |R^h(z+he_\alpha)e_3|^2\Big] = 0.\end{aligned}$$

With this result we easily obtain

$$\begin{aligned}\mathrm{II} :=& \eta_2\sum_{z\in\mathcal{Z}}\big|\big\langle R^h(z)e_3 - R^h(z+he_\alpha)e_3, R^h(z+he_\alpha^\perp)e_3 + R^h(z+h(e_\alpha+e_\alpha^\perp))e_3\big\rangle\big|\\ \leq& \eta_2\sum_{z\in\mathcal{Z}}\big|\big\langle R^h(z)e_3 - R^h(z+he_\alpha)e_3, R^h(z+he_\alpha^\perp)e_3 - R^h(z)e_3\big\rangle\big|\\ &+ \eta_2\sum_{z\in\mathcal{Z}}\big|\big\langle R^h(z)e_3 - R^h(z+he_\alpha)e_3, R^h(z+h(e_\alpha+e_\alpha^\perp))e_3 - R^h(z+he_\alpha)e_3\big\rangle\big| < \infty\end{aligned}$$

and analogously also that

$$\mathrm{III} := \eta_3\sum_{z\in\mathcal{Z}}\big|\big\langle R^h(z)e_3 - R^h(z-he_\alpha)e_3, R^h(z+he_\alpha^\perp)e_3 + R^h(z+h(e_\alpha-e_\alpha^\perp))e_3\big\rangle\big|$$

is uniformly bounded. Obviously

$$\sum_{z\in\mathcal{Z}}\left|\left\langle R^h(z)e_3 - R^h(z+he_\alpha)e_3, \frac{1}{|\Gamma_z^{\rm pos}|}\int_{\Gamma_z^{\rm pos}}\widetilde{R}^h e_3\right\rangle\right| \leq \mathrm{I} + \mathrm{II} + \mathrm{III},$$

and we conclude that

$$\frac{1}{h}\Big(\langle R^h e_3, \widetilde{R}^h e_3\rangle - 1\Big) \xrightharpoonup{2} \frac{1}{\gamma} w_3^0 + \frac{1}{\gamma}\langle Re_3, \phi^{\widetilde{R}e_3}\rangle. \qquad \square$$

Remark 3.11. *In the case when $\varepsilon(h)^2 \ll h \ll \varepsilon(h)$, one would not need to obtain the additional compactness given in Lemma 3.9 and Lemma 3.10. Namely, in that case, the corrector in the cell formula (11) allows in the third column and row arbitrary functions in $L^2(\Omega \times S \times I)$ (see [Vel15] for the periodic case). As already shown in [Vel15], due to this fact, the regime $\varepsilon(h)^2 \ll h \ll \varepsilon(h)$ does not need the notion of oscillatory convergence, introduced in [HNV14], for the regimes $h \sim \varepsilon(h)$ and $h \gg \varepsilon(h)$.*
Lemma 3.9 and Lemma 3.10, together with Lemma 3.12 below (already established in the periodic setting in [Vel15]), completely avoid the notion of oscillatory convergence and rely completely on the duality arguments. The idea of Lemma 3.10 is to join the members $(3,\alpha)$ and $(\alpha,3)$, for $\alpha \in \{1,2\}$ (since only their sum is visible in the limit; see the proof of the lower bound) and to use further cancellation effects; and for the proof of Lemma 3.10 one has to do the additional computations.

Lemma 3.12. *Let $S' \subset \mathbf{R}^2$ be a bounded Lipschitz domain and let $(\widetilde{u}^h) \subset W^{2,2}(S', \mathbf{R}^3)$, $(\widetilde{R}^h) \subset W^{1,2}(S', \mathbf{R}^{3\times 3})$ and $(R^h) \subset L^\infty(S', \mathrm{SO}(3))$ be such that for each $h > 0$ the map R^h is piecewise constant on each square $x' + h[0,1)^2$ with $x' \in h\mathbf{Z}^2$, and for each $\xi \in \mathbf{R}^2$ with $|\xi|_\infty \le h$ we have*

$$\begin{aligned} &h^2\|\widetilde{u}^h\|^2_{W^{2,2}(S')} + \|\nabla'\widetilde{u}^h - (\widetilde{R}^h e_1, \widetilde{R}^h e_2)\|^2_{L^2(S')} + \|R^h - \widetilde{R}^h\|^2_{L^2(S')} \\ &\quad + h^2\|R^h - \widetilde{R}^h\|^2_{L^\infty(S')} + h^2\|\nabla'\widetilde{R}^h\|^2_{L^2(S')} + \|R^h(\cdot + \xi) - R^h\|^2_{L^2(S^h)} \le Ch^2, \end{aligned}$$

for some $C > 0$ and for each sequence of subdomains $S^h \subset S'$ which satisfy $\mathrm{dist}(S^h, \partial S') \ge h$. Then there exists $M_0 \in L^2(S', \mathbf{R}^{2\times 2}_{\mathrm{sym}})$ and $\zeta \in L^2(S', \mathcal{W}^{1,2}_{\mathrm{sym}}(\Omega, \mathbf{R}^2))$ with

$$\mathrm{sym}\,\frac{(R^h e_1, R^h e_2)^T \nabla'\widetilde{u}^h - I_{2\times 2}}{h} \xrightharpoonup{2} M_0 + \mathrm{sym}\,\nabla_\omega \zeta.$$

Proof. Using Theorem A.2 the proof is identical to [Vel15][Lemma 3.7]. $\square$

3.2.2 Proof of Theorem 3.6

Proof. Let (u^h) be as in the claim, and let $S' \subset S$ be open with $C^{1,1}$ boundary. For every h we apply Lemma 3.4 to u^h and obtain (R^h), $(\widetilde{R}^h)$ and $(\widetilde{u}^h)$ as stated in the lemma. Define z^h by the decomposition of u^h into

$$u^h(x', x_3) = \overline{u}^h(x') + hx_3\widetilde{R}^h(x')e_3 + hz^h(x', x_3),$$

where once more $\overline{u}^h(x') = \int_I u^h(x', x_3)\mathrm{d}x_3$. Clearly we have $z^h \in W^{1,2}(S' \times I, \mathbf{R}^3)$ with $\int_I z(x_3)\mathrm{d}x_3 = 0$.
We define the approximate strain

$$G^h := \frac{(R^h)^T \nabla_h u^h - I_{3\times 3}}{h}$$

and split it into

$$\begin{aligned} G^h = &\frac{\iota\big((R^h e_1, R^h e_2)^T \nabla' \widetilde{u}^h - I_{2\times 2}\big)}{h} + \frac{1}{h}\sum_{\alpha=1,2} \langle R^h e_3, \partial_\alpha \widetilde{u}^h\rangle e_3 \otimes e_\alpha \\ &+ \frac{1}{h}(R^h)^T(\nabla' \overline{u}^h - \nabla' \widetilde{u}^h, 0) + \frac{1}{h}\Big((R^h)^T \widetilde{R}^h e_3 \otimes e_3 - e_3 \otimes e_3\Big) \\ &+ x_3 (R^h)^T (\nabla' \widetilde{R}^h e_3, 0) + (R^h)^T \nabla_h z^h. \end{aligned} \tag{15}$$

Since G^h is uniformly bounded in L^2, we may take a subsequence such that $G^h \xrightarrow{2} G$ for some $G \in L^2(\Omega \times S \times I, \mathbf{R}^{3\times 3})$. We study $\operatorname{sym} G$ by computing the possible two-scale limits of the terms in $\operatorname{sym} G^h$. For this we will readily take further subsequences if needed, without denoting them explicitly.
By applying Lemma 3.12 we obtain

$$\operatorname{sym} \frac{\iota\big((R^h e_1, R^h e_2)^T \nabla' \widetilde{u}^h - I_{2\times 2}\big)}{h} \xrightarrow{2} \iota(M_0 + \operatorname{sym} \nabla_\omega \zeta)$$

for some $M_0 \in L^2(S', \mathbf{R}^{2\times 2}_{\mathrm{sym}})$ and $\zeta \in L^2(S', \mathcal{W}^{1,2}_{\mathrm{sym}}(\Omega, \mathbf{R}^2))$. From Lemma 2.12 and Prop. 2.10 we get

$$x_3 (R^h)^T (\nabla' \widetilde{R}^h e_3) \xrightarrow{2} x_3 (\mathrm{II}^u, 0)^T + x_3 R^T \nabla_\omega \phi^{\widetilde{R} e_3},$$

as well as

$$\frac{1}{h}(R^h)^T(\nabla' \overline{u}^h - \nabla' \widetilde{u}^h) \xrightarrow{2} R^T\Big(\theta + \nabla_\omega v\Big)$$

for some $\theta \in L^2(S', \mathbf{R}^{3\times 2})$ and $v, \phi^{\widetilde{R} e_3} \in L^2(S', \mathcal{W}^{1,2}(\Omega, \mathbf{R}^3))$.
For

$$\begin{aligned} &\frac{1}{h}\Big((R^h)^T \widetilde{R}^h e_3 \otimes e_3 - e_3 \otimes e_3\Big) + \frac{1}{h}\sum_{\alpha=1,2} \langle R^h e_3, \partial_\alpha \widetilde{u}^h\rangle e_3 \otimes e_\alpha \\ &= \frac{1}{h}\begin{pmatrix} 0 & 0 & \langle R^h e_1, \widetilde{R}^h e_3\rangle \\ 0 & 0 & \langle R^h e_2, \widetilde{R}^h e_3\rangle \\ \langle R^h e_3, \partial_1 \widetilde{u}^h\rangle & \langle R^h e_3, \partial_2 \widetilde{u}^h\rangle & \langle R^h e_3, \widetilde{R}^h e_3\rangle - 1 \end{pmatrix} \end{aligned}$$

we obtain from Lemma 3.9 and Lemma 3.10 that

$$\operatorname{sym}\left[\frac{1}{h}\Big((R^h)^T\widetilde{R}^h e_3\otimes e_3 - e_3\otimes e_3\Big) + \frac{1}{h}\sum_{\alpha=1,2}\langle R^h e_3, \partial_\alpha \widetilde{u}^h\rangle e_3\otimes e_\alpha\right]$$
$$\xrightharpoonup{2} \frac{1}{\gamma}\operatorname{sym}(w^0\otimes e_3) + \frac{1}{\gamma}\operatorname{sym}(R^T\phi^{\widetilde{R}e_3}e_3\otimes e_3) + \frac{1}{\gamma}\operatorname{sym}(R^T\nabla_\omega\phi^{\widetilde{u}}e_3\otimes e_3)$$

for some $w^0 \in L^2(S', \mathbf{R}^3)$, $\phi^{\widetilde{u}} \in L^2(S', \mathcal{W}^{2,2}(\Omega, \mathbf{R}^3))$. For the last term $(\nabla_h z^h)$ notice that (15) yields an uniform L^2 bound. By Lemma 2.19 we thus get

$$(R^h)^T\nabla_h z^h \xrightharpoonup{2} R^T\left(\nabla_\omega\phi^z, \frac{1}{\gamma}D_{x_3}\phi^z\right)$$

for some $\phi^z \in L^2(S', \mathcal{W}^{1,2}(\Omega\times I, \mathbf{R}^3))$. We conclude that

$$\begin{aligned}\operatorname{sym} G^h \xrightharpoonup{2} &\iota(M_0 + \operatorname{sym}\nabla_\omega\zeta) + \frac{1}{\gamma}\operatorname{sym}(w^0\otimes e_3) + \frac{1}{\gamma}\operatorname{sym}(R^T\phi^{\widetilde{R}e_3}e_3\otimes e_3)\\ &+ \frac{1}{\gamma}\operatorname{sym}(R^T\nabla_\omega\phi^{\widetilde{u}}e_3\otimes e_3) + \operatorname{sym}\Big(R^T\theta + R^T\nabla_\omega v\Big)\\ &+ x_3\operatorname{sym}\Big(\iota(\mathrm{II}^u) + \big(R^T(\nabla_\omega\phi^{\widetilde{R}e_3}), 0\big)\Big) + \operatorname{sym}\Big(R^T\big(\nabla_\omega\phi^z, \frac{1}{\gamma}D_{x_3}\phi^z\big)\Big).\end{aligned}$$

We rewrite this as

$$\operatorname{sym} G = \iota\Big(\operatorname{sym}(\widetilde{B} + x_3\,\mathrm{II}^u)\Big) + \operatorname{sym}\left(\nabla_\omega\phi, \frac{1}{\gamma}D_{x_3}\phi\right),$$

where $\widetilde{B} = M_0 + [R^T\theta_{ij}]_{1\le i,j\le 2}$ as well as

$$\phi(x,\omega) := R^T(x')\widetilde{\phi}(x,\omega) + \zeta(x',\omega) + \gamma x_3\begin{pmatrix}b_1(x'), & b_2(x'), & 0\end{pmatrix}^T,$$

with

$$\begin{aligned}\widetilde{\phi}(x,\omega) &= \phi^z(x,\omega) + v(x',\omega) + x_3\phi^{\widetilde{R}e_3}(x',\omega) + x_3 w_0(x') + \frac{1}{\gamma}\phi^{\widetilde{u}}(x',\omega),\\ b_i &= [R^T\theta(x')]_{3i} \quad \text{for } i = 1,2.\end{aligned}$$

Notice that $\phi \in \mathcal{W}^{1,2}_{\mathrm{sym},\gamma}(\Omega\times I, \mathbf{R}^3)$.
After exhausting S by $S' \subset S$ open with $C^{1,1}$ boundary, using Lemma 3.4 and Remark 3.7 as well as the quadraticity of the form $\mathcal{Q}^\gamma$, the lower bound follows by using c from Assumption 3.1 and lower semicontinuity of the quadratic form Q^0 with respect to the stochastic two-scale convergence, see Lemma 2.21 and Remark 2.23 (see also [HNV14] for the details in the periodic case). $\square$

3.3 Proof of upper bound

In this section we prove the upper bound statement. We recall some issues from the periodic homogenization (see [HNV14]). As in [Sch07] and other related results, the key ingredient here is the density result for $W^{2,2}(S)$ isometric immersions established in [Hor11a, Hor11b] (cf. also [Pak04] for an earlier result in this direction). It is the need for the results in [Hor11a] that forces us to restrict ourselves to domains S which are not only Lipschitz but also piecewise C^1. More precisely, we only need that the outer unit normal be continuous away from a subset of ∂S with length zero.
For a given $u \in W^{2,2}_{\text{iso}}(S)$ and for a displacement $V \in W^{2,2}(S, \mathbf{R}^3)$ we denote by q^u_V the quadratic form

$$q^u_V = \text{sym}\left((\nabla u)^T (\nabla V)\right).$$

We denote by $\mathcal{A}(S)$ the set of all $u \in W^{2,2}_{\text{iso}}(S, \mathbf{R}^3) \cap C^\infty(\overline{S}, \mathbf{R}^3)$ with the property that

$$\mathcal{S} := \left\{ B \in C^\infty(\overline{S}, \mathbf{R}^{2\times 2}_{\text{sym}}) : B = 0 \text{ in a neighborhood of } \{x' \in S : \text{II}^u(x') = 0\} \right\} \\ \subset \{q^u_V : V \in C^\infty(\overline{S}; \mathbf{R}^3)\}.$$

In other words, if $u \in \mathcal{A}(S)$ and $B \in C^\infty(\overline{S}, \mathbf{R}^{2\times 2}_{\text{sym}})$ is a matrix field which vanishes in a neighborhood of $\{\text{II}^u = 0\}$, then there exists a displacement $V \in C^\infty(\overline{S}; \mathbf{R}^3)$ such that $q^u_V = B$. The necessary lemma for the proof of upper bound is the following lemma, whose proof is given in [Sch07, HNV14].

Lemma 3.13. *The set $\mathcal{A}(S)$ is dense in $W^{2,2}_{\text{iso}}(S)$ with respect to the strong $W^{2,2}(S, \mathbf{R}^2)$ topology.*

Proof of Theorem 3.8. Fix some typical $\widetilde{\omega} \in \Omega$. By Lemma 3.13 it suffices to show the claim for $u \in \mathcal{A}(S)$. Fix $B \in \mathcal{S}$ and $V \in C^\infty(\overline{S}, \mathbf{R}^3)$ such that $q^y_V = B$, and define the unit normal $n^u = \partial_1 u \wedge \partial_2 u$. Next we divide the domain S into small squares $(D^\eta_i)^n_{i=1}$, $D^\eta_i \subset S$ of size η such that $|S \setminus \cup^n_{i=1} D^\eta_i| \to 0$ as $\eta \to 0$. On each square we define $A^\eta_i, B^\eta_i \in \mathbf{R}^{2\times 2}_{\text{sym}}$ as the averages

$$A^\eta_i = \frac{1}{|D^\eta_i|} \int_{D^\eta_i} \text{II}^u(x')\text{d}x', \quad B^\eta_i = \frac{1}{|D^\eta_i|} \int_{D^\eta_i} B(x')\text{d}x'.$$

For each $i = 1, \ldots, n$ and $\delta < \frac{\eta}{2}$ we define

$$D^{\eta,\delta}_i = \{x' \in D^\eta_i : \text{dist}(x', \partial D^\eta_i) > \delta\}.$$

For each $i = 1, \dots, n$ let $(g_i^{\eta,k}) \subset (\mathcal{C}^\infty(\Omega) \otimes C^\infty(I))^3$ be a minimizing sequence of $\mathcal{Q}^\gamma$, in the sense that

$$\int_{\Omega\times I} Q\big(\omega, \iota(B_i^\eta + x_3 A_i^\eta) + \mathrm{sym}(D_1 g_i^{\eta,k}, D_2 g_i^{\eta,k}, \tfrac{1}{\gamma} D_{x_3} g_i^{\eta,k})\big) \mathrm{d}x_3 \mathrm{d}P(\omega) - \frac{1}{k}$$
$$\leq \inf_{\phi \in W^{1,2}(\Omega\times I, \mathbf{R}^3)} \int_{\Omega\times I} Q\big(\omega, \iota(B_i^\eta + x_3 A_i^\eta) + \mathrm{sym}(D_1\phi, D_2\phi, \tfrac{1}{\gamma} D_{x_3}\phi)\big) \mathrm{d}x_3 \mathrm{d}P(\omega).$$

We start with the Kirchhoff-Love ansatz, augmented by its linearization induced by the displacement V:

$$v^h(x', x_3) := u(x') + h x_3 n^u(x') + h\big(V(x') + h x_3 \mu(x')\big),$$

where μ is given by

$$\mu = (I_{3\times 3} - n^u \otimes n^u)(\partial_1 V \wedge \partial_2 u + \partial_1 u \wedge \partial_2 V).$$

We set $R(x') = \big(\nabla' u(x'),\ n^u(x')\big)$. A straightforward computation shows that

$$\nabla_h v^h = R + h\Big((\nabla' V, \mu) + x_3(\nabla' n, 0)\Big) + h^2 x_3 (\nabla' \mu, 0). \tag{16}$$

The actual recovery sequence u^h is obtained by adding to v^h the oscillating correction of order $\varepsilon = \varepsilon(h)$:

$$u^{\eta,k,\delta,h}(x', x_3) := v^h(x', x_3) + h\varepsilon \sum_{i=1}^n \chi_i^{\eta,\delta}(x') R(x') g_i^{\eta,k}(T_{\varepsilon^{-1}x'}\widetilde{\omega}, x_3). \tag{17}$$

Here $\chi_i^{\eta,\delta} \in C^1(S)$ are smooth cut-off functions that satisfy

$$\chi_i^{\eta,\delta} = 1 \text{ on } D_i^{\eta,\delta}, \quad \chi_i^{\eta,\delta} = 0 \text{ on } (D_i^\eta)^c \ \text{ and } \ |\nabla \chi_i^{\eta,\delta}| \leq \frac{C}{\delta} \text{ for some } C > 0.$$

Equations (16) and (17) imply together with $n^u \cdot \mu = 0$ that

$$\begin{aligned} R^T \nabla_h u^{\eta,k,\delta,h} = I_{3\times 3} &+ h\iota\left((\nabla' u)^T (\nabla' V) + x_3\, \mathrm{II}^u\right) + h(e_3 \otimes (\mu \cdot \nabla' u), 0)^T \\ &+ h(e_3 \otimes (n \cdot \nabla' V), 0) \\ &+ h \sum_{i=1}^n \chi_i^{\eta,\delta} (D_1 g_i^{\eta,k}, D_2 g_i^{\eta,k}, \tfrac{\varepsilon}{h} D_{x_3} g_i^{\eta,k}) \\ &+ h^2 x_3 R^T (\nabla' \mu, 0) + h\varepsilon \sum_{i=1}^n \Big(\chi_i^{\eta,\delta} (R^T \nabla' R) g_i^{\eta,k}, 0\Big) \\ &+ h\varepsilon \sum_{i=1}^n \Big(\nabla' \chi_i^{\eta,\delta} g_i^{\eta,k}, 0\Big)\,; \end{aligned} \tag{18}$$

the arguments of $g_i^{\eta,k}$ and of $(D_1 g_i^{\eta,k}, D_2 g_i^{\eta,k}, \frac{\varepsilon}{h} D_{x_3} g_i^{\eta,k})$ are $(T_{\varepsilon^{-1}x'}\widetilde{\omega}, x_3)$. From (17) and (18) we conclude that

$$\|u^{\eta,k,\delta,h} - u\|_{W^{1,2}(S\times I,\mathbf{R}^3)} \xrightarrow{h\to 0} 0.$$

Defining

$$G^{\eta,k,\delta,h} = \frac{1}{h}\Big(R^T \nabla_h u^{\eta,k,\delta,h} - I_{3\times 3}\Big)$$

and using the fact that $n \cdot \nabla V + \mu \cdot \nabla' u = 0$, we deduce from (18) that

$$\begin{aligned}
\operatorname{sym} G^{\eta,k,\delta,h} &= \iota\,(q_V^u + x_3\,\mathrm{II}^u) \\
&\quad + \sum_{i=1}^n \chi_i^{\eta,\delta} \operatorname{sym}(D_1 g_i^{\eta,k}, D_2 g_i^{\eta,k}, \tfrac{\varepsilon}{h} D_{x_3} g_i^{\eta,k}) + h \operatorname{sym}\big(x_3 R^T(\nabla'\mu, 0)\big) \\
&\quad + \varepsilon \sum_{i=1}^n \operatorname{sym}\Big(\chi_i^{\eta,\delta}(R^T\nabla' R) g_i^{\eta,k}, 0\Big) \\
&\quad + \varepsilon \sum_{i=1}^n \operatorname{sym}\Big(\nabla'\chi_i^{\eta,\delta} g_i^{\eta,k}, 0\Big)\,;
\end{aligned}$$

using the objectivity property we obtain

$$W(\widetilde{\omega}, x'/\varepsilon, \nabla_h u^{\eta,k,\delta,h}) = W(\widetilde{\omega}, x'/\varepsilon, I_{3\times 3} + hG^{\eta,k,\delta,h}).$$

It is also easy to see from (9) that

$$\left|\frac{1}{h^2} W(\widetilde{\omega}, x'/\varepsilon, \nabla_h u^{\eta,k,\delta,h}) - Q(\widetilde{\omega}, x'/\varepsilon, G^{\eta,k,\delta,h})\right| \to 0,$$

uniformly in x' for $h \to 0$. It is not difficult to conclude

$$\begin{aligned}
&\lim_{\eta\to 0}\lim_{k\to\infty}\lim_{\delta\to 0}\lim_{h\to 0} \frac{1}{h^2}\int_{S\times I} Q(\widetilde{\omega}, x'/\varepsilon, G^{\eta,k,\delta,h})\mathrm{d}x'\mathrm{d}x_3 \\
&= \int_S \inf_\phi \int_{\Omega\times I} Q\big(\omega, \iota(B + x_3\,\mathrm{II}^u) + \operatorname{sym}(D_1\phi, D_2\phi, \tfrac{1}{\gamma} D_{x_3}\phi)\big)\mathrm{d}x_3\mathrm{d}P(\omega)\mathrm{d}x',
\end{aligned}$$

where we minimize over $\phi \in \mathcal{W}^{1,2}_{\mathrm{sym},\gamma}(\Omega\times I, \mathbf{R}^3)$. By choosing appropriate B we get

$$\lim_{\eta\to 0}\lim_{k\to\infty}\lim_{\delta\to 0}\lim_{h\to 0} \frac{1}{h^2}\int_{S\times I} Q(\widetilde{\omega}, x'/\varepsilon, G^{\eta,k,\delta,h})\mathrm{d}x'\mathrm{d}x_3 = \int_S \mathcal{Q}^\gamma(\mathrm{II}^u(x'))\mathrm{d}x'.$$

The claim now follows by the lemma of Attouch and by a classical diagonal argument for Γ-convergence. $\square$

4 Examples for the probability space

The first example is the standard one and is already given in [PV81]. It covers the case of periodic homogenization.

Example 4.1. We take $W : \mathbf{R}^2 \times \mathbf{R}^{3\times3} \to [0,\infty]$ that is 1-periodic in the first component and that satisfies the property a, b, c from Assumption 3.1. Next we take $\Omega = T^2$ the 2-dimensional unit torus with the Lebesgue measurable sets as the σ-algebra and the probability P as Lebesgue measure on T^2. The measure is invariant under translations, e.g., from the dynamical system $T_x\omega = \omega + x \pmod 1$. The infinitesimal generators are the usual partial derivatives, for $i = 1, 2$

$$D_i = \frac{\partial}{\partial\omega_i} \; (\omega = (\omega_1, \omega_2)).$$

In the end we define $W(\omega, x', F) = W(\omega + x', F)$. In this case we obtain for $\mathcal{Q}^\gamma$ the following formula

$$\mathcal{Q}^\gamma(G) = \inf_{\phi,B} \int_{T^2\times I} Q\big(x', \iota(B + x_3G) + \mathrm{sym}(D_1\phi, D_2\phi, \tfrac{1}{\gamma}D_{x_3}\phi)\big)\mathrm{d}x'\mathrm{d}x_3,$$

where the infimum is taken over $\phi \in W^{1,2}(T^2 \times I, \mathbf{R}^3)$, $B \in \mathbf{R}^{2\times2}$.

In [PV81] it is also shown how almost periodic case can be covered with the abstract approach of stochastic homogenization. Periodic (or almost periodic) homogenization naturally destroys the isotropic character of the energy density, even if the constituents are isotropic. In the next example we want to show how we can obtain the isotropic energy density out of isotropic constituents by stochastic homogenization.

Example 4.2. We would like to construct the probability space that consists of some subset of functions on $\mathbf{R}^2$ (taking values in some finite or countable set) that is invariant under rotations of the coordinate system and moreover that the probability measure is invariant under these rotations. One possible construction can be made using Poisson processes. We construct the probability space that consists of piecewise constant functions. Namely, the construction of the probability space goes as follows: we take the Poisson point process in $\mathbf{R}^2$ with the Lebesgue measure and the sequence of independent and identically distributed random variables (J_n) (independent of the Poisson process), taking values in the set $\mathbf{N}$. We then construct the marked Poisson point process, i.e., to every point (of some realization) we give a mark according to the realization of the sequence (J_n) [1]. For $i \in \mathbf{N}$ we take energy density functions $W^i : \mathbf{R}^2 \times \mathbf{R}^{3\times 3} \to [0, \infty]$ that satisfies the properties a, b, c from the Assumption 3.1 and that is isotropic, i.e., it satisfies

$$W^i(\omega, x', FR) = W^i(\omega, x', F), \text{ for all } F \in \mathbf{R}^{3\times 3}, R \in \mathrm{SO}(3),$$
$$\text{for a.e. } (\omega, x') \in \Omega \times \mathbf{R}^2 \text{ and all } i \in \mathbf{N}.$$

This implies that the same property is valid for the appropriate quadratic forms Q^i. Out of each realization of the marked Poisson point process we make the material mixture in the following way: the point $x' \in \mathbf{R}^2$ is occupied by the material i if the point x' is in the Voronoi cell of the point that is marked with the number i. In that way we obtain the probability space where

$$\Omega = \{\text{piecewise constant functions that take values in the set } \mathbf{N} \text{ and}$$
$$\text{that is constructed using the marked Poisson process}\}.$$

For the σ-algebra we take the one generated by the sets

$$\big\{ f \in \Omega \mid f(x_l) = i_l; x_l \in \mathbf{Q}^2, i_l \in \mathbf{N}, \text{ for } l = 1, \dots, n \big\}.$$

For the probability measure we take the pushforward of the measure given by the marked Poisson processes. The action $T_{x'}$ is simply given by the translation $T_{x'}\omega(y') = \omega(x' + y')$. It is easily seen that this action is measure preserving (since the distribution of the marked Poisson process is translation invariant) and ergodic. The ergodicity follows from the facts that the marked Poisson process (with independent marks) is ergodic and that the probability measure is just a pushforward. The energy density we define in the following way:

$$W(\omega, x', F) = W^i(\omega, x', F), \quad \text{if } \omega(x') = i.$$

[1] The definition of the marked Poisson process can be given as follows: it is a point process that takes values in the metric space $\mathbf{R}^2 \times \mathbb{N}$ and that satisfies:

- for every $A \subset \mathbf{R}^2$, $M \subset \mathbf{N}$ the random variable $N(A \times M)$ that denotes the number of points that belong to the set $A \times M$ has the Poisson distribution with the parameter $\lambda|A|P(J_1 \in M)$, for some fixed $\lambda > 0$ (called the intensity of the Poisson process);
- for $A_1, \dots, A_m \subset \mathbf{R}^2 \times \mathbf{N}$, the random variables $N(A_1), \dots, N(A_m)$ that denote the number of points that belong to the sets $A_1, \dots, A_m$ respectively are independent.

Thus, each realization of such a process can be seen as a collection of points in $\mathbf{R}^2$ that additionally have a mark (a natural number). For the general theory of point processes, see , e.g., [DV03].

For $R \in \mathrm{SO}(2)$ we define the rotational transformation $R_t : \Omega \to \Omega$ in the following way

$$R_t(\omega)(x') = \omega(Rx'), \text{ for all } x' \in \mathbf{R}^2.$$

Notice that R_t is measure preserving. By $\widetilde{R}$ we denote the matrix in SO(3) given by $\widetilde{R} = \iota(R)+e_3\otimes e_3$. From the properties of the infinitesimal generator we easily conclude for $f \in \mathcal{C}^\infty(\Omega)$ that

$$(D_1(f \circ R_t), D_2(f \circ R_t)) = (D_1 f \circ R_t, D_2 f \circ R_t)R.$$

From the cell formula (11) and the isotropy property of Q^0 we conclude

$$\begin{aligned}
\mathcal{Q}^\gamma(GR) &= \inf_{\phi,B} \int_{\Omega\times I} Q^0\big(\omega, \iota(B + x_3 GR) + (D_1\phi, D_2\phi, \tfrac{1}{\gamma}D_{x_3}\phi)\big)\mathrm{d}P(\omega)\mathrm{d}x_3 \\
&= \inf_{\phi,B} \int_{\Omega\times I} Q^0\big(\omega, \iota(BR^T + x_3 G) + (D_1\phi, D_2\phi, \tfrac{1}{\gamma}D_{x_3}\phi)\widetilde{R}^T\big)\mathrm{d}P(\omega)\mathrm{d}x_3 \\
&= \inf_{\phi,B} \int_{\Omega\times I} Q^0\big(\omega, \iota(B + x_3 G) + (D_1\phi, D_2\phi, \tfrac{1}{\gamma}D_{x_3}\phi)\big)\mathrm{d}P(\omega)\mathrm{d}x_3 \\
&= \mathcal{Q}^\gamma(G).
\end{aligned}$$

The infima are taken for $\phi \in W^{1,2}(\Omega \times I, \mathbf{R}^3)$, $B \in \mathbf{R}^{2\times 2}$. This proves the isotropy.

Remark 4.3. *Notice that in the first example we can write the cell formula in the following way*

$$\mathcal{Q}^\gamma(G) = \inf_{\phi,B} \int_{\gamma T^2\times I} Q^0\big(x', \iota(B + x_3 G) + \mathrm{sym}(D_1\phi, D_2\phi, D_{x_3}\phi)\big)\mathrm{d}x'\mathrm{d}x_3,$$

where the infimum is taken over $\phi \in W^{1,2}(\gamma T^2 \times I, \mathbf{R}^3)$, $B \in \mathbf{R}^{2\times 2}$. In the second example we can write the cell formula in the following way

$$\mathcal{Q}^\gamma(G) = \inf_{\phi,B} \int_{\Omega^d\times I} Q^0\big(\omega, \iota(B + x_3 G) + (D_1\phi, D_2\phi, D_{x_3}\phi)\big)\mathrm{d}P(\omega)\mathrm{d}x_3,$$

where the infimum is taken for $\phi \in W^{1,2}(\Omega^d \times I, \mathbf{R}^3)$, $B \in \mathbf{R}^{2\times 2}$. Here, Ω^d is the transformed set of functions, where the transformation is given by

$$\omega^d(x') = \omega(\gamma^{-1}x'), \text{ for every } \omega \in \Omega.$$

This transformation changes the intensity of the Poisson in the background. This explains the meaning of the parameter γ. Namely, although it seems that $\varepsilon(h)$ has not a clear physical meaning, its meaning is incorporated in the probability space in the background. In the first case it is the size of the cell of the periodicity, while in the second it is connected to the intensity of the Poisson process in the background.

A Appendix

A.1 Decompositions of L^2

The decomposition of L^2 into a 'gradient' part and a 'divergence-free' part, known as Helmholtz-decomposition, is a classical result in real space, and has since been generalized in various aspects.
The aim in this section is to present possible decompositions of L^2, once a probability space is involved. From now on let $(\Omega, \mathcal{F}, P)$ be a probability space and an N-dimensional ergodic system $(T_x) : \Omega \to \Omega$ (see Definitions 2.1 and 2.2) as well as D_i, $i = 1, \dots N$ the infinitesimal generators of T (see the discussion after Remark 2.3.) We first recall a known result for $L^2(\Omega, \mathbf{R}^N)$, we state a similar decomposition of $L^2(\Omega, \mathbf{R}^{2\times 2}_{\text{sym}})$ into second gradients parts and a remainder, and finally we derive the decomposition for the 'mixed' space $L^2(\Omega \times S, \mathbf{R}^{N+M})$, where $S \subset \mathbf{R}^M$.

A.2 Decomposition of purely random spaces (first order)

The Helmholtz-decomposition for stochastic L^2-spaces were already known. These results can be also easily proved by using the spectral decomposition and the methods in [BMW94]. We recall the results given in [ZKO94] and [DG16] (the definitions of potential and solenoidal fields given in [ZKO94] are given by using realizations, however these definitions can be equivalently defined on the probability space; see [DG16, Rel (3.4), Rel. (3.5)]):
Let

$$
\begin{aligned}
L^2_{\text{pot}}(\Omega) &:= \{f \in L^2(\Omega, \mathbf{R}^N) : \int_\Omega (f_i D_j g - f_j D_i g) = 0, \\
&\qquad\qquad\qquad\qquad \text{for all } g \in W^{1,2}(\Omega), i, j = 1, \dots, N\}, \\
L^2_{\text{sol}}(\Omega) &:= \{f \in L^2(\Omega, \mathbf{R}^N) : \int_\Omega f \cdot (\nabla_\omega g) = 0, \quad \text{for all } g \in W^{1,2}(\Omega)\},
\end{aligned}
$$

as well as

$$
F^2_{\text{pot}}(\Omega) := \left\{ f \in L^2_{\text{pot}}(\Omega) : \int_\Omega f = 0 \right\}, \quad F^2_{\text{sol}}(\Omega) := \left\{ f \in L^2_{\text{sol}}(\Omega) : \int_\Omega f = 0 \right\}.
$$

Theorem A.1. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and an N-dimensional ergodic system $(T_x) : \Omega \to \Omega$. Then*

$$
L^2(\Omega, \mathbf{R}^N) = F^2_{\text{pot}}(\Omega) \oplus F^2_{\text{sol}}(\Omega) \oplus \mathbf{R}^N.
$$

Furthermore we can characterize the spaces by

$$F^2_{\mathrm{pot}}(\Omega) = \mathrm{adh}_{L^2}\{\nabla_\omega g : \quad g \in W^{1,2}(\Omega)\},$$

and for $N = 2, 3$ we get respectively

$$F^2_{\mathrm{sol}}(\Omega) = \mathrm{adh}_{L^2}\{(-D_2, D_1)g : \quad g \in W^{1,2}(\Omega, \mathbf{R})\},$$
$$F^2_{\mathrm{sol}}(\Omega) = \mathrm{adh}_{L^2}\{\nabla_\omega \times g : \quad g \in W^{1,2}(\Omega, \mathbf{R}^3)\}.$$

A.3 Decomposition of purely stochastic spaces (second order)

For the second order decomposition we define the spaces

$$L^2_{\mathrm{ppot}}(\Omega) := \left\{ A \in L^2(\Omega, \mathbf{R}^{2\times 2}_{\mathrm{sym}}) \,\Big|\, \int_\Omega A : \operatorname{cof} \nabla_\omega h = 0 \quad \text{for all } h \in W^{1,2}(\Omega, \mathbf{R}^2) \right\},$$

and

$$L^2_{\mathrm{ssol}}(\Omega) := \left\{ B \in L^2(\Omega, \mathbf{R}^{2\times 2}_{\mathrm{sym}}) \,\Big|\, \int_\Omega B : \nabla^2_\omega h = 0 \quad \text{for all } h \in W^{2,2}(\Omega, \mathbf{R}) \right\}.$$

Denote further

$$F^2_{\mathrm{ppot}}(\Omega) = \left\{ A \in L^2_{\mathrm{ppot}}(\Omega) \,\Big|\, \int_\Omega A = 0 \right\}; \; F^2_{\mathrm{ssol}}(\Omega) = \left\{ B \in L^2_{\mathrm{ssol}}(\Omega) \,\Big|\, \int_\Omega B = 0 \right\}.$$

We obtain the following decomposition and density result:

Theorem A.2. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and a 2-dimensional ergodic system $(T_x) : \Omega \to \Omega$. Then*

$$L^2(\Omega, \mathbf{R}^{2\times 2}_{\mathrm{sym}}) = F^2_{\mathrm{ppot}}(\Omega) \oplus F^2_{\mathrm{ssol}}(\Omega) \oplus \mathbf{R}^{2\times 2}_{\mathrm{sym}},$$

as well as

$$F^2_{\mathrm{ppot}}(\Omega) = \mathrm{adh}_{L^2}\{\nabla^2_\omega b \mid b \in \mathcal{C}^\infty(\Omega)\},$$
$$F^2_{\mathrm{ssol}}(\Omega) = \mathrm{adh}_{L^2}\{\operatorname{cof} \operatorname{sym} \nabla_\omega b \mid b \in \mathcal{C}^\infty(\Omega, \mathbf{R}^2)\}.$$

The theorem can be proved by the same methods as in [ZKO94, Lemma 7.3].

A.4 Decomposition of mixed spaces

Let $S \subset \mathbf{R}^M$ be a bounded Lipschitz domain, and let $L = M + N$. By ∇ we will denote the operator $(\partial_1, \ldots, \partial_L)$, for maps with the domain $\mathbf{R}^N \times S$, and for maps with the domain $\Omega \times S$ the operator $(D_1, \ldots, D_N, \partial_{N+1}, \ldots, \partial_L)$; since from the context the definition that is used is clear, we will not distinguish them in notation. Additionally we define in both cases div $= \nabla\cdot$.

A.4.1 Trace Theorems

In this section we briefly discuss a generalization of the trace operator for functions with the domain $\Omega \times S$. The statements and proofs are analogous results to the classical results for Sobolev functions (see [GR12]). First we define $W^{1,2}(\Omega \times S)$, analogously to (4), by

$$W^{1,2}(\Omega \times S) = W^{1,2}(S, L^2(\Omega)) \cap L^2(S, W^{1,2}(\Omega)).$$

On this space we define the extended trace.

$$\begin{aligned} \gamma : W^{1,2}(\Omega \times S) &\to L^2(\Omega \times \partial S), \\ \gamma(\psi)(\omega, y) &= \widetilde{\gamma}(\psi(\omega, \cdot))(y), \end{aligned}$$

where

$$\widetilde{\gamma} : W^{1,2}(S) \to L^2(\partial S)$$

is the classical trace. By definition $W^{1,2}(\Omega \times S) \subset W^{1,2}(S, L^2(\Omega))$ and so by Fubini's Theorem we obtain $\psi(\omega, \cdot) \in W^{1,2}(S)$ for a.e. $\omega \in \Omega$, if $\psi \in W^{1,2}(\Omega \times S)$. It is easily seen that the map γ is linear and continuous. Furthermore the space $\gamma(W^{1,2}(\Omega \times S))$ is a closed subspace of $L^2(\Omega \times \partial S)$, which we will denote by

$$W^{1/2}(\Omega \times \partial S) := \gamma(W^{1,2}(\Omega \times S)).$$

Together with the norm

$$\|\mu\|_{W^{1/2}(\Omega \times \partial S)} = \inf_{\psi \in W^{1,2}(\Omega \times S), \gamma(\psi) = \mu} \|\psi\|_{W^{1,2}(\Omega \times S)}$$

the space is complete. To extend the trace, we define on $(\mathcal{C}^\infty(\Omega) \otimes C^\infty(\overline{S}))^L$ the norm

$$\|g\|^2_{W^{1,2}_{\mathrm{div}}(\Omega \times S)} = \|g\|^2_{L^2(\Omega \times S)} + \|\operatorname{div} g\|^2_{L^2(\Omega \times S)}$$

and denote the completion of the space as

$$W^{1,2}_{\mathrm{div}}(\Omega \times S) := \operatorname{adh}_{W^{1,2}_{\mathrm{div}}} (\mathcal{C}^\infty(\Omega) \otimes C^\infty(\overline{S}))^L,$$

analogously to the real-variant $W^{1,2}_{\mathrm{div}}(\mathbf{R}^N \times S)$.
Furthermore we split $g = (g^s, g^r)$ into $g^s = (g_1, \ldots, g_N)$ and $g^r = (g_{N+1}, \ldots, g_L)$. Let $\Gamma = \Omega \times \partial S$ be the boundary of $\Omega \times S$, and let ν be the outward-normal of S.

Lemma A.3 (Normal Trace Theorem). *The mapping $\gamma_\nu : g \mapsto g^r\big|_\Gamma \cdot \nu$ defined for $g \in (\mathcal{C}^\infty(\Omega) \otimes C^\infty(\overline{S}))^L$ extends uniquely to a continuous, linear mapping $\gamma_\nu : W^{1,2}_{\mathrm{div}}(\Omega \times S) \to (W^{1/2}(\Omega \times \partial S))'$.*

Proof. Using integration by parts for smooth functions $\varphi \in \mathcal{C}^\infty(\Omega) \otimes C^\infty(\overline{S})$, $g \in (\mathcal{C}^\infty(\Omega) \otimes C^\infty(\overline{S}))^L$ we have

$$\int_{\Omega\times S} g(\omega,x)\cdot\nabla\varphi(\omega,x)\mathrm{d}P(\omega)\mathrm{d}x + \int_{\Omega\times S}(\operatorname{div} g)(\omega,x)\varphi(\omega,x)\mathrm{d}P(\omega)\mathrm{d}x = \int_\Omega\int_{\partial S}\nu(x)\cdot g^r(\omega,x)\varphi(\omega,x)\mathrm{d}x\,\mathrm{d}P(\omega).$$

Applying the Cauchy-Schwarz inequality we thus obtain

$$\left|\int_\Omega\int_{\partial S}\nu(x)\cdot g^r(\omega,x)\varphi(\omega,x)\mathrm{d}x\,\mathrm{d}P(\omega)\right| \le \|g\|_{W^{1,2}_{\mathrm{div}}(\Omega\times S)}\cdot\|\varphi\|_{W^{1,2}(\Omega\times S)},$$

for all $\varphi \in \mathcal{C}^\infty(\Omega)\otimes C^\infty(\overline{S})$, and by density for all $\varphi \in W^{1,2}(\Omega\times S)$.
Now fix some $\psi \in W^{1/2}(\Omega\times\partial S)$. For any such ψ there exists by definition $\varphi \in W^{1,2}(\Omega\times S)$ such that $\gamma(\varphi) = \psi$ and $\|\varphi\|_{W^{1,2}} \le 2\|\psi\|_{W^{1/2}}$. Thus we obtain

$$\left|\int_\Omega\int_{\partial S}\nu(x)\cdot g^r(\omega,x)\psi(\omega,x)\mathrm{d}x\,\mathrm{d}P(\omega)\right| \le 2\|g\|_{W^{1,2}_{\mathrm{div}}(\Omega\times S)}\cdot\|\psi\|_{W^{1/2}(\Omega\times\partial S)}.$$

By definition of the $(W^{1/2}(\Omega\times\partial S))'$ norm we finally obtain

$$\begin{aligned}\left\|g^r\big|_\Gamma\cdot\nu\right\|_{W^{1/2}(\Omega\times\partial S)'} &= \sup_{\substack{\varphi\in W^{1/2}(\Omega\times\partial S),\\ \|\varphi\|_{W^{1/2}}\le 1}}\int_\Omega\int_{\partial S}\nu(x)\cdot g^r(\omega,x)\varphi(\omega,x)\mathrm{d}x\,\mathrm{d}P(\omega)\\ &\le 2\|g\|_{W^{1,2}_{\mathrm{div}}(\Omega\times S)}.\end{aligned}$$

Thus γ_ν can be continuously extended to a map $W^{1,2}_{\mathrm{div}}(\Omega\times S)\to(W^{1/2}(\Omega\times S))'$, and by density the extension is unique. □

To simplify the notation we will write $\psi\big|_{\Omega\times\partial S}$ for $\gamma(\psi)$ and $g^r\cdot\nu$ for $\gamma_\nu(g)$.

A.4.2 Mixed differential equations

We define the spaces of test functions $\mathcal{D} = C_0^\infty(\mathbf{R}^N\times S)$ and $X := C_0^\infty(\mathbf{R}^N)\otimes C^\infty(\overline{S})$, and introduce for $f, g \in L^2_{\mathrm{loc}}(\mathbf{R}^N, L^2(S,\mathbf{R}^L))$ the notation

$$\begin{aligned}\nabla\times f = 0 \text{ in } \mathcal{D}' &:\Longleftrightarrow \int_{\mathbf{R}^N\times S}\big(f_j\partial_i\varphi - f_i\partial_j\varphi\big) = 0 \quad &&\text{for all } i,j \text{ and } \varphi\in\mathcal{D},\\ \operatorname{div} g = 0 \text{ in } \mathcal{D}' &:\Longleftrightarrow \int_{\mathbf{R}^N\times S}\Big\langle g,\nabla\varphi\Big\rangle = 0 &&\text{for all } \varphi\in\mathcal{D}.\end{aligned}$$

If additionally $g \in L^2_{\text{loc}}(\mathbf{R}^N, W^{1,2}_{\text{div}}(S))$, then $g^r \cdot \nu$ is almost everywhere well-defined on ∂S and we set

$$\operatorname{div} g = 0 \text{ in } X' \;:\Longleftrightarrow\; \int_{\mathbf{R}^N \times \mathbf{R}^M} \Big\langle g, \nabla\varphi \Big\rangle = \int_{\mathbf{R}^N} \int_{\partial S} \langle \varphi g^r, \nu \rangle \text{ for all } \varphi \in X.$$

Since $\mathcal{D} \subset X$ the conditions immediately split into

$$\operatorname{div} g = 0 \text{ in } X' \iff \operatorname{div} g = 0 \text{ in } \mathcal{D}' \quad \text{and} \quad g^r \cdot \nu = 0 \quad \text{a.e. on } \mathbf{R}^N \times \partial S.$$

Similar to before we define

$$\widetilde{\mathcal{D}} = \mathcal{C}^\infty(\Omega) \otimes C^\infty_0(S) \quad \text{and} \quad \widetilde{X} = \left\{ \psi \in \mathcal{C}^\infty(\Omega) \otimes C^\infty(\overline{S}) : \int_{\Omega \times S} \nabla\psi = 0 \right\},$$

and for $f, g \in L^2(\Omega \times S, \mathbf{R}^L)$ we denote

$$\begin{aligned} \nabla \times f = 0 \text{ in } \widetilde{\mathcal{D}'} \;&:\Longleftrightarrow\; \int_{\Omega \times S} \big(f_j \partial_i \psi - f_i \partial_j \psi \big) = 0 \quad && \text{for all } i, j \text{ and } \psi \in \widetilde{\mathcal{D}}, \\ \operatorname{div} g = 0 \text{ in } \widetilde{\mathcal{D}'} \;&:\Longleftrightarrow\; \int_{\Omega \times S} \langle g, \nabla\psi \rangle = 0 && \text{for all } \psi \in \widetilde{\mathcal{D}}. \end{aligned}$$

Next we define for $g \in W^{1,2}_{\text{div}}(\Omega \times S)$:

$$\operatorname{div} g = 0 \quad \text{in } \widetilde{X}' :\Longleftrightarrow \operatorname{div} g \text{ in } \widetilde{\mathcal{D}}' \quad \text{and } g^r \cdot \nu = 0 \quad \text{a.e. on } \Omega \times \partial S.$$

Note that $\nabla \times f = 0$ in $\widetilde{\mathcal{D}}'$ holds, iff. almost all realization satisfy $\nabla \times f = 0$ in $\mathcal{D}'$, and similarly for $\operatorname{div} g$.
Finally define the sets

$$\begin{aligned} L^2_{\text{pot}}(\Omega \times S) &= \{ f \in L^2(\Omega \times S, \mathbf{R}^L) : \quad \nabla \times f = 0 \text{ in } \widetilde{\mathcal{D}}' \}, \\ L^2_{\text{sol}}(\Omega \times S) &= \{ g \in L^2(\Omega \times S, \mathbf{R}^L) : \quad \operatorname{div} g = 0 \text{ in } \widetilde{X}' \}, \end{aligned}$$

and

$$\begin{aligned} F^2_{\text{pot}}(\Omega \times S) &= \left\{ f \in L^2_{\text{pot}}(\Omega \times S) : \int_{\Omega \times S} f(\omega, x) \mathrm{d}P(\omega) \mathrm{d}x = 0 \right\}, \\ F^2_{\text{sol}}(\Omega \times S) &= \left\{ g \in L^2_{\text{sol}}(\Omega \times S) : \int_{\Omega \times S} g(\omega, x) \mathrm{d}P(\omega) \mathrm{d}x = 0 \right\}. \end{aligned}$$

We are now able to state the decomposition theorem for the mixed-spaces:

Theorem A.4. *Let $(\Omega, \mathcal{F}, P)$ be a probability space with $\mathcal{F}$ countably generated and a N-dimensional ergodic system $(T_x) : \Omega \to \Omega$ and let $S \subset \mathbf{R}^M$ be a bounded Lipschitz domain. Then:*

(i) $L^2(\Omega \times S, \mathbf{R}^L) = F^2_{\mathrm{pot}}(\Omega \times S) \oplus F^2_{\mathrm{sol}}(\Omega \times S) \oplus \mathbf{R}^L$.

(ii) $F^2_{\mathrm{pot}}(\Omega \times S) = \mathrm{adh}_{L^2(\Omega\times S,\mathbf{R}^L)}\{\nabla g : g \in W^{1,2}(\Omega \times S)\}$.

(iii) *If* $L = 3$*, i.e.* $M = 1, N = 2$ *or* $M = 2, N = 1$ *then*

$$F^2_{\mathrm{sol}}(\Omega \times S) = \mathrm{adh}_{L^2(\Omega\times S,\mathbf{R}^L)}\{\nabla \times g : g \in W^{1,2}(\Omega, \mathbf{R}^L)\}.$$

A.4.3 Orthogonality of div and ∇

The decomposition will follow easily, once we have proved the following lemma:

Lemma A.5. *Let* $f \in L^2_{\mathrm{pot}}(\Omega \times S)$ *and* $g \in L^2_{\mathrm{sol}}(\Omega \times S)$*. Then*

$$\begin{aligned}\int_{\Omega\times S} & f(\omega, y) \cdot g(\omega, y)\mathrm{d}P(\omega)\mathrm{d}y \\ &= \frac{1}{|S|}\left(\int_{\Omega\times S} f(\omega, y)\mathrm{d}P(\omega)\mathrm{d}y\right) \cdot \left(\int_{\Omega\times S} g(\omega, y)\mathrm{d}P(\omega)\mathrm{d}y\right).\end{aligned}$$

Especially if additionally $\int f = 0$ *or* $\int g = 0$*, then* $\langle f, g\rangle_{L^2(\Omega\times S)} = 0$.

Before proving the lemma, we first show that 'multiplying' an ergodic system with a periodic system yields once more an ergodic system.

Lemma A.6. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space with an* N*-dimensional ergodic system* $(T_x) : \Omega \to \Omega$*. Let* $\widetilde{T} : \mathbf{R}^M \times [0, 1)^M \to [0, 1)^M$ *be defined by* $\widetilde{T}_y(\omega^M) = \omega^M + y \pmod 1$*. Define the product dynamical system*

$$T \times \widetilde{T} : (\mathbf{R}^N \times \mathbf{R}^M) \times (\Omega \times [0, 1)^M) \to \Omega \times [0, 1)^M$$

by

$$(T \times \widetilde{T})_{(x,y)}(\omega^N, \omega^M) = (T_x\omega^N, T_y\omega^M).$$

If T *is ergodic, then* $T \times \widetilde{T}$ *is ergodic as well.*

Here we use the weaker formulation of ergodicity, explained in Remark 2.3.

Proof. Let $B \subset \Omega \times [0, 1)^M$ be measurable and invariant under $T \times \widetilde{T}$, i.e.

$$(T \times \widetilde{T})_{(x,y)}(B) = B \quad \text{for all } (x, y) \in \mathbf{R}^N \times \mathbf{R}^M.$$

Choosing $x = 0$ we get

$$\bigcup_{y\in\mathbf{R}^M} (T \times \widetilde{T})_{(0,y)}(B) = B.$$

Thus B can be written as $B' \times [0,1)^M$ with $B' \subset \Omega$ measurable. We have

$$(P \otimes \mathscr{L}^M)(B) = P(B').$$

By ergodicity of T we have $P(B') \in \{0,1\}$ and thus $(P \otimes \mathscr{L}^M)(B) \in \{0,1\}$. $\square$

Proof of Lemma A.5. By translating and scaling it suffices to show it for domains $S \subset Q := \frac{1}{2}(-1,1)^M$. Fix some $f, g \in L^2(\Omega \times S)$ and extend them to $f, g \in L^2(\Omega \times Q)$ with their corresponding mean-value on $\Omega \times S$. Finally extend both function Q-periodically onto $\mathbf{R}^M$. Assuming $\nabla \times f = 0$ in $\mathcal{D}'(\Omega \times S)$ and $\nabla \cdot g = 0$ in $\widetilde{X}'(\Omega \times S)$, the extended functions satisfy the PDE clearly on $\Omega \times (\mathbf{Z}^M + S)$. For some $\widetilde{\omega} \in \Omega$ typical we define the sequence of functions

$$f^\varepsilon(x,y) = f(T_{\varepsilon^{-1}x}\widetilde{\omega}, \varepsilon^{-1}y), \quad g^\varepsilon(x,y) = g(T_{\varepsilon^{-1}x}\widetilde{\omega}, \varepsilon^{-1}y). \tag{19}$$

We prove this lemma by showing that

$$f^\varepsilon \cdot g^\varepsilon \overset{\star}{\rightharpoonup} \overline{f} \cdot \overline{g} \quad \text{in } \mathcal{D}'(\mathbf{R}^N \times Q), \tag{20}$$

where $\overline{f}, \overline{g}$ are the corresponding weak limits for $f^\varepsilon, g^\varepsilon$ as defined in (19). By Birkhoff's Theorem we have

$$\overline{f} = \frac{1}{|S|}\int_{\Omega \times S} f(\omega,z)\mathrm{d}P(\omega)\mathrm{d}z, \quad \overline{g} = \frac{1}{|S|}\int_{\Omega \times S} g(\omega,z)\mathrm{d}P(\omega)\mathrm{d}z,$$

as well as

$$f^\varepsilon \cdot g^\varepsilon \overset{\star}{\rightharpoonup} \overline{fg} \quad \text{in } L^1_{\text{loc}}(\mathbf{R}^N \times Q) \tag{21}$$

by Lemma A.6. By uniqueness of the limit both have to agree, which is the claim of the lemma.
From (21) we deduce, that convergence holds for every $\varepsilon \to 0$, and to identify the limit in terms of $\overline{f}, \overline{g}$ it suffices to choose the specific sequence $\varepsilon_n = n^{-1}$, where we suppress the index n and still write $\varepsilon \to 0$ instead of $n \to \infty$.
The proof of (20) is motivated by the proof of the classical div-curl lemma (see e.g., [All02, Lemma 1.3.1]). By locality of the statement we can reduce ourselves to the case $K \subset\subset \mathbf{R}^N \times Q$ and define $K^S = K \cap (\mathbf{R}^N \times S)$. We can assume that K^S has Lipschitz boundary. Furthermore we may assume that the weak limits of $f^\varepsilon, g^\varepsilon$ are zero.
Define ψ to be the primitive of f on the domain $\mathbf{R}^N \times S$ with the property $\int_{K^S} \psi(x,y) = 0$. Extend ψ onto $\mathbf{R}^N \times (\mathbf{Z}^M + S)$ periodically. Furthermore define

$$\psi^\varepsilon(x,y) = \varepsilon\psi(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}) + c^\varepsilon \quad \text{on} \quad \mathbf{R}^N \times \varepsilon(\mathbf{Z}^M + S),$$

with the constant

$$c^\varepsilon = -\int_{K^S} \varepsilon\psi(\frac{x}{\varepsilon}, y)\mathrm{d}(x,y).$$

By construction we have

$$\nabla\psi^\varepsilon = f^\varepsilon \text{ on } \mathbf{R}^N \times \varepsilon(\mathbf{Z}^M + S) \quad \text{as well as} \quad \int_{K^S} \psi^\varepsilon(x, \varepsilon y)\mathrm{d}(x,y) = 0.$$

We define the finite set $Z_\varepsilon := \mathbf{Z}^M \cap (-\varepsilon^{-1}, \varepsilon^{-1})^M$ and partition Q (up to a null set) into

$$Q = \bigcup_{k\in Z_\varepsilon} \varepsilon(Q + k).$$

Fix some $\varphi \in C_0^\infty(\mathbf{R}^n \times Q)$ and we compute

$$\begin{aligned}
\int_{\mathbf{R}^n\times Q} \varphi(x,y)\mathrm{d}(x,y) &= \sum_{k\in Z_\varepsilon} \int_{\mathbf{R}^n\times(\varepsilon Q+\varepsilon k)} \varphi(x,y)\mathrm{d}(x,y) \\
&= \int_{\mathbf{R}^n\times(\varepsilon Q)} \left[\sum_{k\in Z_\varepsilon} \varphi(x, y+\varepsilon k)\right] \mathrm{d}(x,y).
\end{aligned}$$

Using additionally the periodicity of $f^\varepsilon, g^\varepsilon$, we thus get

$$\begin{aligned}
&\int_{\mathbf{R}^N\times Q} f^\varepsilon(x,y)\cdot g^\varepsilon(x,y)\varphi(x,y)\mathrm{d}(x,y) \\
&= \int_{\mathbf{R}^N\times(\varepsilon Q)} f^\varepsilon(x,y)\cdot g^\varepsilon(x,y)\left[\sum_{k\in Z_\varepsilon} \varphi(x,y+\varepsilon k)\right]\mathrm{d}(x,y) \\
&= \int_{\mathbf{R}^N\times[\varepsilon(Q\setminus S)]} f^\varepsilon(x,y)\cdot g^\varepsilon(x,y)\left[\sum_{k\in Z_\varepsilon} \varphi(x,y+\varepsilon k)\right]\mathrm{d}(x,y) \\
&\quad + \int_{\mathbf{R}^N\times(\varepsilon S)} f^\varepsilon(x,y)\cdot g^\varepsilon(x,y)\left[\sum_{k\in Z_\varepsilon} \varphi(x,y+\varepsilon k)\right]\mathrm{d}(x,y) \\
&= \int_{\mathbf{R}^N\times[\varepsilon(Q\setminus S)]} \overline{f}\cdot\overline{g}\cdot\left[\sum_{k\in Z_\varepsilon} \varphi(x,y+\varepsilon k)\right]\mathrm{d}(x,y) \\
&\quad + \int_{\mathbf{R}^N\times\partial(\varepsilon S)} \langle (g^r)^\varepsilon(x,y), \nu\rangle \left[\psi^\varepsilon(x,y)\sum_{k\in Z_\varepsilon} \varphi(x,y+\varepsilon k)\right]\mathrm{d}(x,y) \\
&\quad - \int_{\mathbf{R}^N\times(\varepsilon S)} \psi^\varepsilon(x,y)\cdot \operatorname{div}\left[g^\varepsilon(x,y)\sum_{k\in Z_\varepsilon} \varphi(x,y+\varepsilon k)\right]\mathrm{d}(x,y).
\end{aligned}$$

The first term vanishes by the assumption on the weak limits of $f^\varepsilon, g^\varepsilon$, while the second one vanishes by the boundary condition on g. In the last term we apply the product rule: the term, where the divergence falls on g^ε vanishes, by using the PDE and density.
We are left with

$$\begin{aligned}
&\int_{\mathbf{R}^N\times Q} f^\varepsilon(x,y)\cdot g^\varepsilon(x,y)\varphi(x,y)\mathrm{d}(x,y)\\
&= -\int_{\mathbf{R}^N\times(\varepsilon S)} \psi^\varepsilon(x,y)\cdot\left\langle g^\varepsilon(x,y), \sum_{k\in Z_\varepsilon}\nabla\varphi(x,y+\varepsilon k)\right\rangle\mathrm{d}(x,y)\\
&= -\int_{K^S} \psi^\varepsilon(x,\varepsilon y)\cdot\left\langle g^\varepsilon(x,\varepsilon y), \varepsilon^M\sum_{k\in Z_\varepsilon}\nabla\varphi(x,\varepsilon y+\varepsilon k)\right\rangle\mathrm{d}(x,y).
\end{aligned}$$

We note that $(x,y)\mapsto\psi^\varepsilon(x,\varepsilon y)$ is uniformly bounded in $L^2(K^S)$. Indeed, using Poincaré's inequality, recalling $\int_{K^S}\psi^\varepsilon(x,\varepsilon y)\mathrm{d}(x,y)=0$, yields

$$\|\psi^\varepsilon(x,\varepsilon y)\|_{L^2(K^S)}\le C_{K^S}\|f^\varepsilon(x,\varepsilon y)\|_{L^2(K^S)}=C_{K^S}\|f(T_{\varepsilon^{-1}x}\widetilde{\omega},y)\|_{L^2(K^S)}.$$

Furthermore the sequence

$$(x,y)\mapsto f^\varepsilon(x,\varepsilon y)=f(T_{\varepsilon^{-1}x}\widetilde{\omega},y)$$

is uniformly bounded in $L^2(K^S)$ for almost all $\widetilde{\omega}\in\Omega$. To see this define for every $x\in\mathbf{R}^N$ the cross sections

$$K_x^S:=\left\{y\in\mathbf{R}^M:(x,y)\in K^S\right\}\subset S,$$

and thus

$$\begin{aligned}
\int_{K^S}|f(T_{\varepsilon^{-1}x}\widetilde{\omega},y)|^2\mathrm{d}(x,y) &= \int_{\mathbf{R}^N}\int_{K_x^S}|f(T_{\varepsilon^{-1}x}\widetilde{\omega},y)|^2\mathrm{d}y\mathrm{d}x\\
&\le \int_{\{x\in\mathbf{R}^N:K_x^S\neq\emptyset\}}\left(\int_S|f(\cdot,y)|^2\mathrm{d}y\right)(T_{\varepsilon^{-1}x}\widetilde{\omega})\mathrm{d}x.
\end{aligned}$$

By the Ergodic Theorem the integrand converges for almost every $\widetilde{\omega}$ to $C\|f\|^2_{L^2(\Omega\times S)}$, for some constant $C>0$ depending only on K^S. Thus for almost every $\widetilde{\omega}\in\Omega$ we have

$$\limsup_{\varepsilon\downarrow 0}\int_{K^S}|f(T_{\varepsilon^{-1}x}\widetilde{\omega},y)|^2\mathrm{d}(x,y)<\infty,$$

and therefore the left-hand side is uniformly bounded for almost every $\widetilde{\omega}$.

Noticing also that

$$\nabla((x,y)\mapsto\psi(x,\varepsilon y))=(f_1,\dots,f_N,\varepsilon f_{N+1},\dots,\varepsilon f_L),$$

we have a uniform bound on $(x,y)\mapsto\psi^\varepsilon(x,\varepsilon y)$ in $W^{1,2}(K^S)$ and thus a subsequence converging weakly to some $\Psi\in W^{1,2}(K^S)$. By Rellich's Theorem we have also strong convergence in $L^2(K^S)$. Additionally Ψ does not depend on y: to see this, we apply Poincaré's inequality once more and obtain

$$\begin{aligned}\|\partial_y\Psi\|_{L^2(K^S)}&\le\liminf_{\varepsilon\downarrow 0}\|\partial_y((x,y)\mapsto\psi^\varepsilon(x,\varepsilon y))\|_{L^2(K^S)}\\&\le\liminf_{\varepsilon\downarrow 0}\varepsilon\|(x,y)\mapsto f^\varepsilon(x,\varepsilon y)\|_{L^2(K^S)}=0.\end{aligned}$$

The sequence of functions $(x,y)\mapsto g^\varepsilon(x,\varepsilon y)=g(T_{\varepsilon^{-1}x}\widetilde{\omega},y)$ converges weakly to $(x,y)\mapsto(\int_\Omega g(\omega,y)\mathrm{d}\omega)$ in $L^2(Q)$, a function independent of x. Finally observe that

$$\left[(x,y)\mapsto\varepsilon^M\sum_{k\in Z_\varepsilon}\nabla\varphi(x,\varepsilon y+\varepsilon k)\right]\to\left[x\mapsto\int_Q\nabla\varphi(x,\hat{y})\mathrm{d}\hat{y}\right]$$

uniformly in x. We thus have

$$\begin{aligned}&\int_{\mathbf{R}^N\times S}\left\langle g^\varepsilon(x,\varepsilon y),\psi^\varepsilon(x,\varepsilon y)\Big(\varepsilon^M\sum_{k\in Z_\varepsilon}\nabla\varphi(x,\varepsilon y+\varepsilon k)\Big)\right\rangle\mathrm{d}(x,y)\\&\to\int_{\mathbf{R}^N\times S}\left\langle\int_\Omega g(\omega,y)\mathrm{d}P(\omega)\ ,\ \Psi(x)\int_Q\nabla\varphi(x,\hat{y})\mathrm{d}\hat{y}\right\rangle\mathrm{d}(x,y),\end{aligned}$$

since the first factor converges weakly and the second strongly. Rearranging the integrals yields

$$\begin{aligned}&\int_{\mathbf{R}^N\times S}\left\langle\int_\Omega g(\omega,y)\mathrm{d}P(\omega)\ ,\ \Psi(x)\int_Q\nabla\varphi(x,\hat{y})\mathrm{d}\hat{y}\right\rangle\mathrm{d}(x,y)\\&\qquad=\left\langle|S|\cdot\overline{g}\ ,\ \int_{\mathbf{R}^N\times Q}\Psi(x)\cdot\nabla\varphi(x,\hat{y})\mathrm{d}\hat{y}\mathrm{d}x\right\rangle=0,\end{aligned}$$

since $\overline{g}=0$. This finishes the proof. □

A.4.4 The decomposition

We will prove Theorem A.4 (i) similar to [ZKO94][Lemma 7.3]. For this we introduce a mollifier in the mixed setting. Let $K_1\in C_0^\infty(\mathbf{R}^N),K_2\in$

$C_0^\infty(\mathbf{R}^M)$ be standard mollifier, i.e., K_1, K_2 are even functions in the sense that $K_i(x) = K_i(-x)$ for all x and $i = 1, 2$, and

$$K_1, K_2 \geq 0, \quad \int_{\mathbf{R}^N} K_1 = \int_{\mathbf{R}^M} K_2 = 1.$$

Define for $\delta > 0$ the sequences

$$K_1^\delta(s) = \frac{1}{\delta^N} K_1(\delta^{-1}s), \quad K_2^\delta(y) = \frac{1}{\delta^M} K_2(\delta^{-1}y),$$

and further the mollification-operators $\mathcal{J}^\delta$ for $g \in L^2(\Omega \times \mathbf{R}^M)$ by

$$(\mathcal{J}^\delta g)(\omega, x) = \int_{\mathbf{R}^N} \int_{\mathbf{R}^M} K_1^\delta(s) K_2^\delta(x-y) g(T_s\omega, y) \mathrm{d}y \mathrm{d}s.$$

It is easily seen that $\mathcal{J}^\delta g$ is a continuous, linear, symmetric operator $L^2(\Omega \times \mathbf{R}^M) \to L^2(\Omega \times \mathbf{R}^M)$ with

$$\lim_{\delta \downarrow 0} \mathcal{J}^\delta g = g \quad \text{strongly in } L^2(\Omega \times \mathbf{R}^M).$$

Furthermore $\mathcal{J}^\delta g \in W^{1,2}(\Omega \times S, \mathbf{R}^M)$ for $g \in L^2(\Omega \times \mathbf{R}^M)$ and

$$\nabla(\mathcal{J}^\delta g) = \mathcal{J}^\delta \nabla g \quad \text{for all } g \in W^{1,2}(\Omega \times \mathbf{R}^M).$$

Proof of Theorem A.4 (i). The orthogonality between $F^2_{\mathrm{pot}}(\Omega \times S)$ and $L^2_{\mathrm{sol}}(\Omega \times S)$ follows from Lemma A.5. Therefore

$$L^2_{\mathrm{sol}}(\Omega \times S) \subset [F^2_{\mathrm{pot}}(\Omega \times S)]^\perp.$$

For the reverse inclusion let $g \in [F^2_{\mathrm{pot}}(\Omega \times S)]^\perp$, i.e.

$$\langle g, f \rangle_{L^2} = 0 \quad \text{for all } f \in F^2_{\mathrm{pot}}(\Omega \times S).$$

Fix some $\varphi \in \widetilde{\mathcal{D}}$, and note that $\nabla\varphi \in F^2_{\mathrm{pot}}(\Omega \times S)$. Extend both φ and g by 0 to functions defined on $\Omega \times \mathbf{R}^M$
Fix some $\delta > 0$. By using the elementary properties of the mollification operator $\mathcal{J}^\delta$ we have

$$0 = \langle g, \nabla \mathcal{J}^\delta \varphi \rangle_{L^2} = \langle g, \mathcal{J}^\delta(\nabla\varphi) \rangle_{L^2} = \langle \mathcal{J}^\delta g, \nabla\varphi \rangle_{L^2} = -\langle \operatorname{div} \mathcal{J}^\delta g, \varphi \rangle_{L^2}.$$

By the density of $\widetilde{\mathcal{D}} \subset \widetilde{X} \subset L^2(\Omega \times S)$ we get $\operatorname{div}(\mathcal{J}^\delta g) = 0$ a.e., thus

$$\operatorname{div} \mathcal{J}^\delta g = 0 \quad \text{in } \widetilde{\mathcal{D}'}.$$

By the strong convergence $\mathcal{J}^\delta g \to g$ in $L^2(\Omega \times S, \mathbf{R}^L)$ we get

$$\operatorname{div} g = 0 \quad \text{in } \widetilde{\mathcal{D}'}.$$

Furthermore for any $\psi \in \widetilde{X}$ we have $\nabla\psi \in F^2_{\text{pot}}$ as well and thus

$$\begin{aligned} 0 = \int_{\Omega\times S} \langle \mathcal{J}^\delta g, \nabla\psi\rangle &= \int_{\Omega\times S} \operatorname{div}\left(\mathcal{J}^\delta(g)\cdot\psi\right) \\ &= \int_{\Omega\times S} \sum_{k=N+1}^{L} \partial_k(\mathcal{J}^\delta(g)_k \cdot \psi) = \int_\Omega \int_{\partial S} \langle \psi \mathcal{J}^\delta(g^r), \nu\rangle. \end{aligned}$$

Note that $\mathcal{J}^\delta g \to g$ in $L^2(\Omega\times S)$ together with $\operatorname{div} g = \operatorname{div} \mathcal{J}^\delta g = 0$ implies that $\mathcal{J}^\delta g \to g$ in $W^{1,2}_{\text{div}}(\Omega\times S)$. Together with the equality

$$\int_\Omega \int_{\partial S} \langle \psi \mathcal{J}^\delta(g''), \nu\rangle = 0,$$

following from Lemma A.3, the strong convergence $\mathcal{J}^\delta g \to g$ in $W^{1,2}_{\text{div}}(\Omega\times S)$ is enough to conclude that $\operatorname{div} g = 0$ in $\widetilde{X}'$, thus $g \in L^2_{\text{sol}}(\Omega\times S)$. □

For the proof of Theorem A.4 (ii) we follow [DG16]:

Proof of Theorem A.4 (ii). From classical Hilbert space theory follows

$$L^2(\Omega\times S, \mathbf{R}^L) = \operatorname{adh}_{L^2}\{\nabla\chi : \chi \in \widetilde{X}\} \oplus \left[\operatorname{adh}_{L^2}\{\nabla\chi : \chi\in\widetilde{X}\}\right]^\perp.$$

By the previous orthogonal decompositions it is enough to show that

$$\left[\operatorname{adh}_{L^2}\{\nabla\chi : \chi\in\widetilde{X}\}\right]^\perp = L^2_{\text{sol}}(\Omega\times S),$$

since then

$$\operatorname{adh}_{L^2}\{\nabla\chi : \chi\in\widetilde{X}\} = F^2_{\text{pot}}(\Omega\times S)$$

follows trivially from

$$L^2(\Omega\times S, \mathbf{R}^L) = F^2_{\text{pot}}(\Omega\times S) \oplus L^2_{\text{sol}}(\Omega\times S).$$

But

$$\left[\operatorname{adh}_{L^2}\{\nabla\chi : \chi\in\widetilde{X}\}\right]^\perp = L^2_{\text{sol}}(\Omega\times S)$$

was just the definition of the space $L^2_{\text{sol}}(\Omega\times S)$. □

The claim of Theorem A.4 (iii) can be proven almost identically.

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