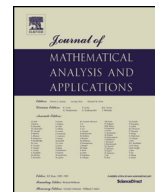




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Solving some q -trigonometric conjectures of Gosper

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ABSTRACT

Gosper in 2001 introduced the functions $\sin_q z$ and $\cos_q z$ as q -analogues for the trigonometric functions $\sin z$ and $\cos z$ respectively. He stated, but did not prove, many identities involving the functions $\sin_q z$ and $\cos_q z$ along with some other related functions and constants. In this note we shall provide proofs for some of these identities by means of the classical theory of elliptic functions.

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1. Introduction

Throughout the paper let $q = e^{\pi i \tau}$ with $\text{Im}(\tau) > 0$, let $\tau' = -\frac{1}{\tau}$, and let $p = e^{\pi i \tau'}$. Note that the assumption $\text{Im}(\tau) > 0$ guarantees that $|q| < 1$ and $|p| < 1$. For a complex variable a , the q -shifted factorials are defined by

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i), \quad (a; q)_\infty = \lim_{n \rightarrow \infty} (a; q)_n.$$

For brevity of notation we write

$$(a_1, \dots, a_k; q)_n = (a_1; q)_n \cdots (a_k; q)_n, \quad (a_1, \dots, a_k; q)_\infty = (a_1; q)_\infty \cdots (a_k; q)_\infty.$$

The first Jacobi theta function is defined as follows:

$$\theta_1(z, q) = \theta_1(z | \tau) = 2 \sum_{n=0}^{\infty} (-1)^n q^{(2n+1)^2/4} \sin(2n+1)z,$$

which by Jacobi's triple product identity has the following infinite product representation

$$\theta_1(z | \tau) = iq^{\frac{1}{4}} e^{-iz} (q^2 e^{-2iz}, e^{2iz}, q^2; q^2)_\infty.$$

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See Whittaker and Watson [11, p. 469] and Gasper and Rahman [3, p. 15]. For the purpose of this work we will need the following basic properties of the function $\theta_1(z | \tau)$ which can be derived straightforwardly by the definitions.

$$\begin{aligned}\theta_1(k\pi | \tau) &= 0 \quad (k \in \mathbb{Z}), \\ \theta_1(z + \pi | \tau) &= -\theta_1(z | \tau) = \theta_1(-z | \tau), \\ \theta_1\left(z - \frac{\pi}{2} | \tau\right) &= -\theta_1\left(z + \frac{\pi}{2} | \tau\right), \\ \theta_1(z + \pi\tau | \tau) &= -q^{-1}e^{-2iz}\theta_1(z | \tau).\end{aligned}\tag{1}$$

Moreover, one can show that for any positive integer k we have

$$\theta_1\left(z + \pi\tau \mid \frac{\tau}{k}\right) = (-1)^k q^{-k} e^{-2kiz} \theta_1\left(z \mid \frac{\tau}{k}\right).\tag{2}$$

Gasper [4] introduced q -analogues of $\sin(z)$ and $\cos(z)$ respectively as follows:

$$\begin{aligned}\sin_q(\pi z) &= q^{(z-1/2)^2} \prod_{n=1}^{\infty} \frac{(1 - q^{2n-2z})(1 - q^{2n+2z-2})}{(1 - q^{2n-1})^2} = q^{(z-\frac{1}{2})^2} \frac{(q^{2z}, q^{2-2z}; q^2)_{\infty}}{(q; q^2)_{\infty}^2}, \\ \cos_q(\pi z) &= q^{z^2} \prod_{n=1}^{\infty} \frac{(1 - q^{2n-2z-1})(1 - q^{2n+2z-1})}{(1 - q^{2n-1})^2} = q^{z^2} \frac{(q^{1+2z}, q^{1-2z}; q^2)_{\infty}}{(q; q^2)_{\infty}^2}.\end{aligned}$$

Gasper [4, pp. 98, 99] proved that the functions \sin_q and \cos_q are related to Jacobi's theta function $\theta_1(z | \tau)$ as follows:

$$\sin_q(z) = \frac{\theta_1(z, p)}{\theta_1\left(\frac{\pi}{2}, p\right)} \quad \text{where } (\ln p)(\ln q) = \pi^2,$$

which is readily seen to be equivalent to

$$\sin_q(z) = \frac{\theta_1(z | \tau')}{\theta_1\left(\frac{\pi}{2} | \tau'\right)}.\tag{3}$$

Clearly the formula (3) combined with the evident relations $\cos_q(z) = \sin_q(\pi/2 - z)$ and $\theta_1(z + \pi) = -\theta_1(z)$ yield

$$\cos_q(z) = \frac{\theta_1\left(z + \frac{\pi}{2}, p\right)}{\theta_1\left(\frac{\pi}{2}, p\right)} = \frac{\theta_1\left(z + \frac{\pi}{2} | \tau'\right)}{\theta_1\left(\frac{\pi}{2} | \tau'\right)}.\tag{4}$$

The author also introduced the following related two functions

$$\text{ccs}_q \pi z := \frac{\cos_{q^2} \pi z}{\cos_q \pi z} \quad \text{and} \quad \text{ssn}_q \pi z := \frac{\sin_{q^2} \pi z}{\sin_q \pi z}.$$

Since $\sin_q z$ and $\cos_q z$ are q -analogues of $\sin z$ and $\cos z$ respectively, it is natural to ask the question which well-known trigonometric relations have q -analogues in terms of $\sin_q z$ and $\cos_q z$. For instance, Gasper established the following formula:

$$\sin_q(2z) = q^{\frac{-1}{4}} \frac{(q^2; q^2)_{\infty}^2 (q^2; q^4)_{\infty}^2}{(q; q^2)_{\infty}^2 (q^4; q^4)_{\infty}^2} \sin_{q^2} z \cos_{q^2} z, \tag{q-Double2}$$

which can be seen to be a q -analogue for the famous identity $\sin 2z = 2 \sin z \cos z$. In an attempt to give q -analogues for more trigonometric formulas, Gosper [4] stated many identities involving $\sin_q(z)$ and $\cos_q(z)$. However, the author did not provide any proofs for most of his formulas. Gosper based his conjectures on a computer algebra program called MACSYMA and asked the question whether his formulas are true. Among these identities we find the following which we mark by labels similar to the ones used by Gosper.

$$\cos_q(2z) = (\cos_{q^2} z)^2 - (\sin_{q^2} z)^2, \quad (q\text{-Double}_3)$$

$$\sin_q(x+y) = \frac{\sin_q(x-y)}{\sin_{q^2}(x-y)} (\sin_{q^2} x \cos_{q^2} y + \cos_{q^2} x \sin_{q^2} y), \quad (q\text{-Add})$$

$$\operatorname{ccs}_q(z) = \sin_{q^2}^2 \frac{z}{2} + \cos_{q^2}^2 \frac{z}{2}, \quad (q\text{-Add}_1)$$

$$\cos_q(x+y) = \frac{\cos_{q^2} x \cos_{q^2} y - \sin_{q^2} x \sin_{q^2} y}{\operatorname{ccs}_q(x-y)}, \quad (q\text{-Add}_2)$$

and

$$\begin{aligned} \sin_{q^3} x \sin_q(2y-x) - \sin_{q^3} y \sin_q(2x-y) \\ = \cos_{q^3} y \cos_q(2x-y) - \cos_{q^3} x \cos_q(2y-x). \end{aligned} \quad (q\text{-Add}_3)$$

See Gosper [4, pp. 97–101]. Recently, Mezö [8] gave a different proof for $(q\text{-Double}_2)$ by an analysis of its left-hand side and its right-hand side through logarithmic derivatives with respect to z . Alternatively, taking into account the relations (3) and (4), formula $(q\text{-Double}_3)$ can be written as

$$\frac{\theta_1\left(2z + \frac{\pi}{2} \mid \tau'\right)}{\theta_1\left(\frac{\pi}{2} \mid \tau'\right)} = \left(\frac{\theta_1\left(z + \frac{\pi}{2} \mid \frac{\tau'}{2}\right)}{\theta_1\left(\frac{\pi}{2} \mid \frac{\tau'}{2}\right)} \right)^2 - \left(\frac{\theta_1\left(z \mid \frac{\tau'}{2}\right)}{\theta_1\left(\frac{\pi}{2} \mid \frac{\tau'}{2}\right)} \right)^2,$$

which after rearrangement becomes

$$\begin{aligned} \theta_1\left(2z + \frac{\pi}{2} \mid \tau'\right) \theta_1^2\left(\frac{\pi}{2} \mid \frac{\tau'}{2}\right) &= \theta_1\left(\frac{\pi}{2} \mid \tau'\right) \theta_1^2\left(z + \frac{\pi}{2} \mid \frac{\tau'}{2}\right) \\ &\quad - \theta_1\left(\frac{\pi}{2} \mid \tau'\right) \theta_1^2\left(z \mid \frac{\tau'}{2}\right). \end{aligned} \quad (5)$$

Furthermore, again by virtue of (3) and (4) note that formula $(q\text{-Double}_2)$ means

$$\frac{\theta_1(2z \mid \tau')}{\theta_1\left(\frac{\pi}{2} \mid \tau'\right)} = C(q) \frac{\theta_1\left(z \mid \frac{\tau'}{2}\right)}{\theta_1\left(\frac{\pi}{2} \mid \frac{\tau'}{2}\right)} \cdot \frac{\theta_1\left(z + \frac{\pi}{2} \mid \frac{\tau'}{2}\right)}{\theta_1\left(\frac{\pi}{2} \mid \frac{\tau'}{2}\right)},$$

or equivalently,

$$\theta_1(2z \mid \tau') \theta_1^2\left(\frac{\pi}{2} \mid \frac{\tau'}{2}\right) = C(q) \theta_1\left(\frac{\pi}{2} \mid \tau'\right) \theta_1\left(z \mid \frac{\tau'}{2}\right) \theta_1\left(z + \frac{\pi}{2} \mid \frac{\tau'}{2}\right),$$

which by observing that

$$C(q) = \frac{\theta_1^2\left(\frac{\pi}{2} \mid \frac{\tau'}{2}\right)}{\theta_1^2\left(\frac{\pi}{4} \mid \frac{\tau'}{2}\right)},$$

means that

$$\theta_1\left(z \mid \frac{\tau'}{2}\right) \theta_1\left(z + \frac{\pi}{2} \mid \frac{\tau'}{2}\right) \theta_1\left(\frac{\pi}{2} \mid \tau'\right) = \theta_1(2z \mid \tau') \theta_1^2\left(\frac{\pi}{4} \mid \frac{\tau'}{2}\right). \quad (6)$$

After recognising the equivalent forms (5) and (6) as three-term addition formulas involving theta functions, this author in [2] proved both (*q-Double*₂) and (*q-Double*₃) by using the theory of elliptic functions. In fact, we can easily check that each one of Gosper's conjectures (*q-Add*), (*q-Add*₁), (*q-Add*₂), and (*q-Add*₃) is a three-term addition formulas involving theta functions. See below Theorem 1, Theorem 2, Theorem 3, and Theorem 4 along with their corollaries. The theory of elliptic functions proved to be a powerful tool to study this type of addition formulas. For recent papers dealing with addition formulas using elliptic functions, we refer to Liu [6,7]. See also Whittaker and Watson [11], Lawden [5], and Shen [9,10] for more additive formulas involving theta functions and applications. For more information on elliptic functions we refer to Apostol [1]. In this paper we will confirm Gosper's conjectures (*q-Add*), (*q-Add*₁), (*q-Add*₂), and (*q-Add*₃) by employing the theory of elliptic functions.

2. Main results

Theorem 1. *For all complex numbers x and y we have*

$$\begin{aligned} & \theta_1(x-y \mid \tau) \theta_1\left(x \mid \frac{\tau}{2}\right) \theta_1\left(y + \frac{\pi}{2} \mid \frac{\tau}{2}\right) + \theta_1(x-y \mid \tau) \theta_1\left(x + \frac{\pi}{2} \mid \frac{\tau}{2}\right) \theta_1\left(y \mid \frac{\tau}{2}\right) \\ &= \theta_1(x+y \mid \tau) \theta_1\left(x-y \mid \frac{\tau}{2}\right) \theta_1\left(\frac{\pi}{2} \mid \frac{\tau}{2}\right). \end{aligned}$$

Corollary 1. *For all complex numbers x and y we have*

$$\sin_q(x+y) = \frac{\sin_q(x-y)}{\sin_{q^2}(x-y)} (\sin_{q^2} x \cos_{q^2} y + \cos_{q^2} x \sin_{q^2} y).$$

Proof. By the relations (3) and (4), we can readily see that this result is equivalent to Theorem 1. \square

Theorem 2. *For all complex number z we have*

$$\begin{aligned} & \theta_1^2\left(z \mid \frac{\tau}{2}\right) \theta_1\left(2z + \frac{\pi}{2} \mid \tau\right) + \theta_1^2\left(z + \frac{\pi}{2} \mid \frac{\tau}{2}\right) \theta_1\left(2z + \frac{\pi}{2} \mid \tau\right) \\ &= \theta_1\left(2z + \frac{\pi}{2} \mid \frac{\tau}{2}\right) \theta_1\left(\frac{\pi}{2} \mid \frac{\tau}{2}\right) \theta_1\left(\frac{\pi}{2} \mid \tau\right). \end{aligned}$$

Corollary 2. *For all complex number z we have*

$$\operatorname{ccs}_q(z) = \sin_{q^2}^2 \frac{z}{2} + \cos_{q^2}^2 \frac{z}{2}.$$

Proof. By the relations (3) and (4), we can readily see that this result is an equivalent form of Theorem 2. \square

Theorem 3. *For all complex numbers x and y we have*

$$\begin{aligned} & \theta_1\left(x-y + \frac{\pi}{2} \mid \tau\right) \theta_1\left(x + \frac{\pi}{2} \mid \frac{\tau}{2}\right) \theta_1\left(y + \frac{\pi}{2} \mid \frac{\tau}{2}\right) \\ & \quad - \theta_1\left(x-y + \frac{\pi}{2} \mid \tau\right) \theta_1\left(x \mid \frac{\tau}{2}\right) \theta_1\left(y \mid \frac{\tau}{2}\right) \\ &= \theta_1\left(x+y + \frac{\pi}{2} \mid \tau\right) \theta_1\left(x-y + \frac{\pi}{2} \mid \frac{\tau}{2}\right) \theta_1\left(\frac{\pi}{2} \mid \frac{\tau}{2}\right). \end{aligned}$$

Corollary 3. For all complex numbers x and y we have

$$\cos_q(x+y) = \frac{\cos_{q^2} x \cos_{q^2} y - \sin_{q^2} x \sin_{q^2} y}{\cos_q(x-y)}.$$

Proof. By the relations (3) and (4), it easily seen that the requested formula is an equivalent form of the identity in Theorem 3. \square

Theorem 4. For all complex numbers x and y we have

$$\begin{aligned} & \theta_1\left(x \mid \frac{\tau}{3}\right) \theta_1(2y-x \mid \tau) - \theta_1\left(y \mid \frac{\tau}{3}\right) \theta_1(2x-y \mid \tau) \\ &= \theta_1\left(y+\frac{\pi}{2} \mid \frac{\tau}{3}\right) \theta_1\left(2x-y+\frac{\pi}{2} \mid \tau\right) - \theta_1\left(x+\frac{\pi}{2} \mid \frac{\tau}{3}\right) \theta_1\left(2y-x+\frac{\pi}{2} \mid \tau\right). \end{aligned}$$

Corollary 4. For all complex numbers x and y we have

$$\sin_{q^3} x \sin_q(2y-x) - \sin_{q^3} y \sin_q(2x-y) = \cos_{q^3} y \cos_q(2x-y) - \cos_{q^3} x \cos_q(2y-x).$$

Proof. By the relations (3) and (4), we have that the desired formula is an equivalent to the identity in Theorem 3. \square

3. General results

In this section we collect the main results from which our identities follow.

Theorem 5. Let n be a positive integer and let $f(u)$ be an entire function such that

$$f(u+\pi) = -f(u) \quad \text{and} \quad f\left(u + \frac{\pi\tau}{n}\right) = (-1)^n q^{\frac{-1}{n}} e^{-2iu} f(u).$$

Then for all complex numbers x_1, x_2, \dots, x_{n+1} we have:

$$\sum_{j=1}^{n+1} \frac{\theta_1((n-1)x_j - x_1 - x_2 - \dots - x_{j-1} - x_{j+1} - x_{j+2} - \dots - x_{n+1} \mid \tau) f(x_j)}{\prod_{\substack{k=1 \\ k \neq j}}^n \theta_1(x_j - x_k \mid \frac{\tau}{n})} = 0.$$

Corollary 5. For all complex numbers w, x, y , and z we have:

$$\begin{aligned} & \theta_1(2z-w-y-x \mid \tau) \theta_1\left(w-x \mid \frac{\tau}{3}\right) \theta_1\left(w-y \mid \frac{\tau}{3}\right) \theta_1\left(x-y \mid \frac{\tau}{3}\right) \theta_1\left(z \mid \frac{\tau}{3}\right) \\ &= \theta_1(2w-x-y-z \mid \tau) \theta_1\left(x-y \mid \frac{\tau}{3}\right) \theta_1\left(x-z \mid \frac{\tau}{3}\right) \theta_1\left(y-z \mid \frac{\tau}{3}\right) \theta_1\left(w \mid \frac{\tau}{3}\right) \\ & \quad - \theta_1(2x-w-y-z \mid \tau) \theta_1\left(w-y \mid \frac{\tau}{3}\right) \theta_1\left(w-z \mid \frac{\tau}{3}\right) \theta_1\left(y-z \mid \frac{\tau}{3}\right) \theta_1\left(x \mid \frac{\tau}{3}\right) \\ & \quad + \theta_1(2y-w-x-z \mid \tau) \theta_1\left(w-x \mid \frac{\tau}{3}\right) \theta_1\left(w-z \mid \frac{\tau}{3}\right) \theta_1\left(x-z \mid \frac{\tau}{3}\right) \theta_1\left(y \mid \frac{\tau}{3}\right). \end{aligned}$$

Proof. It easy to verify that the summation in Theorem 5 for $n=3$ becomes

$$\begin{aligned} & \theta_1(2z-w-y-x \mid \tau) \theta_1\left(w-x \mid \frac{\tau}{3}\right) \theta_1\left(w-y \mid \frac{\tau}{3}\right) \theta_1\left(x-y \mid \frac{\tau}{3}\right) f(z) \\ &= \theta_1(2w-x-y-z \mid \tau) \theta_1\left(x-y \mid \frac{\tau}{3}\right) \theta_1\left(x-z \mid \frac{\tau}{3}\right) \theta_1\left(y-z \mid \frac{\tau}{3}\right) f(w) \end{aligned}$$

$$\begin{aligned}
& -\theta_1(2x-w-y-z|\tau)\theta_1\left(w-y\left|\frac{\tau}{3}\right.\right)\theta_1\left(w-z\left|\frac{\tau}{3}\right.\right)\theta_1\left(y-z\left|\frac{\tau}{3}\right.\right)f(x) \\
& +\theta_1(2y-w-x-z|\tau)\theta_1\left(w-x\left|\frac{\tau}{3}\right.\right)\theta_1\left(w-z\left|\frac{\tau}{3}\right.\right)\theta_1\left(x-z\left|\frac{\tau}{3}\right.\right)f(y).
\end{aligned} \tag{7}$$

It is also easy to check by the identities (1) and (2) that the function $f(u) = \theta_1\left(u\left|\frac{\tau}{3}\right.\right)$ satisfies the conditions of Theorem 5 for $n = 3$. Then we deduce the desired identity by taking $f(u) = \theta_1\left(u\left|\frac{\tau}{3}\right.\right)$ in identity (7). \square

Corollary 6. For all complex number x, y , and z we have

$$\begin{aligned}
& \theta_1(z-x-y|\tau)\theta_1\left(x-y\left|\frac{\tau}{2}\right.\right)\theta_1\left(z-\frac{\pi}{2}\left|\frac{\tau}{2}\right.\right) \\
& = \theta_1(y-x-z|\tau)\theta_1\left(x-z\left|\frac{\tau}{2}\right.\right)\theta_1\left(y-\frac{\pi}{2}\left|\frac{\tau}{2}\right.\right) \\
& - \theta_1(x-y-z|\tau)\theta_1\left(y-z\left|\frac{\tau}{2}\right.\right)\theta_1\left(x-\frac{\pi}{2}\left|\frac{\tau}{2}\right.\right).
\end{aligned}$$

Proof. Note that the sum in Theorem 5 for $n = 2$ gives

$$\begin{aligned}
& \theta_1(z-x-y|\tau)\theta_1\left(x-y\left|\frac{\tau}{2}\right.\right)f(z) = \theta_1(y-x-z|\tau)\theta_1\left(x-z\left|\frac{\tau}{2}\right.\right)f(y) \\
& - \theta_1(x-y-z|\tau)\theta_1\left(y-z\left|\frac{\tau}{2}\right.\right)f(x).
\end{aligned} \tag{8}$$

Further, it is readily seen by the identities (1) and (2) that the function $f(u) = \theta_1\left(u-\frac{\pi}{2}\left|\frac{\tau}{2}\right.\right)$ satisfies the conditions of Theorem 5 for $n = 2$. Now put $f(u) = \theta_1\left(u-\frac{\pi}{2}\left|\frac{\tau}{2}\right.\right)$ in the formula (8) to conclude the desired result. \square

4. Proof of Theorem 1

Letting in Corollary 6, $z = 0$ yields

$$\begin{aligned}
& \theta_1(y-x|\tau)\theta_1\left(x\left|\frac{\tau}{2}\right.\right)\theta_1\left(y-\frac{\pi}{2}\left|\frac{\tau}{2}\right.\right) - \theta_1(x-y|\tau)\theta_1\left(y\left|\frac{\tau}{2}\right.\right)\theta_1\left(x-\frac{\pi}{2}\left|\frac{\tau}{2}\right.\right) \\
& = \theta_1(-x-y|\tau)\theta_1\left(x-y\left|\frac{\tau}{2}\right.\right)\theta_1\left(\frac{-\pi}{2}\left|\frac{\tau}{2}\right.\right),
\end{aligned}$$

which by virtue of the properties (1) is equivalently to the desired formula.

5. Proof of Theorem 2

Letting in Corollary 6, $x - z = y - \frac{\pi}{2}$ gives

$$\begin{aligned}
& \theta_1\left(\frac{\pi}{2}-2z|\tau\right)\theta_1^2\left(x-z\left|\frac{\tau}{2}\right.\right) - \theta_1\left(-\frac{\pi}{2}|\tau\right)\theta_1\left(x-2z+\frac{\pi}{2}\left|\frac{\tau}{2}\right.\right)\theta_1\left(x-\frac{\pi}{2}\left|\frac{\tau}{2}\right.\right) \\
& = \theta_1\left(2z-2x-\frac{\pi}{2}|\tau\right)\theta_1^2\left(z-\frac{\pi}{2}\left|\frac{\tau}{2}\right.\right).
\end{aligned}$$

Now let in the previous identity $x = -\pi$ and use the basic properties in (1) to conclude that

$$\begin{aligned}
& \theta_1\left(2z+\frac{\pi}{2}|\tau\right)\theta_1^2\left(z\left|\frac{\tau}{2}\right.\right) - \theta_1\left(2z+\frac{\pi}{2}\left|\frac{\tau}{2}\right.\right)\theta_1\left(\frac{\pi}{2}\left|\frac{\tau}{2}\right.\right)\theta_1\left(\frac{\pi}{2}|\tau\right) \\
& = -\theta_1\left(2z+\frac{\pi}{2}|\tau\right)\theta_1^2\left(z+\frac{\pi}{2}\left|\frac{\tau}{2}\right.\right),
\end{aligned}$$

which is clearly equivalent to the desired formula. This completes the proof.

6. Proof of Theorem 3

Letting in Corollary 6, $z = 0$, $x \rightarrow -x - \frac{\pi}{2}$, and $y \rightarrow -y$ gives

$$\begin{aligned} & \theta_1\left(-y+x+\frac{\pi}{2} \mid \tau\right) \theta_1\left(-x-\frac{\pi}{2} \mid \frac{\tau}{2}\right) \theta_1\left(-y-\frac{\pi}{2} \mid \frac{\tau}{2}\right) \\ & \quad - \theta_1\left(-x-\frac{\pi}{2}+y \mid \tau\right) \theta_1\left(-y \mid \frac{\tau}{2}\right) \theta_1\left(-x-\frac{\pi}{2}-\frac{\pi}{2} \mid \frac{\tau}{2}\right), \\ & = \theta_1\left(x+\frac{\pi}{2}+y \mid \tau\right) \theta_1\left(-x-\frac{\pi}{2}+y \mid \frac{\tau}{2}\right) \theta_1\left(\frac{-\pi}{2} \mid \frac{\tau}{2}\right), \end{aligned}$$

or equivalently,

$$\begin{aligned} & \theta_1\left(x-y+\frac{\pi}{2} \mid \tau\right) \theta_1\left(x+\frac{\pi}{2} \mid \frac{\tau}{2}\right) \theta_1\left(y+\frac{\pi}{2} \mid \frac{\tau}{2}\right) \\ & \quad - \theta_1\left(x-y+\frac{\pi}{2} \mid \tau\right) \theta_1\left(y \mid \frac{\tau}{2}\right) \theta_1\left(x \mid \frac{\tau}{2}\right), \\ & = \theta_1\left(x+y+\frac{\pi}{2} \mid \tau\right) \theta_1\left(x-y+\frac{\pi}{2} \mid \frac{\tau}{2}\right) \theta_1\left(\frac{\pi}{2} \mid \frac{\tau}{2}\right), \end{aligned}$$

as desired.

7. Proof of Theorem 4

Let in Corollary 5, $w = 0$ and simplify to deduce

$$\begin{aligned} & \theta_1(2x-y-z \mid \tau) \theta_1\left(y-z \mid \frac{\tau}{3}\right) - \theta_1(2y-x-z \mid \tau) \theta_1\left(x-z \mid \frac{\tau}{3}\right) \\ & = \theta_1(2z-x-y \mid \tau) \theta_1\left(x-y \mid \frac{\tau}{3}\right). \end{aligned}$$

Next let in the previous identity, $z \rightarrow 0$ to obtain

$$\begin{aligned} & \theta_1\left(x \mid \frac{\tau}{3}\right) \theta_1(2y-x \mid \tau) - \theta_1\left(y \mid \frac{\tau}{3}\right) \theta_1(2x-y \mid \tau) \\ & = \theta_1(x+y \mid \tau) \theta_1\left(x-y \mid \frac{\tau}{3}\right). \end{aligned} \quad (9)$$

Further, performing in the previous formula the substitutions $x \rightarrow x + \frac{\pi}{2}$ and $y \rightarrow y + \frac{\pi}{2}$ and using the basic properties (1) yield

$$\begin{aligned} & \theta_1\left(x+\frac{\pi}{2} \mid \frac{\tau}{3}\right) \theta_1(2y-x+\frac{\pi}{2} \mid \tau) - \theta_1\left(y+\frac{\pi}{2} \mid \frac{\tau}{3}\right) \theta_1(2x-y+\frac{\pi}{2} \mid \tau) \\ & = -\theta_1(x+y \mid \tau) \theta_1\left(x-y \mid \frac{\tau}{3}\right). \end{aligned} \quad (10)$$

Finally, comparing the relations (9) and (10) we clearly conclude the desired formula.

8. Proof of Theorem 5

Let

$$g(u) = \frac{\theta_1(nu - x_1 - x_2 - \cdots - x_{n+1} \mid \tau) f(u)}{\prod_{j=1}^{n+1} \theta_1(u - x_j \mid \frac{\tau}{n})},$$

where x_1, x_2, \dots, x_{n+1} are different from the zeros of $\theta_1(nu - x_1 - x_2 - \dots - x_{n+1} \mid \tau)f(u)$. Suppose for the moment that $0 < x_1, x_2, \dots, x_{n+1} < \pi$. Then by the properties of the function θ_1 and the assumptions on the function $f(u)$ we can easily check that

$$g(u + \pi) = g(u) \quad \text{and} \quad g\left(u + \frac{\pi\tau}{n}\right) = g(u),$$

showing that $g(u)$ is an elliptic function with periods π and $\frac{\pi\tau}{n}$. Clearly, the function $g(u)$ has simple poles at x_1, x_2, \dots, x_{n+1} in the fundamental parallelogram $0, \pi, \frac{\pi\tau}{n}, \pi + \frac{\pi\tau}{n}$. We have for any $j = 1, 2, \dots, n+1$

$$\begin{aligned} \text{Res}(g; x_j) &= \lim_{u \rightarrow x_j} \frac{u - x_j}{\theta_1\left(u - x_j \mid \frac{\tau}{n}\right)} \\ &\quad \cdot \frac{\theta_1\left((n-1)x_j - x_1 - x_2 - \dots - x_{j-1} - x_{j+1} - x_{j+2} - \dots - x_{n+1} \mid \tau\right)f(x_j)}{\prod_{\substack{k=1 \\ k \neq j}}^n \theta_1\left(x_j - x_k \mid \frac{\tau}{n}\right)} \\ &= \frac{1}{\theta_1'\left(0 \mid \frac{\tau}{n}\right)} \frac{\theta_1\left((n-1)x_j - x_1 - x_2 - \dots - x_{j-1} - x_{j+1} - x_{j+2} - \dots - x_{n+1} \mid \tau\right)f(x_j)}{\prod_{\substack{k=1 \\ k \neq j}}^n \theta_1\left(x_j - x_k \mid \frac{\tau}{n}\right)}. \end{aligned}$$

Hence by the residue theorem for elliptic functions (see [1]), we have

$$\sum_{j=1}^{n+1} \frac{\theta_1\left((n-1)x_j - x_1 - x_2 - \dots - x_{j-1} - x_{j+1} - x_{j+2} - \dots - x_{n+1} \mid \tau\right)f(x_j)}{\prod_{\substack{k=1 \\ k \neq j}}^n \theta_1\left(x_j - x_k \mid \frac{\tau}{n}\right)} = 0$$

we obtain the desired identity for $0 < x_1, x_2, \dots, x_{n+1} < \pi$. But this result can be extended to any complex numbers x_1, x_2, \dots, x_{n+1} by the principle of analytic continuation. This completes the proof.

References

- [1] T.M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Springer, 1990.
- [2] M. El Bachraoui, Confirming a q -trigonometric conjecture of Gosper, Proc. Amer. Math. Soc. (2017), <https://doi.org/10.1090/proc/13830>, in press.
- [3] G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, 2004.
- [4] R.W. Gosper, Experiments and discoveries in q -trigonometry, in: F.G. Garvan, M.E.H. Ismail (Eds.), *Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics*, Kluwer, Dordrecht, Netherlands, 2001, pp. 79–105.
- [5] D.F. Lawden, *Elliptic Functions and Applications*, Springer-Verlag, 1989.
- [6] Z.-G. Liu, A theta function identity and its implications, Trans. Amer. Math. Soc. 357 (2) (2005) 825–835.
- [7] Z.-G. Liu, An addition formula for the Jacobian theta function and its applications, Adv. Math. 212 (1) (2007) 389–406.
- [8] I. Mezö, Duplication formulae involving Jacobi theta functions and Gosper's q -trigonometric functions, Proc. Amer. Math. Soc. 141 (7) (2013) 2401–2410.
- [9] L.-C. Shen, On the additive formulae of the theta functions and a collection of Lambert series pertaining to the modular equations of degree 5, Trans. Amer. Math. Soc. 345 (1) (1994) 323–345.
- [10] L.-C. Shen, On some modular equations of degree 5, Proc. Amer. Math. Soc. 123 (5) (1995) 1521–1526.
- [11] E.T. Whittaker, G.N. Watson, *A Course of Modern Analysis*, Cambridge University Press, 1996.