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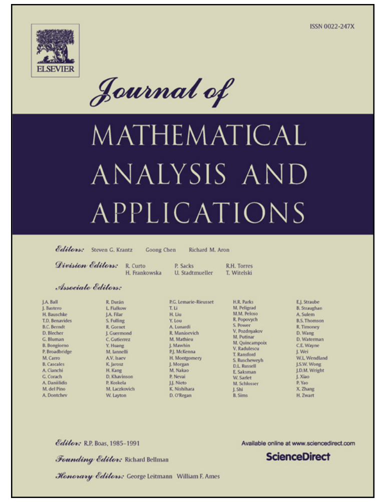

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Local Martingale Solutions to the Stochastic Two Layer Shallow Water Equations with Multiplicative White Noise

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Abstract

We study the two layers shallow water equations on a bounded domain $\mathcal{M} \subset \mathbb{R}^2$ driven by a multiplicative white noise, and obtain the existence and uniqueness of a maximal pathwise solution for a limited period of time, the time of existence being strictly positive almost surely. The proof makes use of anisotropic estimates and stopping time arguments, of the Skorohod representation theorem, and the Gyöngy-Krylov theorem which is an infinite dimensional analogue of the Yamada-Watanabe theorem.

Contents

1 Introduction

In this article, we study the two layer shallow water equations in space dimensions two, on a bounded domain $\mathcal{M}$, forced by multiplicative noise:

$$d\mathbf{v}_1 - \nu_1\Delta\mathbf{v}_1 dt + (\mathbf{v}_1\cdot\nabla)\mathbf{v}_1 dt + g\nabla h_1 dt + g\frac{\rho_2}{\rho_1}\nabla h_2 dt + f\mathbf{k}\times\mathbf{v}_1 dt = F dt + \sigma_1(\mathbf{v}_1,\mathbf{v}_2,h_1,h_2)dW_1 \qquad \text{in } \mathcal{M}\times(0,T), \tag{1.1a}$$

$$d\mathbf{v}_2 - \nu_2\Delta\mathbf{v}_2 dt + (\mathbf{v}_2\cdot\nabla)\mathbf{v}_2 dt + g\nabla h_2 dt + g\nabla h_1 + f\mathbf{k}\times\mathbf{v}_2 dt = G dt + \sigma_2(\mathbf{v}_1,\mathbf{v}_2,h_1,h_2)dW_2 \qquad \text{in } \mathcal{M}\times(0,T), \tag{1.1b}$$

$$dh_1 + \nabla\cdot(h_1\mathbf{v}_1)dt - \delta_1\Delta h_1 dt = \sigma_3(\mathbf{v}_1,\mathbf{v}_2,h_1,h_2)dW_3 \qquad \text{in } \mathcal{M}\times(0,T), \tag{1.1c}$$

$$dh_2 + \nabla\cdot(h_2\mathbf{v}_2)dt - \delta_2\Delta h_2 dt = \sigma_4(\mathbf{v}_1,\mathbf{v}_2,h_1,h_2)dW_4 \qquad \text{in } \mathcal{M}\times(0,T), \tag{1.1d}$$

supplemented with the following initial conditions and Dirichlet boundary conditions

$$\mathbf{v}_i(t=0) = \mathbf{v}_i^0(x,y) \qquad \text{in } \mathcal{M},\ i=1,2, \tag{1.1e}$$

$$h_i(t=0) = {h_i}^0(x,y) > 0 \qquad \text{in } \mathcal{M},\ i=1,2, \tag{1.1f}$$

$$\mathbf{v}_i = 0 \qquad \text{on } \partial\mathcal{M}\times(0,T),\ i=1,2, \tag{1.1g}$$

$$h_i = 0 \qquad \text{on } \partial\mathcal{M} \times (0,T),\ i = 1,2. \tag{1.1h}$$

Here, $\mathbf{v}_1 = (u_1, v_1)$ where $u_1 := u_1(x,y,\omega,t)$ and $v_1 := v_1(x,y,\omega,t)$ denote the velocity in the x direction and y direction of the upper layer, respectively. Similarly, $\mathbf{v}_2 = (u_2, v_2)$ where $u_2 := u_2(x,y,\omega,t)$ and $v_2 := v_2(x,y,\omega,t)$ corresponds to the lower layer. We will assume that $h_1 = H_1 + \mathring{h}_1$, where $H_1 > 0$ is the average depth of the upper layer, a constant, and $\mathring{h}_1$ is the deviation from this average height. Similarly, $h_2 = H_2 + \mathring{h}_2$, where $H_2 > 0$ is the average depth of the lower layer, a constant, and $\mathring{h}_2$ is the deviation from this average height. Also, ν_1 and ν_2 are the viscosities, δ_1, and δ_2 are given positive constants, g is the gravitational constant, f is the Coriolis parameter assumed to be constant, ρ_1 and ρ_2 are the densities of the top fluid and bottom fluid respectively, and both of them are assumed to be constants as well. Finally, $F := F(x,y,t)$, $G := G(x,y,t)$, $\mathbf{v}_1^0(x,y)$, $\mathbf{v}_2^0(x,y)$, $h_1^0(x,y)$ and $h_2^0(x,y)$ are given. Typically, F represents a wind force at the surface, G represents an interfacial surfacic Force.

The system (1.1) describes the motion of the two superposed layers of fluids with different densities so that no mixing occurs. A typical example is the superposition in an estuary of the fresh water coming from a river and of the heavier salted water from the sea. More generally, the stratified salted water in a deep ocean is often modeled as the superposition of a number of layers of fluid, see e.g [26]. In an earlier article [21] we investigated the case of a single layer shallow water. In the present article we will emphasize the aspects of the study which are different from [21].

The addition of white noise driven terms to the basic governing equations for a physical system is natural for both practical and theoretical applications. For examples, the stochastically forced terms can be used to account for numerical and empirical uncertainties and thus provide a mean to study the robustness of a basic model. Particularly, in the context of fluids, complex phenomena related to turbulence may also be modeled by stochastic perturbations.

Although the mathematical literature for the deterministic shallow water equations is extensive, to the best of our knowledge, no one has addressed yet the stochastic shallow water equations before [21]. In the deterministic context, one must assume that the initial data is small or, otherwise, the solution is only known to exist for a short period of time. In the stochastic context we consider the shallow water equations forced by a multiplicative white noise representing e.g. random wind perturbations at the surface and we opt to focus on the latter situation that is we will look for a solution up to a small stopping time. The preceding paper [21] addressed the single layer model proposed by Orenga et al in [11] and [25]. Orenga's model omits the Coriolis term and assumes the external force is zero in the momentum equation. In the deterministic context see e.g. [11] and [25], who omit the Coriolis term and assume the external force to be zero in the momentum equation. In [8] and [37] the model is similar to that of Orenga et al., but it has an additional term with $\frac{1}{h}$ in the momentum equation. The model most closely related to the present article can be found in [31]. It does include a Coriolis term, but it still assumes no external forcing and it contains the $\frac{1}{h}$ term. For convenience, we choose a model which omits the $\frac{1}{h}$ term (which amounts to a linearization $h_i \sim H_i$) and adds the term $-\delta\Delta h$ to the continuity equation in order to absorb some of the terms involving the gradient of the height of the water. Due to this extra viscous term, we require boundary conditions on h, that are specified below. We also choose to include an external force that is independent of the solution. A realistic formulation of the external force can be found in e.g. [35], but this adds more unnecessary difficulties to the problem. For more about the physical derivation of these equations, see e.g. [34]. For the two layer model we are investigating in this article, we began with the models proposed in [22] and [27] but ultimately decided on a model similar to [30]. As in the single layer problem, the momentum equations lack any external force. We choose to include the external forces F and G which are both independent of the solution.

When we consider the stochastic two-layer system, several difficulties arise. First, we do not have the cancellation property for the nonlinear term, as is the case in e.g. [1], [17], [18], and [19]. We also do not have the assumption that $\mathbf{v}_1$ or $\mathbf{v}_2$ is divergence free, as in the Navier-Stokes system (see e.g. [2]). In the deterministic case, this implies that, in general, one can only obtain local in time a priori estimates for the solution, and hence local in time existence of solutions. As we will see below, the same holds in the stochastic context. Few results are known regarding of local in time existence of solutions of stochastic partial differential equations. Local in time solutions of the Navier-Stokes equations have been obtained in [1]. In this article, the mapping defining the solutions is "randomized" to account for a white noise forcing. In partly related directions, we would like to mention the lecture notes [15] in which the author studies the role that white noises may have in preventing blow up. See also [6], [7] in which the author derives results of blow-up in finite time for solutions of stochastic pdes. See also [4] in which the authors study the two layer quasi-geostrophic equation; these equations have some similarity with the shallow water equations but, unlike the shallow water equations that we consider, well-posedness is granted for all time in the deterministic context and then in the stochastic context.

Let us emphasize again the general motivation for studying the present system. Multi-layered shallow water equations are commonly used in oceanography, to model the motion of the highly stratified flow in the earth's ocean. Also stochastic perturbations of geophysical equations are commonly used nowadays to parametrize the many uncertainties in the models. In the present case one can think at the topography of the bottom, at the height of the upper surface (mini waves), at wind forces, as well as biological factors and /or salinity.

As we are working in the intersection of two fields, we note that some confusion may arise due to the terminology. In the literature for stochastic differential equations the term" weak solution" is referred to " martingale solution" while the designation " strong solution " may be used for a" pathwise solution". In the former notion, one constructs a probabilistic basis as part of the solution while in the latter case, the existence of solutions can be established on a preordained probability space. For more details about the two types of solutions, we refer the reader to e.g. [12], [14], [13], and [24]. Unlike the study of deterministic nonlinear evolutionary partial differential equations, the study of well-posedness in the stochastic setting gives new difficulties due to the addition of the probabilistic parameter. We will overcome the difficulty by utilizing a different compactness result based on fractional Sobolev spaces that allows us to treat nonlinear stochastic equations in a way similar to the deterministic case; see [13], [32]. Proofs of other compactness embedding theorems can be found in [3], [5], [29], and [33].

In this work, we will use the same approach introduced in [9] and [21] to establish the existence of both martingale and pathwise solutions. We derive the estimates for the nonlinear terms closer to those currently available for the three dimensional Navier Stokes equations. Due to the lack of cancellation property for the nonlinear terms, the results are obtained up to a finite stopping time only.

The structure of this article is organized as follows: In Section 2, we review the basic setting, defining the relevant function spaces and introducing various notions of solutions. In Section 3, we provide some a priori estimates on the moment of solutions of any order up to a stopping time, whereas in our previous one [21], a priori estimates of solutions are only available up to order two. Furthermore, in this section, we deduce the local bound of pathwise solutions in $L^\infty(0, T, H^2)$ if the initial datum belong to the same space. As discussed in the aforementioned paper, the positiveness of random stopping times are not granted. The absence of a lower bound on the stopping times leads to further difficulties later on when deriving the compactness result and passing to the limit. In order to conquer these difficulties, we will construct a modified system which truncates the nonlinear terms in order to obtain the existence of global solutions for this system and obtain the existence of local solutions of the original system by introducing an appropriate positive stopping time which we show to be strictly positive almost surely afterward.

Therefore, Sections 4 and 5 are aimed to establish the existence of both global martingale and pathwise solutions of the modified system. In Section 6, we establish the existence of local martingale solutions, pathwise solutions and maximal pathwise solution by defining an appropriate stopping time. Finally, the Appendices collects some useful lemmas and theorems, among the other existing results which are used throughout the article. We believe that these results are very widely applicable for the study of well posedness of other nonlinear stochastic partial differential equations and therefore hold independent interest.

Remark 1.1. *Different boundary conditions on $\mathbf{v}_i$ and h_i for $i = 1, 2$ appear also in the literature, such as:*

$$\mathbf{v}_i \cdot n = 0 \qquad \text{and} \qquad \operatorname{curl}(\mathbf{v}_i) = 0 \qquad \text{on } \partial\mathcal{M} \times (0, T), \tag{1.2}$$

$$\nabla \mathring{h}_i \cdot n = 0 \qquad \text{on } \partial\mathcal{M} \times (0, T). \tag{1.3}$$

This set of boundary conditions yields the same type of results but it requires more technical work.

Remark 1.2. *Physically the depths of each layer of water are necessarily positive. For a proof of the positivity of h_i, see Appendix B in [21].*

2 Analytic tools

2.1 The abstract functional analytic setting

We begin by reviewing some basic function spaces associated with (1.1). We will work with the spaces $H = H_1 \times H_1 \times H_2 \times H_2, V = V_1 \times V_1 \times V_2 \times V_2$ where

$$H_1 := L^2(\mathcal{M})^2,\ V_1 := (H_0^1(\mathcal{M}))^2,\ H_2 := L^2(\mathcal{M}),\ V_2 := H_0^1(\mathcal{M}). \tag{2.1}$$

the spaces H_1 and H_2 are endowed with the usual inner product and norm denoted by $(\cdot, \cdot)$ and $|\cdot|$, respectively, while on V_1 and V_2, we will use $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, which are the usual inner product and norm of the gradients.

We also consider fractional powers of the $(-\Delta)$ operator with the boundary conditions (1.1g) and (1.1h). By the classical spectral theory, there is an orthonormal basis $\{\psi_k\}_{k\geq 1}$ of H and an unbounded increasing sequence of eigenvalues $\{\lambda_k\}_{k\geq 1}, \lambda_k \to \infty$ as $k \to \infty$, such that $-\Delta\psi_k = \lambda_k\psi_k$. We define $D(-\Delta) = V \cap (H^2(\mathcal{M}))^6$ and for $\alpha \geq 0$ we take:

$$D((-\Delta)^\alpha) = \left\{ u \in H_1 : \sum_{k=1}^{\infty} \lambda_k^{2\alpha} |u_k|^2 < \infty \right\}, \tag{2.2}$$

endowed with the Hilbertian norm

$$|u|_\alpha := |(-\Delta)^\alpha u| = \left(\sum_{k=1}^{\infty} \lambda_k^{2\alpha} |u_k|^2 \right)^{1/2}. \tag{2.3}$$

Here, $u = \sum_{k=1}^\infty u_k\psi_k$ with $|u|^2 = \sum_{k=1}^\infty |u_k|^2 < \infty$.

For the Galerkin scheme below, we introduce the finite dimensional spaces $H_n = \text{span}\{\psi_1, \ldots, \psi_n\}$ and let $P_n, Q_n = I - P_n$ be the projection operators onto H_n and onto its orthogonal complement. By abuse of notation we will also use the operator P_n to denote $P_n \mathbf{v}_i = P_n(\mathbf{v}_i, 0)$ and $P_n h_i = P_n(\mathbf{0}, h_i)$. We have the generalized and reverse Poincaré inequalities which hold for any $\alpha_1 < \alpha_2$:

$$|P_n u|_{\alpha_2} \leq \lambda_n^{\alpha_2-\alpha_1} |P_n u|_{\alpha_1} \qquad \text{and} \qquad |Q_n u|_{\alpha_1} \leq \frac{1}{\lambda_n^{\alpha_2-\alpha_1}} |Q_n u|_{\alpha_2}. \tag{2.4}$$

2.2 Stochastic preliminaries

In this section, we discuss the stochastic framework on which much of the subsequent analysis relies. For an extended treatment of this topic, we refer the reader to [12].

To begin with, we define a stochastic basis $\mathcal{S} := \left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, \{W_i^k\}_{k\geq 1}\right)$ that is a filtered probability space and $(\Omega, \mathcal{F}, \mathbb{P})$ is the underlying probability space, $\{\mathcal{F}_t\}_{t\geq 0}$ is a complete right continuous filtration, and for $i = 1, 2, 3, 4$, $\{W_i^k\}_{k\geq 1}$ is a sequence of independent one-dimensional Brownian motions adapted to $\mathcal{F}_t$. Expectation is taken with respect to $\mathbb{P}$ and is denoted by $\mathbb{E}$.

Let $\mathfrak{U}$ be an auxiliary separable real Hilbert space endowed with a Hilbert basis $\{e_j\}_{j\geq 1}$. We then consider the stochastically driven terms in (1.1) $W_i(t, \cdot, \omega)$, the $\mathfrak{U}-$valued stochastic processes, formally represented for the moment, by the following series:

$$W_i(t, \cdot, \omega) = \sum_{\ell=1}^{\infty} W_i^\ell(t, \omega) e_\ell(\cdot). \tag{2.5}$$

This expression makes each W_i a cylindrical Brownian motion evolving over a separable space $\mathfrak{U}$ with orthogonal basis e_k.

We next recall some basic definitions and properties of spaces of Hilbert-Schmidt operators. To this end, we suppose that X and Y are two separable Hilbert spaces with the associated norms and inner products given by $|\cdot|_X, |\cdot|_Y$ and $\langle \cdot, \cdot \rangle_X, \langle \cdot, \cdot \rangle_Y$, respectively.

We denote by $L_2(\mathfrak{U}, H) := \{R \in \mathcal{L}(\mathfrak{U}, H) : \sum_{k=1}^{\infty} |Re_k|_X^2 < \infty\}$ the collection of Hilbert-Schmidt operators mapping from $\mathfrak{U}$ into X. This space $L_2(\mathfrak{U}, H)$ is a Hilbert space equipped with the following inner product and norm

$$\langle R, S \rangle_{L_2(\mathfrak{U},H)} = \sum_{k=1}^{\infty} \langle Re_k, Se_k \rangle_H \text{ and } \|R\|^2_{L^2(\mathfrak{U},H)} = \sum_{k=1}^{\infty} |Re_k|_H^2 .$$

We also define another auxiliary space $\mathfrak{U}_0 \supset \mathfrak{U}$ as

$$\mathfrak{U}_0 := \left\{ u = \sum_{k=1}^{\infty} a_k e_k : \sum_{k=1}^{\infty} \frac{a_k^2}{k^2} < \infty \right\},$$

endowed with the norm

$$|v|_{\mathfrak{U}_0}^2 := \sum_{k=1}^{\infty} \frac{a_k^2}{k^2}, \text{ for } v = \sum_{k=1}^{\infty} a_k e_k.$$

Note that the embedding of $\mathfrak{U} \subset \mathfrak{U}_0$ is Hilbert-Schmidt.
Next, given an $X-$valued predictable process $G \in L^2(\Omega; L^2_{loc}([0,\infty); L_2(\mathfrak{U}, X)))$ one may define the (Itô) stochastic integral

$$M_t := \int_0^t G dW, \tag{2.6}$$

which belongs to $\mathcal{M}_X^2$, the space of all $X-$valued square integrable martingales (see e.g.[28]).
For a.e. t and a.s., $G \in L_2(\mathfrak{U}, H)$ so that $G_k = G \cdot e_k \in H$, where $\{e_k\}$ is the basis of $\mathfrak{U}$. Then (2.6) can be represented as

$$M_t = \sum_k \int_0^t G_k dW^k.$$

The martingale $\{M_t\}_{t \geq 0}$ has many desirable properties. Most notably for the analysis here, the Burkholder-Davis-Gundy inequality holds which in the present context takes the form,

$$\mathbb{E}\left(\sup_{t\in[0,T]} \left| \int_0^t G dW \right|_X^r \right) \leq C_1 \mathbb{E}\left(\int_0^T \|G\|^2_{L_2(\mathfrak{U}_0, X)} dt \right)^{\frac{r}{2}}, \tag{2.7}$$

valid for $r \geq 1$. With $G_k = G \cdot e_k$, (2.7) can be rewritten as

$$\mathbb{E}\left(\sup_{t\in[0,T]} \left| \int_0^t \sum_{k=1}^{\infty} G \cdot e_k dW^k \right|_X^r \right) \leq C_1 \mathbb{E}\left(\int_0^T \sum_{k=1}^{\infty} \|G \cdot e_k\|_X^2 dt \right)^{\frac{r}{2}}. \tag{2.8}$$

Here C_1 is an absolute constant depending on r. We shall also make use of a variation of inequality (2.7), which applies to fractional derivatives of M_t. For $p \geq 2$ and $\alpha \in [0, 1/2)$ we have

$$\mathbb{E}\left(\sup_{t\in[0,T]} \left| \int_0^t G dW \right|^p_{W^{\alpha,p}([0,T];X)} \right) \leq C\, \mathbb{E}\left(\int_0^T \|G\|^p_{L_2(\mathfrak{U}_0, X)} dt \right), \tag{2.9}$$

which holds for all $X-$valued predictable $G \in L^2(\Omega; L^p_{loc}([0,\infty); L_2(\mathfrak{U}_0, X)))$.

For the convenience of the reader, we shall recall the definition of the spaces $W^{\alpha,p}([0,T]; X)$ in Section 7 below.

We can express (2.9) in a similar form as in (2.8) as

$$\mathbb{E}\left(\sup_{t\in[0,T]}\left|\sum_k \int_0^t Ge_k dW^k\right|^p_{W^{\alpha,p}([0,T];X)}\right) \leq C\,\mathbb{E}\left(\int_0^T \sum_k |Ge_k|^p_X\,dt\right). \tag{2.10}$$

We will also make use of the decomposition $\mathbf{u} = \sum_{j=1}^{\infty} \xi_j\phi_j$ where $\xi_j = \xi_j(t,\omega)$ and the ϕ_j are the eigenfunctions of $A = -\Delta$ in $D(A) \subset H$ so that $A\mathbf{u}$ becomes $\sum_{j=1}^{\infty} \xi_j\lambda_j\phi_j$; and if $b \in H$, $b = \sum_{j=1}^{\infty} b_j\phi_j$ with $b_j = (b, \phi_j)$.

In what follows, in our estimates made henceforth C is a generic constant that can change in its value from line to line, and sometimes within same line, if so required. We will frequently use the notation $\preceq$ to mean multiplicative up to a constant.

Next, our standing assumptions on the external forcing and the noise are that $F, G \in L^\infty(\Omega\times[0,T];H_1)$ and $L^\infty(\Omega\times[0,T];H_2)$ respectively.[1]

$$\sigma_i : V_i \times [0,T] \times \Omega \to L_2(\mathfrak{U}_0, V_i)$$

are measurable, essentially bounded in time and L^2 in Ω, adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, and satisfies

$$\|\sigma_i(\mathbf{v}_1,\mathbf{v}_2,h_1,h_2,t,\omega)\|^2_{L_2(\mathfrak{U}_0,V_i)} \leq K_V(1 + \|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + \|h_1\|^2 + \|h_2\|^2) \tag{2.11}$$

$$\begin{aligned}\sup_{t\in[0,T]} \|\sigma_i(\mathbf{v}_1,\mathbf{v}_2,h_1,h_2,t,\omega) - \sigma_i(\bar{\mathbf{v}}_1,\bar{\mathbf{v}}_2,\bar{h}_1,\bar{h}_2,t,\omega)\|^2_{L_2(\mathfrak{U}_0,V)} & \\ \leq K_V\Big(\|\mathbf{v}_1-\bar{\mathbf{v}}_1\|^2 + \|\mathbf{v}_2-\bar{\mathbf{v}}_2\|^2 + \|h_1-\bar{h}_1\|^2 + \|h_2-\bar{h}_2\|^2\Big) & \\ \forall \mathbf{v}_1,\mathbf{v}_2,\bar{\mathbf{v}}_1,\bar{\mathbf{v}}_2,h_1,h_2,\bar{h}_1,\bar{h}_2 \in V,\ \mathbb{P}\text{- a.s.} &\end{aligned} \tag{2.12}$$

We will also need regularity of σ_i in $D(-\Delta)$, in the sense that

$$\sigma_i : D(-\Delta) \times [0,T] \times \Omega \to L_2(\mathfrak{U}_0, D(-\Delta))$$

is measurable, adapted to $\{\mathcal{F}_t\}_{t\geq 0}$, and satisfies

$$\|\sigma_i(\mathbf{v}_1,\mathbf{v}_2,h_1,h_2,t,\omega)\|^2_{L_2(\mathfrak{U}_0,D(-\Delta))} \leq K_1\Big(|\Delta\mathbf{v}_1|^2 + |\Delta\mathbf{v}_2|^2 + |\Delta h_1|^2 + |\Delta h_2|^2\Big) \tag{2.13}$$

[1]One can also assume F and G to be random,but we choose F and G to be deterministic here, or else it will be unnecessarily tricky for the proof of the existence of the martingale solutions later on.

Remark 2.1 (Notation)**.** *For $i = 1, 2, 3, 4$, for the sake of simplicity, we set $\sigma_i(U) = \sigma_i(\mathbf{v}_1, \mathbf{v}_2, h_1, h_2, t, \omega)$, $W_i = \sum_{k=1}^{\infty} e_k W_i^k$, and we then have:*

$$\begin{aligned}\sigma_i(U)dW_i &= \sum_{k=1}^{\infty} \sigma_i^k(U) \cdot e_k dW_i^k = \sum_{k,\ell=1}^{\infty} \langle \sigma_i(U)e_k, \phi_\ell \rangle \phi_\ell dW_i^k \\ &= \sum_{k,\ell=1}^{\infty} \sigma_i^{k\ell} \phi_\ell dW_i^k, \end{aligned} \tag{2.14}$$

where

$$\sigma_i(U) \cdot e_k = \sum_\ell \sigma_i^{k\ell} \phi_\ell, \;\; \sigma_i^{k\ell} = \langle \sigma_i(U) \cdot e_k, \phi_\ell \rangle,$$

which makes sense since $\sigma_i(U) \cdot e_k \in H$ and $\{\phi_\ell\}$ is a Hilbert basis of H.

We shall assume furthermore that if $\tilde{\mathbf{v}} : [0, T] \times \Omega \to H_1$ is predictable, then so is $\sigma_i(U)$. Given a H_1-valued predictable process $\tilde{\mathbf{v}} \in L^2(\Omega; L^2(0, T; H_1))$, the series expansions (2.14) can be shown to be well defined as stochastic integrals, e.g., for $i = 1, 2, 3, 4$

$$\left\langle \int_0^\tau \sigma_i(U)dW_1, \tilde{\mathbf{v}} \right\rangle = \left\langle \sum_k \int_0^\tau \sigma_i^k(U)dW_i^k, \tilde{\mathbf{v}} \right\rangle = \sum_k \int_0^\tau \langle \sigma_i^k(U), \tilde{\mathbf{v}} \rangle dW_i^k, \tag{2.15}$$

for all $\tilde{\mathbf{v}} \in H_1$, and stopping time τ. In this context, the four equations from (1.1a) to (1.1d) fully make sense as Itô integrals with values in the spaces V_1', V_2' after (Itô) integration from 0 to t, for a.e. $t \in [0, T]$.

2.3 Definitions of solutions

Here we define the notion of strong and weak solutions to problem (1.1) from the probabilistic view. First, we recall what it means for a stochastic process to be predictable:

Definition 2.1. *For a given stochastic basis $\mathcal{S}$, let $\Phi = \Omega \times [0, \infty)$ and take $\mathcal{G}$ to be the σ-algebra generated by sets of the form*

$$(s, t] \times \mathfrak{F}, \quad 0 \le s < t < \infty, \quad \mathfrak{F} \in \mathcal{F}_s; \qquad \{0\} \times \mathfrak{F}, \quad \mathfrak{F} \in \mathcal{F}_0. \tag{2.16}$$

An X-valued process U is called predictable *w.r.t. $\mathcal{S}$ if it is measurable from $(\Phi, \mathcal{G})$ into $(X, \mathcal{B}(X))$ where $\mathcal{B}(X)$ is the family of Borel sets of X.*

We next give the definitions of local and global solutions of (1.1) for both martingale and pathwise solutions. Before that, we make some assumptions for the initial condition $(\mathbf{v}_1(0), \mathbf{v}_2(0), h_1(0), h_2(0))$, which may be random in general. For the case of martingale solutions, since the stochastic basis is unknown, we are only able to specify

$(\mathbf{v}_1(0), \mathbf{v}_2(0), h_1(0), h_2(0))$ as an initial probability measure μ_0 on $V_1 \times V_1 \times V_2 \times V_2$. For the case of pathwise solutions where the stochastic basis $\mathcal{S}$ is fixed in advance, we assume that relative to this basis $(\mathbf{v}_0, h_0)$ is a $V_1 \times V_1 \times V_2 \times V_2$ valued random variable such that $(\mathbf{v}_1(0), \mathbf{v}_2(0), h_1(0), h_2(0)) \in L^2(\Omega, V_1 \times V_1 \times V_2 \times V_2)$ and is $\mathcal{F}_0$-measurable and in addition we assume that

$$\mathbf{v}_i(0) \in L^p(\Omega, \mathcal{F}_0, V_1), h_i(0) \in L^p(\Omega, \mathcal{F}_0, V_2) \text{ for } i = 1, 2. \tag{2.17}$$

Definition 2.2 (Local and global martingale solutions). *Suppose that μ_0 is a probability measure on $V_1 \times V_2$ and for $i = 1, 2, 3, 4$, $\sigma_i(U)$ satisfies the Lipschitz conditions in (2.11) and (2.12), is predictable, and $\mathcal{F}_t$-adapted. Then we say that $\left(\tilde{\mathcal{S}}, \tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{h}_1, \tilde{h}_2, \tilde{\tau}\right)$ is a* local Martingale solution *of problem (1.1) if*
$\tilde{\mathcal{S}} := \left(\tilde{\Omega}, \tilde{\mathcal{F}}, \left\{\tilde{\mathcal{F}}_t\right\}_{t\geq 0}, \tilde{\mathbb{P}}, \tilde{W}_1, \tilde{W}_2, \tilde{W}_3, \tilde{W}_4\right)$ is a stochastic basis, $\tilde{\tau}$ is a strictly positive stopping time (i.e. $\tilde{\tau} > 0$ almost surely) relative to $\tilde{\mathcal{F}}_t$, and for $i = 1, 2$ $\tilde{\mathbf{v}}_i(\cdot \wedge \tilde{\tau}), \tilde{h}_i(\cdot \wedge \tilde{\tau})$ are $\tilde{\mathcal{F}}_t$-adapted processes in V_1, V_2, respectively, so that

$$\tilde{\mathbf{v}}_i(\cdot \wedge \tilde{\tau}) \in L^2(\tilde{\Omega}; L^\infty([0, T]; V_1)), \tag{2.18a}$$

$$\tilde{h}_i(\cdot \wedge \tilde{\tau}) \in L^2(\tilde{\Omega}; L^\infty([0, T]; V_2)), \tag{2.18b}$$

$$\tilde{\mathbf{v}}_i(t)\mathbb{1}_{t\leq\tilde{\tau}} \in L^2(\tilde{\Omega}; L^2(0, T; D(-\Delta)), \tag{2.18c}$$

$$\tilde{h}_i(t)\mathbb{1}_{t\leq\tilde{\tau}} \in L^2(\tilde{\Omega}; L^2(0, T; D(-\Delta)). \tag{2.18d}$$

Furthermore, the law of $\left(\tilde{\mathbf{v}}_1(0), \tilde{\mathbf{v}}_2(0), \tilde{h}_1(0), \tilde{h}_2(0)\right)$ is μ_0, i.e.
$\mu_0(E) = \tilde{\mathbb{P}}\left(\left(\tilde{\mathbf{v}}_1(0), \tilde{\mathbf{v}}_2(0), \tilde{h}_1(0), \tilde{h}_2(0)\right) \in E\right)$ for all Borel subsets $E \subset V_1 \times V_1 \times V_2 \times V_2$, and $\left(\tilde{\mathbf{v}}, \tilde{h}\right)$ must satisfy almost surely for every $t \geq 0$, every $v \in V_1$, every $\eta \in V_2$ and for $i = 1, 2$

$$\begin{aligned}(\tilde{\mathbf{v}}_1(t \wedge \tilde{\tau}), v) + \int_0^{t\wedge\tilde{\tau}} \left(-\nu_1\Delta\tilde{\mathbf{v}}_1 + (\tilde{\mathbf{v}}_1 \cdot \nabla)\tilde{\mathbf{v}}_1 + g\nabla\tilde{h}_1 + g\frac{\rho_2}{\rho_1}\nabla\tilde{h}_2 + f\mathbf{k} \times \tilde{\mathbf{v}}_1 - F, v\right) ds \\ = (\tilde{\mathbf{v}}_1(0), v) + \int_0^{t\wedge\tilde{\tau}} \sum_{k=1}^{\infty}(\sigma_1(U)e_k, v)d\tilde{W}_1^k,\end{aligned} \tag{2.19}$$

$$\begin{aligned}(\tilde{\mathbf{v}}_2(t \wedge \tilde{\tau}), v) + \int_0^{t\wedge\tilde{\tau}} \left(-\nu_2\Delta\tilde{\mathbf{v}}_2 + (\tilde{\mathbf{v}}_2 \cdot \nabla)\tilde{\mathbf{v}}_2 + g\nabla\tilde{h}_1 + g\nabla\tilde{h}_2 + f\mathbf{k} \times \tilde{\mathbf{v}}_2 - G, v\right) ds \\ = (\tilde{\mathbf{v}}_2(0), v) + \int_0^{t\wedge\tilde{\tau}} \sum_{k=1}^{\infty}(\sigma_2(U)e_k, v)d\tilde{W}_2^k,\end{aligned} \tag{2.20}$$

$$\left(\tilde{h}_1(t \wedge \tilde{\tau}), \eta\right) + \int_0^{t\wedge\tilde{\tau}} \left(\nabla \cdot \left(\tilde{h}_1\tilde{\mathbf{v}}_1\right) - \delta\Delta\tilde{h}_1, \eta\right) ds = \left(\tilde{h}_1(0), \eta\right) + \int_0^{t\wedge\tilde{\tau}} \sum_{k=1}^{\infty}(\sigma_3(U)e_k, \eta)d\tilde{W}_3^k, \tag{2.21}$$

$$\left(\tilde{h}_2(t\wedge\tilde{\tau}),\eta\right)+\int_0^{t\wedge\tilde{\tau}}\left(\nabla\cdot\left(\tilde{h}_2\tilde{\mathbf{v}}_2\right)-\delta\Delta\tilde{h}_2,\eta\right)ds=\left(\tilde{h}_2(0),\eta\right)+\int_0^{t\wedge\tilde{\tau}}\sum_{k=1}^{\infty}(\sigma_4(U)e_k,\eta)d\tilde{W}_4^k. \tag{2.22}$$

We say that the martingale solution $\left(\tilde{\mathcal{S}},\tilde{\mathbf{v}},\tilde{h},\tilde{\tau}\right)$ *is global if* $\tilde{\tau}=\infty$ *a.s.*

Definition 2.3 (Local, maximal and global pathwise solutions)**.** *Suppose that* $\mathcal{S}=(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P},W_1,W_2)$ *is a fixed stochastic basis and that* $(\mathbf{v}_1(0),\mathbf{v}_2(0),h_1(0),h_2(0))$ *is a* $(V_1)^2\times(V_2)^2$ *valued random variable (relative to* $\mathcal{S}$*) satisfying* (2.18) *and the same conditions hold for* F,G *and* σ_i*,* $i=1,2$*.*

(i) A quintuplets $(\mathbf{v}_1,\mathbf{v}_2,h_1,h_2,\tau)$ *is a local pathwise solution to* (1.1) *if* τ *is a strictly positive stopping time,* $\mathbf{v}_1(\cdot\wedge\tau)$ *,*$\mathbf{v}_2(\cdot\wedge\tau)$ $\mathcal{F}_t$*-adapted processes in* V_1*, and* $h_1(\cdot\wedge\tau)$*,*$h_2(\cdot\wedge\tau)$ $\mathcal{F}_t$*-adapted processes in* V_2 *(relative to the fixed basis* $\mathcal{S}$*) such that* (2.18)–(2.20) *hold.*

(ii) Pathwise solutions of (1.1) *are said to be* unique up to a stopping time $\tau>0$ *if given any pair of pathwise solutions* $(\mathbf{v}_1,\mathbf{v}_2,h_1,h_2,\tau)$ *and* $(\tilde{\mathbf{v}}_1,\tilde{\mathbf{v}}_2,\tilde{h}_1,\tilde{h}_2,\tau)$ *which coincide at* $t=0$ *on a subset* Ω_0 *of* Ω*:*

$$\Omega_0:=\left\{\mathbf{v}_1(0)=\tilde{\mathbf{v}}_1(0),\mathbf{v}_1(0)=\tilde{\mathbf{v}}_1(0),h_1(0)=\tilde{h}_1(0),h_2(0)=\tilde{h}_2(0)\right\}\subset\Omega, \tag{2.23}$$

then

$$\mathbb{P}\left(\mathbb{1}_{\Omega_0}(\mathbf{v}_1(t\wedge\tau)-\tilde{\mathbf{v}}_1(t\wedge\tau))=0,\forall t\geq 0\right)=1, \tag{2.24}$$

$$\mathbb{P}\left(\mathbb{1}_{\Omega_0}(\mathbf{v}_2(t\wedge\tau)-\tilde{\mathbf{v}}_2(t\wedge\tau))=0,\forall t\geq 0\right)=1, \tag{2.25}$$

$$\mathbb{P}\left(\mathbb{1}_{\Omega_0}\left(h_1(t\wedge\tau)-\tilde{h}_1(t\wedge\tau)\right)=0,\forall t\geq 0\right)=1, \tag{2.26}$$

$$\mathbb{P}\left(\mathbb{1}_{\Omega_0}\left(h_2(t\wedge\tau)-\tilde{h}_2(t\wedge\tau)\right)=0,\forall t\geq 0\right)=1. \tag{2.27}$$

(iii) Suppose we have $\{\tau_n\}_{n\geq 1}$*, a strictly increasing sequence of stopping times that converge to a stopping time* ξ*, and assume that* $\mathbf{v}_1$*,*$\mathbf{v}_2$*,* h_1 *and* h_2 *are predictable continuous* $\mathcal{F}_t$*-adapted processes in* H_1 *and* H_2*, respectively. We say that* $(\mathbf{v}_1,\mathbf{v}_2,h_1,h_2,\xi):=(\mathbf{v}_1,\mathbf{v}_2,h_1,h_2,\xi,\{\tau_n\}_{n\geq 1})$ *is a* maximal pathwise solution *if* $(\mathbf{v},\mathbf{v}_2,h_1,h_2,\tau_n)$ *is a local pathwise solution for each* n *and*

$$\begin{aligned}&\sup_{t\in[0,\xi]}||\mathbf{v}_1||^2+\sup_{t\in[0,\xi]}||\mathbf{v}_2||^2+\int_0^{\xi}|\Delta\mathbf{v}_1|^2ds+\int_0^{\xi}|\Delta\mathbf{v}_2|^2ds+\\&\sup_{t\in[0,\xi]}||h_1||^2+\sup_{t\in[0,\xi]}||h_2||^2+\int_0^{\xi}|\Delta h|^2ds+\int_0^{\xi}|\Delta h_2|^2ds=\infty,\end{aligned} \tag{2.28}$$

a.s. on the set $\{\xi<\infty\}$*. If we have*

$$\begin{aligned}&\sup_{t\in[0,\xi]}||\mathbf{v}_1||^2+\sup_{t\in[0,\xi]}||\mathbf{v}_2||^2+\int_0^{\xi}|\Delta\mathbf{v}_1|^2ds+\int_0^{\xi}|\Delta\mathbf{v}_2|^2ds+\\&\sup_{t\in[0,\xi]}||h_1||^2+\sup_{t\in[0,\xi]}||h_2||^2+\int_0^{\xi}|\Delta h|^2ds+\int_0^{\xi}|\Delta h_2|^2ds=n,\end{aligned} \tag{2.29}$$

for almost every $\omega \in \{\xi < \infty\}$, then the sequence τ_n announces *a finite blow-up time.*

(iv) *If $(\mathbf{v}_1, \mathbf{v}_2, h_1, h_2, \xi)$ is a maximal pathwise solution and $\xi = \infty$ almost surely, then we say that the solution is* global.

We now state the main results in this work:

Theorem 2.1. *We are given μ_0 as a probability measure on V, $F, G \in L^\infty(0,T;H_1)$ and $\sigma_i(U)$, $i = 1,2,3,4$ satisfying the Lipschitz conditions $\left(2.11\right)$ and $\left(2.12\right)$, predictable, and $\mathcal{F}_t$-adapted. Then there exists a local martingale solution $(\tilde{\mathcal{S}}, \tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{h_1}, \tilde{h_2}, \tau)$ to $\left(1.1\right)$.*

Theorem 2.2. *Assume we are working relative to a given fixed stochastic basis and let $F \in L^\infty(0,T;H_1)$, $G \in L^\infty(0,T;H_2)$ and $\sigma_i(U)$, $i = 1,2,3,4$ satisfying the Lipschitz conditions $\left(2.11\right)$ and $\left(2.12\right)$, predictable, and $\mathcal{F}_t$-adapted. Suppose furthermore that $\left(2.17\right)$ also holds. Then there exists a unique, maximal pathwise solution $(\mathbf{v}_1, \mathbf{v}_2, h_1, h_2, \xi, (\tau_n)_{n\geq 1})$ to $\left(1.1\right)$.*

We begin by establishing a priori estimates on the moments of solutions of $\left(1.1\right)$. Below we show how regular such a solution must be depending on the space from where the initial data is taken.

3 Formal a priori estimates

We now state a lemma that enables us to derive a L^p- norm on ∇U for all $p \geq 2$ and L^2-norm on ΔU.

Lemma 3.1. *(Local a priori estimates)*

We fix a stochastic basis $\mathcal{S} := (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W_1, W_2)$ and let $U = (\mathbf{v}_1, \mathbf{v}_2, h_1, h_2)$ is a pathwise solution of $\left(1.1\right)$ and let $F, G \in L^p(0,T,H)$ for some $p \geq 2$. Then

1) For $U_0 \in L^p(0,T,\mathcal{F}_0,V)$

$$\mathbb{E}\Bigg(\sup_{0\leq t\leq t*<T}(\|\mathbf{v}_1\|^p + \|\mathbf{v}_2\|^p + \|h_1\|^p + \|h_2\|^p) + 2\nu_1\int_0^{t*}|\Delta\mathbf{v}_1|^2\|\mathbf{v}_1\|^{p-2}dt +$$
$$2\nu_2\int_0^{t*}|\Delta\mathbf{v}_2|^2\|\mathbf{v}_2\|^{p-2}dt + 2\delta_1\int_0^{t*}|\Delta h_1|^2\|h_1\|^{p-2}dt + 2\delta_2\int_0^{t*}|\Delta h_2|^2\|h_2\|^{p-2}dt\Bigg)$$
$$\preceq \mathbb{E}(\|\mathbf{v}_1(0)\|^p + \|\mathbf{v}_1(0)\|^p + \|\mathbf{v}_1(0)\|^p + \|\mathbf{v}_1(0)\|^p) + \int_0^T |F|^p\,dt + |G|^p\,dt. \quad \left(3.1\right)$$

2) For $U_0 \in L^2(0,T,\mathcal{F}_0,D(-\Delta))$ *and we further assume that* $F,G \in L^2(0,T,H_0^1(\mathcal{M}^2))$

$$\mathbb{E}\Big(\sup_{0\le t\le \hat{t}<T}(|\Delta\mathbf{v}_1|^2+|\Delta\mathbf{v}_2|^2+|\Delta h_1|^2+|\Delta h_2|^2)+2\nu_1\int_0^{\hat{t}}\|\Delta\mathbf{v}_1\|^2dt$$
$$+2\nu_2\int_0^{\hat{t}}\|\Delta\mathbf{v}_2\|^2dt+2\delta_1\int_0^{\hat{t}}\|\Delta h_1\|^2dt+2\delta_2\int_0^{\hat{t}}\|\Delta h_2\|^2dt\Big)$$
$$\preceq \mathbb{E}(|\Delta\mathbf{v}_1(0)|^2+|\Delta\mathbf{v}_2(0)|^2+|\Delta h_1(0)|^2+|\Delta h_2(0)|^2)+\int_0^T|\Delta F|^2\,dt+|\Delta G|^2\,dt. \tag{3.2}$$

In all cases, the implicit constants depend only on the initial datum and both $t*$ *and* $\hat{t}$ *will be specified later.*

Proof. 1. We apply the Itô lemma to the map $U\mapsto|\nabla U|^p$ in (1.1) and this yields

$$\begin{aligned}
&d\|\mathbf{v}_1\|^p+p\nu_1|\Delta\mathbf{v}_1|^2\|\mathbf{v}_1\|^{p-2}dt+d\|\mathbf{v}_2\|^p+p\nu_2|\Delta\mathbf{v}_2|^2\|\mathbf{v}_2\|^{p-2}dt+d\|h_1\|^p+p\delta_1|\Delta h_1|^2\|h_1\|^{p-2}dt+\\
&d\|h_2\|^p+p\delta_2|\Delta h_2|^2\|h_2\|^{p-2}dt=p\langle F,\Delta\mathbf{v}_1\rangle\|\mathbf{v}_1\|^{p-2}dt+p\langle G,\Delta\mathbf{v}_2\rangle\|\mathbf{v}_2\|^{p-2}dt\\
&-pg\langle\nabla h_1,\Delta\mathbf{v}_1\rangle\|\mathbf{v}_1\|^{p-2}dt-pg\langle\nabla h_2,\Delta\mathbf{v}_2\rangle\|\mathbf{v}_2\|^{p-2}dt-pg\langle\nabla h_1,\Delta\mathbf{v}_2\rangle\|\mathbf{v}_2\|^{p-2}dt\\
&-pg\frac{\rho_1}{\rho_2}\langle\nabla h_2,\Delta\mathbf{v}_1\rangle\|\mathbf{v}_1\|^{p-2}dt-p\langle f\mathbf{k}\times\mathbf{v}_1,\Delta\mathbf{v}_1\rangle\|\mathbf{v}_1\|^{p-2}dt-p\langle f\mathbf{k}\times\mathbf{v}_2,\Delta\mathbf{v}_2\rangle\|\mathbf{v}_2\|^{p-2}dt\\
&-p\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_1,\Delta\mathbf{v}_1\rangle\|\mathbf{v}_1\|^{p-2}dt-\langle(\mathbf{v}_2\cdot\nabla)\mathbf{v}_2,\Delta\mathbf{v}_2\rangle\|\mathbf{v}_2\|^{p-2}dt\\
&-p\langle\nabla\cdot(h_1\mathbf{v}_1),\Delta h_1\rangle\|h_1\|^{p-2}dt-p\langle\nabla\cdot(h_2\mathbf{v}_2),\Delta h_2\rangle\|h_2\|^{p-2}dt\\
&+\frac{p}{2}\sum_{k=1}^\infty||\sigma_1(U)e_k||^2\|\mathbf{v}_1\|^{p-2}dt+\frac{p}{2}\sum_{k=1}^\infty||\sigma_2(U)e_k||^2\|\mathbf{v}_2\|^{p-2}dt\\
&+\frac{p}{2}\sum_{k=1}^\infty||\sigma_3(U)e_k||^2\|h_1\|^{p-2}dt+\frac{p}{2}\sum_{k=1}^\infty||\sigma_4(U)e_k||^2\|h_2\|^{p-2}dt\\
&+\frac{p(p-2)}{2}\sum_{k=1}^\infty\langle\sigma_1(U)e_k,\Delta\mathbf{v}_1\rangle^2||\mathbf{v}_1||^{p-4}dt+\frac{p(p-2)}{2}\sum_{k=1}^\infty\langle\sigma_2(U)e_k,\Delta\mathbf{v}_2\rangle^2||\mathbf{v}_2||^{p-4}dt\\
&+\frac{p(p-2)}{2}\sum_{k=1}^\infty\langle\sigma_3(U)e_k,\Delta h_1\rangle^2||h_1||^{p-4}dt+\frac{p(p-2)}{2}\sum_{k=1}^\infty\langle\sigma_4(U)e_k,\Delta h_2\rangle^2||h_2||^{p-4}dt\\
&+p\sum_{k=1}^\infty\langle\sigma_1(U)e_k,\Delta\mathbf{v}_1\rangle\|\mathbf{v}_1\|^{p-2}dW_1^k+p\sum_{k=1}^\infty\langle\sigma_2(U)e_k,\Delta\mathbf{v}_2\rangle\|\mathbf{v}_2\|^{p-2}dW_2^k\\
&+p\sum_{k=1}^\infty\langle\sigma_3(U)e_k,\Delta h_1\rangle\|h_1\|^{p-2}dW_3^k+p\sum_{k=1}^\infty\langle\sigma_4(U)e_k,\Delta h_2\rangle\|h_2\|^{p-2}dW_4^k.
\end{aligned}\tag{3.3}$$

We integrate (3.28) in time over $[0,r]$ for $0\le r\le s\le T$, take the supremum in r over

$[0, s]$; we deduce that:

$$\sup_{0\le t\le s}\left[\|\mathbf{v}_1\|^p+\|\mathbf{v}_2\|^p+\|h_1\|^p+\|h_2\|^p\right]+p\nu_1\int_0^s|\Delta\mathbf{v}_1|^2\|\mathbf{v}_1\|^{p-2}dt+p\nu_2\int_0^s|\Delta\mathbf{v}_2|^2\|\mathbf{v}_2\|^{p-2}dt$$

$$\le 8(\|\mathbf{v}_1(0)\|^p+\|\mathbf{v}_2(0)\|^p+\|h_1(0)\|^p+\|h_2(0)\|^p)+\sum_{i=1}^{20}M_i+$$

$$+8p\sup_{0\le r\le s}\left|\int_0^r\sum_{k=1}^{\infty}\langle\sigma_1(U)e_k,\Delta\mathbf{v}_1\rangle\|\mathbf{v}_1\|^{p-2}dW_1^k\right|+8p\sup_{0\le r\le s}\left|\int_0^r\sum_{k=1}^{\infty}\langle\sigma_2(U)e_k,\Delta\mathbf{v}_2\rangle\|\mathbf{v}_2\|^{p-2}dW_2^k\right|$$

$$+8p\sup_{0\le r\le s}\left|\int_0^r\sum_{k=1}^{\infty}\langle\sigma_3(U)e_k,\Delta h_1\rangle\|h_1\|^{p-2}dW_3^k\right|+8p\sup_{0\le r\le s}\left|\int_0^r\sum_{k=1}^{\infty}\langle\sigma_4(U)e_k,\Delta h_2\rangle\|h_2\|^{p-2}dW_4^k\right|$$

We will now estimate each of the quantities on the right hand side of the above inequality. We proceed with the terms M_1 through M_6. By using the Cauchy Schwarz inequality, the Young inequality and the Poincaré inequality, we obtain:

$$M_1:=8p\int_0^s|\langle F,\Delta\mathbf{v}_1\rangle|\,\|\mathbf{v}_1\|^{p-2}dt\le C\int_0^s(|F|^pdt+\|\mathbf{v}_1\|^p)dt+\frac{p\nu_1}{10}\int_0^s|\Delta\mathbf{v}_1|^2\,\|\mathbf{v}_1\|^{p-2}dt. \tag{3.4}$$

In the same manner, we obtain the following bounds:

$$M_2:=8p\int_0^s|\langle G,\Delta\mathbf{v}_2\rangle|\,\|\mathbf{v}_2\|^{p-2}dt\le C(\int_0^s|G|^pdt+\|\mathbf{v}_2\|^p)dt+\frac{p\nu_2}{10}\int_0^s|\Delta\mathbf{v}_2|^2\,\|\mathbf{v}_2\|^{p-2}dt. \tag{3.5}$$

For $\alpha=3,4,5,6, i=1,2, j=1,2$, we find:

$$M_\alpha:=\varrho\int_0^s|\langle\nabla h_i,\Delta\mathbf{v}_j\rangle|\,\|\mathbf{v}_j\|^{p-2}dt\le C(\int_0^s\|h_i\|^p+\|\mathbf{v}_j\|^p)dt+\frac{p\nu_j}{10}\int_0^s|\Delta\mathbf{v}_j|^2\,\|\mathbf{v}_j\|^{p-2}dt. \tag{3.6}$$

where ϱ either equals $8pg$ or $8pg\frac{\rho_2}{\rho_1}$. For $\beta=7,8, i=1,2$, we obtain:

$$M_\beta\ :=\ 8p\int_0^s|\langle f\mathbf{k}\times\mathbf{v}_i,\Delta\mathbf{v}_i\rangle|\,\|\mathbf{v}_i\|^{p-2}dt\ \le\ C\int_0^s\|\mathbf{v}_1\|^pdt+\frac{p\nu_1}{10}\int_0^s|\Delta\mathbf{v}_i|^2\,\|\mathbf{v}_i\|^{p-2}dt. \tag{3.7}$$

The nonlinear terms are bounded by utilizing Hölder's, Agmon's, the Young and the Poincaré inequalities

$$\begin{aligned}M_9:=8p\int_0^s|\langle(\mathbf{v}_1\cdot\nabla)\mathbf{v}_1,\Delta\mathbf{v}_1\rangle|\,\|\mathbf{v}_1\|^{p-2}dt&\le C\int_0^s|\mathbf{v}_1|_{L^\infty}|\nabla\mathbf{v}_1|\,|\Delta\mathbf{v}_1|\,\|\mathbf{v}_1\|^{p-2}dt\\ \le C\int_0^s|\mathbf{v}_1|^{\frac12}|\Delta\mathbf{v}_1|^{\frac12}|\nabla\mathbf{v}_1|\,|\Delta\mathbf{v}_1|\,\|\mathbf{v}_1\|^{p-2}dt&\le C\int_0^s|\nabla\mathbf{v}_1|^{\frac32}|\Delta\mathbf{v}_1|^{\frac32}\|\mathbf{v}_1\|^{p-2}dt\\ \le C\int_0^s|\nabla\mathbf{v}_1|^6\|\mathbf{v}_1\|^{p-2}dt&+\frac{p\nu_1}{10}\int_0^s|\Delta\mathbf{v}_1|^2\|\mathbf{v}_1\|^{p-2}dt\\ \le C\int_0^s\|\mathbf{v}_1\|^{p+4}dt&+\frac{p\nu_1}{10}\int_0^s|\Delta\mathbf{v}_1|^2\|\mathbf{v}_1\|^{p-2}dt.\end{aligned} \tag{3.8}$$

Similarly,

$$M_{10} := 8p\int_0^s |\langle(\mathbf{v}_2\cdot\nabla)\mathbf{v}_2,\Delta\mathbf{v}_2\rangle|\,\|\mathbf{v}_2\|^{p-2}dt \le C\int_0^s \|\mathbf{v}_2\|^{p+4}dt + \frac{p\nu_2}{10}\int_0^s |\Delta\mathbf{v}_2|^2\,\|\mathbf{v}_2\|^{p-2}dt. \tag{3.9}$$

We estimate the term M_{11} by first splitting it as follows:

$$\begin{aligned} M_{11} &:= 8p\int_0^s |\langle\nabla\cdot(h_1\mathbf{v}_1),\Delta h_1\rangle|\,\|h_1\|^{p-2}dt \\ &\le 8p\int_0^s |\langle\nabla\cdot\mathbf{v}_1 h_1,\Delta h_1\rangle|\|h_1\|^{p-2}dt + 8p\int_0^s |\langle\nabla h_1\mathbf{v}_1,\Delta h_1\rangle|\|h_1\|^{p-2}dt \\ &:= M_{11}^1 + M_{11}^2. \end{aligned} \tag{3.10}$$

M_{11}^1 is treated by first using Hölder's inequality:

$M_{11}^1 := \int_0^s |\langle\nabla\cdot\mathbf{v}_1 h_1,\Delta h_1\rangle|\|h_1\|^{p-2}dt \le C\int_0^s |\nabla\mathbf{v}_1|\,|h_1|_{L^\infty}|\Delta h_1|\,\|h_1\|^{p-2}dt.$

By utilizing Agmon's inequality in space dimension two for the second term, we obtain:

$M_{11}^1 \le C\int_0^s \|\mathbf{v}_1\|\,|h_1|^{\frac{1}{2}}\,|\Delta h_1|^{\frac{3}{2}}\,\|h_1\|^{p-2}dt.$

By applying the Poincaré inequality and the Young inequality to the first three terms, with $p=6, q=12, r=\frac{4}{3}$, we obtain:

$$\begin{aligned} M_{11}^1 &\le \int_0^s (C\|\mathbf{v}_1\|^6 + C\|h_1\|^6 + \frac{p\delta_1}{4}|\Delta h_1|^2)\|h_1\|^{p-2}dt \\ &= \frac{p\delta_1}{4}\int_0^s |\Delta h_1|^2\,\|h_1\|^{p-2}dt + C\int_0^s (\|h_1\|^{p+4} + \|\mathbf{v}_1\|^6\|h_1\|^{p-2})dt. \end{aligned} \tag{3.11}$$

We derive the estimate for M_{11}^2 as follows:

$M_{11}^2 := \int_0^s |\langle\nabla h_1\mathbf{v}_1,\Delta h_1\rangle|\|h_1\|^{p-2}dt \le C\int_0^s |\nabla h_1|_{L^4}|\mathbf{v}_1|_{L^4}|\Delta h_1|\,\|h_1\|^{p-2}dt.$

By using Ladyzhenskya's inequality which is $|\mathbf{u}|_{L^4} \le C|\mathbf{u}|^{\frac{1}{2}}|\nabla\mathbf{u}|^{\frac{1}{2}}$ in space dimension two for the first term and the embedding $H_0^1 \hookrightarrow L^4$ in space dimension two for the second term, the next line follows

$M_{11}^2 \le C\int_0^s |\nabla h_1|^{\frac{1}{2}}\,|\Delta h_1|^{\frac{3}{2}}\,|\nabla\mathbf{v}_1|\,\|h_1\|^{p-2}dt.$

By utilizing the Young inequality for the first three terms of the RHS, we obtain:

$$M_{11}^2 \le C\int_0^s \|h_1\|^{p+4} + \frac{p\delta_1}{4}\int_0^s |\Delta h_1|^2\,\|h_1\|^{p-2}dt + C\int_0^s \|\mathbf{v}_1\|^6\|h_1\|^{p-2})dt. \tag{3.12}$$

By combining (3.11) and (3.12), the bound for M_{11} results as follows

$$M_{11} \le C\int_0^s \|h_1\|^{p+4} + \frac{p\delta_1}{2}\int_0^s |\Delta h_1|^2\,\|h_1\|^{p-2}dt + C\int_0^s \|\mathbf{v}_1\|^6\|h_1\|^{p-2})dt. \tag{3.13}$$

Analogously, we obtain:

$$\begin{aligned} M_{12} :&= \int_0^s |\langle \nabla \cdot (h_2\mathbf{v}_2), \Delta h_2\rangle|\, dt \qquad (3.14)\\ &\le C\int_0^s \left(\|\mathbf{v}_2\|^6\|h_2\|^{p-2}dt + \|h_2\|^{p+4}dt\right) + \frac{p\delta_2}{2}\int_0^s |\Delta h_2|^2\,\|h_2\|^{p-2}dt. \end{aligned}$$

Again, by simply using the Lipschitz assumptions (2.11) and the Young inequality, we obtain:

$$\begin{aligned} M_{13}+M_{14}+M_{15}+M_{16} &:= 8p\Bigg[\int_0^s\sum_{k=1}^{\infty}\|\sigma_1(U)e_k\|^2\|\mathbf{v}_1\|^{p-2}dt+\int_0^s\sum_{k=1}^{\infty}\|\sigma_2(U)e_k\|^2\|\mathbf{v}_2\|^{p-2}dt\\ &+\int_0^s\sum_{k=1}^{\infty}\|\sigma_3(U)e_k\|^2\|h_1\|^{p-2}dt+\int_0^s\sum_{k=1}^{\infty}\|\sigma_4(U)e_k\|^2\|h_2\|^{p-2}dt\Bigg]\\ \le 8K_Vp\int_0^s&(1+\|\mathbf{v}_1\|^2+\|\mathbf{v}_2\|^2+\|h_1\|^2+\|h_2\|^2)(\|\mathbf{v}_1\|^{p-2}+\|\mathbf{v}_1\|^{p-2}+\|\mathbf{v}_2\|^{p-2}+\|h_1\|^{p-2}+\|h_2\|^{p-2})\\ &\le C\int_0^s(\|\mathbf{v}_1\|^p+\|\mathbf{v}_2\|^p+\|h_1\|^p+\|h_2\|^p)dt+CT. \qquad (3.15) \end{aligned}$$

We observe that since $V = H_0^1(\mathcal{M})^6$, we have for $\sigma_i(\mathbf{v},h)e_k \in H$, $i = 1,2$, $\Delta u \in H$,

$$\langle \sigma_i(\mathbf{v},h)e_k, \Delta u\rangle = \int_{\mathcal{M}} \sigma_i(\mathbf{v},h)e_k\cdot \Delta u d\mathcal{M}.$$

By integrating by parts, this is equal to

$$-\int_{\mathcal{M}}\nabla\sigma_i(U)e_k\cdot\nabla u d\mathcal{M}+\int_{\partial\mathcal{M}}\sigma_i(U)e_k(\nabla u\cdot n)dS = -\int_{\mathcal{M}}\nabla\sigma_i(U)e_k\cdot\nabla u d\mathcal{M}. \qquad (3.16)$$

The next estimate is obtained via Lipschitz assumptions (2.11) along with the above expression,

$$\begin{aligned} &M_{17}+M_{18}+M_{19}+M_{20} :=\\ &4p(p-2)\Bigg[\int_0^s\sum_{k=1}^{\infty}\langle\sigma_1(U)e_k,\Delta\mathbf{v}_1\rangle^2||\mathbf{v}_1||^{p-4}dt+\int_0^s\sum_{k=1}^{\infty}\langle\sigma_2(U)e_k,\Delta\mathbf{v}_2\rangle^2||\mathbf{v}_2||^{p-4}dt\\ &+\int_0^s\sum_{k=1}^{\infty}\langle\sigma_3(U)e_k,\Delta h_1\rangle^2||h_1||^{p-4}dt+\int_0^s\sum_{k=1}^{\infty}\langle\sigma_4(U)e_k,\Delta h_2\rangle^2||h_2||^{p-4}dt\Bigg]\\ &=4p(p-2)\Bigg[\int_0^s\sum_{k=1}^{\infty}\langle\nabla\sigma_1(U)e_k,\nabla\mathbf{v}_1\rangle^2||\mathbf{v}_1||^{p-4}dt+\int_0^s\sum_{k=1}^{\infty}\langle\nabla\sigma_2(U)e_k,\nabla\mathbf{v}_2\rangle^2||\mathbf{v}_2||^{p-4}dt\\ &+\int_0^s\sum_{k=1}^{\infty}\langle\nabla\sigma_3(U)e_k,\nabla h_1\rangle^2||h_1||^{p-4}dt+\int_0^s\sum_{k=1}^{\infty}\langle\nabla\sigma_4(U)e_k,\nabla h_2\rangle^2||h_2||^{p-4}dt\Bigg]\\ &\le C\int_0^s(1+\|\mathbf{v}_1\|^2+\|\mathbf{v}_2\|^2+\|h_1\|^2+\|h_1\|^2)(\|\mathbf{v}_1\|^{p-2}+\|\mathbf{v}_2\|^{p-2}+\|h_1\|^{p-2}+\|h_2\|^{p-2})dt\\ &\le C\int_0^s(\|\mathbf{v}_1\|^p+\|\mathbf{v}_2\|^p+\|h_1\|^p+\|h_2\|^p)+CT. \qquad (3.17) \end{aligned}$$

The last line follows thanks to the Young inequality.

Combining (3.4)–(3.17), multiplying by 2, and finally taking the mathematical expectation on both sides and this yields:

$$
\begin{aligned}
&\mathbb{E}\Bigg(\sup_{0\le r\le s} (\|\mathbf{v}_1(r)\|^p + \|\mathbf{v}_2(r)\|^p + \|h_1(s)\|^p + \|h_2(s)\|^p) + p\nu_1 \int_0^s |\Delta \mathbf{v}_1(t)|^2 \|\mathbf{v}_1(t)\|^{p-2} dt \\
&+ p\nu_2 \int_0^s |\Delta \mathbf{v}_2(t)|^2 \|\mathbf{v}_2\|^{p-2} dt + p\delta_1 \int_0^s |\Delta h_1(t)|^2 \|h_1\|^{p-2} dt + p\delta_2 \int_0^s |\Delta h_2(t)|^2 \|h_2\|^{p-2} dt \Bigg) \\
&\le 8\mathbb{E}\left(\|\mathbf{v}_1(0)\|^p + \|\mathbf{v}_2(0)\|^p + \|h_1(0)\|^p + \|h_2(0)\|^p\right) + C\bigg(\int_0^s |F|^p + |G|^p \, dt \bigg) + CT \\
&+ \mathbb{E}\bigg(\int_0^s (\|\mathbf{v}_1\|^p + \|\mathbf{v}_2\|^p + \|h_1\|^p + \|h_2\|^p)(\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + \|h_1\|^2 + \|h_2\|^2)^2 \bigg) + \\
&\mathbb{E}\left(\sup_{r\in[0,s]} \left| \int_0^r \sum_{k=1}^\infty \langle \sigma_1(U)e_k, \Delta \mathbf{v}_1\rangle \|\mathbf{v}_1\|^{p-2} dW_1^k \right| \right) + \mathbb{E}\left(\sup_{r\in[0,s]} \left| \int_0^r \sum_{k=1}^\infty \langle \sigma_2(U)e_k, \Delta \mathbf{v}_2\rangle \|\mathbf{v}_2\|^{p-2} dW_2^k \right| \right) \\
&+ \mathbb{E}\left(\sup_{r\in[0,s]} \left| \int_0^r \sum_{k=1}^\infty \langle \sigma_3(U)e_k, \Delta h_1\rangle \|h_1\|^{p-2} dW_3^k \right| \right) + \mathbb{E}\left(\sup_{r\in[0,s]} \left| \int_0^r \sum_{k=1}^\infty \langle \sigma_4(U)e_k, \Delta h_2\rangle \|h_2\|^{p-2} dW_4^k \right| \right).
\end{aligned} \tag{3.18}
$$

For $i = 1, 2$, by making use of (3.16), the BDG inequality, and the Young inequality, the two stochastically forced terms are addressed as follows:

$$
\begin{aligned}
&\mathbb{E}\bigg(\sup_{r\in[0,s]} \left| \int_0^r \sum_{k=1}^\infty \langle \sigma_i(U)e_k, \Delta \mathbf{v}_i\rangle \|\mathbf{v}_i\|^{p-2} dW_i^k \right| \bigg) \\
&\le (\text{with } Ge_k = G_k = \langle \sigma_i(U)e_k, \Delta \mathbf{v}_i\rangle \|\mathbf{v}_i\|^{p-2}) \\
&\le C_1 \mathbb{E}\bigg(\int_0^s \sum_{k=1}^\infty \langle \sigma_i(U)e_k, \Delta \mathbf{v}_i\rangle^2 \|\mathbf{v}_i\|^{2(p-2)} dt \bigg)^{\frac{1}{2}} \\
&= C_1 \mathbb{E}\bigg(\int_0^s \sum_{k=1}^\infty \langle \nabla\sigma_i(U)e_k, \nabla \mathbf{v}_i\rangle^2 \|\mathbf{v}_i\|^{2(p-2)} dt \bigg)^{\frac{1}{2}} \\
&\le C_1 \mathbb{E}\bigg(\int_0^s \sum_{k=1}^\infty |\nabla\sigma_i(U)e_k|^2 |\nabla \mathbf{v}_i|^2 \|\mathbf{v}_i\|^{2(p-2)} dt \bigg)^{\frac{1}{2}} \\
&\le C_1 \mathbb{E}\bigg(\int_0^s \sum_{k=1}^\infty |\nabla\sigma_i(U)e_k|^2 \|\mathbf{v}_i\|^{2(p-1)} dt \bigg)^{\frac{1}{2}} \le C\mathbb{E}\left[\bigg(\sup_{r\in[0,s]} \|\mathbf{v}_i\|^{p-1} \bigg) \bigg(\int_0^s \sum_{k=1}^\infty |\nabla\sigma_i(U)e_k|^2 \, dt \bigg)^{\frac{1}{2}} \right] \\
&\le \frac{1}{2} \mathbb{E}\bigg(\sup_{r\in[0,s]} \|\mathbf{v}_i\|^p \bigg) + C\mathbb{E}\bigg(\int_0^s \sum_{k=1}^\infty |\nabla\sigma_i(U)e_k|^2 \, dt \bigg)^{\frac{p}{2}} \\
&\le \frac{1}{2} \mathbb{E}\left(\sup_{r\in[0,s]} \|\mathbf{v}_i(r)\|^p \right) + C\mathbb{E}\bigg(\int_0^s (1 + \|\mathbf{v}_1\|^p + \|\mathbf{v}_2\|^p + \|h_1\|^p + \|h_2\|^p) dt \bigg),
\end{aligned} \tag{3.19}
$$

The last line holds true due to the Lipschitz assumptions (2.11) and Hölder's inequality.

Analogously, for $i = 3, 4, j = 1, 2$ the following estimates hold

$$\begin{aligned}
&\mathbb{E}\left(\sup_{r\in[0,s]}\left|\int_0^r\sum_{k=1}^{\infty}\langle\sigma_i(U)e_k,\Delta h_j\rangle\|h_j\|^{p-2}dW_i^k\right|\right)\\
&\leq\frac{1}{2}\mathbb{E}\left(\sup_{r\in[0,s]}\|h_j(r)\|^p\right)+C\mathbb{E}\left(\int_0^s(1+\|\mathbf{v}_1\|^p+\|\mathbf{v}_2\|^p+\|h_1\|^p+\|h_2\|^p)dt\right). \qquad (3.20)
\end{aligned}$$

Collecting all the estimates in (3.18)–(3.20) and multiplying by 2, we obtain:

$$\begin{aligned}
&\mathbb{E}\Bigg(\sup_{0\leq r\leq s}(\sum_{i=1}^{2}\|\mathbf{v}_i(r)\|^p+\sum_{i=1}^{2}\|h_i(s)\|^p)+p\nu_1\int_0^s|\Delta\mathbf{v}_1(t)|^2\|\mathbf{v}_1(t)\|^{p-2}dt\\
&+p\nu_2\int_0^s|\Delta\mathbf{v}_2(t)|^2\|\mathbf{v}_2(t)\|^{p-2}dt+p\delta_1\int_0^s|\Delta h_1(t)|^2\|h_1\|^{p-2}dt+p\delta_2\int_0^s|\Delta h_2(t)|^2\|h_2(t)\|^{p-2}dt\Bigg)\\
&\leq\mathbb{E}\left(16\|\mathbf{v}_1(0)\|^p+\|\mathbf{v}_2(0)\|^p+\|h_1(0)\|^p+\|h_2(0)\|^p\right)+C\mathbb{E}\left(\int_0^s(|F|^p+|G|^p)dt\right)+C\\
&\leq\mathbb{E}\left(16\|\mathbf{v}_1(0)\|^p+\|\mathbf{v}_2(0)\|^p+\|h_1(0)\|^p+\|h_2(0)\|^p\right)+C\mathbb{E}\left(\int_0^s(|F|^p+|G|^p)dt\right)\\
&+\mathbb{E}\left(\int_0^s(\sum_{i=1}^{2}\|\mathbf{v}_i\|^p+\sum_{i=1}^{2}\|h_i\|^p)(\|\mathbf{v}_1\|^2+\|\mathbf{v}_2\|^2+\|h_1\|^2+\|h_2\|^2)^2dt\right). \qquad (3.21)
\end{aligned}$$

Now, we assume that $M > 1$ and define the stopping time

$$\tau=\tau_M:=\inf_{r\geq 0}\left\{\left(\|\mathbf{v}_1(r)\|^2+\|h_2(r)\|^2+\|\mathbf{v}_2(r))\|^2+\|h_2(r))\|^2\right)>M\right\} \qquad (3.22)$$

Replacing s by $s\wedge\tau$ in (3.21) yields

$$\begin{aligned}
&\mathbb{E}\Bigg(\sup_{r\in[0,s\wedge\tau]}(\|\mathbf{v}_1(r)\|^p+\|h_1(r)\|^p+\|\mathbf{v}_2(r)\|^p+\|h_2(r)\|^p)+p\nu_1\int_0^{s\wedge\tau}|\Delta\mathbf{v}_1|^2\|\mathbf{v}_1\|^{p-2}dr+\\
&+p\delta_1\int_0^{s\wedge\tau}|\Delta h_1|^2\|h_1\|^{p-2}dr+p\nu_2\int_0^{s\wedge\tau}|\Delta\mathbf{v}_2|^2\|\mathbf{v}_2\|^{p-2}dr+p\delta_2\int_0^{s\wedge\tau}|\Delta h_2|^2\|h_2\|^{p-2}dr\Bigg)\\
&\leq 16\mathbb{E}\left((\|\mathbf{v}_1(0)\|^p+\|\mathbf{v}_2(0)\|^p+\|h_1(0)\|^p+\|h_2(0)\|^p)+\int_0^T(|F|^p+|G|^p)dt+C\right)\\
&+CM^2\mathbb{E}\left(\int_0^{s\wedge\tau}\sup_{0\leq r\leq t}(\|\mathbf{v}_1(r)\|^p+\|h_1(r)\|^p+\|\mathbf{v}_2(r)\|^p+\|h_2(r)\|^p)dt\right)+C. \qquad (3.23)
\end{aligned}$$

Now, we define

$$\mathcal{Y}(t):=\mathbb{E}\left(\int_0^{t\wedge\tau}\sup_{0\leq r\leq s}(\|\mathbf{v}_1(r)\|^p+\|h_1(r)\|^p+\|\mathbf{v}_2(r)\|^p+\|h_2(r)\|^p)ds\right), \qquad (3.24)$$

and

$$\mathcal{K}_0 := \mathbb{E}\left(16(\|\mathbf{v}_1(0)\|^p+\|h_1(0)\|^p+\|\mathbf{v}_2(0)\|^p+\|h_2(0)\|^p)\right)+\int_0^T(|F|^p+|G|^p)dt+C. \quad (3.25)$$

From (3.23) and (3.25), we obtain:

$$\mathcal{Y}'(s) \leq \mathcal{K}_0 + CM^2\mathcal{Y}(s).$$

This gives

$$\mathcal{Y}(s) \leq \frac{\mathcal{K}_0}{CM^2}\left(e^{CM^2s}-1\right). \quad (3.26)$$

Along with (3.23) and (3.26), we deduce that

$$\mathbb{E}\Bigg(\sup_{r\in[0,s\wedge\tau]}(\|\mathbf{v}_1(r)\|^p+\|h_1(r)\|^p+\|\mathbf{v}_2(r)\|^p+\|h_2(r)\|^p)+p\nu_1\int_0^{s\wedge\tau}|\Delta\mathbf{v}_1|^2\|\mathbf{v}_1\|^{p-2}dr+$$
$$p\nu_2\int_0^{s\wedge\tau}|\Delta\mathbf{v}_2|^2\|\mathbf{v}_2\|^{p-2}dr+p\delta_1\int_0^{s\wedge\tau}|\Delta h_1|^2\|h_1\|^{p-2}dr+p\delta_2\int_0^{s\wedge\tau}|\Delta h_2|^2\|h_2\|^{p-2}dr\Bigg)$$
$$\leq \mathcal{K}_0+\frac{\mathcal{K}_0}{CM^2}e^{CM^2s}CM^2 \leq \mathcal{K}_0+\mathcal{K}_0e^{CM^2s} \quad (3.27)$$

The right hand side of (3.27) is bounded by M if

$$s \leq \frac{1}{CM^2}\log\frac{M-\mathcal{K}_0}{\mathcal{K}_0} := s_M.$$

As long as M is large enough such that $M-\mathcal{K}_0 > \mathcal{K}_0$ or $M > 2\mathcal{K}_0$, the local existence in time of solution is granted on $[0, s_M\wedge\tau^M]$.
In other words,

$$\mathbb{E}\Bigg(\sup_{r\in[0,s\wedge\tau]}\|\mathbf{v}_1(r)\|^p+\sup_{r\in[0,s\wedge\tau]}\|\mathbf{v}_2(r)\|^p+\sup_{r\in[0,s\wedge\tau]}\|h(r)\|^p+\sup_{r\in[0,s\wedge\tau]}\|h(r)\|^p$$
$$+p\nu_1\int_0^{s\wedge\tau}|\Delta\mathbf{v}_1|^2\|\mathbf{v}_1\|^{p-2}dr+p\nu_2\int_0^{s\wedge\tau}|\Delta\mathbf{v}_2|^2\|\mathbf{v}_2\|^{p-2}dr+p\delta_1\int_0^{s\wedge\tau}|\Delta h_1|^2\|h_1\|^{p-2}dr+$$
$$p\delta_2\int_0^{s\wedge\tau}|\Delta h_2|^2\|h_2\|^{p-2}dt\Bigg)\leq M$$

for $0 < s < \dfrac{1}{CM^2}\log\dfrac{M-\mathcal{K}_0}{\mathcal{K}_0}$, with $M > 2\mathcal{K}_0$. □

2. Applying the Itô lemma to the map $U \mapsto |\Delta U|^2$ in (1.1) and noting that due to the

Poincaré inequality, both of norms $|\nabla \cdot (\nabla^2 \mathbf{u})|$ and $\|\Delta \mathbf{u}\|$ are equivalent, we obtain

$$
\begin{aligned}
&d\,|\Delta \mathbf{v}_1|^2 + 2\nu_1\|\Delta \mathbf{v}_1\|^2 dt + d\,|\Delta \mathbf{v}_2|^2 + 2\nu_2\|\Delta \mathbf{v}_2\|^2 dt + d\,|\Delta h_1|^2 + 2\delta_1\|\Delta h_1\|^2 dt + d\,|\Delta h_2|^2 + \\
&2\delta_2\|\Delta h_2\|^2 dt = 2\langle F, \Delta^2 \mathbf{v}_1\rangle dt + 2\langle G, \Delta^2 \mathbf{v}_2\rangle dt - 2g\langle \nabla h_1, \Delta^2 \mathbf{v}_1\rangle dt - 2g\langle \nabla h_1, \Delta^2 \mathbf{v}_2\rangle dt \\
&- 2g\langle \nabla h_2, \Delta^2 \mathbf{v}_2\rangle dt - 2g\frac{\rho_2}{\rho_1}\langle \nabla h_2, \Delta^2 \mathbf{v}_1\rangle dt - 2\langle f\mathbf{k}\times \mathbf{v}_1, \Delta^2 \mathbf{v}_1\rangle dt - 2\langle f\mathbf{k}\times \mathbf{v}_2, \Delta^2 \mathbf{v}_2\rangle dt \\
&-2\langle (\mathbf{v}_1\cdot\nabla)\mathbf{v}_1, \Delta^2 \mathbf{v}_1\rangle dt - 2\langle (\mathbf{v}_2\cdot\nabla)\mathbf{v}_2, \Delta^2 \mathbf{v}_2\rangle dt - 2\langle \nabla\cdot(h_1\mathbf{v}_1), \Delta^2 h_1\rangle dt - 2\langle \nabla\cdot(h_2\mathbf{v}_2), \Delta^2 h_2\rangle dt \\
&+ \sum_{k=1}^{\infty} |\Delta\sigma_1(U)e_k|^2 dt + \sum_{k=1}^{\infty} |\Delta\sigma_2(U)e_k|^2 dt + \sum_{k=1}^{\infty} |\Delta\sigma_3(U)e_k|^2 dt + \sum_{k=1}^{\infty} |\Delta\sigma_4(U)e_k|^2 dt \\
&\qquad 2\sum_{k=1}^{\infty}\langle \sigma_1(U)e_k, \Delta^2 \mathbf{v}_1\rangle dW_1^k + 2\sum_{k=1}^{\infty}\langle \sigma_2(U)e_k, \Delta^2 \mathbf{v}_2\rangle dW_2^k \\
&\qquad + 2\sum_{k=1}^{\infty}\langle \sigma_3(U)e_k, \Delta^2 h_1\rangle dW_3^k + 2\sum_{k=1}^{\infty}\langle \sigma_4(U)e_k, \Delta^2 h_2\rangle dW_4^k. \qquad (3.28)
\end{aligned}
$$

Integrating (3.28) in time over $[0, r]$ and taking the supremum over $[0, s]$ for $0 \le r \le s \le T$ yield:

$$
\begin{aligned}
&\sup_{0\le r\le s} |\Delta \mathbf{v}_1(r)|^2\, dr + \sup_{0\le r\le s} |\Delta \mathbf{v}_2(r)|^2 + \sup_{0\le r\le s} |\Delta h_1(r)|^2 + \sup_{0\le r\le s} |\Delta h_2(r)|^2 \\
&+ 2\nu_1\int_0^s \|\Delta \mathbf{v}_1(r)\|^2 dr + 2\nu_2\int_0^s \|\Delta \mathbf{v}_2(t)\|^2 dt + 2\delta_1\int_0^s \|\Delta h_1(t)\|^2 dt + 2\delta_2\int_0^s \|\Delta h_2(t)\|^2 dt \\
&\le 8(|\Delta \mathbf{v}_1(0)|^2 + |\Delta \mathbf{v}_2(0)|^2 + |\Delta h_1(0)|^2 + |\Delta h_2(0)|^2) + \sum_{i=1}^{16} K_i \\
&+ 16\sup_{0\le r\le s}\left|\int_0^r \sum_{k=1}^{\infty}\langle \sigma_1(U)e_k, \Delta^2 \mathbf{v}_1\rangle dW_1^k\right| + 16\sup_{0\le r\le s}\left|\int_0^r \sum_{k=1}^{\infty}\langle \sigma_2(U)e_k, \Delta^2 \mathbf{v}_2\rangle dW_2^k\right| \\
&+ 16\sup_{0\le r\le s}\left|\int_0^r \sum_{k=1}^{\infty}\langle \sigma_3(U)e_k, \Delta^2 h_1\rangle dW_3^k\right| + 16\sup_{0\le r\le s}\left|\int_0^r \sum_{k=1}^{\infty}\langle \sigma_4(U)e_k, \Delta^2 h_2\rangle dW_4^k\right|. \qquad (3.29)
\end{aligned}
$$

By utilizing integration by parts, the Cauchy-Schwarz inequality and the Poincaré inequality, the estimates for all linear terms follow

$$K_1 := 16\int_0^s \left|\langle F, \Delta^2 \mathbf{v}_1\rangle\right| dt \le \frac{24}{\nu_1}\int_0^s |\nabla F|^2\, dt + \frac{\nu_1}{5}\int_0^s \|\Delta \mathbf{v}_1\|^2 dt. \qquad (3.30)$$

$$K_2 := 16\int_0^s \left|\langle G, \Delta^2 \mathbf{v}_2\rangle\right| dt \le \frac{24}{\nu_2}\int_0^s |\nabla G|^2\, dt + \frac{\nu_2}{5}\int_0^s \|\Delta \mathbf{v}_2\|^2 dt. \qquad (3.31)$$

For $\alpha = 3, 4, 5, 6, i = 1, 2, j = 1, 2$, the following estimates hold

$$K_\alpha := \eta\int_0^s \left|\langle \nabla h_i, \Delta^2 \mathbf{v}_j\rangle\right| dt \le \frac{3\eta}{\nu_j}\int_0^s |\Delta h_i(t)|^2\, dt + \frac{\nu_j}{5}\int_0^s \|\Delta \mathbf{v}_j\|^2 dt, \qquad (3.32)$$

where η either equals $16g$ or $16g\frac{\rho_2}{\rho_1}$.
For $\beta = 7, 8, i = 1, 2$, we have:

$$K_\beta := 16\int_0^s \left|\langle f\mathbf{k}\times \mathbf{v}_i, \Delta^2 \mathbf{v}_i\rangle\right| dt \le \frac{32\lambda_1 g^2}{\nu_i}\int_0^s |\Delta \mathbf{v}_i(t)|^2\, dt + \frac{\nu_j}{5}\int_0^s \|\Delta \mathbf{v}_j\|^2 dt. \qquad (3.33)$$

By integration by parts and product rule, we split K_9 as follows:

$$K_9 := 16\int_0^s \left|\langle (\mathbf{v}_1\cdot\nabla)\mathbf{v}_1, \Delta^2\mathbf{v}_1\rangle\right| dt \le 16\int_0^s \left|\langle \nabla\cdot[(\mathbf{v}_1\cdot\nabla)\mathbf{v}_1], \nabla\cdot(\nabla^2)\mathbf{v}_1\rangle\right| dt \quad (3.34)$$
$$\le C\int_0^s \left|\langle \nabla\cdot\mathbf{v}_1\cdot\nabla\mathbf{v}_1, \nabla\cdot(\nabla^2)\mathbf{v}_1\rangle\right| + C\int_0^s \left|\langle \mathbf{v}_1\Delta\mathbf{v}_1, \nabla\cdot(\nabla^2\mathbf{v}_1)\rangle\right| dt$$
$$:= K_9^1 + K_9^2.$$

We evaluate K_9^1 by first using Hölder's inequality and the Poincaré inequality,

$$K_9^1 := \int_0^s \left|\langle \nabla\cdot\mathbf{v}_1\cdot\nabla\mathbf{v}_1, \nabla\cdot(\nabla^2)\mathbf{v}_1\rangle\right| \le C\int_0^s |\nabla\mathbf{v}_1|_{L^4}^2 \,\|\Delta\mathbf{v}_1\| dt$$
$$\le C\int_0^s |\mathbf{v}_1|\,|\Delta\mathbf{v}_1|\,\|\Delta\mathbf{v}_1\| dt \le C\int_0^s |\nabla\mathbf{v}_1|\,|\nabla\mathbf{v}_1|^{\frac{1}{2}}\,\|\Delta\mathbf{v}_1\|^{\frac{1}{2}}\|\Delta\mathbf{v}_1\| dt$$
$$= C\int_0^s |\nabla\mathbf{v}_1|^{\frac{3}{2}}\,\|\Delta\mathbf{v}_1\|^{\frac{3}{2}} dt, \quad (3.35)$$

where we have used Ladyzhenskya's inequality to obtain the LHS and interpolation inequality (Lemma 7.4) to achieve the RHS on the second line.

Finally, by applying the Young inequality with $p = 4, q = \frac{4}{3}$ to the last relation, we obtain:

$$K_9^1 \le \frac{\nu_1}{10}\int_0^s \|\Delta\mathbf{v}_1\|^2 dt + C\int_0^s \|\mathbf{v}_1\|^6 dt. \quad (3.36)$$

The term K_9^2 is estimated as follows:

$$K_9^2 := \int_0^s \left|\langle \mathbf{v}_1\Delta\mathbf{v}_1, \nabla\cdot(\nabla^2\mathbf{v}_1)\rangle\right| dt \le C\int_0^s |\mathbf{v}_1|_{L^\infty}\,|\Delta\mathbf{v}_1|\,\|\Delta\mathbf{v}_1\| dt$$
$$\le C\int_0^s |\mathbf{v}_1|^{\frac{1}{2}}\,|\Delta\mathbf{v}_1|^{\frac{3}{2}}\,\|\Delta\mathbf{v}_1\| dt \le C\int_0^s |\mathbf{v}_1|^{\frac{1}{2}}\,|\nabla\mathbf{v}_1|^{\frac{3}{4}}\,\|\Delta\mathbf{v}_1\|^{\frac{3}{4}}\|\Delta\mathbf{v}_1\| dt = C\int_0^s |\nabla\mathbf{v}_1|^{\frac{5}{4}}\,\|\Delta\mathbf{v}_1\|^{\frac{7}{4}} dt,$$

where Agmon's inequality is used to obtain the first relation and the interpolation inequality (Lemma 7.4) is used to accomplish the second inequality. Then, in virtue of the Young inequality with $p = 8, q = \frac{8}{7}$, we finally find:

$$K_9^2 \le \frac{\nu_1}{10}\int_0^s \|\Delta\mathbf{v}_1\|^2 dt + C\int_0^s \|\mathbf{v}_1\|^{10} dt. \quad (3.37)$$

Hence, by (3.35) and (3.37),

$$K_9 \le \frac{\nu_1}{5}\int_0^s \|\Delta\mathbf{v}_1\|^2 dt + C\int_0^s (\|\mathbf{v}_1\|^6 + \|\mathbf{v}_1\|^{10}) dt. \quad (3.38)$$

Similarly,

$$K_{10} := 16\int_0^s |\langle (\mathbf{v}_2\cdot\nabla)\mathbf{v}_2, \Delta\mathbf{v}_2\rangle|\, dt$$
$$\le \frac{\nu_2}{3}\int_0^s \|\Delta\mathbf{v}_2\|^2 dt + C\Big(\int_0^s \|\mathbf{v}_2\|^6 dt + \int_0^s \|\mathbf{v}_2\|^{10} dt\Big). \quad (3.39)$$

We estimate K_{11} by first splitting the term as follows

$$\begin{aligned} K_{11} :=& 16\int_0^s \left|\langle \nabla\cdot(h_1\mathbf{v}_1), \Delta^2\mathbf{v}_1\rangle\right| dt = 16\int_0^s \left|\langle \nabla\cdot(\nabla\cdot(h_1\mathbf{v}_1)), \nabla\cdot\nabla^2 h_1\rangle\right| dt \qquad (3.40)\\ &\le C\int_0^s \left|\langle \Delta h_1\mathbf{v}_1 + \Delta\mathbf{v}_1 h_1 + 2\nabla\cdot\mathbf{v}_1\nabla h_1, \nabla\cdot(\nabla^2 h_1)\rangle\right| := K_{11}^1 + K_{11}^2 + K_{11}^3. \end{aligned}$$

K_{11}^1 is estimated by the Hölder's, Agmon's inequalities as follows:

$$K_{11}^1 := C\int_0^s \left|\langle \Delta h_1\mathbf{v}_1, \nabla\cdot(\nabla^2 h_1)\rangle\right| dt \le C\int_0^s |\Delta h_1|_{L^4} |\mathbf{v}_1|_{L^4} \|\Delta h_1\| dt$$

By applying the interpolation inequality (Lemma 7.4) to the first term and the embedding $H_0^1 \hookrightarrow L^4$ in space dimension 2, we obtain:

$$\begin{aligned} K_{11}^1 &\le C\int_0^s |\nabla h_1|^{\frac{1}{2}} \|\Delta h_1\|^{\frac{1}{2}} \|\mathbf{v}_1\| \|\Delta h_1\| dt = C\int_0^s |\nabla h_1|^{\frac{1}{2}} \|\Delta h_1\|^{\frac{3}{2}} \|\mathbf{v}_1\| dt \\ &\le C\int_0^s |\nabla h_1|^4 dt + C\int_0^s \|\mathbf{v}_1\|^8 dt + \frac{\delta_1}{3}\int_0^s \|\Delta h_1\|^2 dt \qquad (3.41) \end{aligned}$$

The last line holds due to the Young inequality with $p = 8, q = 8, r = \frac{4}{3}$.

K_{11}^2 is evaluated by using Hölder's inequality:

$$K_{11}^2 := \int_0^s \left|\langle \Delta\mathbf{v}_1 h_1, \nabla\cdot(\nabla^2 h_1)\rangle\right| \le C\int_0^s |\Delta\mathbf{v}_1| |h_1|_{L^\infty} \|\Delta h_1\| dt.$$

By using Agmon's inequality to control the second term of the RHS, we obtain:

$$K_{11}^2 \le C\int_0^s |\Delta\mathbf{v}_1| |h_1|^{\frac{1}{2}} |\Delta h_1|^{\frac{1}{2}} \|\Delta h_1\| dt$$

By applying the interpolation inequality (Lemma 7.4) to the third term, along with the Poincaré inequality to the second term of the RHS, we obtain:

$$\begin{aligned} K_{11}^2 &\le C\int_0^s |\Delta\mathbf{v}_1| \|h_1\|^{\frac{1}{2}} \|h_1\|^{\frac{1}{4}} \|\Delta h_1\|^{\frac{1}{4}} \|\Delta h_1\| dt = C\int_0^s |\Delta\mathbf{v}_1| \|h_1\|^{\frac{3}{4}} \|\Delta h_1\|^{\frac{5}{4}} dt \\ &\le C\int_0^s |\Delta\mathbf{v}_1|^4 dt + \int_0^s \|h_1\|^6 dt + \frac{\delta_1}{3}\int_0^s \|\Delta h_1\|^2 dt, \end{aligned}$$

The last line follows thanks to the Young inequality with $p = 4, q = 8, r = \frac{8}{5}$. (3.42)

In the same manner, the treatment for the term K_{11}^3 is proceeded as follows:

$$\begin{aligned} K_{11}^3 := \int_0^s \left|\langle \nabla\cdot\mathbf{v}_1\nabla h_1, \nabla\cdot(\nabla^2 h_1)\rangle\right| &\le C\int_0^s |\nabla\cdot\mathbf{v}_1|_{L^4} |\nabla h_1|_{L^4} \|\Delta h_1\| dt \\ &\le C\int_0^s |\nabla h_1|^{\frac{1}{2}} |\Delta h_1|^{\frac{1}{2}} |\Delta\mathbf{v}_1| \|\Delta\mathbf{v}_1\| dt \\ &\le C\int_0^s \|h_1\|^{\frac{1}{2}} \|h_1\|^{\frac{1}{4}} \|\Delta h_1\|^{\frac{1}{4}} |\Delta\mathbf{v}_1| \|\Delta h_1\| dt \end{aligned}$$

where both Agmon's inequality and the interpolation inequality (Lemma 7.4) are applied to accomplish the second line. We then use the Young inequality with $p = 8, q = 4, r = \frac{8}{5}$

to derive:

$$K_{11}^3 \leq \frac{\delta_1}{3}\int_0^s \|\Delta h_1\|^2 dt + C\int_0^s \|h_1\|^6 dt + C\int_0^s |\Delta \mathbf{v}_1|^4\, dt. \tag{3.43}$$

Gathering all the estimates (3.41), (3.42) and (3.43), we find

$$\begin{aligned} K_{11} &:= 16\int_0^s \left|\langle \nabla\cdot(\mathbf{v}_1 h_1), \nabla\cdot(\nabla^2 h_1\rangle\right| \\ &\leq C\bigg(\int_0^s \|h_1\|^4 + \|h_1\|^6 + \|\mathbf{v}_1\|^8 + |\Delta \mathbf{v}_1|^4\, dt\bigg) + \delta_1\int_0^s \|\Delta h_1\|^2 dt. \end{aligned} \tag{3.44}$$

We obtain the similar bound for the term K_{12} as follows

$$\begin{aligned} K_{12} &:= 16\int_0^s \left|\langle \nabla\cdot(\mathbf{v}_2 h_2), \nabla\cdot(\nabla^2 h_2\rangle\right| \\ &\leq C\bigg(\int_0^s \|h_2\|^4 + \|h_2\|^6 + \|\mathbf{v}_2\|^8 + |\Delta \mathbf{v}_2|^4\, dt\bigg) + \delta_2\int_0^s \|\Delta h_2\|^2 dt. \end{aligned} \tag{3.45}$$

By utilizing the Lipschitz assumptions (2.13), we find

$$\begin{aligned} &K_{13}+K_{14}+K_{15}+K_{16} := \\ &\quad 32\int_0^s\sum_{k=1}^{\infty}\bigg(|\Delta\sigma_1(U)e_k|^2 + |\Delta\sigma_2(U)e_k|^2 + |\Delta\sigma_3(U)e_k|^2 + |\Delta\sigma_4(U)e_k|^2\, dt\bigg) \\ &\qquad\qquad \leq 32K_1\int_0^s (1+|\Delta\mathbf{v}_1|^2 + |\Delta h_1|^2 + |\Delta \mathbf{v}_2|^2 + |\Delta h_2|^2) dt. \end{aligned} \tag{3.46}$$

Accumulating all the estimates from (3.30) to (3.46) and taking the mathematical expectation on both sides yield

$$\begin{aligned} &\mathbb{E}\bigg(\sup_{0\leq r\leq s}(|\Delta\mathbf{v}_1|^2 + |\Delta\mathbf{v}_2|^2 + |\Delta h_1|^2 + |\Delta h_2|^2) + \nu_1\int_0^s \|\Delta\mathbf{v}_1(r)\|^2 dr + \nu_2\int_0^s \|\Delta\mathbf{v}_2(t)\|^2 dt \\ &+\delta_1\int_0^s\|\Delta h_1(t)\|^2 dt + \delta_2\int_0^s \|\Delta h_2(t)\|^2 dt\bigg) \leq 8\mathbb{E}\bigg(|\Delta\mathbf{v}_1(0)|^2 + |\Delta\mathbf{v}_2(0)|^2 + |\Delta h_1(0)|^2 + |\Delta h_2(0)|^2\bigg) \\ &+\mathbb{E}\mathcal{K}_0\bigg(|\nabla F|^2 + |\nabla G|^2\bigg) + 32TK_1 + \mathcal{K}_1\mathbb{E}\int_0^s\bigg(|\Delta\mathbf{v}_1|^2 + |\Delta\mathbf{v}_2|^2 + |\Delta h_1|^2 + |\Delta h_2|^2\bigg)dt \\ &+C\int_0^s(\|\mathbf{v}_1\|^8 + \|\mathbf{v}_2\|^8)dt + C\int_0^s(\|h_1\|^4 + \|h_2\|^4 + \|h_1\|^6 + \|h_2\|^6 + |\Delta\mathbf{v}_1|^4 + |\Delta\mathbf{v}_2|^4)dt \\ &+\mathbb{E}\bigg(\sup_{0\leq r\leq s}\bigg|\int_0^r\sum_{k=1}^{\infty}\langle\sigma_1(U)e_k, \Delta^2\mathbf{v}_1\rangle dW_1^k\bigg|\bigg) + \mathbb{E}\bigg(\sup_{0\leq r\leq s}\bigg|\int_0^r\sum_{k=1}^{\infty}\langle\sigma_2(U)e_k, \Delta^2\mathbf{v}_2\rangle dW_2^k\bigg|\bigg) \\ &+\mathbb{E}\bigg(\sup_{0\leq r\leq s}\bigg|\int_0^r\sum_{k=1}^{\infty}\langle\sigma_3(U)e_k, \Delta^2 h_1\rangle dW_3^k\bigg|\bigg) + \mathbb{E}\bigg(\sup_{0\leq r\leq s}\bigg|\int_0^r\sum_{k=1}^{\infty}\langle\sigma_1(U)e_k, \Delta^2 h_2\rangle dW_4^k\bigg|\bigg), \end{aligned} \tag{3.47}$$

where

$$\mathcal{K}_0 = \max\left(\frac{32}{\nu_1}, \frac{32}{\nu_2}\right), \mathcal{K}_1 = 32\max\left(\frac{g^2}{\nu_1}, \frac{g^2}{\nu_2}, \frac{\rho_2^2 g^2}{\nu_1\rho_1}, \frac{g^2\lambda_1}{\nu_1}, \frac{g^2\lambda_1}{\nu_1}\right). \tag{3.48}$$

The stochastic terms are estimated by using integration by parts, the BDG inequality, and the Lipschitz assumptions (2.12):

$$\mathbb{E}\left(\sup_{0\le r\le s}\left|\int_0^r \sum_{k=1}^\infty \langle\sigma_i(U)e_k, \Delta^2\mathbf{v}_i\rangle dW_i^k\right|\right) \le (\text{with } Ge_k = G_k = \langle\sigma_i(U)e_k, \Delta\mathbf{v}_i\rangle, i = 1,2)$$

$$\le C_1\mathbb{E}\left(\int_0^s \sum_{k=1}^\infty \langle\nabla\sigma_i(U)e_k, \nabla\cdot(\nabla)^2\mathbf{v}_i\rangle^2 dt\right)^{\frac{1}{2}} \le C_1\mathbb{E}\left(\int_0^s \sum_{k=1}^\infty \langle\Delta\sigma_i(U)e_k, (\nabla)^2\mathbf{v}_i\rangle^2 dt\right)^{\frac{1}{2}}$$

$$\le C_1\mathbb{E}\left(\int_0^s \sum_{k=1}^\infty |\Delta\sigma_i(U)e_k|\,|\Delta\mathbf{v}_1|^2\, dt\right)^{\frac{1}{2}} \le C_1\mathbb{E}\left(\sup_{0\le r\le s}|\Delta\mathbf{v}_i(r)|^2\int_0^s \sum_{k=1}^\infty |\Delta\sigma_i(U)e_k|^2\, dt\right)^{\frac{1}{2}}$$

$$\le \frac{1}{2}\mathbb{E}\left(\sup_{0\le r\le s}|\Delta\mathbf{v}_i(r)|^2\right) + \frac{C_1^2}{2}\mathbb{E}\left(\int_0^s \sum_{k=1}^\infty |\Delta\sigma_i(U)e_k|^2\, dt\right)$$

$$\le \frac{1}{2}\mathbb{E}\left(\sup_{0\le r\le s}|\Delta\mathbf{v}_i(r)|^2\right)$$

$$+\frac{K_1C_1^2}{2}\mathbb{E}\left(\int_0^s (1 + |\Delta\mathbf{v}_1(t)|^2 + |\Delta\mathbf{v}_2(t)|^2 + |\Delta h_1(t)|^2 + |\Delta h_2(t)|^2)dt\right). \tag{3.49}$$

Similarly, for $i = 3, 4$, $j = 1, 2$, we obtain:

$$\mathbb{E}\left(\sup_{0\le r\le s}\left|\int_0^r \sum_{k=1}^\infty \langle\sigma_i(U)e_k, \Delta^2 h_j\rangle dW_i^k\right|\right) \le \frac{1}{2}\mathbb{E}\left(\sup_{0\le r\le s}|\Delta h_j(r)|^2\right) \tag{3.50}$$

$$+\frac{K_1C_1^2}{2}\mathbb{E}\left(\int_0^s (1 + |\Delta\mathbf{v}_1(t)|^2 + |\Delta\mathbf{v}_2(t)|^2 + |\Delta h_1(t)|^2 + |\Delta h_2(t)|^2)dt\right).$$

Combining (3.47) to (3.50) and multiplying by 2, we find

$$\mathbb{E}\left(\sup_{0\le r\le s}(|\Delta\mathbf{v}_1|^2 + |\Delta\mathbf{v}_2|^2 + |\Delta h_1|^2 + |\Delta h_2|^2\right) + \mathbb{E}\left(\nu_1\int_0^s \|\Delta\mathbf{v}_1(r)\|^2 dr + \nu_2\int_0^s \|\Delta\mathbf{v}_2(t)\|^2 dt\right.$$

$$\left.+\,\delta_1\int_0^s \|\Delta h_1(t)\|^2 dt + \delta_2\int_0^s \|\Delta h_2(t)\|^2 dt\right)$$

$$\le 8\mathbb{E}\left(|\Delta\mathbf{v}_1(0)|^2 + |\Delta\mathbf{v}_2(0)|^2 + |\Delta h_1(0)|^2 + |\Delta h_2(0)|^2\right) + 32TK_1$$

$$+\mathbb{E}\mathcal{K}_0\int_0^s \left(|\nabla F|^2 + |\nabla G|^2\right)dt + \mathcal{K}_2\mathbb{E}\int_0^s \left(|\Delta\mathbf{v}_1|^2 + |\Delta\mathbf{v}_2|^2 + |\Delta h_1|^2 + |\Delta h_2|^2\right)dt$$

$$+C\int_0^s (\|h_1\|^4 + \|h_2\|^4 + \|h_1\|^6 + \|h_2\|^6)dt + C\int_0^s (\|\mathbf{v}_1\|^8 + \|\mathbf{v}_2\|^8 + |\Delta\mathbf{v}_1|^4 + |\Delta\mathbf{v}_2|^4)dt, \tag{3.51}$$

where

$$\mathcal{K}_2 := \mathcal{K}_1 + 4C_1K_1.$$

Now, we assume that $N > 1$ and consider the stopping time

$$\tau = \tau_N := \inf_{s\geq 0}\Big\{\Big(|\Delta\mathbf{v}_1(s)|^2 + |\Delta\mathbf{v}_2(s))|^2 + |\Delta h_1(s)|^2 + |\Delta h_2(s)|^2\Big) > N\Big\} \tag{3.52}$$

Finally, we consider the stopping time

$$\tau^{M,N} = \tau_N \wedge \tau_M \tag{3.53}$$

where τ_M is defined at (3.22).
Replacing s by $s \wedge \tau^{M,N}$ in (3.51) gives

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{0\leq r\leq s}(|\Delta\mathbf{v}_1|^2 + |\Delta\mathbf{v}_2|^2 + |\Delta h_1|^2 + |\Delta h_2|^2\bigg) + \mathbb{E}\bigg(\nu_1\int_0^s \|\Delta\mathbf{v}_1(r)\|^2 dr + \\
&\qquad \nu_2\int_0^s \|\Delta\mathbf{v}_2(t)\|^2 dt\bigg) + \mathbb{E}\bigg(\delta_1\int_0^{s\wedge\tau^{M,N}} \|\Delta h_1(t)\|^2 dt + \delta_2\int_0^s \|\Delta h_2(t)\|^2 dt\bigg) \\
&\qquad\qquad \leq 16\mathbb{E}\bigg(|\Delta\mathbf{v}_1(0)|^2 + |\Delta\mathbf{v}_2(0)|^2 + |\Delta h_1(0)|^2 + |\Delta h_2(0)|^2\bigg) \\
&+\mathbb{E}\mathcal{K}_0\int_0^s\bigg(|\nabla F|^2 + |\nabla G|^2\bigg)dt + 32TK_1 + \mathcal{K}_2\mathbb{E}\int_0^s\bigg(|\Delta\mathbf{v}_1|^2 + |\Delta\mathbf{v}_2|^2 + |\Delta h_1|^2 + |\Delta h_2|^2\bigg)dt \\
&\quad + CN\mathbb{E}\int_0^s (|\Delta\mathbf{v}_1|^2 + |\Delta\mathbf{v}_2|^2 + |\Delta h_1|^2 + |\Delta h_2|^2)dt + C\int_0^s (M^2 + M^3 + M^4)dt.
\end{aligned} \tag{3.54}$$

We define

$$\mathcal{Y}(s) = \int_0^{s\wedge\tau^{M\wedge N}}\bigg(|\Delta\mathbf{v}_1(t)|^2 + |\Delta\mathbf{v}_2(t)|^2 + |\Delta h_1(t)|^2 + |\Delta h_2(t)|^2\bigg)dt \tag{3.55}$$

$$\begin{aligned}
\mathcal{K}_3 = 16\mathbb{E}(|\Delta\mathbf{v}_1(0)|^2 + |\Delta\mathbf{v}_1(0)|^2 + |\Delta h_1(0)|^2 + |\Delta h_2(0)|^2) + \\
\mathcal{K}_0\int_0^T (\|F\|^2 + \|G\|^2)dt + CT(M^2 + M^3 + M^4) + 32TK_1.
\end{aligned} \tag{3.56}$$

From (3.54), (3.55) and (3.56), we obtain

$$\mathcal{Y}'(s) \leq \mathcal{K}_3 + (CN + \mathcal{K}_2)\mathcal{Y}(s) \tag{3.57}$$

This gives

$$\mathcal{Y}(s) \leq \frac{\mathcal{K}_3}{CN + \mathcal{K}_2}\left(e^{(CN+\mathcal{K}_2)s} - 1\right). \tag{3.58}$$

In conjugation with (3.54), we obtain

$$\mathbb{E}\Bigg(\sup_{0\le r\le s\wedge\tau^{M,N}}\Big(|\Delta\mathbf{v}_1|^2+|\Delta\mathbf{v}_2|^2+|\Delta h_1|^2+|\Delta h_2|^2\Big)+2\nu_1\int_0^{s\wedge\tau^{M,N}}\|\Delta\mathbf{v}_1(t)\|^2dt+$$
$$2\nu_1\int_0^{s\wedge\tau^{M,N}}\|\Delta\mathbf{v}_1(t)\|^2dt+2\delta_1\int_0^{s\wedge\tau^{M,N}}\|\Delta h_1(t)\|^2dt+2\delta_2\int_0^{s\wedge\tau^{M,N}}\|\Delta h_2(t)\|^2dt\Bigg)$$
$$\le\mathcal{K}_3+\frac{\mathcal{K}_3}{CN+\mathcal{K}_2}e^{(CN+\mathcal{K}_2)s}.(CN+\mathcal{K}_2)\quad(3.59)$$

The right hand side of (3.59) is bounded by N if

$$\mathcal{K}_3+e^{(CN+\mathcal{K}_2)s}\mathcal{K}_3\le N$$

or

$$0\le s\le\frac{1}{CN+\mathcal{K}_2}\log\frac{N-\mathcal{K}_3}{\mathcal{K}_3}:=s_N\text{ for }N\ge 2\mathcal{K}_3\quad(3.60)$$

As long as we can choose $N>2\mathcal{K}_3$, the local existence of the solution is obtained on $[0,s_N\wedge\tau^{M,N}]$.

4 The modified system with a cut-off function

This section is focused on the study of the martingale solutions of the following modified system

$$d\mathbf{v}_1+\left(-\nu_1\Delta\mathbf{v}_1+\theta(\|\mathbf{v}_1\|+\|h_1\|+\|\mathbf{v}_2\|+\|h_2\|)g\nabla h_1+g\frac{\rho_2}{\rho_1}\nabla h_2+f\mathbf{k}\times\mathbf{v}_1\right)dt$$
$$=F+\sum_{k=1}^{\infty}\sigma_1(U)e_kdW_1^k,\quad(4.1a)$$

$$d\mathbf{v}_2+(-\nu_2\Delta\mathbf{v}_2+\theta(\|\mathbf{v}_1\|+\|h_1\|+\|\mathbf{v}_2\|+\|h_2\|)g\nabla h_2+g\nabla h_2+f\mathbf{k}\times\mathbf{v}_2)$$
$$=G+\sum_{k=1}^{\infty}\sigma_2(U)e_kdW_2^k,\quad(4.1b)$$

$$dh_1+(-\delta_1\Delta h_1+\theta(\|\mathbf{v}_1\|+\|h_1\|+\|\mathbf{v}_2\|+\|h_2\|)\nabla\cdot(h_1\mathbf{v}_1))\,dt=\sum_{k=1}^{\infty}\sigma_3(U)e_kdW_3^k,\quad(4.1c)$$

$$dh_2+(-\delta_2\Delta h_2+\theta(\|\mathbf{v}_1\|+\|h_1\|+\|\mathbf{v}_2\|+\|h_2\|)\nabla\cdot(h_2\mathbf{v}_2))\,dt=\sum_{k=1}^{\infty}\sigma_4(U)e_kdW_4^k.\quad(4.1d)$$

For simplicity, we denote $\theta(\|\mathbf{v}_1\|+\|h_1\|+\|\mathbf{v}_2\|+\|h_2\|)$ by $\theta(\|U\|)$ for $\|U\|=\|\mathbf{v}_1\|+\|h_1\|+\|\mathbf{v}_2\|+\|h_2\|$.

Here $\theta : \mathbb{R} \to [0,1]$ is a $\mathcal{C}^\infty$ cut-off function satisfies

$$\theta(\epsilon) = \begin{cases} 1 \text{ if } |\epsilon| \le \mathcal{K}, \\ 0 \text{ if } |\epsilon| \ge 2\mathcal{K}. \end{cases}$$

where $\mathcal{K}$ is any positive number and is independent of n. The specific choice for $\mathcal{K}$ will be made more evidently in the next section.

Theorem 4.1 (Global existence of martingale solutions to the modified system)**.** *With the same assumptions as in Theorem 2.1, there exists a global martingale solution to* (4.1).

Theorem 4.2 (Global existence of pathwise solutions to the modified system)**.** *Under the same assumptions as in Theorem 2.2, there exists a global pathwise solution to* (4.1) *relative to given probability space* $\mathcal{S} = (\Omega, \mathcal{F}, \mathbb{P})$.

4.1 The Galerkin scheme

Considering the projection P_n defined as in (2.4), we introduce the Galerkin approximation $U^n := (\mathbf{v}_1^n, \mathbf{v}_2^n, h_1^n, h_2^n)$ associated to the modified system (4.1), with $\mathbf{v}_1^n, \mathbf{v}_2^n, h_1^n$ and h_2^n functions from some interval $(0, \tau_n)$ into $P_n(V_1 \times V_1 \times V_2 \times V_2)$, namely,

$$d\mathbf{v}_1^n - \nu_1 \Delta \mathbf{v}_1^n dt + P_n\Big[\theta\left(\|U^n\|\right)(\mathbf{v}_1^n \cdot \nabla)\mathbf{v}_1^n + \mathrm{g}\nabla \mathrm{h}_1^n + g\frac{\rho_2}{\rho_1}\nabla \mathrm{h}_2^n + \mathrm{f}\,\mathbf{k}^n \times \mathbf{v}_1^n\Big]dt = P_n \mathrm{F} dt + \sum_{k=1}^{\infty} P_n \sigma_1(U^n) e_k dW_1^k, \tag{4.2a}$$

$$d\mathbf{v}_2^n - \nu_2 \Delta \mathbf{v}_2^n dt + P_n\Big[\theta\left(\|U^n\|\right)(\mathbf{v}_2^n \cdot \nabla)\mathbf{v}_2^n + \mathrm{g}\nabla \mathrm{h}_2^n + g\nabla \mathrm{h}_1^n + \mathrm{f}\,\mathbf{k}^n \times \mathbf{v}_2^n\Big]dt = P_n \mathrm{G} dt + \sum_{k=1}^{\infty} P_n \sigma_2(U^n) e_k dW_2^k, \tag{4.2b}$$

$$d\mathrm{h}_1^n + P_n\Big[-\delta_1 \Delta \mathrm{h}_1^n + \theta\left(\|U^n\|\right)\nabla \cdot (\mathrm{h}_1^n \mathbf{v}_1^n)\Big]dt = \sum_{k=1}^{\infty} P_n \sigma_3(U^n) e_k dW_3^k, \tag{4.2c}$$

$$d\mathrm{h}_2^n + P_n\Big[-\delta_2 \Delta \mathrm{h}_2^n + \theta\left(\|U^n\|\right)\nabla \cdot (\mathrm{h}_2^n \mathbf{v}_2^n)\Big]dt = \sum_{k=1}^{\infty} P_n \sigma_4(U^n) e_k dW_4^k, \tag{4.2d}$$

$$\mathbf{v}_i^n(0) = \mathbf{v}_{i0}^n = P_n \mathbf{v}_i(0), \qquad h_i^n(0) = h_{i0}^n = P_n h_i(0), \text{ for } i = 1, 2. \tag{4.2e}$$

4.2 Uniform estimates for the Galerkin system

The essential estimate for our study below is the following:

Lemma 4.1. *Let $\mathbf{v}_1^n, \mathbf{v}_2^n, h_1^n$ and h_2^n be the solutions of (4.2) and assume that $\mathbf{v}_1(0), \mathbf{v}_2(0) \in L^p(\Omega, \mathcal{F}_0, H_1), h_1(0), h_2(0) \in L^p(\Omega, \mathcal{F}_0, H_2), F, G \in L^p(\Omega \times [0,T], H)$ for some $p \geq 2$. Then we have the following estimates*

$$\mathbb{E}\left(\sup_{0\leq r\leq T} \|\mathbf{v}_1^n(r)\|^p + \sup_{0\leq r\leq T} \|h_1^n(r)\|^p + \sup_{0\leq r\leq T} \|\mathbf{v}_2^n(r)\|^p + \sup_{0\leq r\leq T} \|h_2^n(r)\|^p\right) \leq \mathcal{K}_4, \tag{4.3a}$$

and

$$\mathbb{E}\left(\int_0^T |\Delta \mathbf{v}_1^n|^2\, dt + \int_0^T |\Delta h_1^n|^2\, dt + \int_0^T |\Delta \mathbf{v}_2^n|^2\, dt + \int_0^T |\Delta h_2^n|^2\, dt\right) \leq \mathcal{K}_5, \tag{4.3b}$$

where $\mathcal{K}_3$ and $\mathcal{K}_4$ depend only on the data and are independent of n.

Proof. Since P_n and A commute with each other and $H_n \subset A$, the same proof of Lemma 3.1 carries over to (4.2) with a slightly modification on the nonlinear terms. Thanks to the presence of the cut off in front of the nonlinear terms, we can derive the global bounds instead of local bounds as in Lemma 3.1. We only provide the details for the estimates of the non-linear terms. The bounds for those terms are derived as follows:

$$\begin{aligned}
\tilde{I}_9 &:= \int_0^s |\langle \theta(\|U^n\|)(\mathbf{v}_1^n \cdot \nabla)\mathbf{v}_1^n, \Delta \mathbf{v}_1^n\rangle|\, \|\mathbf{v}_1^n\|^{p-2} dt \\
&\leq C \int_0^s |\mathbf{v}_1^n|_{L^\infty} |\theta(\|U^n\|)\nabla \cdot \mathbf{v}_1^n|_{L^2} |\Delta \mathbf{v}_1^n|_{L^2} \|\mathbf{v}_1^n\|^{p-2} dt \\
&\leq C \int_0^s |\mathbf{v}_1^n|^{\frac{1}{2}} |\Delta \mathbf{v}_1^n|^{\frac{1}{2}} |\Delta \mathbf{v}_1^n|_{L^2} \|\mathbf{v}_1^n\|^{p-2} dt \\
&\leq C \int_0^s \|\mathbf{v}_1^n\|^2 \|\mathbf{v}_1^n\|^{p-2} dt + \frac{p\nu_1}{10}\int_0^s |\Delta \mathbf{v}_1^n|^2\, \|\mathbf{v}_1^n\|^{p-2} dt \\
&\leq C \int_0^s \|\mathbf{v}_1^n\|^p dt + \frac{p\nu_1}{10}\int_0^s |\Delta \mathbf{v}_1^n|^2\, \|\mathbf{v}_1^n\|^{p-2} dt,
\end{aligned} \tag{4.4}$$

where the third and the fourth lines hold true due to the Agmon's inequality, the definition of the cut-off function in (4) and the Young inequality.

Similarly,

$$\tilde{I}_{10} := \int_0^s |\langle \theta(\|U^n\|)(\mathbf{v}_2^n \cdot \nabla)\mathbf{v}_2^n, \Delta \mathbf{v}_2^n\rangle|\, \|\mathbf{v}_2^n\|^{p-2} dt \leq C \int_0^s \|\mathbf{v}_2^n\|^p dt + \frac{p\nu_2}{10}\int_0^s |\Delta \mathbf{v}_2^n|^2\, \|\mathbf{v}_2^n\|^{p-2} dt. \tag{4.5}$$

$$\begin{aligned}
\tilde{I}_{11} &:= \int_0^s |\langle \theta(\|U^n\|)\nabla \cdot (\mathbf{v}_1^n h_1^n), \Delta h_1^n\rangle| \|h_1^n\|^{p-2} dt \leq \int_0^s |\langle \theta(U^n)\nabla \cdot \mathbf{v}_1^n h_1^n, \Delta h_1^n\rangle| \|h_1^n\|^{p-2} dt \\
&\quad + \int_0^s |\langle \theta(U^n)\nabla h_1^n \mathbf{v}_1^n, \Delta h_1^n\rangle| \|h_1^n\|^{p-2} dt := \tilde{I}_{11}^1 + \tilde{I}_{11}^2.
\end{aligned}$$

We estimate the first term by making use of Hölder's inequality:

$$\tilde{I}_{11}^1 := \int_0^s |\langle \theta(U^n)\nabla \cdot \mathbf{v}_1^n h_1^n, \Delta h_1^n\rangle| \|h_1^n\|^{p-2} dt \leq C \int_0^t |h_1^n|_{L^\infty} |\theta(\|U^n\|)\nabla \mathbf{v}_1^n| \, |\Delta h_1^n| \, \|h_1^n\|^{p-2} dt.$$

Using Agmon's inequality to control the first term and the definition of the cut-off function to control the second term of the RHS, the next line follows

$$\tilde{I}_{11}^1 \leq C \int_0^s |h_1^n|^{\frac{1}{2}} |\Delta h_1^n|^{\frac{1}{2}} |\Delta h_1^n| \, \|h_1^n\|^{p-2} dt = C \int_0^s |h_1^n|^{\frac{1}{2}} |\Delta h_1^n|^{\frac{3}{2}} \|h_1^n\|^{p-2} dt$$
$$\leq C \int_0^s \|h_1^n\|^p dt + \frac{p\delta_1}{4} \int_0^s |\Delta h_1^n|^2 \, \|h_1^n\|^{p-2} dt, \tag{4.6}$$

The last line holds true thanks to the Young inequality.
We obtain the similar bound for $\tilde{I}_{11}^2$ as follows:

$$\tilde{I}_{11}^2 \leq C \int_0^s \|h_1^n\|^p dt + \frac{p\delta_1}{4} \int_0^s |\Delta h_1^n| \, \|h_1^n\|^{p-2} dt. \tag{4.7}$$

Combining (4.6) and (4.7), we obtain:

$$\tilde{I}_{11} := \int_0^s |\langle \theta(\|U^n\|)\nabla \cdot (h_1^n \mathbf{v}_2^n), \Delta h_1^n\rangle| \, dt \leq C \int_0^s \|h_1^n\|^p + \frac{p\delta_1}{2} \int_0^s |\Delta h_1^n|^2 \, \|h_1^n\|^{p-2} dt. \tag{4.8}$$

Almost identically, we obtain:

$$\tilde{I}_{12} := \int_0^s |\langle \theta(\|U^n\|)\nabla \cdot (h_2^n \mathbf{v}_2^n), \Delta h_2^n\rangle| \, dt \leq C \int_0^s \|h_2^n\|^p + \frac{p\delta_2}{2} \int_0^s |\Delta h_2^n|^2 \, \|h_2^n\|^{p-2} dt. \tag{4.9}$$

All the estimates for the linear terms and stochastic terms are carried out in the same way as in the previous section. Combining those estimates with slightly changes on the constants and all of the relations from (4.4) through (4.9), we deduce that:

$$\begin{aligned}
&\mathbb{E}\Bigg(\sup_{0\leq r\leq s} (\|\mathbf{v}_1^n(r)\|^p + \|\mathbf{v}_2^n(r)\|^p + \|h_1^n(s)\|^p + \|h_2^n(s)\|^p) + p\nu_1 \int_0^s |\Delta \mathbf{v}_1^n(t)|^2 \, \|\mathbf{v}_1^n(t)\|^{p-2} dt \\
&+ p\nu_2 \int_0^s |\Delta \mathbf{v}_2^n(t)|^2 \, \|\mathbf{v}_2^n(t)\|^{p-2} dt + \delta_1 \int_0^s |\Delta h_1^n(t)|^2 \, \|h_1^n\|^{p-2} dt + p\delta_2 \int_0^s |\Delta h_2^n(t)|^2 \, \|h_2^n(t)\|^{p-2} dt \Bigg) \\
&\preceq \mathbb{E}\left(\|\mathbf{v}_1^n(0)\|^p + \|\mathbf{v}_2^n(0)\|^p + \|h_1^n(0)\|^p + \|h_2^n(0)\|^p\right) + \mathbb{E}\left(\int_0^s (|F|^p + |G|^p) dt\right) \\
&\quad + \mathbb{E}\left(\int_0^s (\|\mathbf{v}_1^n\|^p + \|\mathbf{v}_2^n\|^p + \|h_1^n\|^p + \|h_2^n\|^p) dt\right) \\
&\preceq \mathbb{E}\left(\|\mathbf{v}_1^n(0)\|^p + \|\mathbf{v}_2^n(0)\|^p + \|h_1^n(0)\|^p + \|h_2^n(0)\|^p\right) + \mathbb{E}\left(\int_0^s (|F|^p + |G|^p) dt\right) + C \\
&+ \mathbb{E}\left(\int_0^s \left[\sup_{0\leq r\leq t} \|\mathbf{v}_1^n(r)\|^p + \sup_{0\leq r\leq t} \|\mathbf{v}_2^n(r)\|^p + \sup_{0\leq r\leq t} \|h_1^n(r)\|^p + \sup_{0\leq r\leq t} \|h_2^n(r)\|^p\right] dt\right) + C.
\end{aligned} \tag{4.10}$$

By applying the deterministic Gronwall inequality to

$$Y(s) = E\Bigg(\sup_{r\in[0,s]}\|\mathbf{v}_1^n(r)\|^p + \sup_{r\in[0,s]}\|\mathbf{v}_2^n(r)\|^p + \sup_{s\in[0,s]}\|h_1^n(r)\|^p + \sup_{r\in[0,s]}\|h_2^n(r)\|^p\Bigg),$$

we obtain:

$$\begin{aligned}
&E\Bigg(\sup_{r\in[0,s]}\|\mathbf{v}_1^n(r)\|^p + \sup_{r\in[0,s]}\|\mathbf{v}_2^n(r)\|^p + \sup_{s\in[0,s]}\|h_1^n(r)\|^p + \sup_{r\in[0,s]}\|h_2^n(r)\|^p\Bigg)\\
&\quad\preceq (\|\mathbf{v}_1^n(0)\|^p + \|\mathbf{v}_2^n(0)\|^p + \|h_1^n(0)\|^p + \|h_2^n(0)\|^p) + \mathbb{E}\left(\int_0^s |F|^p + |G|^p\,dt\right).\\
&\quad\preceq (\|\mathbf{v}_1(0)\|^p + \|\mathbf{v}_2(0)\|^p + \|h_1(0)\|^p + \|h_2(0)\|^p) + \left(\int_0^s |F|^p + |G|^p\,dt\right). \qquad (4.11)
\end{aligned}$$

From (4.10) and (4.11), the lemma is proved. □

Our goal now is to derive some estimates in fractional Sobolev spaces which are crucial for establishing the existence of both martingale and pathwise solutions.

Lemma 4.2 (Estimates in Fractional Sobolev spaces)**.** *Under the same assumptions as in Theorem 2.1, we consider the associated sequence of solutions* $\{(\mathbf{v}_1^n, \mathbf{v}_2^n, h_1^n, h_2^n)\}_{n\geq 1}$ *of the Galerkin system* (4.1)*. Let* $p > 2$ *and assume that* $\mathbb{E}(\|\mathbf{v}_1(0)\|^p + \|\mathbf{v}_2(0)\|^p + \|h_1(0)\|^p + \|h_2(0)\|^p) < \infty$*. Then there exists a finite number* $\mathcal{K} > 0$ *(depending only on the data) such that*

$$\mathbb{E}\left(\left|\int_0^t \sum_{k=1}^\infty P_n\sigma_i(U^n)e_k dW_i^k\right|^p_{W^{\alpha,p}([0.T];H_1)}\right) \leq \mathcal{K},\ \ for\ i = 1, 2, \qquad (4.12a)$$

$$\mathbb{E}\left(\left|\int_0^t \sum_{k=1}^\infty P_n\sigma_i^k(U^n)e_k dW_i^k\right|^p_{W^{\alpha,p}([0.T];H_2)}\right) \leq \mathcal{K},\ \ for\ i = 3, 4, \qquad (4.12b)$$

$$\mathbb{E}\left(\left|\mathbf{v}_i^n(t) - \int_0^t \sum_{k=1}^\infty P_n\sigma_i(\mathbf{v}^n, h^n) dW_i^k\right|^2_{W^{1,2}([0.T];H_1)}\right) \leq \mathcal{K},\ \ for\ i = 1, 2, \qquad (4.12c)$$

$$\mathbb{E}\left(\left|h_i^n(t) - \int_0^t \sum_{k=1}^\infty P_n\sigma_j^k(\mathbf{v}^n, h^n) dW_j^k\right|^2_{W^{1,2}([0.T];H_2)}\right) \leq \mathcal{K},\ \ for\ i = 1, 2,\ j = 3, 4. \qquad (4.12d)$$

Proof. The proofs can be followed in exact the same way as in [21] so we omit them. □

4.3 Compactness arguments

We fix a stochastic basis, $\mathcal{S} = (\Omega, \mathcal{F}, (\mathcal{F})_{t\geq 0}, \mathbb{P}, W_1, W_2, W_3, W_4)$, and given $(\mathbf{v}_1^0, \mathbf{v}_2^0, h_1^0, h_2^0)$ which is $\mathcal{F}_0$-measurable and has distribution μ_0. Then we go back to the finite dimensional approximations relative to $\mathcal{S}$ and $(\mathbf{v}_1^0, \mathbf{v}_2^0, h_1^0, h_2^0)$. We define the phase space

$$\mathcal{X} = \mathcal{X}_{\mathbf{v}_1} \times \mathcal{X}_{\mathbf{v}_2} \times \mathcal{X}_{h_1} \times \mathcal{X}_{h_2} \times \prod_{i=1}^4 \mathcal{X}_{W_i},$$

where

$$
\begin{aligned}
&\mathcal{X}_{\mathbf{v}_1} = \mathcal{X}_{\mathbf{v}_2} = L^2(0,T;V_1) \cap \mathcal{C}([0,T];V_1'),\\
&\mathcal{X}_{h_1} = \mathcal{X}_{h_2} = L^2(0,T;V_2) \cap \mathcal{C}([0,T];V_2'),\\
&\mathcal{X}_{W_1} = \mathcal{X}_{W_2} = \mathcal{X}_{W_3} = \mathcal{X}_{W_4} = \mathcal{C}([0,T];\mathfrak{U}_0).
\end{aligned} \tag{4.13}
$$

We consider the probability measures

$$\mu^n_{\mathbf{v}_1}(\cdot) = \mu^n_{\mathbf{v}_2}(\cdot) = \mathbb{P}\left(\mathbf{v}^n \in \cdot\right) \in \mathbb{P}(L^2(0,T;V_1) \cap \mathcal{C}([0,T];V_1')), \tag{4.14}$$

$$\mu^n_{h_1}(\cdot) = \mu^n_{h_2}(\cdot) = \mathbb{P}\left(h^n \in \cdot\right) \in \mathbb{P}(L^2(0,T;V_2) \cap \mathcal{C}([0,T];V_2')), \tag{4.15}$$

$$\text{and} \quad \mu_{W_i}(\cdot) = \mu^n_{W_i}(\cdot) = \mathbb{P}(W_i \in \cdot) \in \mathbb{P}(\mathcal{C}([0,T];\mathfrak{U}_0)), \text{ for } i = 1,2,3,4. \tag{4.16}$$

This defines a sequence of probability measures $\mu^n = \mu^n_{\mathbf{v}_1} \times \mu^n_{\mathbf{v}_2} \times \mu^n_{h_1} \times \mu^n_{h_2} \times \prod_{i=1}^4 \mu^n_{W_i}$ on $\mathcal{X}$. Then we have the following tightness result:

Lemma 4.3. *Consider the measure μ^n on $\mathcal{X}$ defined as above in (4.14)–(4.16). Then the sequence $\{\mu^n\}_{n\geq 1}$ is tight and therefore weakly compact on the phase space $\mathcal{X}$.*

Proof. The reader is referred to our previous work [21] for a detailed proof. □

4.4 Passage to the limit

Suppose μ_0 is a probaility measure on $V_1 \times V_1 \times V_2 \times V_2$ satisfying

$$\int_{V_1\times V_1\times V_2\times V_2} \|\mathbf{u}_0\|^2 \mu_0(d\mu) < \infty \tag{4.17}$$

where $\mathbf{u}_0 = (\mathbf{v}_1^0, \mathbf{v}_2^0, h_1^0, h_2^0)$; in the previous lemma, we have shown that the sequence of measures $\{\mu^n\}_{n\geq 1}$ associated with the Galerkin sequence $\{\mathbf{v}_1^n, \mathbf{v}_2^n, h_1^n, h_2^n, W_1, W_2, W_3, W_4\}$ is weakly compact over χ. This implies the existence of a subsequence μ^{n_j} and to simplify writing, we write j for n_j. We now apply the Skorohod embedding theorem to infer the following theorem.

Theorem 4.3. *Let μ_0 be a probability measure on $V_1 \times V_2$ satisfying (4.17). Then there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ with the associated expectation denoted by $\tilde{\mathbb{E}}$ a sequence of $\mathcal{X}$-valued random variables $(\tilde{\mathbf{v}}_1^{n_j}, \tilde{\mathbf{v}}_1^{n_j}, \tilde{h}_1^{n_j}, \tilde{h}_2^{n_j}, \tilde{W}_1^{n_j}, \tilde{W}_2^{n_j}, \tilde{W}_3^{n_j}, \tilde{W}_4^{n_j})$, such that*

1. *$(\tilde{\mathbf{v}}_1^j, \tilde{\mathbf{v}}_2^j, \tilde{h}_1^j, \tilde{h}_2^j, \tilde{W}_1^j, \tilde{W}_2^j, \tilde{W}_3^j, \tilde{W}_4^j)$ has the same law $(\mathbf{v}_1^n, \mathbf{v}_2^n, h_1^n, h_2^n, W_1, W_2, W_3, W_4)$.*
2. *$(\tilde{\mathbf{v}}_1^{n_j}, \tilde{\mathbf{v}}_2^{n_j}, \tilde{h}_1^{n_j}, \tilde{h}_2^{n_j}, \tilde{W}_1^{n_j}, \tilde{W}_2^{n_j}, \tilde{W}_3^{n_j}, \tilde{W}_4^{n_j})$ converges almost surely in the topology of $\mathcal{X}$ to an element $(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{h}_1, \tilde{h}_2, \tilde{W}_1, \tilde{W}_2, \tilde{W}_3, \tilde{W}_4)$ i.e.*

$$\tilde{\mathbf{v}}_i^{n_j} \to \tilde{\mathbf{v}}_i \text{ in } L^2(0,T;V_1) \cap \mathcal{C}([0,T];V_1') \ \tilde{\mathbb{P}} - a.s \text{ for } i = 1,2, \tag{4.18a}$$

$$\tilde{h}_i^{n_j} \to \tilde{h}_i \text{ in } L^2(0,T;V_2) \cap \mathcal{C}([0,T];V_2') \ \tilde{\mathbb{P}} - a.s \text{ for } i = 1,2, \tag{4.18b}$$

$$\tilde{W}_i^{n_j} \to \tilde{W}_i \text{ in } \mathcal{C}([0,T];\mathfrak{U}_0) \ \tilde{\mathbb{P}} - a.s \text{ for } i = 1,2,3,4. \tag{4.18c}$$

3. *Each* $\tilde{W}_i^{n_j}$, $i = 1,2,3,4$ *is a cylindrical Wiener process relative to the filtration* $\tilde{F}_t^{n_j}$ *given by*

$$\tilde{\mathcal{F}}_t^j := \sigma(\tilde{W}_1^j(s), \tilde{W}_2^j(s), \tilde{W}_3^j(s), \tilde{W}_4^j(s), \tilde{\mathbf{v}}_1^j(s), \tilde{\mathbf{v}}_2^j(s), \tilde{h}_1^{n_j}(s), \tilde{h}_2^{n_j}(s), s \le t).$$

4. *Each* $(\tilde{\mathbf{v}}_1^{n_j}, \tilde{\mathbf{v}}_2^{n_j}, \tilde{h}_1^{n_j}, \tilde{h}_2^{n_j}, \tilde{W}_1^{n_j}, \tilde{W}_2^{n_j}, \tilde{W}_3^{n_j}, \tilde{W}_4^{n_j})$ *satisfies:*

$$\begin{aligned} d\tilde{\mathbf{v}}_1^j - \nu_1 \Delta \tilde{\mathbf{v}}_1^j dt + P_n[\theta(\|\tilde{\mathbf{v}_1}^j\|^2 + \|\tilde{\mathbf{v}_2}^j\|^2 + \|\tilde{h_1}^j\|^2 + \|\tilde{h_2}^j\|^2)(\tilde{\mathbf{v}}_1^j \cdot \nabla)\tilde{\mathbf{v}}_1^j + \\ g\frac{\rho_2}{\rho_1}\nabla \tilde{h}_1^j + f\mathbf{k} \times \tilde{\mathbf{v}}_1^j]dt = P_j F dt + P_j \sigma_1(\tilde{U}^j) d\tilde{W}_1^{kj}, \end{aligned} \tag{4.19}$$

$$\begin{aligned} d\tilde{\mathbf{v}}_2^j - \nu_2 \Delta \tilde{\mathbf{v}}_2^j dt + P_n[\theta(\|\tilde{\mathbf{v}_1}^j\|^2 + \|\tilde{\mathbf{v}_2}^j\|^2 + \|\tilde{h_1}^j\|^2 + \|\tilde{h_2}^j\|^2)(\tilde{\mathbf{v}}_2^j \cdot \nabla)\tilde{\mathbf{v}}_2^j \\ + g\nabla \tilde{h_2}^j + f\mathbf{k} \times \tilde{\mathbf{v}}_2^j]dt = P_n G dt + \sum_{k=1}^{\infty} P_n \sigma_2(\tilde{U}^{n_j}) d\tilde{W}_2^{k,j}, \end{aligned} \tag{4.20}$$

$$\begin{aligned} d\tilde{h}_1^{n_j} + P_n[\theta(\|\tilde{\mathbf{v}_1}^{n_j}\| + \|\tilde{\mathbf{v}_2}^{n_j}\| + \|\tilde{h_1}^{n_j}\| + \|\tilde{h_2}^{n_j}\|)\nabla \cdot (\tilde{h}_1^{n_j} \tilde{\mathbf{v}}_1^{n_j}) - \delta_1 \Delta \tilde{h}_1^{n_j}]dt = \\ \sum_{i=1}^{\infty} P_n \sigma_3(\tilde{U}^{n_j}) d\tilde{W}_3^{n_j}, \\ d\tilde{h}_2^{n_j} + P_n[\theta(\|\tilde{\mathbf{v}_1}^j\|^2 + \|\tilde{\mathbf{v}_2}^j\|^2 + \|\tilde{h_1}^j\|^2 + \|\tilde{h_2}^j\|^2)\nabla \cdot (\tilde{h}_2^{n_j} \tilde{\mathbf{v}}_2^{n_j}) - \delta_2 \Delta \tilde{h}_2^{n_j}]dt = \\ \sum_{i=1}^{\infty} P_{n_j} \sigma_4(\tilde{U}^{n_j}) d\tilde{W}_4^{n_j}, \end{aligned} \tag{4.21}$$

$$\tilde{\mathbf{v}}_i^{n_j}(0) = P_{n_j}\tilde{\mathbf{v}}_i(0), \ \tilde{h}_i^{n_j}(0) = P_{n_j}\tilde{h}_i^{n_j}(0) > 0, \ i = 1,2. \tag{4.22}$$

Let $\tilde{\mathcal{S}} = (\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\ge 0}, \tilde{W}_1, \tilde{W}_2, \tilde{W}_3, \tilde{W}_4)$, *where* $\tilde{\mathcal{F}}_t := \bigcap_{s>t} \tilde{\mathcal{F}}_s^0, t \in [0,T]$ *and* $\tilde{\mathcal{F}}_s^0$ *is defined as follows*

$$\mathcal{N} := \{A \in \tilde{\mathcal{F}} | \tilde{\mathbb{P}}(A) = 0\}, \ \bar{\mathcal{F}}_t = \sigma(\tilde{W}_1(s), \tilde{W}_2(s), \tilde{W}_3(s), \tilde{W}_4(s)\tilde{\mathbf{v}}_1(s), \tilde{\mathbf{v}}_2(s), \tilde{h}_1(s), \tilde{h}_2(s), s \le t), \tag{4.23}$$

and $\tilde{\mathcal{F}}_t^0 = \sigma(\bar{\mathcal{F}}_t \cup \mathcal{N})$.

Then $(\tilde{\mathcal{S}}, \tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{h}_1, \tilde{h}_2)$ *is a global martingale solution in the sense of Definition 2.2.*

Proof: The proofs from (1) through (3) are direct consequences of the Skorohod Representation Theorem.

By utilizing the technique as in [1], the proof of (4) follows without any major modification.

From (4), it is easy to see that all the statistical estimates for $\mathbf{v}_i^n$ and h_i^n, $i = 1, 2$ are valid for $\tilde{\mathbf{v}}_i^{n_j}$ and $\tilde{h}_i^{n_j}$. Hence $(\tilde{\mathbf{v}}_i^{n_j})$, $i = 1, 2$ belong to a bounded set of $L^2(\tilde{\Omega}; L^\infty(0, T; V_1) \cap L^2(\tilde{\Omega}; L^2(0, T; D(-\Delta))$, there are $\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2$ in this intersection space such that

$$\tilde{\mathbf{v}}_i^{n_j} \rightharpoonup \tilde{\mathbf{v}}_i \text{ weak-star in } L^2(\tilde{\Omega}; L^\infty(0, T; V_1)), \tag{4.24}$$

$$\text{and } \tilde{\mathbf{v}}_i^{n_j} \rightharpoonup \tilde{\mathbf{v}}_i \text{ weakly in } L^2(\tilde{\Omega}; L^2(0, T; D(-\Delta)). \tag{4.25}$$

Similarly, there exist $\tilde{h}_1, \tilde{h}_2$ in $L^2(\tilde{\Omega}; L^\infty(0, T; V_2) \cap L^2(\tilde{\Omega}; L^2(0, T; D(-\Delta))$ such that

$$\tilde{h}_i^{n_j} \rightharpoonup \tilde{h}_i \text{ weak-star in } L^2(\tilde{\Omega}; L^\infty(0, T; V_2)), \tag{4.26}$$

$$\text{and } \tilde{h}_i^{n_j} \rightharpoonup \tilde{h}_i\text{weakly in } L^2(\tilde{\Omega}; L^2(0, T; D(-\Delta)). \tag{4.27}$$

Our task now is to show that $\tilde{\mathbf{v}}_i, \tilde{h}_i, i = 1, 2$, satisfy the system (4.1).

Due to Lemma 4.1, for $i = 1, 2$, we readily obtain the following estimates:

$$\sup_j \mathbb{E}\left(\int_0^T \|\tilde{\mathbf{v}_i}^{n_j}\|^2 dt\right) \leq \sup_j C\mathbb{E}\left(\sup_{0\leq t\leq T} \|\tilde{\mathbf{v}_i}^{n_j}\|^2\right) < \infty, \tag{4.28}$$

$$\sup_j \mathbb{E}\left(\int_0^T \|\tilde{h_i}^{n_j}\|^2 dt\right) \leq \sup_j C\mathbb{E}\left(\sup_{0\leq t\leq T} \|\tilde{h_i}^{n_j}\|^2\right) < \infty. \tag{4.29}$$

Combining (4.18a), (4.18b),(4.28) and (4.33) we infer by applying the Vitali convergence theorem that

$$\tilde{\mathbf{v}_i}^{n_j} \to \tilde{\mathbf{v}}_i \text{ in } L^2(\tilde{\Omega}; L^2(0, T; V_1)) \tag{4.30}$$

$$\tilde{h_i}^{n_j} \to \tilde{h}_i \text{ in } L^2(\tilde{\Omega}; L^2(0, T; V_2)) \tag{4.31}$$

By thinning the sequence j, if necessary, we conclude that

$$\|\tilde{\mathbf{v}}^{n_j} - \tilde{\mathbf{v}}\| \to 0 \text{ and } \|\tilde{h}^{n_j} - \tilde{h}\| \to 0, \qquad i = 1, 2. \tag{4.32}$$

for almost every $(t, \omega) \in [0, T] \times \tilde{\Omega}$.

Since $\mathbf{v}_i^j \to \mathbf{v}_i$ in $\mathcal{C}([0, T], H_1)$ a.s., we can deduce the existence of set $\Omega_1, i = 1, 2$such that $\tilde{\mathbb{P}}(\Omega_i) = 1$ and on these set, the following convergence hold

$$\lim_{j\to\infty} \langle \mathbf{v}_i^{n_j} - \tilde{\mathbf{v}}_i(0), \phi\rangle_{L^2} = 0, i = 1, 2 \tag{4.33}$$

Similarly, there exist two sets $\Omega_i \subset \tilde{\Omega}, i = 3, 4$ of full measure such that

$$\lim_{j\to\infty} \langle h_i^{n_j}(0) - \tilde{h}_i(0), \psi\rangle_{L^2} = 0, \ i = 3, 4 \tag{4.34}$$

Set $\bar{\Omega} = \tilde{\Omega} \setminus \bigcup_{i=1}^{4} \Omega_i$ and we now show that the convergence of the other terms holds in $L^2(\bar{\Omega} \times [0,T]$. Due to the strong convergence in (4.18a) and the estimates for $\mathbf{v}_i^{n_j}, i = 1,2$ by using the Vitali Convergence Theorem, we find that $\mathbf{v}_i^{n_j}$ converges to $\tilde{\mathbf{v}}_i$ in $L^2(\bar{\Omega}, L^2(0,T,V_1))$ and $h_i^{n_j}$ converges to $\tilde{h}_i$ in $L^2(\bar{\Omega}, L^2(0,T,V_2))$, for $i = 1,2$. Hence, by extracting some subsequences, we deduce that $\mathbf{v}_i^{n_j} \to \tilde{\mathbf{v}}_i$ a.e and $\tilde{\mathbb{P}}$- a.s.in V_1 and $h_i^{n_j} \to \tilde{h}_i$ a.e and $\tilde{\mathbb{P}}$-a.s. in V_2, that is, there exist $\Omega_T^i \subset \bar{\Omega} \times [0,T]$, for $i = 1,2,3,4$ with full measure such that $\forall(\omega,t) \in \Omega_T^1, \Omega_T^2$

$$\lim_{j\to\infty} \|\mathbf{v}_i^{n_j} - \tilde{\mathbf{v}}_i\|_{V_1} = 0 \tag{4.35}$$

Analogously, $\forall(\omega,t) \in \Omega_T^3, \Omega_T^4$

$$\lim_{j\to\infty} \|h_i^{n_j} - \tilde{h}_i\|_{V_2} = 0 \tag{4.36}$$

From which, we imply that

$$\lim_{j\to\infty} \langle \mathbf{v}_i^{n_j}(t) - \tilde{\mathbf{v}}_i(t), \psi\rangle = 0, \text{ and } \lim_{j\to\infty} \langle h_i^{n_j}(t) - \tilde{h}_i(t), \psi\rangle = 0 \tag{4.37}$$

The convergence for the linear terms are straightforward. Indeed, due to (4.35) and (4.36), there exist sets $\Omega_T^i, i = 5,...,15$ of full measure w.r.t $d\tilde{\mathbb{P}} \otimes dt$ and some extracted subsequences still denoted by $\mathbf{v}_i^{n_j}, h^{n_j}$ such that for all $(\omega,t) \in \Omega_T^i, i = 5,...,14$, the following convergences hold as $j \to \infty$,

$$\left|\int_0^t \nu_i \langle \Delta(\tilde{\mathbf{v}}_i^j - \tilde{\mathbf{v}}_i), \psi\rangle ds\right| \preceq \|\psi\| \left(\int_0^T \|\tilde{\mathbf{v}}_i^j - \tilde{\mathbf{v}}_i\|^2 ds\right)^{\frac{1}{2}} \to 0, \tag{4.38a}$$

$$\left|\int_0^t g\langle \nabla(\tilde{h}_i^j - \tilde{h}_i), \psi\rangle ds\right| \preceq \|\psi\| \left(\int_0^T \|\tilde{h}_i^j - \tilde{h}_i\|^2 ds\right)^{\frac{1}{2}} \to 0, \tag{4.38b}$$

$$\left|\int_0^t \langle g\frac{\rho_2}{\rho_1} \nabla(\tilde{h}_2^j - \tilde{h}_2), \mathbf{v}\rangle ds\right| \leq C \sup_{0\leq r\leq T} \|\mathbf{v}(r)\| \left(\int_0^T \|\tilde{h}_i^j - \tilde{h}_i\|^2 ds\right)^{\frac{1}{2}} \to 0, \tag{4.38c}$$

$$\left|\int_0^t \delta_i \langle \Delta(\tilde{h}_i^j - \tilde{h}_i), \psi\rangle ds\right| \preceq \|\psi\| \left(\int_0^T \|\tilde{h}_i^j - \tilde{h}_i\|^2 ds\right)^{\frac{1}{2}} \to 0, \tag{4.38d}$$

$$\left|\int_0^t \langle f\mathbf{k} \times (\tilde{\mathbf{v}}_i^j - \tilde{\mathbf{v}}_i), \psi\rangle ds\right| \preceq \|\psi\| \left(\int_0^T \|\tilde{\mathbf{v}}_i^j - \tilde{\mathbf{v}}_i\|^2 ds\right)^{\frac{1}{2}} \to 0. \tag{4.38e}$$

Furthermore, in virtue of Lemma 4.1, the following estimates can be easily obtained

$$\tilde{\mathbb{E}} \int_0^t \left|\int_0^t \nu_i \langle \Delta \mathbf{v}_i^{n_j}, \psi\rangle ds\right|^2 dt \preceq \|\psi\| \tilde{\mathbb{E}}\Big(\sup_{0\leq t\leq T} \|\mathbf{v}_i^{n_j}\|^2\Big) \leq \kappa. \tag{4.39a}$$

$$\tilde{\mathbb{E}} \int_0^T \left|\int_0^t g\langle \nabla(\tilde{h}_i^{n_j}), \psi\rangle ds\right|^2 dt \preceq \|\psi\| \tilde{\mathbb{E}}\Big(\sup_{0\leq t\leq T} \|h_i^{n_j}\|^2\Big) \leq \kappa. \tag{4.39b}$$

$$\tilde{\mathbb{E}}\int_0^T\left|\int_0^t g\frac{\rho_2}{\rho_1}\langle\nabla\tilde{h}_i^{n_j},\psi\rangle ds\right|^2 dt \preceq \|\psi\|\tilde{\mathbb{E}}\left(\sup_{0\le t\le T}\|h_2^{n_j}\|^2\right)\le\kappa. \tag{4.39c}$$

$$\tilde{\mathbb{E}}\int_0^T\left|\int_0^t \delta_i\langle\Delta\tilde{h}_i^{n_j},\psi\rangle ds\right|^2 dt \preceq \|\psi\|\tilde{\mathbb{E}}\left(\sup_{0\le t\le T}\|h_i^{n_j}\|^2\right)\le\kappa. \tag{4.39d}$$

$$\tilde{\mathbb{E}}\int_0^T\left|\int_0^t \langle f\mathbf{k}\times\mathbf{v}_i^{n_j},\psi\rangle ds\right|^2 dt \preceq \|\psi\|\tilde{\mathbb{E}}\left(\sup_{0\le t\le T}\|\mathbf{v}_i^{n_j}\|^2\right)\le\kappa. \tag{4.39e}$$

Collecting all the above estimates and by Lebesgue Dominated Convergence Theorem, we conclude that

$$\lim_{j\to\infty}\left\|\mu_i\int_0^t\langle\Delta\mathbf{v}_i^{n_j}-\Delta\tilde{\mathbf{v}}_i,\psi\rangle ds\right\|_{L^2(\bar{\Omega}\times[0,T])}=0 \tag{4.40a}$$

$$\lim_{j\to\infty}\left\|g\int_0^t\frac{\rho_2}{\rho_1}\langle\nabla h_2^{n_j}-\nabla\tilde{h}_2,\psi\rangle ds\right\|_{L^2(\bar{\Omega}\times[0,T])}=0 \tag{4.40b}$$

$$\lim_{j\to\infty}\left\|g\int_0^t\langle\nabla h_i^{n_j}-\nabla\tilde{h}_i,\psi\rangle ds\right\|_{L^2(\bar{\Omega}\times[0,T])}=0 \tag{4.40c}$$

$$\lim_{j\to\infty}\left\|\delta_i\int_0^t\langle\Delta h_i^{n_j}-\Delta\tilde{h}_i,\psi\rangle ds\right\|_{L^2(\bar{\Omega}\times[0,T])}=0 \tag{4.40d}$$

$$\lim_{j\to\infty}\left\|\int_0^t\langle f\mathbf{k}\times(\mathbf{v}_i^{n_j}-\tilde{\mathbf{v}}_i),\psi\rangle ds\right\|_{L^2(\bar{\Omega}\times[0,T])}=0 \tag{4.40e}$$

Par extraction of some subsequences, there exist subsets $\Omega_T^i, i=15,..,23$ with full measure such that on these sets, the convergence in (4.40) hold pointwise.
Now for the nonlinear terms, we first denote $\theta(\|\mathbf{v}_1^{n_j}\|^2+\|\mathbf{v}_2^{n_j}\|^2+\|h_1^{n_j}\|^2+\|h_2^{n_j}\|^2)$ by $\theta(\|\mathbf{v}_j\|)$ and $\theta(\|\tilde{\mathbf{v}}_1\|^2+\|\tilde{\mathbf{v}}_2\|^2+\|\tilde{h}_1\|^2+\|\tilde{h}_2\|^2)$ by $\theta(\|\tilde{\mathbf{v}}\|)$ to simplify the exposition.
Next, for $i=1,2$, we have:

$$\begin{aligned}
&\left|\int_0^t\langle P_n[\theta(\|\mathbf{v}_j\|)(\mathbf{v}_i^{n_j}\cdot\nabla)\mathbf{v}_i^j]-\theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_i\cdot\nabla)\tilde{\mathbf{v}}_i,\psi\rangle ds\right|\\
&\le\int_0^t|\langle P_n[\theta(\|\mathbf{v}_j\|)(\mathbf{v}_i^{n_j}\cdot\nabla)\mathbf{v}_i^{n_j}]-P_n[\theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_i\cdot\nabla)\tilde{\mathbf{v}}_i,]\rangle\psi ds|+\int_0^t|\langle Q_n\theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_i\cdot\nabla)\tilde{\mathbf{v}}_i,]\rangle\psi ds|.\\
&\le\int_0^t|\langle\theta(\|\mathbf{v}_j\|)(\mathbf{v}_i^{n_j}\cdot\nabla)\mathbf{v}_i^{n_j}-\theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_i\cdot\nabla)\tilde{\mathbf{v}}_i,\psi\rangle ds|+\int_0^t|\langle Q_n\theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_i\cdot\nabla)\tilde{\mathbf{v}}_i,]\rangle\psi ds|.\\
&=: I_1+I_2.
\end{aligned}$$

Thanks to (4.36) and (4.37), we see for all $(x,\omega,t)\in\mathcal{M}\times\Omega_T^3$ and $(x,\omega,t)\in\mathcal{M}\times\Omega_T^4$

$$\lim_{j\to\infty}\theta(\|\mathbf{v}_j\|)(\mathbf{v}_i^{n_j}\cdot\nabla)\mathbf{v}_i^{n_j}=\theta(\|\tilde{\mathbf{v}}_i\|)(\tilde{\mathbf{v}}_i\cdot\nabla)\tilde{\mathbf{v}}_i. \tag{4.41}$$

Next, because of Lemma 4.1 and Hölder's inequality, we are able to derive the following bounds

$$\int_0^t |\langle \theta(\|\mathbf{v}_j\|)(\mathbf{v}_i^{n_j} \cdot \nabla)\mathbf{v}^{n_j}, \psi\rangle| ds \leq C \int_0^t |\theta(\|\mathbf{v}_j\|)^{\frac{1}{2}} \nabla \cdot \mathbf{v}_i^{n_j}|_{L^\infty} |\theta(\|\mathbf{v}_j\|)^{\frac{1}{2}} \mathbf{v}_i^{n_j}|_{L^2} \|\psi\| \preceq \|\psi\|. \tag{4.42a}$$

and

$$\tilde{\mathbb{E}} \int_0^T \left| \int_0^t |\langle \theta(\|\mathbf{v}_j\|)(\mathbf{v}_i^{n_j} \cdot \nabla)\mathbf{v}^{n_j}, \psi\rangle ds \right|^2 dt \preceq \|\psi\|^2 \tilde{\mathbb{E}} \left(\sup_{0 \leq t \leq T} \|\mathbf{v}_i^{n_j}\|^2 \right) \leq C. \tag{4.42b}$$

The term I_2 is estimated in the same way as I_1. More precisely, we infer from Lemma 4.1 and Hölder's inequality that

$$\begin{aligned} I_2 &:= |\langle Q_{n_j} \theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_i \cdot)\tilde{\mathbf{v}}_i, \psi\rangle| \leq |Q_{n_j} \theta(\|\tilde{\mathbf{v}}\|)^{\frac{1}{2}} \tilde{\mathbf{v}}_i|_{L^2} |\theta(\|\tilde{\mathbf{v}}\|)^{\frac{1}{2}} \nabla \tilde{\mathbf{v}}_i|_{L^\infty}^{\frac{1}{2}} ds \\ &\leq \|\psi\| \frac{1}{\lambda_{n_j}^{\frac{1}{2}}} |Q_{n_j} \theta(\|\tilde{\mathbf{v}}\|)^{\frac{1}{2}} \tilde{\mathbf{v}}_i|_{H^1} |\theta(\|\tilde{\mathbf{v}}\|)^{\frac{1}{2}} \nabla \tilde{\mathbf{v}}_i|_{L^\infty}^{\frac{1}{2}} ds \preceq \|\psi\| \frac{1}{\lambda_{n_j}^{\frac{1}{2}}} \to 0 \text{ as } j \to \infty. \end{aligned} \tag{4.43}$$

We can deduce the similar estimates as in (4.42)

$$\int_0^t |\langle Q_{n_j} \theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_i \cdot)\tilde{\mathbf{v}}_i, \psi\rangle| ds \preceq \|\psi\| \tag{4.44a}$$

$$\tilde{\mathbb{E}} \int_0^T \left| \int_0^t |\langle Q_{n_j} \theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_i \cdot \nabla)\tilde{\mathbf{v}}_i, \psi\rangle ds \right|^2 dt \preceq \tilde{\mathbb{E}} \left(\sup_{0 \leq t \leq T} \|\tilde{\mathbf{v}}_i\|^2 \right) \leq C. \tag{4.44b}$$

From (4.41) to (4.44) and with the Lebesgue Dominated Convergence Theorem, we imply that

$$\lim_{j \to \infty} \left\| \int_0^t \langle P_n[\theta(\|\mathbf{v}_j\|)(\mathbf{v}_i^{n_j} \cdot \nabla)\mathbf{v}_i^j] - \theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_i \cdot \nabla)\tilde{\mathbf{v}}_i, \psi\rangle ds \right\|_{L^2(\bar{\Omega} \times [0,T])} = 0 \tag{4.45}$$

By extracting subsequences, we infer that there exist Ω_T^{24} and Ω_T^{25} such that for all $(\omega, t) \in \Omega_T^{24}$, we have

$$\lim_{j \to \infty} \int_0^t \langle P_n[\theta(\|\mathbf{v}_j\|)(\mathbf{v}_1^{n_j} \cdot \nabla)\mathbf{v}_1^j] - \theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_1 \cdot \nabla)\tilde{\mathbf{v}}_1, \psi\rangle ds = 0 \tag{4.46}$$

And for all $(\omega, t) \in \Omega_T^{25}$, the below convergence holds

$$\lim_{j \to \infty} \int_0^t \langle P_n[\theta(\|\mathbf{v}_j\|)(\mathbf{v}_2^{n_j} \cdot \nabla)\mathbf{v}_2^j] - \theta(\|\tilde{\mathbf{v}}\|)(\tilde{\mathbf{v}}_2 \cdot \nabla)\tilde{\mathbf{v}}_2, \psi\rangle ds = 0 \tag{4.47}$$

It is not difficult to deduce the following convergence

$$\lim_{j \to \infty} \left\| \int_0^t \langle P_n[\theta(\|\tilde{\mathbf{u}}^j\|)\nabla(\tilde{h}_i^j \tilde{\mathbf{v}}_i^{n_j})] - \theta(\|\tilde{\mathbf{u}}\|)\nabla(\tilde{h}_i \tilde{\mathbf{v}}_i), \psi\rangle ds \right\|_{L^2(\bar{\Omega} \times [0,T])} = 0. \tag{4.48}$$

We can extract two sets Ω_T^{26} and Ω_T^{27} and extracted subsequences still denoted by $\mathbf{v}_1^{n_j}, h_1^{n_j}$ and $\mathbf{v}_2^{n_j}, h_2^{n_j}$ such that on these sets, the following convergence hold respectively

$$\lim_{j\to\infty}\int_0^t \langle P_n[\theta(\|\tilde{\mathbf{u}}^j\|)\nabla(\tilde{h}_i^j\tilde{\mathbf{v}}_i^{n_j})] - \theta(\|\tilde{\mathbf{u}}\|)\nabla(\tilde{h}_i\tilde{\mathbf{v}}_i), \psi\rangle ds = 0. \tag{4.49a}$$

We address the stochastic term by using Lemma 7.5. We first simplify the expositions by introducing $U^{n_j} = (\tilde{\mathbf{v}}_1^{n_j}, \tilde{\mathbf{v}}_2^{n_j}, \tilde{h}_1^{n_j}, \tilde{h}_2^{n_j})$ and $U = (\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{h}_1, \tilde{h}_2)$.
From (4.18c), we know that $\tilde{W}_i^{n_j} \to \tilde{W}_i, \forall i = 1,2,3,4$ in probability in $\mathcal{C}(0,T;\mathfrak{U}_0)$ and thus it suffices to show that $P_{n_j}\sigma_i(U^{n_j}) \to \sigma_i(U)$ in $L^2(0,T;L_2(\mathfrak{U},V))$ except on a set of measure zero of $\bar{\Omega}$ and hence in probability. We utilize the Poincaré inequality, the hypotheses (2.4), (2.11), (4.18a) and (4.18b), we estimate:

$$\begin{aligned}\|P_{n_j}\sigma_i(U^{n_j}) - \sigma_i(U)\|^2_{L_2(\mathfrak{U},V)} &\le \|P_{n_j}\sigma(U^{n_j}) - P_{n_j}\sigma(U)\|^2_{L_2(\mathfrak{U},V)} + \|Q_{n_j}\sigma_i(U)\|^2_{L_2(\mathfrak{U},V)}\\ &\preceq \|U^{n_j} - U\|_V^2 + \frac{1}{\lambda_{n_j}}(1+\|U\|^2) \longrightarrow 0 \text{ as } n_j\to\infty. \end{aligned}\tag{4.50}$$

Thus, we conclude that $\|P_j\sigma_1(U^j) - \sigma_1(U)\|_{L_2(\mathfrak{U},V_1)} \to 0, \forall(\omega,t)\in\Omega_T^0,$ as $j\to\infty$.
On the other hand, noting that due to (2.11), (2.12) along with (4.1), we find

$$\tilde{\mathbb{E}}\bigg(\int_0^T P_j\|\sigma_1(\tilde{\mathbf{v}}^j)\|^2_{L_2(\mathfrak{U},V_1)}dt\bigg) \le C\mathbb{E}\bigg(\int_0^T (1+\|\tilde{\mathbf{v}}^j\|^2\bigg) \le C. \tag{4.51}$$

With (4.50), (4.51) in hand and Vitali Convergence Theorem, we infer that

$$P_{n_j}\sigma_i(\tilde{\mathbf{u}}^{n_j}) \to \sigma_i(\tilde{\mathbf{u}}) \text{ in } L^2(\bar{\Omega}; L^2([0,T], L_2(\mathfrak{U},V)). \tag{4.52}$$

This implies that the following convergence holds almost surely and in particular, it holds in probability:

$$P_{n_j}\sigma_i(\tilde{\mathbf{u}}^{n_j}) \to \sigma_i(\tilde{\mathbf{u}}) \text{ in } L^2([0,T], L_2(\mathfrak{U},V)). \tag{4.53}$$

Combining with (4.18c), Lemma 7.5 is applied and we infer that

$$\int_0^t P_{n_j}\sigma_i(\tilde{\mathbf{u}}^{n_j})d\tilde{W}^{n_j} \to \int_0^t \sigma_i(\tilde{\mathbf{u}})d\tilde{W} \text{ in } L^2([0,T],V). \tag{4.54}$$

By making use of the Burkholder- Davis-Gundy inequality and the uniform bounds in Lemma (4.1), we can easily obtain the following estimate:

$$\begin{aligned}&\tilde{\mathbb{E}}\bigg(\bigg\|\int_0^t P_{n_j}\sigma_i(\tilde{\mathbf{u}}^{n_j})d\tilde{W}^{n_j}\bigg\|_V^2\bigg)\\ &\qquad\le C\tilde{\mathbb{E}}\bigg(\int_0^T \|P_{n_j}\sigma_i(\tilde{\mathbf{u}}^{n_j})\|^2_{L_2(\mathfrak{U},V)}dt\bigg) \le C\tilde{\mathbb{E}}\bigg(\int_0^T \|\sigma_i(\tilde{\mathbf{u}}^{n_j})\|^2_{L_2(\mathfrak{U},V)}dt\bigg)\\ &\qquad\le C\tilde{\mathbb{E}}\bigg(\int_0^T (1+\|\tilde{\mathbf{u}}^{n_j}\|^2\bigg)dt \le C. \end{aligned}\tag{4.55}$$

By utilizing the Lebesgue Dominated Convergence Theorem one more time, we obtain that the convergence in (4.52) holds further in $L^2(\bar{\Omega}; L^2([0,T], L_2(\mathfrak{U}, V))$. Hence, by the stochastic Fubini theorem, we can extract a subsequence and we find a set of full measure $\Omega_T^6 \subset \bar{\Omega} \times [0,T]$ such that the convergence of the stochastic term holds for all $(\omega, t) \in \Omega_T^6$.

4.5 Global pathwise uniqueness

Now we prove that the global martingale solution for the modified system is pathwise unique.

Proposition 4.1. *Suppose that* $(\mathcal{S}, \hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{h}_1, \hat{h}_2)$ *and* $(\mathcal{S}, \tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{h}_1, \tilde{h}_2)$ *are two global martingale solutions of* (4.1) *relative to the same stochastic basis* $\mathcal{S} := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}, W_1, W_2, W_3, W_4)$. *Pathwise uniqueness means that if we define* $\Omega_0 := \{\hat{\mathbf{v}}_1(0) = \tilde{\mathbf{v}}_1(0), \hat{\mathbf{v}}_2(0) = \tilde{\mathbf{v}}_2(0), \hat{h}_1(0) = \tilde{h}_1(0), \hat{h}_2(0) = \tilde{h}_2(0)\}$*, then* $(\hat{\mathbf{v}}_1, \hat{\mathbf{v}}_2, \hat{h}_1, \hat{h}_2)$ *and* $(\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2, \tilde{h}_1, \tilde{h}_2))$ *are indistinguishable on* Ω_0 *in the sense that*

$$\mathbb{P}(\mathbb{1}_{\Omega_0}(\hat{\mathbf{v}}_i(t) - \tilde{\mathbf{v}}_i(t)) = 0,\ \forall t \geq 0) = 1,\ \textit{for } i = 1, 2$$

$$\mathbb{P}(\mathbb{1}_{\Omega_0}(\hat{h}_i(t) - \tilde{h}_i(t)) = 0,\ \forall t \geq 0) = 1.\ \textit{for } i = 1, 2$$

Proof. For $i = 1, 2$ we will let $\mathbf{v}_i = \hat{\mathbf{v}}_i - \tilde{\mathbf{v}}_i, h_i = \hat{h}_i - \tilde{h}_i, \bar{\mathbf{v}}_i = \mathbb{1}_{\Omega_0}\mathbf{v}_i$ and $\bar{h}_i = \mathbb{1}_{\Omega_0} h_i$.

We will also need the following stopping times

$$\begin{aligned}
\tau^m :&= \inf_{t\geq 0}\Bigg\{ \int_0^t |\Delta\tilde{\mathbf{v}}_1|^2 + 3|\Delta\hat{\mathbf{v}}_1|^2 + 2|\nabla\hat{\mathbf{v}}_1|^6 + |\nabla\hat{\mathbf{v}}_1|^2 + 2|\nabla\tilde{\mathbf{v}}_1|^2|\Delta\tilde{\mathbf{v}}_2|^2 + \\
&3|\Delta\hat{\mathbf{v}}_2|^2 + 2|\nabla\hat{\mathbf{v}}_2|^6 + |\nabla\hat{\mathbf{v}}_2|^2 + 2|\nabla\tilde{\mathbf{v}}_2|^2 + 3\left|\Delta\hat{h}_1\right|^2 + 6\left|\nabla\hat{h}_1\right|^4 + 6|\nabla\hat{\mathbf{v}}_1|^4 + 4|\nabla\tilde{\mathbf{v}}_1|^8 + \\
&4\left|\nabla\hat{h}_1\right|^4 + 3\left|\Delta\hat{h}_2\right|^2 + 6\left|\nabla\hat{h}_2\right|^4 + 6|\nabla\hat{\mathbf{v}}_2|^4 + 4|\nabla\tilde{\mathbf{v}}_2|^8 + 4\left|\nabla\hat{h}_2\right|^4 \geq m\Bigg\}.
\end{aligned} \tag{4.56}$$

To simplify our notation, we also set

$$\begin{aligned}
&\|\tilde{\mathbf{v}}_1\|^2 + \|\tilde{\mathbf{v}}_2\|^2 + \|\tilde{h}_1\|^2 + \|\tilde{h}_2\|^2 = \|\tilde{U}\|, \\
&\|\hat{\mathbf{v}}_1\|^2 + \|\hat{\mathbf{v}}_2\|^2 + \|\hat{h}_1\|^2 + \|\hat{h}_2\|^2 = \|\hat{U}\|, \\
&\|\bar{\mathbf{v}}_1\|^2 + \|\bar{\mathbf{v}}_2\|^2 + \|\bar{h}_1\|^2 + \|\bar{h}_2\|^2 = \|\bar{U}\|.
\end{aligned} \tag{4.57}$$

Substituting $\mathbf{v}_1$ and $\mathbf{v}_2$ into (4.1) and taking the difference between these equations, we arrive at the following equations:

$$\begin{aligned} &d\mathbf{v}_1 - \nu_1 \Delta \mathbf{v}_1 dt + f\mathbf{k} \times \mathbf{v}_1 dt + g\nabla h_1 dt - g\frac{\rho_2}{\rho_1}\nabla h_2 dt + \theta(\|\hat{U}\|)(\hat{\mathbf{v}}_1 \cdot \nabla)\hat{\mathbf{v}}_1 dt \\ &- \theta(\|\tilde{U}\|)(\tilde{\mathbf{v}}_1 \cdot \nabla)\tilde{\mathbf{v}}_1 dt = \sum_{k=1}^{\infty} \sigma_1(\hat{U}) e_k dW_1^k - \sum_{k=1}^{\infty} \sigma_1(\tilde{U}) e_k dW_1^k, \\ &\mathbf{v}_1(0) = \hat{\mathbf{v}}_1(0) - \tilde{\mathbf{v}}_1(0) \end{aligned} \tag{4.58}$$

$$\begin{aligned} &d\mathbf{v}_2 - \nu_1 \Delta \mathbf{v}_2 dt + f\mathbf{k} \times \mathbf{v}_2 dt + g\nabla h_2 dt - g\nabla h_1 dt + \theta(\|\hat{U}\|)(\hat{\mathbf{v}}_2 \cdot \nabla)\hat{\mathbf{v}}_2 dt - \theta(\|\tilde{U}\|)(\tilde{\mathbf{v}}_2 \cdot \nabla)\tilde{\mathbf{v}}_2 dt \\ &= \sum_{k=1}^{\infty} \sigma_2(\hat{U}) e_k dW_2^k - \sum_{k=1}^{\infty} \sigma_2(\tilde{U}) e_k dW_2^k, \\ &\mathbf{v}_2(0) = \hat{\mathbf{v}}_2(0) - \tilde{\mathbf{v}}_2(0) \end{aligned} \tag{4.59}$$

$$\begin{aligned} &dh_1 - \delta_1 \Delta h_1 dt = \theta(\hat{U})\nabla \cdot (\hat{h}_1 \hat{\mathbf{v}}_1) dt - \theta(\tilde{U})\nabla \cdot (\tilde{h}_1 \tilde{\mathbf{v}}_1) dt + \sum_{k=1}^{\infty} \sigma_3(\tilde{U}) e_k dW_2 - \sum_{k=1}^{\infty} \sigma_3(\hat{U}) e_k dW_3. \\ &h_1(0) = \hat{h}_1(0) - \tilde{h}_1(0) \end{aligned} \tag{4.60}$$

$$\begin{aligned} &dh_2 - \delta_1 \Delta h_2 dt = \theta(\hat{U})\nabla \cdot (\hat{h}_2 \hat{\mathbf{v}}_2) dt - \theta(\tilde{U})\nabla \cdot (\tilde{h}_2 \tilde{\mathbf{v}}_2) dt + \sum_{k=1}^{\infty} \sigma_4(\tilde{U}) e_k dW_4^k - \sum_{k=1}^{\infty} \sigma_4(\hat{U}) e_k dW_4^k. \\ &h_2(0) = \hat{h}_2(0) - \tilde{h}_2(0) \end{aligned} \tag{4.61}$$

Applying the Itô formula to the map $\mathbf{u} \mapsto |\nabla \mathbf{u}|^2$ in (4.58) and (4.61) and adding the corresponding relations together yield

$$\begin{aligned} &d\|\mathbf{v}_1\|^2 + 2\nu_1 |\Delta \mathbf{v}_1|^2 dt + d\|\mathbf{v}_2\|^2 + 2\nu_2 |\Delta \mathbf{v}_2|^2 dt + d\|h_1\|^2 + 2\delta_1 |\Delta h_1|^2 dt + d\|h_2\|^2 + \\ &2\delta_2 |\Delta h_2|^2 dt = -2g\langle \nabla h_1, \Delta \mathbf{v}_1\rangle dt - 2g\frac{\rho_2}{\rho_1}\langle \nabla h_2, \Delta \mathbf{v}_1\rangle - 2g\langle \nabla h_2, \Delta \mathbf{v}_2\rangle dt - \\ &2g\langle \nabla h_1, \Delta \mathbf{v}_2\rangle dt - 2\langle \theta(\hat{U})(\hat{\mathbf{v}}_1 \cdot \nabla)\hat{\mathbf{v}}_1, \Delta \mathbf{v}_1\rangle + 2\langle \theta(\tilde{U})(\tilde{\mathbf{v}}_2 \cdot \nabla)\tilde{\mathbf{v}}_2, \Delta \mathbf{v}_2\rangle + 2\langle \theta(\tilde{U})(\tilde{\mathbf{v}}_1 \cdot \nabla)\tilde{\mathbf{v}}_1, \Delta \mathbf{v}_1\rangle \\ &- 2\langle \theta(\hat{U})(\hat{\mathbf{v}}_2 \cdot \nabla)\hat{\mathbf{v}}_2, \Delta \mathbf{v}_2\rangle - 2\langle \theta(\tilde{U})\nabla \cdot (\tilde{h}_1 \tilde{\mathbf{v}}_1), \Delta h_1\rangle dt + 2\langle \theta(\hat{U})\nabla \cdot (\hat{h}_2 \hat{\mathbf{v}}_2), \Delta h_1\rangle dt \\ &+ 2\langle f\mathbf{k} \times \mathbf{v}_1, \Delta \mathbf{v}_1\rangle dt + 2\langle f\mathbf{k} \times \mathbf{v}_2, \Delta \mathbf{v}_2\rangle dt \\ &+ \sum_{k=1}^{\infty} ||\sigma_1(\hat{U}) - \sigma_1(\tilde{U}) e_k||^2 dt + \sum_{k=1}^{\infty} ||\sigma_2(\hat{U}) - \sigma_2(\tilde{U}) e_k||^2 dt \\ &+ \sum_{k=1}^{\infty} ||\sigma_3(U) - \sigma_3(\tilde{U}) e_k||^2 dt + \sum_{k=1}^{\infty} ||\sigma_4(\hat{U}) - \sigma_4(\tilde{U}) e_k||^2 dt \\ &+ 2\sum_{k=1}^{\infty} \langle \sigma_1(\hat{U}) - \sigma_1(\tilde{U}), \Delta \mathbf{v}_1\rangle dW_1^k + 2\sum_{k=1}^{\infty} \langle \sigma_2(\hat{U}) - \sigma_2(\tilde{U})] dW_2, \Delta \mathbf{v}_2\rangle dW_2^k + \\ &2\sum_{k=1}^{\infty} \langle \sigma_3(\hat{U}) e_k - \sigma_3(\tilde{U}) e_k, \Delta h_1\rangle dW_3^k + 2\sum_{k=1}^{\infty} \langle \sigma_4(\hat{U}) e_k - \sigma_4(\tilde{U}) e_k, \Delta h_2\rangle dW_4^k. \end{aligned} \tag{4.62}$$

Integrating (4.62) in time over $[0, t \wedge \tau^m], 0 \leq t \leq T$, multiplying by $\mathbb{1}_{\Omega_0}$ and finally taking the expected value of the supremum in $t \in [0, T]$ yield

$$\mathbb{E}\mathbb{1}_{\Omega_0}\left(\sup_{s\in[0,t\wedge\tau^m]}\|\bar{\mathbf{v}}\|^2 + 2\nu\int_0^{t\wedge\tau^m}|\Delta\bar{\mathbf{v}}|^2\,ds + \sup_{s\in[0,t\wedge\tau^m]}\|\bar{h}\|^2 + 2\delta\int_0^{t\wedge\tau^m}\left|\Delta\bar{h}\right|^2 ds\right)$$
$$\leq 8\mathbb{E}\mathbb{1}_{\Omega_0}\left(\|\bar{\mathbf{v}}(0)\|^2 + \|\bar{h}(0)\|^2\right) + \sum_{i=1}^{18}\tilde{J}_i, \tag{4.63}$$

where

$$\sup_{t\in[0,t\wedge\tau^m]}\|\bar{\mathbf{v}}\|^2 := \sup_{t\in[0,t\wedge\tau^m]}\|\bar{\mathbf{v}_1}\|^2 + \sup_{t\in[0,t\wedge\tau^m]}\|\bar{\mathbf{v}_2}\|^2.$$
$$\sup_{t\in[0,\tau\wedge m]}\|\bar{h}\|^2 := \sup_{t\in[0,t\wedge\tau^m]}\|\bar{h_1}\|^2 + \sup_{t\in[0,t\wedge\tau^m]}\|\bar{h_2}\|^2. \tag{4.64}$$
$$2\nu\int_0^{s\wedge\tau_m}|\Delta\bar{\mathbf{v}}|^2\,dt := 2\nu_1\int_0^{t\wedge\tau^m}|\Delta\bar{\mathbf{v}}_1|^2\,ds + 2\nu_2\int_0^{t\wedge\tau^m}|\Delta\bar{\mathbf{v}}_2|^2\,ds \tag{4.65}$$
$$2\delta\int_0^{s\wedge\tau^m}\left|\Delta\bar{h}\right|^2 dt := 2\delta_1\int_0^{t\wedge\tau^m}\left|\Delta\bar{h}_1\right|^2 ds + 2\delta_2\int_0^{t\wedge\tau^m}\left|\Delta\bar{h}_2\right|^2 ds \tag{4.66}$$

For $\alpha = 1, 2, 3, 4, i = 1, 2, j = 1, 2$, by simply using the Cauchy-Schwarz inequality, the following estimates hold:

$$\tilde{J}_\alpha := \kappa_0\mathbb{E}\mathbb{1}_{\Omega_0}\left(\int_0^{t\wedge\tau^m}|\langle\nabla\bar{h}_i, \Delta\bar{\mathbf{v}}_j\rangle|ds\right) \leq C\mathbb{E}\mathbb{1}_{\Omega_0}\left(\int_0^{t\wedge\tau^m}\|\bar{h}_i\|^2 ds\right) + \frac{\nu_j}{4}\mathbb{E}\mathbb{1}_{\Omega_0}\left(\int_0^{t\wedge\tau^m}|\Delta\bar{\mathbf{v}}_j|^2\,ds\right), \tag{4.67}$$

where κ_0 equals either $8g$ or $8g\frac{\rho_2}{\rho_1}$.

For $\beta = 5, 6; i = 1, 2$, we obtain:

$$\tilde{J}_\beta := 8g\mathbb{E}\mathbb{1}_{\Omega_0}\left(\int_0^{t\wedge\tau^m}|\langle f\mathbf{k}\times\mathbf{v}_i, \Delta\mathbf{v}_i\rangle|ds\right) \leq C\mathbb{E}\mathbb{1}_{\Omega_0}\left(\int_0^{t\wedge\tau^m}\|\bar{\mathbf{v}}_i\|^2 ds\right) + \frac{\nu_i}{4}\mathbb{E}\mathbb{1}_{\Omega_0}\left(\int_0^{t\wedge\tau^m}|\Delta\bar{\mathbf{v}}_i|^2\,ds\right). \tag{4.68}$$

Next, we estimate

$$\tilde{J}_7 := 8\mathbb{E}\mathbb{1}_{\Omega_0}\left(\int_0^{t\wedge\tau^m}|\langle\theta(\|\hat{U}\|)(\hat{\mathbf{v}}_1\cdot\hat{\mathbf{v}}_1)\hat{\mathbf{v}}_1 - \theta(||\tilde{U}||)(\tilde{\mathbf{v}}_1\cdot\tilde{\mathbf{v}}_1)\tilde{\mathbf{v}}_1, \Delta\bar{\mathbf{v}_1}\rangle|ds\right)$$
$$\leq 8\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m}|\langle(\theta(\|\hat{U}\|) - \theta(\|\tilde{U}\|))(\hat{\mathbf{v}}_1\cdot\nabla)\hat{\mathbf{v}}_1, \Delta\bar{\mathbf{v}_1}\rangle|ds +$$
$$8\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m}|\langle(\hat{\mathbf{v}}_1\cdot\nabla)\hat{\mathbf{v}}_1 - (\tilde{\mathbf{v}}_1\cdot\nabla)\tilde{\mathbf{v}}_1, \Delta\bar{\mathbf{v}_1}\rangle|ds := \tilde{J}_7^1 + \tilde{J}_7^2.$$

The estimate for the term $\tilde{J}_7^1$ is proceeded by using Holdër's inequality and the fact that the cut-off function defined in $\big(4\big)$ is Lipschitz

$$\tilde{J}_7^1 \leq C\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m}(\|\bar{\mathbf{v}}_1\|^2+\|\bar{\mathbf{v}}_2\|^2+\|\bar{h}_1\|^2+\|\bar{h}_2\|^2)\,|\mathbf{v}_1|_{L^\infty}\,|\nabla\hat{\mathbf{v}}_1|\,|\Delta\bar{\mathbf{v}}_1|\,dt\bigg) \tag{4.69}$$

$$\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m}(\|U\|)\,|\nabla\mathbf{v}_1|^{\frac{1}{2}}\,|\nabla\mathbf{v}_1|^{\frac{1}{2}}\,|\Delta\hat{\mathbf{v}}_1|\,dt\bigg)$$

$$+\,C\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m}(\|U\|)\,|\nabla\mathbf{v}_1|^{\frac{1}{2}}\,|\nabla\mathbf{v}_1|^{\frac{1}{2}}\,|\Delta\hat{\mathbf{v}}_1|\,dt\bigg)$$

$$\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m}(\|U\|)(|\Delta\tilde{\mathbf{v}}_1|^2+2\,|\Delta\hat{\mathbf{v}}_1|^2+2\,|\nabla\hat{\mathbf{v}}_1|^6)dt \tag{4.70}$$

The last line follows thanks to Poincaré's inequality and the Young inequality.

The estimate for $\tilde{J}_7^2$ is derived by utilizing Holdër's inequality, Ladyzhenskya's inequality in space dimension two and the Young inequality

$$\tilde{J}_7^2 \leq C\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m}|\langle(\hat{\mathbf{v}}_1\cdot\nabla)\hat{\mathbf{v}}_1-(\tilde{\mathbf{v}}_1\cdot\nabla)\tilde{\mathbf{v}}_1,\Delta\bar{\mathbf{v}}_1\rangle|\,dt\bigg) \tag{4.71}$$

$$\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m}|\langle[(\hat{\mathbf{v}}_1-\tilde{\mathbf{v}}_1)\cdot\nabla]\hat{\mathbf{v}}_1-(\tilde{\mathbf{v}}_1\cdot\nabla)(\tilde{\mathbf{v}}_1-\hat{\mathbf{v}}_1),\Delta\bar{\mathbf{v}}_1\rangle|\,dt\bigg)$$

$$\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m}|\langle[\bar{\mathbf{v}}_1\cdot\nabla]\hat{\mathbf{v}}_1-(\tilde{\mathbf{v}}_1\cdot\nabla)\bar{\mathbf{v}}_1,\Delta\bar{\mathbf{v}}_1\rangle|\,ds\bigg)$$

$$\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m}(|\bar{\mathbf{v}}_1|_{L^\infty}\,|\nabla\hat{\mathbf{v}}_1|_{L^4}+|\tilde{\mathbf{v}}_1|_{L^4}\,|\nabla\bar{\mathbf{v}}_1|_{L^4})\,|\Delta\bar{\mathbf{v}}_1|_{L^2}\,dt\bigg)$$

$$\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m}\Big(|\bar{\mathbf{v}}_1|^{\frac{1}{2}}\,|\nabla\bar{\mathbf{v}}_1|^{\frac{1}{2}}\,|\hat{\mathbf{v}}_1|^{\frac{1}{2}}\,|\nabla\hat{\mathbf{v}}_1|^{\frac{1}{2}}+|\tilde{\mathbf{v}}_1|^{\frac{1}{2}}\,|\nabla\tilde{\mathbf{v}}_1|^{\frac{1}{2}}\,|\bar{\mathbf{v}}_1|^{\frac{1}{2}}\,|\nabla\bar{\mathbf{v}}_1|^{\frac{1}{2}}\Big)\,|\Delta\bar{\mathbf{v}}_1|_{L^2}\,dt\bigg)$$

$$\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m}\Big(|\bar{\mathbf{v}}_1|^{\frac{1}{2}}\,|\nabla\bar{\mathbf{v}}_1|^{\frac{1}{2}}\,|\hat{\mathbf{v}}_1|^{\frac{1}{2}}\,|\nabla\hat{\mathbf{v}}_1|^{\frac{1}{2}}\,|\Delta\bar{\mathbf{v}}_1|_{L^2}+|\tilde{\mathbf{v}}_1|^{\frac{1}{2}}\,|\nabla\tilde{\mathbf{v}}_1|^{\frac{1}{2}}\,|\bar{\mathbf{v}}_1|^{\frac{1}{2}}\,|\nabla\bar{\mathbf{v}}_1|^{\frac{1}{2}}\,|\Delta\bar{\mathbf{v}}_1|_{L^2}^{\frac{3}{2}}\,dt\Big)$$

By applying the Young inequality to the first term with $p=2, q=2$ and the second term, with $p=\dfrac{4}{3}, q=4$ to the second term, we obtain:

$$\tilde{J}_7^2 \leq \frac{\nu_1}{8}\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m}|\Delta\bar{\mathbf{v}}_1|^2\,dt+C\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m}|\nabla\bar{\mathbf{v}}_1|^2\,|\nabla\hat{\mathbf{v}}_1|\,|\Delta\hat{\mathbf{v}}_1|\,dt+|\nabla\bar{\mathbf{v}}_1|^2\,(|\tilde{\mathbf{v}}_1|^2\,|\nabla\tilde{\mathbf{v}}_1|^2)dt$$

$$=\frac{\nu_1}{8}\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m}|\Delta\bar{\mathbf{v}}_1|^2\,dt+C\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m}\|\bar{\mathbf{v}}_1\|^2(|\nabla\hat{\mathbf{v}}_1|\,|\Delta\hat{\mathbf{v}}_1|+|\tilde{\mathbf{v}}_1|^2\,|\nabla\tilde{\mathbf{v}}_1|^2)dt.$$

Combining the estimates (4.69) and (4.71), we see that

$$\tilde{J}_7 := 8C\mathbb{E}\mathbb{1}_{\Omega_0}\left(\int_0^{t\wedge\tau^m} |\langle\theta(\|\hat{U}\|)(\hat{\mathbf{v}}_1\cdot\mathbf{v}_1)\hat{\mathbf{v}}_1 - \theta(\|\tilde{U}\|)(\tilde{\mathbf{v}}_1\cdot\mathbf{v}_1)\tilde{\mathbf{v}}_1, \Delta\bar{\mathbf{v}}_1\rangle|dt\right)$$
$$\leq 8C\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m} |\langle(\theta(\|\hat{U}\|)-\theta(\|\tilde{U}\|))(\hat{\mathbf{v}}_1\cdot\nabla)\hat{\mathbf{v}}_1, \Delta\bar{\mathbf{v}}_1\rangle|dt$$

$$+ 8C\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m} |\langle(\hat{\mathbf{v}}_1\cdot\hat{\nabla})\hat{\mathbf{v}}_1 - (\tilde{\mathbf{v}}_1\cdot\tilde{\nabla})\tilde{\mathbf{v}}_1, \Delta\bar{\mathbf{v}}_1\rangle|dt \tag{4.72}$$
$$C\mathbb{E}\mathbb{1}_{\Omega_0}\left(\int_0^{t\wedge\tau^m} (\|\bar{\mathbf{v}}_1\|^2 + \|\bar{\mathbf{v}}_2\|^2 + \|\bar{h}_1\|^2 + \|\bar{h}_2\|^2)(|\Delta\tilde{\mathbf{v}}_1|^2 + 2|\Delta\hat{\mathbf{v}}_1|^2 + 2|\nabla\hat{\mathbf{v}}_1|^6)dt\right.$$
$$+ \frac{\nu_1}{4}\mathbb{E}\int_0^{t\wedge\tau^m} |\Delta\bar{\mathbf{v}}_1|^2\,dt + C\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m} \|\bar{\mathbf{v}}_1\|^2(|\nabla\hat{\mathbf{v}}_1|^2 + |\Delta\hat{\mathbf{v}}_1|^2\,dt + |\tilde{\mathbf{v}}_1|^2|\nabla\tilde{\mathbf{v}}_1|^2)dt.$$

Similarly,

$$\tilde{J}_8 := 8\mathbb{E}\mathbb{1}_{\Omega_0}\left(\int_0^{t\wedge\tau^m} |\langle\theta(\hat{U})(\hat{\mathbf{v}}_2\cdot\nabla)\hat{\mathbf{v}}_2 - \theta(\tilde{U})(\tilde{\mathbf{v}}_2\cdot\nabla)\tilde{\mathbf{v}}_2, \Delta\bar{\mathbf{v}}_2\rangle|ds\right) \tag{4.73}$$
$$C\mathbb{E}\mathbb{1}_{\Omega_0}\left(\int_0^{t\wedge\tau^m} (\|\bar{\mathbf{v}}_1\|^2 + \|\bar{\mathbf{v}}_2\|^2 + \|\bar{h}_1\|^2 + \|\bar{h}_2\|^2)(|\Delta\tilde{\mathbf{v}}_2|^2 + 2|\Delta\hat{\mathbf{v}}_2|^2 + 2|\nabla\hat{\mathbf{v}}_2|^6)dt\right.$$
$$+ \frac{\nu_2}{4}\mathbb{E}\int_0^{t\wedge\tau^m} |\Delta\bar{\mathbf{v}}_2|^2\,dt + C\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m} \|\bar{\mathbf{v}}_2\|^2(|\nabla\hat{\mathbf{v}}_2|^2 + |\Delta\hat{\mathbf{v}}_2|^2\,dt + |\tilde{\mathbf{v}}_2|^2|\nabla\tilde{\mathbf{v}}_2|^2)dt. \tag{4.74}$$

In the same manner, the next four deterministic terms are controlled as follows:

$$\tilde{J}_9 := 8\mathbb{E}\mathbb{1}_{\Omega_0}\left(\int_0^{t\wedge\tau^m} \left|\langle\theta(\|\hat{U}\|)\nabla\cdot(\hat{h}_1\hat{\mathbf{v}}_1) - \theta(\|\tilde{U}\|)\nabla\cdot(\tilde{h}_1\tilde{\mathbf{v}}_1), \Delta\bar{h}_1\rangle\right| ds\right)$$
$$\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\left(\int_0^{t\wedge\tau^m} |\theta(\|\hat{U}\|)-\theta(\|\tilde{U}\|)|\langle\nabla\cdot(\hat{h}_1\hat{\mathbf{v}}_1), \Delta\bar{h}_1\rangle|ds+\right.$$
$$C\mathbb{E}\mathbb{1}_{\Omega_0}\left(\int_0^{t\wedge\tau^m} \langle\nabla\cdot(\hat{h}_1\hat{\mathbf{v}}_1) - \nabla\cdot(\tilde{h}_1\tilde{\mathbf{v}}_1), \Delta\bar{h}_1\rangle|ds := \tilde{J}_9^1 + \tilde{J}_9^2.\right. \tag{4.75}$$

$\tilde{J}_9^1$ is treated by making use of the Lipschitzian property of the cut-off function (4) and Holdër's inequality

$$\tilde{J}_9^1 := \mathbb{E}\mathbb{1}_{\Omega_0}\left(\int_0^{t\wedge\tau^m} |\theta(\|\hat{U}\|)-\theta(\|\tilde{U}\|)|\langle\nabla\cdot(\hat{h}_1\hat{\mathbf{v}}_1), \Delta\bar{h}_1\rangle|ds\right.$$
$$\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\left(\int_0^{t\wedge\tau^m} (\|\bar{\mathbf{v}}_1\|^2 + \|\bar{\mathbf{v}}_2\|^2 + \|\bar{h}_1\|^2 + \|\bar{h}_2\|^2)(\left|\nabla\hat{h}_1\right|_{L^4}|\hat{\mathbf{v}}_1|_{L^\infty} + \|\nabla\hat{\mathbf{v}}_1\||\hat{\mathbf{v}}_1|_{L^\infty})\left|\Delta\bar{h}_1\right| dt\right.$$

By using Ladyzhenskya's inequality, Agmon's inequality and the Sobolev embedding $H_0^1 \hookrightarrow L^4$ in space dimension two, we obtain:

$$\tilde{J}_9^1 \leq C\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m} \|\bar{U}\|^2(\left|\hat{h}_1\right|^{\frac{1}{2}}(\left|\hat{h}_1\right|^{\frac{1}{2}}|\Delta h_1|^{\frac{1}{2}}|\nabla\hat{\mathbf{v}}_1| + |\nabla\hat{\mathbf{v}}_1|\,|h_1|^{\frac{1}{2}}|\Delta h_1|^{\frac{1}{2}})(\left|\Delta\hat{h}_1\right| + \left|\Delta\tilde{h}_1\right|)dt$$

$$\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m} \|\bar{U}\|^2(3\left|\Delta\hat{h}_1\right|^2 + 2\left|\Delta\tilde{h}_1\right|^2 + 4|\nabla\tilde{\mathbf{v}}_1|^8 + 4\left|\nabla\hat{h}_1\right|^4)dt. \tag{4.76}$$

where the last line is obtained by using the Young inequality several times and the Poincaré inequality

The bound for $\tilde{J}_9^2$ is deduced as follows:

$$\tilde{J}_9^2 := \mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m} \langle\nabla\cdot(\hat{h}_1\hat{\mathbf{v}}_1) - \nabla\cdot(\tilde{h}_1\tilde{\mathbf{v}}_1), \Delta\bar{h}_1\rangle|dt \tag{4.77}$$

$$\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m} |\langle\nabla\bar{h}_1\hat{\mathbf{v}}_1 + \nabla\tilde{h}_1\bar{\mathbf{v}}_1, \Delta\bar{h}_1\rangle|dt + C\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m} |\langle\nabla\cdot\hat{\mathbf{v}}_1\bar{h}_1 + \tilde{h}_1\nabla\cdot\bar{\mathbf{v}}_1, \Delta\bar{h}_1\rangle|dt\bigg)$$

$$:= \tilde{J}_9^{2,1} + \tilde{J}_9^{2,2}.$$

We estimate $\tilde{J}_9^{2,1}$ by using Hölder's inequality, Ladyzhenskya's inequality in 2D, and the embedding $H^1 \hookrightarrow L^4$ and then finally use the Young inequality

$$\begin{aligned}
&\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m} |\langle\nabla\bar{h}_1\hat{\mathbf{v}}_1 + \nabla\tilde{h}_1\bar{\mathbf{v}}_1, \Delta\bar{h}_1\rangle|dt \\
&\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m}\left(\left|\nabla\bar{h}_1\right|_{L^4}|\hat{\mathbf{v}}_1|_{L^4} + \left|\nabla\tilde{h}_1\right|_{L^4}|\bar{\mathbf{v}}_1|_{L^4}\right)\left|\Delta\bar{h}_1\right|dt \\
&\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m}\left|\nabla\bar{h}_1\right|^{\frac{1}{2}}\left|\Delta\bar{h}_1\right|^{\frac{1}{2}}|\nabla\hat{\mathbf{v}}_1| + \left|\nabla\tilde{h}_1\right|^{\frac{1}{2}}\left|\Delta\tilde{h}_1\right|^{\frac{1}{2}}|\nabla\bar{\mathbf{v}}_1|)\left|\Delta\bar{h}_1\right|dt \\
&= C\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m}\left|\nabla\bar{h}_1\right|^{\frac{1}{2}}\left|\Delta\bar{h}_1\right|^{\frac{3}{2}}|\nabla\hat{\mathbf{v}}_1| + \left|\nabla\tilde{h}_1\right|^{\frac{1}{2}}\left|\Delta\tilde{h}_1\right|^{\frac{1}{2}}|\nabla\bar{\mathbf{v}}_1|\left|\Delta\bar{h}_1\right|dt \\
&\leq \frac{\delta_1}{2}\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m}\left|\Delta\bar{h}_1\right|^2dt + C\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m}\left|\nabla\bar{h}_1\right|^2|\nabla\hat{\mathbf{v}}_1|^4dt + |\nabla\bar{\mathbf{v}}_1|^2\left|\nabla\tilde{h}_1\right|\left|\Delta\tilde{h}_1\right|dt
\end{aligned} \tag{4.78}$$

In the same manner, we derive the bound for the term $\tilde{J}_9^{2,2}$ as follows:

$$\tilde{J}_9^{2,2} = \mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m} |\langle\nabla\cdot\hat{\mathbf{v}}_1\bar{h}_1 + \tilde{h}_1\nabla\cdot\bar{\mathbf{v}}_1, \Delta\bar{h}_1\rangle|ds \tag{4.79}$$

$$\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m}(|\nabla\tilde{\mathbf{v}}_1|_{L^4}\left|\bar{h}_1\right|_{L^4} + \left|\tilde{h}_1\right|_{L^\infty}|\nabla\bar{\mathbf{v}}_1|)\left|\Delta\bar{h}_1\right|ds$$

$$\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m}|\nabla\hat{\mathbf{v}}_1|\left|\bar{h}_1\right|^{\frac{1}{2}}\left|\Delta\bar{h}_1\right|^{\frac{3}{2}} + C\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m}\left|\tilde{h}_1\right|^{\frac{1}{2}}\left|\Delta\tilde{h}_1\right|^{\frac{1}{2}}|\nabla\bar{\mathbf{v}}_1|\left|\Delta\bar{h}_1\right|dt$$

By using the Young inequality to the first term with $p = 4q = \dfrac{4}{3}$ and to the second term with $p = 2, q = 2$, we obtain:

$$\tilde{J}_9^{2,2} \leq \frac{\delta_1}{2}\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m}\left|\Delta\bar{h}_1\right|^2dt + C\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m}|\nabla\hat{\mathbf{v}}_1|^4\left|\nabla\bar{h}_1\right|^2 + |\nabla\bar{\mathbf{v}}_1|\,2(\left|\tilde{h}_1\right|^2 + \left|\Delta\tilde{h}_1\right|^2)dt$$

Collecting all the estimates (4.76) through (4.79), we find:

$$\tilde{J}_9 := 8\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m} \Big|\langle\theta(\|\hat{U}\|)\nabla\cdot(\hat{h}_1\hat{\mathbf{v}}_1) - \theta(\|\tilde{U}\|)\nabla\cdot(\tilde{h}_1\tilde{\mathbf{v}}_1), \Delta\bar{h}_1\rangle\Big|\, ds\bigg) \tag{4.80}$$

$$\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m} (\|\bar{U}\|^2)(3\left|\Delta\hat{h}_1\right|^2 + 4\left|\Delta\tilde{h}_1\right|^2 + 2\left|\nabla\tilde{h}_1\right|^2 + 6\left|\nabla\hat{\mathbf{v}}_1\right|^4 + 4\left|\nabla\tilde{\mathbf{v}}_1\right|^8 + 4\left|\nabla\hat{h}_1\right|^4)dt$$

$$+ \delta_1\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m} \left|\Delta\bar{h}_1\right|^2 dt$$

Analogously,

$$\tilde{J}_{10} := 8\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m} \Big|\langle\theta(\|\hat{U}\|)\nabla\cdot(\hat{h}_2\hat{\mathbf{v}}_2) - \theta(\|\tilde{U}\|)\nabla\cdot(\tilde{h}_2\tilde{\mathbf{v}}_2), \Delta\bar{h}_2\rangle\Big|\, ds\bigg) \tag{4.81}$$

$$\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m} (\|\bar{U}\|^2)(2\left|\Delta\hat{h}_2\right|^2 + \left|\Delta\hat{h}_2\right|^2 + \left|\nabla\hat{h}_2\right|^2\left|\nabla\hat{\mathbf{v}}_2\right|^4 + 4\left|\nabla\hat{\mathbf{v}}_2\right|^4 + \left|\nabla\tilde{h}_2\right|^2$$

$$+ 2\left|\Delta\tilde{h}_2\right|^2 + \left|\tilde{h}_2\right|^2) + \delta_2\mathbb{E}\mathbb{1}_{\Omega_0}\int_0^{t\wedge\tau^m} \left|\Delta\bar{h}_2\right|^2 dt$$

For the next four terms, we simply use the Lipschitz assumption (2.12):

$$\tilde{J}_{11} + \tilde{J}_{12} + \tilde{J}_{13} + \tilde{J}_{14} := \sum_{j=1}^{4}\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m} \mathbb{1}_{\Omega_0}\sum_{i=1}^{4}\|\sigma_i(\hat{U})e_k - \sigma_i(\tilde{U})e_k\|^2 ds +$$

$$\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m} (\|\bar{\mathbf{v}}_1\|^2 + \|\bar{h}_1\|^2 + \|\bar{\mathbf{v}}_2\|^2 + \|\bar{h}_2\|^2)ds\bigg). \tag{4.82}$$

The estimates for the last stochastic terms are obtained by using integration by parts, the Lipschitz assumption (2.12) along with the Burkholder-Davis-Gundy inequality

$$\tilde{J}_\gamma := 8\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\sup_{t\in[0,t\wedge\tau^m]}\bigg|\int_0^{t\wedge\tau^m}\sum_{k=1}^{\infty}\langle\sigma_i(\hat{U})e_k - \sigma_i(\tilde{U})e_k, \Delta\bar{\mathbf{v}}_i\rangle dW_i^k\bigg|\bigg) \tag{4.83}$$

$$\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m}\|\bar{\mathbf{v}}_i\|^2\bigg) + C\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m}(\sum_{i=1}^{2}\|\bar{\mathbf{v}}_i\|^2 + \|\bar{h}_i\|^2)dt\bigg),$$

for $\gamma = 15, 16, i = 1, 2$.

Similarly, for $\zeta = 17, 18, i = 1, 2; j = 3, 4$, we find:

$$\tilde{J}_\zeta := 8\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\sup_{s\in[0,t\wedge\tau^m]}\bigg|\int_0^{t\wedge\tau^n}\sum_{k=1}^{\infty}\langle\sigma_j(\hat{U})e_k - \sigma_j(\tilde{U})e_k, \Delta\bar{h}_i\rangle dW_j^k\bigg|\bigg) \tag{4.84}$$

$$\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m}\|\bar{h}_i\|^2\bigg) + C\mathbb{E}\mathbb{1}_{\Omega_0}\bigg(\int_0^{t\wedge\tau^m}(\sum_{i=1}^{2}\|\bar{\mathbf{v}}_i\|^2 + \|\bar{h}_i\|^2)dt\bigg).$$

Collecting all the above estimates between (4.67)-(4.84) and multiplying by two, we arrive at:

$$\mathbb{E}\mathbb{1}_{\Omega_0}\left(\sup_{s\in[0,t\wedge\tau^m]}\|\bar{\mathbf{v}}\|^2+2\nu\int_0^{t\wedge\tau^m}|\Delta\bar{\mathbf{v}}|^2\,ds+\sup_{s\in[0,t\wedge\tau^m]}\|\bar{h}\|^2+2\delta\int_0^{t\wedge\tau^m}\left|\Delta\bar{h}\right|^2 ds\right)$$
$$\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\left(\|\bar{\mathbf{v}}(0)\|^2+\|\bar{h}(0)\|^2\right)$$
$$+C\mathbb{E}\mathbb{1}_{\Omega_0}\Bigg(\int_0^{t\wedge\tau^m}\|U\|^2(|\Delta\tilde{\mathbf{v}}_1|^2+3\,|\Delta\hat{\mathbf{v}}_1|^2+2\,|\nabla\hat{\mathbf{v}}_1|^6+|\nabla\hat{\mathbf{v}}_1|^2+2\,|\nabla\tilde{\mathbf{v}}_1|^2\,|\Delta\tilde{\mathbf{v}}_2|^2$$
$$+3\,|\Delta\hat{\mathbf{v}}_2|^2+2\,|\nabla\hat{\mathbf{v}}_2|^6+|\nabla\hat{\mathbf{v}}_2|^2+2\,|\nabla\tilde{\mathbf{v}}_2|^2+3\left|\Delta\hat{h}_1\right|^2+6\left|\nabla\hat{h}_1\right|^4+6\,|\nabla\hat{\mathbf{v}}_1|^4+$$
$$4\,|\nabla\tilde{\mathbf{v}}_1|^8+4\left|\nabla\hat{h}_1\right|^4+3\left|\Delta\hat{h}_2\right|^2+6\left|\nabla\hat{h}_2\right|^4+6\,|\nabla\hat{\mathbf{v}}_2|^4+4\,|\nabla\tilde{\mathbf{v}}_2|^8+4\left|\nabla\hat{h}_2\right|^4\Bigg). \tag{4.85}$$

Now, we apply the stochastic Gronwall inequality (Lemma 7.3) with

- $X=\|\bar{\mathbf{v}_1}\|^2+\|\bar{\mathbf{v}_2}\|^2+\|\bar{h_1}\|^2+\|\bar{h_2}\|^2$,
- Y$=\nu_1\,|\Delta\bar{\mathbf{v}_1}|^2+\nu_2\,|\Delta\bar{\mathbf{v}_2}|^2+\delta_1\left|\Delta\bar{h_1}\right|^2+\delta_2\left|\Delta\bar{h_2}\right|^2$,
- $Z=0, R=1+|\Delta\tilde{\mathbf{v}}_1|^2+3\,|\Delta\hat{\mathbf{v}}_1|^2+2\,|\nabla\hat{\mathbf{v}}_1|^6+|\nabla\hat{\mathbf{v}}_1|^2+2\,|\nabla\tilde{\mathbf{v}}_1|^2\,|\Delta\tilde{\mathbf{v}}_2|^2+3\,|\Delta\hat{\mathbf{v}}_2|^2+2\,|\nabla\hat{\mathbf{v}}_2|^6+|\nabla\hat{\mathbf{v}}_2|^2+2\,|\nabla\tilde{\mathbf{v}}_2|^2+3\left|\Delta\hat{h}_1\right|^2+6\left|\nabla\hat{h}_1\right|^4+6\,|\nabla\hat{\mathbf{v}}_1|^4+4\,|\nabla\tilde{\mathbf{v}}_1|^8+4\left|\nabla\hat{h}_1\right|^4+3\left|\Delta\hat{h}_2\right|^2+6\left|\nabla\hat{h}_2\right|^4+6\,|\nabla\hat{\mathbf{v}}_2|^4+4\,|\nabla\tilde{\mathbf{v}}_2|^8+4\left|\nabla\hat{h}_2\right|^4$.

and yield

$$\mathbb{E}\mathbb{1}_{\Omega_0}\Bigg(\sup_{s\in[0,t\wedge\tau^m]}(\|\bar{\mathbf{v}_1}\|^2+\|\bar{\mathbf{v}_2}\|^2)+\nu_1\int_0^{t\wedge\tau^m}|\Delta\bar{\mathbf{v}_1}|^2\,ds+\sup_{s\in[0,t\wedge\tau^m]}\|\bar{h_1}\|^2+\|\bar{h_2}\|^2) \tag{4.86}$$
$$+\nu_2\int_0^{t\wedge\tau^m}|\Delta\bar{\mathbf{v}_2}|^2\,dt+\nu_2\int_0^{t\wedge\tau^m}|\Delta\bar{\mathbf{v}_2}|^2\,dt+\delta_1\int_0^{t\wedge\tau^m}\left|\Delta\bar{h_1}\right|^2 ds+\delta_2\int_0^{t\wedge\tau^m}\left|\Delta\bar{h_2}\right|^2 ds\Bigg)$$
$$\leq C\mathbb{E}\mathbb{1}_{\Omega_0}\left(\|\bar{\mathbf{v}_1}(0)\|^2+\|\bar{h_1}(0)\|^2+\|\bar{\mathbf{v}_2}(0)\|^2+\|\bar{h_2}(0)\|^2\right)=0.$$

From the definition of τ^m, it is easy to see that τ^m is an increasing sequence and by Lemma 4.1, we see that $\lim_{m\to\infty}\tau^m=\infty$. Thus we have shown that

$$\mathbb{E}\mathbb{1}_{\Omega_0}\left(\sup_{t\in[0,T]}(\|\bar{\mathbf{v}_1}\|^2+\|\bar{\mathbf{v}_2}\|^2+\|\bar{h_1}\|^2+\|\bar{h_2}\|^2)\right)=0. \tag{4.87}$$

This implies that

$$\mathbb{P}(\mathbb{1}_{\Omega_0}(\mathbf{v}_1(t)-\mathbf{v}_2(t))=0;\forall t\geq 0)=1, \tag{4.88}$$

$$\text{and}\qquad \mathbb{P}(\mathbb{1}_{\Omega_0}(h_1(t)-h_2(t))=0;\forall t\geq 0)=1. \tag{4.89}$$

In other words, $\mathbf{v}_1$ and $\mathbf{v}_2$ are indistinguishable on Ω_0 and so are h_1 and h_2. This proves global pathwise uniqueness. □

5 The existence of global pathwise solutions

Sections (4.4) and (4.5) already established the existence of martingale solutions and pathwise uniqueness for the modified system (4.1). We may now apply the Gyöngy-Krylov theorem (see [20]), which is an infinite dimensional extension of the Yamada-Watanabe Theorem (see [38]), to infer the existence of a global pathwise solution $(\mathbf{v}_1, \mathbf{v}_2, h_1, h_2)$.

In pursuit of the Gyöngy-Krylov theorem, we come back to the sequence $\{(\mathbf{v}_1^n, \mathbf{v}_2^n, h_1^n, h_2^n)\}$ of Galerkin solutions relative to the given stochastic basis $\mathcal{S}$ and consider the collection of joint distributions $\mu_U^{m,n}$ given by (U^m, U^n) where $U^m := (\mathbf{v}_1^m, \mathbf{v}_2^m, h_1^m, h_2^m)$. We then define the extended phase spaces as follows:

$$\mathcal{Z}^{U,W} := \mathcal{Z}_U \times \mathcal{Z}_U \times \mathcal{Z}_{W_1} \times \mathcal{Z}_{W_2} \times \mathcal{Z}_{W_3} \times \mathcal{Z}_{W_4}. \tag{5.1}$$

where

$$\begin{aligned} &\mathcal{Z}_U := \mathcal{Z}_{\mathbf{v}_1} \times \mathcal{Z}_{\mathbf{v}_2} \times \mathcal{Z}_{h_1} \times \mathcal{Z}_{h_2}, \\ &\mathcal{Z}_{\mathbf{v}_1} = \mathcal{Z}_{\mathbf{v}_2} = L^2(0,T;V_1) \cap \mathcal{C}([0,T];V_1'), \\ &\mathcal{Z}_{h_1} = \mathcal{Z}_{h_2} = L^2(0,T;V_2) \cap \mathcal{C}([0,T];V_2'), \\ &\mathcal{Z}_{W_1} = \mathcal{Z}_{W_2} = \mathcal{X}_{W_3} = \mathcal{X}_{W_4} = \mathcal{C}([0,T];\mathfrak{U}_0). \end{aligned} \tag{5.2}$$

We finally take:

$$\begin{aligned} &\mu_U^m(Z_1) = \mathbb{P}(U^m \in Z_1), \forall Z_1 \in \operatorname{Pr}(\mathcal{Z}_U), \\ &\mu_{W_i}^m(Z_2) = \mu_{W_i}(Z_2) = \mathbb{P}(W_i \in Z_3), \forall Z_3 \in \operatorname{Pr}(\mathcal{C}([0,T];\mathfrak{U}_0)), \text{ for } i = 1,2,3,4, \\ &\mu_U^{m,n} := \mu_U^m \times \mu_U^n, \\ &\mu^{m,n,W} := \mu_U^{m,n} \times \mu_{W_1} \times \mu_{W_2} \times \mu_{W_3} \times \mu_{W_4}. \end{aligned}$$

We now state a tightness result.

Lemma 5.1. *The collection $\mu^{m,n,W}$ is tight and hence weakly compact on $\mathcal{Z}^{U,W}$.*

Proof. The proof is nearly identical to Lemma 4.3. Indeed, as in [21], for every $\epsilon > 0$, we can find a set

$$\mathcal{A}_\epsilon^U := \mathcal{A}_\epsilon^{\mathbf{v}_1} \times \mathcal{A}_\epsilon^{\mathbf{v}_2} \times \mathcal{A}_\epsilon^{h_1} \times \mathcal{A}_\epsilon^{h_2}$$

which is compact in $\mathcal{Z}_U$ such that

$$\mu_U^{m,n}(\mathcal{A}_\epsilon^U) \geq \left(1 - \frac{\epsilon}{8}\right)^4. \tag{5.3}$$

For the constant sequences $\{\mu_{W_i}^m\}$, which are weakly compact, we see that they are also tight by Proposition 7.1, and hence there exist compact sets $\tilde{A}_\epsilon^i \subset \mathcal{C}([0,T];\mathfrak{U}_0)$ such that for each n, and for $i = 1,2,3,4$,

$$\mu^m_{W_i}\left(\tilde{A}^i_\epsilon\right) \geq 1 - \frac{\epsilon}{4}. \tag{5.4}$$

We finally take

$$\mathcal{A}^\epsilon := \mathcal{A}^U_\epsilon \times \mathcal{A}^U_\epsilon \times \tilde{A}^1_\epsilon \times \tilde{A}^2_\epsilon \times \tilde{A}^3_\epsilon \times \tilde{A}^4_\epsilon \tag{5.5}$$

which is compact on $\mathcal{Z}^{U,W}$. From (5.3) and (5.4), we obtain:

$$\mu^{m,n,W} \geq \left(1 - \frac{\epsilon}{8}\right)^4 \left(1 - \frac{\epsilon}{4}\right)^4 \geq 1 - \epsilon, \tag{5.6}$$

which holds for all $0 < \epsilon < 1$. The proof of the lemma is complete. □

Proposition 5.1. *There exists a unique pathwise solution of the system* (4.1).

Proof. By Lemma 5.1, in virtue of the Prohorov's Theorem, we imply that the sequence $\mu^{m,n,W}$ is weakly compact over the space $\mathcal{Z}^{U,W}$. Therefore, we can deduce the existence of a subsequence $\mu^{m_k,n_k,W}$ which converges weakly to an element μ'. With the help of the Skorohod Representation Theorem, we infer the existence of a new underlying probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ and a sequence of $\mathcal{Z}^{U,W}$- random variables $(\tilde{U}^{m_k}, \bar{U}^{n_k}, \bar{W}^{k'}_1, \bar{W}^{k'}_2, \bar{W}^{k'}_3, \bar{W}^{k'}_4)$ and $(\tilde{U}, \bar{U}, \bar{W}_1, \bar{W}_2, \bar{W}_3, \bar{W}_4)$ such that

- $$\bar{\mathbb{P}}\left[(\tilde{U}^{m_k}, \bar{U}^{n_k}, W^{k'}_1, \bar{W}^{k'}_2, \bar{W}^{k'}_3, \bar{W}^{k'}_4) \in E\right] = \mu^{m_k,n_k,W}(E), \text{ for } E \in \text{ Pr } (\mathcal{Z}^{U,W}). \tag{5.7}$$

  And
  $$\bar{\mathbb{P}}\left[(\tilde{U}, \bar{U}, W_1, W_2, W_3, W_4) \in E\right] = \mu'(E), \text{ for } E \in \text{ Pr } (\mathcal{Z}^{U,W}). \tag{5.8}$$

- $(\tilde{U}^{m_k}, \bar{U}^{n_k}, \bar{W}^{k'}_1, \bar{W}^{k'}_2, \bar{W}^{k'}_3, \bar{W}^{k'}_4)$ converges with probability 1 in the topology of $\mathcal{Z}^{U,W}$ to $(\tilde{U}, \bar{U}, \bar{W}_1, \bar{W}_2, \bar{W}_3, \bar{W}_4)$.

In particular, we infer the following:

- $(\tilde{U}^{m_k}, \bar{W}^{k'}_1, \bar{W}^{k'}_2, \bar{W}^{k'}_3, \bar{W}^{k'}_4)$ converges almost surely to $(\tilde{U}, \bar{W}_1, \bar{W}_2, \bar{W}_3, \bar{W}_4)$ in the topology of $\mathcal{Z}_U$.
- $(\bar{U}^{n_k}, \bar{W}^{k'}_1, \bar{W}^{k'}_2, \bar{W}^{k'}_3, \bar{W}^{k'}_4)$ converges almost surely to $(\bar{U}, \bar{W}_1, \bar{W}_2, \bar{W}_3, \bar{W}_4)$ in the topology of $\mathcal{Z}_U$.

By the same argument in Section 4.4, we can establish that both $(\tilde{U}, \bar{W}_1, \bar{W}_2, \bar{W}_3, \bar{W}_4)$ and $(\bar{U}, \bar{W}_1, \bar{W}_2, \bar{W}_3, \bar{W}_4)$ are global martingale solutions of the system (4.1). One can readily show that both $\tilde{U}$ and $\bar{U}$ agree with other at time $t=0$ a.s. Hence, by using the result in Section 4.5, we obtain both $\tilde{U}=\bar{U}$ in $\mathcal{Z}_U$ a.s. In other words,

$$\mu'\Big(\{(x,y)\in\mathcal{Z}_U\times\mathcal{Z}_U : x=y\}\Big)=\bar{\mathbb{P}}(\tilde{U}=\bar{U} \text{ in } \mathcal{Z}_U)=1. \tag{5.9}$$

This implies, by Proposition 7.3, that the original sequence $U^n := (\mathbf{v}_1^n,\mathbf{v}_2^n,h_1^n,h_2^n)$ defined on the initial probability space $(\Omega,\mathcal{F},\mathbb{P})$ converges in probability to an element $U := (\mathbf{v}_1,\mathbf{v}_2,h_1,h_2) := \big(\bar{\mathbf{v}}_1,\bar{\mathbf{v}}_2,\bar{h}_1,\bar{h}_2\big)$. Along the subsequence, we further infer that convergence holds almost surely in the topology of $\mathcal{Z}_U$. More precisely, for $i=1,2$,

$$\mathbf{v}_i^n\to\mathbf{v}_i \qquad \text{a.s. in } L^2(0,T;V_1)\cap\mathcal{C}([0,T];H_1), \tag{5.10}$$

$$h_i^n\to h_i \qquad \text{a.s. in } L^2(0,T;V_2)\cap\mathcal{C}([0,T];H_2). \tag{5.11}$$

By the identical argument in Section 4.4, we obtain that $(\mathbf{v}_1,\mathbf{v}_2,h_1,h_2)$ are a global pathwise solution of the equations (4.1) in the sense of Definition 2.3. By using the same technique used in our previous work [21], we are able to show that $(\mathbf{v}_1,\mathbf{v}_2)\in L^2(0,T;D(-\Delta))\cap\mathcal{C}([0,T];V_1)$ and $(h_1,h_2)\in L^2(0,T;(-\Delta))\cap\mathcal{C}([0,T];V_2)$. The proof of the existence of global pathwise solution is complete. □

6 Existence and uniqueness of solutions for the original system

6.1 Local martingale solutions

Theorem 4.3 already shows that $(\mathcal{S},\tilde{\mathbf{v}}_1,\tilde{\mathbf{v}}_2,\tilde{h}_1,\tilde{h}_2)$ is a global martingale solution for (4.1). Now we set

$$\tilde{\tau} := \inf_{t\geq 0}\Big\{\sup_{0\leq r\leq t}(\|\tilde{\mathbf{v}}_1(r)\|^2+\|\tilde{\mathbf{v}}_2(r)\|^2+\|\tilde{h}_1(r)\|^2+\|\tilde{h}_2(r)\|^2)>M\Big\}. \tag{6.1}$$

where

$$M=1+\|\tilde{\mathbf{v}}_1(0)\|^2+\|\tilde{\mathbf{v}}_2(0)\|^2+\|\tilde{h}_1(0)\|^2+\|\tilde{h}_2(0)\|^2) \tag{6.2}$$

By the following lemma, τ is strictly positive almost surely, and we observe that for $i=1,2$

$$\int_0^{t\wedge\tilde{\tau}}\theta(\|\tilde{U}\|)\nabla\cdot(\tilde{h}_i\tilde{\mathbf{v}}_i)ds=\int_0^{t\wedge\tilde{\tau}}\nabla\cdot(\tilde{h}_i\tilde{\mathbf{v}}_i)ds \text{ and } \int_0^{t\wedge\tilde{\tau}}\theta(\|\tilde{U}\|)(\tilde{\mathbf{v}}_i\cdot\nabla)\tilde{\mathbf{v}}_i ds=\int_0^{t\wedge\tilde{\tau}}(\tilde{\mathbf{v}}_i\cdot\nabla)\tilde{\mathbf{v}}_i ds.$$

We obtain that $(\mathcal{S},\tilde{\mathbf{v}}_1,\tilde{\mathbf{v}}_2,\tilde{h}_1,\tilde{h}_2,\tilde{\tau})$ is a local martingale solution.

Lemma 6.1. *The stopping time $\tilde{\tau}$ defined in (6.1) is strictly positive almost surely.*

Proof. Let $\epsilon > 0$ be given. By the definition of τ, it is easy to see that:

$$\{\tilde{\tau} < \epsilon\} \subseteq \left\{ \sup_{s\in[0,\tau\wedge\epsilon]} (\|\tilde{\mathbf{v}}_1(s)\|^2 + \|\tilde{\mathbf{v}}_2(s)\|^2 + \|\tilde{h}_1(s)\|^2 + \|\tilde{h}_2(s)\|^2) - \|\tilde{\mathbf{v}}_1(0)\|^2 \right. \qquad (6.3)$$

$$\left. - \|\tilde{h}_1(0)\|^2 - \|\tilde{\mathbf{v}}_2(0)\|^2 - \|\tilde{h}_2(0)\|^2 > 1 \right\}$$

From which, using Chebyshev's inequality, we obtain:

$$\begin{aligned}\mathbb{P}(\tilde{\tau} = 0) =& \mathbb{P}(\cap_{\epsilon>0}\{\tilde{\tau} < \epsilon\}) = \limsup_{\epsilon\to 0} \mathbb{P}(\{\tilde{\tau} < \epsilon\}) \\ &\preceq \limsup_{\epsilon\to 0} \mathbb{E} \sup_{s\in[0,\tilde{\tau}\wedge\epsilon]} (\|\tilde{\mathbf{v}}_1(s)\|^2 + \|\tilde{\mathbf{v}}_2(s)\|^2 + \\ &\left. \|\tilde{h}_1(s)\|^2 + \|\tilde{h}_2(s)\|^2) - \|\tilde{\mathbf{v}}_1(0)\|^2 - \|\tilde{h}_1(0)\|^2 - \|\tilde{\mathbf{v}}_2(0)\|^2 - \|\tilde{h}_2(0)\|^2 > 1 \right)\end{aligned}$$

Thus, the desired result will be obtained once we can show that

$$\limsup_{\epsilon\to 0} \mathbb{E}\left(\sup_{s\in[0,\tilde{\tau}\wedge\epsilon]} \|\tilde{\mathbf{v}}_1(s)\|^2 + \|\tilde{\mathbf{v}}_2(s)\|^2 + \|\tilde{h}_1(s)\|^2 + \|\tilde{h}_2(s)\|^2 \right. \qquad (6.4)$$

$$\left. - \|\tilde{\mathbf{v}}_1(0)\|^2 - \|\tilde{h}_1(0)\|^2 - \|\tilde{\mathbf{v}}_2(0)\|^2 - \|\tilde{h}_2(0)\|^2 \right) = 0.$$

For that purpose, we let $p = 2$ and replace $\mathbf{v}_1^n, \mathbf{v}_2^n$ by $\tilde{\mathbf{v}}_1, \tilde{\mathbf{v}}_2$, h_1^n, h_2^n by $\tilde{h}_1, \tilde{h}_2$ and correspondingly s by $\tau \wedge \epsilon$ in (4.11) yielding

$$\begin{aligned}&\mathbb{E}\left(\sup_{r\in[0,\tilde{\tau}\wedge\epsilon]} (\|\tilde{\mathbf{v}}_1(r)\|^2 + \|\tilde{\mathbf{v}}_2(r)\|^2 + \|\tilde{h}_1(r)\|^2 + \|\tilde{h}_2(r)\|^2) \right) \qquad (6.5) \\ &\preceq \left(\|\tilde{\mathbf{v}}_1(0)\|^2 + \|\tilde{\mathbf{v}}_2(0)\|^2 + \|\tilde{h}_1(0)\|^2 + \|\tilde{h}_2(0)\|^2 \right) + \left(\int_0^s (|F|^2 + |G|^2) dt \right).\end{aligned}$$

Thus, we arrive at

$$\begin{aligned}&\mathbb{E} \sup_{s\in[0,\tilde{\tau}\wedge\epsilon]} \left(\|\tilde{\mathbf{v}}\|^2 + \|\tilde{h}\|^2 \right) - \mathbb{E}\left(\|\tilde{\mathbf{v}}_1(0)\|^2 + \|\tilde{h}_1(0)\|^2 + \|\tilde{\mathbf{v}}_2(0)\|^2 + \|\tilde{h}_2(0)\| \right) \\ &\preceq \mathbb{E}\left(\int_0^{\tilde{\tau}\wedge\epsilon} (|F|^2 + |G|^2 \right) dt \preceq \limsup_{\epsilon\to 0} \epsilon (|F|^2_{L^\infty} + |G|^2_{L^\infty})) = 0.\end{aligned} \qquad (6.6)$$

□

Therefore, Theorem 2.1 is proved.

6.2 Local pathwise solutions

We let τ be as in (6.1), and use an identical argument to Section 5 to conclude that $(\mathbf{v}, h, \tau)$ is a local pathwise solution of (1.1). We therefore conclude the proof of Theorem 2.2.

6.3 Maximal pathwise solutions

We also see that the local solution can be extended in time to be a maximal solution.

Proposition 6.1. *There exists a unique maximal solution* $(\mathbf{v}, h, \xi)$ *and a sequence* ρ_R *announcing* ξ.

Proof. The reader is referred to [21] for the proof of this proposition. □

7 Appendices

Appendix A

Now, suppose that $\tilde{H}$ is a separable Hilbert space. Given $p \geq 2, \alpha \in (0,1)$, we define the fractional derivative space $W^{\alpha,p}(0,T;\tilde{H})$ as the Sobolev space of all $u \in L^p(0,T;\tilde{H})$ such that

$$\int_0^T \int_0^T \frac{|u(t)-u(s)|_{\tilde{H}}^p}{|t-s|^{1+\alpha p}} dtds < \infty, \tag{7.1}$$

endowed with the norm

$$|u|^p_{W^{\alpha,p}(0,T;\tilde{H})} = \int_0^T |u(t)|^p_{\tilde{H}} dt + \int_0^T \int_0^T \frac{|u(t)-u(s)|_{\tilde{H}}^p}{|t-s|^{1+\alpha p}} dtds. \tag{7.2}$$

We have applied the following lemmas, the proofs of which can be found in e.g. [13]:

Lemma 7.1. *Let* $\mathcal{E}_0 \subset\subset \mathcal{E} \subset \mathcal{E}_1$ *be Banach spaces with the injections being continuous and* $\mathcal{E}_0, \mathcal{E}_1$ *reflexive. For* $p \in (1,\infty), \alpha \in (0,1)$, *we have*

$$L^p(0,T;\mathcal{E}_0) \cap W^{\alpha,p}(0,T;\mathcal{E}_1) \subset\subset L^p(0,T;\mathcal{E}). \tag{7.3}$$

Lemma 7.2. *If* $\mathcal{E} \subset\subset \bar{\mathcal{E}}$ *are Banach spaces and* $p \in (1,\infty), \alpha \in (0,1]$ *are such that* $\alpha p > 1$, *then*

$$W^{\alpha,p}(0,T;\mathcal{E}) \subset\subset C\left([0,T];\bar{\mathcal{E}}\right). \tag{7.4}$$

We additionally often use the following stochastic version of the Gronwall lemma (see e.g. [19]):

Lemma 7.3. *Fix $T > 0$ and assume that $X, Y, Z, R : \Omega \times [0, T) \to \mathbb{R}$ are non-negative stochastic processes. Let $\tau < T$ be a stopping time such that*

$$\mathbb{E}\biggl(\int_0^\tau (RX + Z)ds\biggr) < \infty \qquad \text{and} \qquad \int_0^\tau Rds < \kappa, \qquad a.s.$$

Suppose that for all stopping times $0 \le \tau_a \le \tau_b \le \tau$

$$\mathbb{E}\biggl(\sup_{t\in[\tau_a,\tau_b]} X + \int_{\tau_a}^{\tau_b} Y ds\biggr) \le C_0 \mathbb{E}\biggl(X(\tau_a) + \int_{\tau_a}^{\tau_b} (RX + Z)ds\biggr),$$

where C_0 is independent of τ_a and τ_b. Then

$$\mathbb{E}\biggl(\sup_{t\in[0,\tau]} X + \int_0^\tau Y ds\biggr) \le C\mathbb{E}\biggl(X(0) + \int_0^\tau Z ds\biggr),$$

where C is a constant depending only on C_0, T, and κ.

Finally, we require the Vitali convergence theorem (see e.g. [16]):

Theorem 7.1. *Suppose that a sequence of functions $\{f_n\}$ are L^p integrable on a finite measure space, where $1 \le p < \infty$. Then this sequence converges in L^p to a measurable function f if the following conditions are satisfied:*

(i) $\{f_n\}$ converges to f in measure; and

(ii) the functions $\{|f_n|^p\}$ are uniformly integrable.

Lemma 7.4. *(see [17]) $u \in W^{s_1,p}, 0 \le s_2 \le s_1 \le s \le \infty$. Then there exists a constant C, such that:*

$$\|\mathbf{u}\|_{s,p} \le C\|\mathbf{u}\|^{\alpha}_{s_1,p}\|\mathbf{u}\|^{1-\alpha}_{s_2,p} \tag{7.5}$$

where $s = \alpha s_1 + (1-\alpha)s_2$.

Remark 7.1. *One can easily prove for $p > 1$ and a nonempty family $\mathcal{X}$ of random variables bounded in L^p that if $\sup_{X\in\mathcal{X}} ||X||_{L^p} < \infty$, then $\mathcal{X}$ is uniformly integrable.*

Appendix B

Definition 7.1. *Suppose (X, d) is a complete separable metric space with $\mathcal{B}(X)$ its associated Borel σ-algebra. Let $C_b(X)$ be the set of all real-valued continuous bounded functions on X, and $Pr(X)$ be the set of all probability measures on $(X, \mathcal{B}(X))$. A collection $\Lambda \subset Pr(X)$ is* tight *if for every $\epsilon > 0$ there exists a compact set $K_\epsilon \subset X$ s.t.*

$$\mu(K_\epsilon) \ge 1 - \epsilon \qquad \forall \mu \in \Lambda. \tag{7.6}$$

A sequence $\{\mu_n\}_{n\ge0} \subset Pr(X)$ converges weakly *to a probability measure μ if*

$$\int f d\mu_n \to \int f d\mu \qquad \forall f \in C_b(X). \tag{7.7}$$

The proofs of the following results can be found in e.g. [12].

Proposition 7.1 (Prohorov's Theorem)**.** *A collection $\Lambda \subset Pr(X)$ is weakly compact if and only if it is tight.*

Proposition 7.2 (Skorohod Representation Theorem)**.** *Suppose that a sequence $\{\mu_n\}_{n\geq 0}$ converges weakly to a measure μ. Then there exists a probability space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$ and a sequence of X-valued random variables $\left\{\tilde{Y}_n\right\}_{n\geq 0}$ relative to this space such that $\tilde{Y}_n$ converges a.s. to the random variable $\tilde{Y}$ and such that the laws of $\tilde{Y}_n$ and $\tilde{Y}$ are μ_n and μ, respectively, i.e. $\mu_n(E) = \mathbb{P}\left(\tilde{Y}_n \in E\right), \mu(E) = \mathbb{P}\left(\tilde{Y} \in E\right), \forall E \in \mathcal{B}(X)$.*

Finally, we suppose that $\{Y_n\}_{n\geq 0}$ is a sequence of X-valued random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\{\mu_{m,n}\}_{m,n\geq 0}$ be the collection of joint laws of $\{Y_n\}_{n\geq 0}$, i.e.

$$\mu_{m,n}(E) := \mathbb{P}((Y_m, Y_n) \in E), \qquad \forall E \in \mathcal{B}(X \times X). \tag{7.8}$$

We also need this result from [20]:

Proposition 7.3 (Gyöngy-Krylov Theorem)**.** *A sequence of X-valued random variables $\{Y_n\}_{n\geq 0}$ converges in probability if and only if for every subsequence of joint probability laws, $\{\mu_{m_k,n_k}\}_{k\geq 0}$ there exists a further subsequence which converges weakly to a probability measure μ s.t.*

$$\mu(\{(x, y) \in X \times X : x = y\}) = 1. \tag{7.9}$$

Lemma 7.5. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a fixed probability space, X a separable Hilbert space. Consider a sequence of stochastic bases $\mathcal{S}_n = (\Omega, \mathcal{F}, \{\mathcal{F}_t^n\}_{t\geq 0}, \mathbb{P}, W_1^n, W_2^n)$, where each W_i^n is a cylindrical Brownian motion over $\mathfrak{U}$ with respect to $\mathcal{F}_t^n$. Assume that $\{G^n\}_{n\geq 0}$ are a collection of X-valued $\mathcal{F}_t^n$ predictable processes such that $G^n \in L^2(0, T; L_2(\mathfrak{U}, X))$ a.s. Finally, consider $\mathcal{S} = (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}, W_1, W_2)$ and $G \in L^2(0, T; L_2(\mathfrak{U}_0, X))$ a.s., which is $\mathcal{F}_t$ predictable. If*

$$G^n \to G \qquad \textit{in probability in } L^2(0, T; L_2(\mathfrak{U}_0, X)), \tag{7.10}$$

$$W^n \to W \qquad \textit{in probability in } C([0, T]; \mathfrak{U}_0), \tag{7.11}$$

then

$$\int_0^t G^n dW^n \to \int_0^t G dW \qquad \textit{in probability in } L^2(0, T; X). \tag{7.12}$$

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