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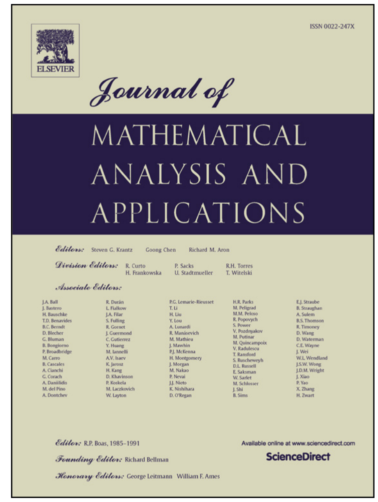

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On a fully parabolic chemotaxis system with Lotka-Volterra competitive kinetics

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Abstract: This paper is devoted to the following fully parabolic chemotaxis system with Lotka-Volterra competitive kinetics

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla w) + \mu_1 u(1 - u - a_1 v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - \chi_2 \nabla \cdot (v \nabla w) + \mu_2 v(1 - v - a_2 u), & x \in \Omega,\ t > 0, \\ w_t = \Delta w - \lambda w + b_1 u + b_2 v, & x \in \Omega,\ t > 0, \end{cases}$$

under homogeneous Neumann boundary conditions, where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. We mainly consider the global existence and boundedness of classical solutions in the three dimensional case, which extends and partially improves the results of Bai-Winkler(Indiana Univ. Math. J., 2016), Xiang(J. Math. Anal. Appl.,2018), as well as Lin-Mu-Wang(Math. Meth. Appl. Sci. 2015),etc.

Keywords: chemotaxis; global existence; boundedness; Lotka-Volterra competitive kinetics
AMS (2000) Subject Classifications: 35K55; 35Q92; 35Q35; 92C17

1 Introduction and main results

In this paper, we study the following chemotaxis system with Lotka-Volterra competitive kinetics:

$$\begin{cases} u_t = \Delta u - \chi_1 \nabla \cdot (u \nabla w) + \mu_1 u(1 - u - a_1 v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - \chi_2 \nabla \cdot (v \nabla w) + \mu_2 v(1 - v - a_2 u), & x \in \Omega,\ t > 0, \\ w_t = \Delta w - \lambda w + b_1 u + b_2 v, & x \in \Omega,\ t > 0. \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x \in \partial\Omega,\ t > 0 \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$; $\partial\nu$ denotes the differentiation with respect to the outward normal on $\partial\Omega$; χ_1, χ_2, μ_1, μ_2, a_1, a_2, b_1, b_2 and λ are positive constants; the initial data $u_0(x)$, $v_0(x)$, and $w_0(x)$ are given nonnegative functions satisfying:

$$u_0(x) \in C(\overline{\Omega}),\ v_0(x) \in C(\overline{\Omega}),\ w_0(x) \in W^{1,q}(\Omega)(q > n),\ u_0 \not\equiv 0,\ v_0 \not\equiv 0,\ w_0 \not\equiv 0 \text{ in } \overline{\Omega}. \tag{1.2}$$

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System (1.1) models the spatio-temporal evolution of two populations with densities u and v, respectively, and in which, besides random diffusion, both species are able to move toward the gradient of a chemical signal with concentration w, jointly produced by themselves. In fact, system (1.1) with $v = 0$ becomes:

$$\begin{cases} u_t = \Delta u - \chi\nabla \cdot (u\nabla w) + \mu u(1-u), & x \in \Omega,\ t > 0, \\ w_t = \Delta w - \lambda w + bu, & x \in \Omega,\ t > 0, \\ \partial_\nu u = \partial_\nu w = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x),\ w(x,0) = w_0(x), & x \in \Omega. \end{cases} \tag{1.3}$$

The outstanding feature of (1.3) with $\mu = 0$ is that solutions may blow up in finite time or infinite time if $n \geq 2$ (see, e.g., [6, 7, 4, 5, 22]). However, the blowup phenomena may be prevented for arbitrarily small $\mu > 0$ in one or two dimensional space (see [14, 13, 15], etc.); and for sufficiently large μ in bounded convex domains in higher dimensional space [23], which was improved in [12], [25], [21], etc. In particular, if $n = 3$, certain global weak solutions exist for any $\mu > 0$ (see [24]). We remark that there are some recent works investing system (1.3) coupled with fluids(see for instance [3, 10, 17, 18, 19] and references therein.

As for two-species chemotaxis model, the global existence and uniformly boundedness of solution were established in [26] for a variant model of (1.1), which allowed the chemotactic sensitivities depending on the chemical concentration w and decaying fast to zero as w goes to infinity. And, for system (1.1) with $b_1 = b_2 = 1$, the unique global bounded classical solution was established in [11], for any $n \geq 3$ and for a range of parameters (e.g., $\lambda \geq \frac{1}{2}$), by a comparison principle which heavily relies on the assumption that the domain is convex. In [1], the unique global bounded solution was established for $n \leq 2$, and the large time behavior for any $n \geq 1$ was obtained by means of the construction of suitable energy functionals. As we know, in the two dimensional case, the Gagliardo-Nirenberg inequality plays a great role for the derivation of the L^2 estimates on u and v in [1, Lemma 2.5]; however, in the higher dimensional case, it does not work well.

The goal of this work is to establish the global existence and boundedness of solutions to system (1.1) in the physical relevant domain $\Omega \subset \mathbb{R}^3$. To this end, we shall explore the restrictions on parameters as that in [25], and extend the corresponding statements on global existence in [1] and [25] to the two-species systems in the three dimensional case.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, and let χ_1, χ_2, μ_1, μ_2, a_1, a_2, b_1, b_2 and λ be some positive constants satisfying*

$$\frac{\mu_1}{b_1} > \frac{(9\chi_1^2 + 3\chi_2^2)(\sqrt{5} + \sqrt{2})}{2\sqrt{\chi_1^2 + \chi_2^2}} \tag{1.4}$$

and

$$\frac{\mu_2}{b_2} > \frac{(9\chi_2^2 + 3\chi_1^2)(\sqrt{5} + \sqrt{2})}{2\sqrt{\chi_1^2 + \chi_2^2}}. \tag{1.5}$$

Then for any initial data satisfying (1.2)*, the problem* (1.1) *possesses a unique global bounded classical solution fulfilling*

$$\begin{aligned} &u \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)), \\ &v \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)), \\ &w \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)) \cap L^\infty_{loc}([0,\infty); W^{1,q}(\Omega)), \end{aligned}$$

which is bounded in $\bar{\Omega} \times (0,\infty)$.

Remark 1.1 *We remark that our results extend the statement on global existence in [1] to the three-dimensional case; moreover, the convexity of domain, the assumptions of $\lambda \geq \frac{1}{2}$, as well as $b_1 = b_2 = 1$ required in [11] are all deleted.*

Remark 1.2 *We emphasize that Theorem 1.1 also holds for the chemo-repulsion case, i.e., $\chi_1, \chi_2 < 0$ is allowed in* (1.4) *and* (1.5)*, respectively. Furthermore, it is easy to find that, if $\chi_1 = \chi_2$, then both the lower bounds of $\frac{\mu_1}{b_1}$ and $\frac{\mu_2}{b_2}$ are equal to $3(\sqrt{10}+2)|\chi|$, which is about the same as the one-species case considered in [25].*

2 Preliminaries

The local existence and extensibility criterion of classical solutions is established in [1, Lemma 2.1], which can be proved by applying the standard methods in the local existence theory for chemotaxis problems, see [20, Lemma 2.1] or [2, Lemma 3.1], etc.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded domain with smooth boundary, and χ_1, χ_2, μ_1, μ_2, a_1, a_2, b_1, b_2 and λ be positive constants. Then for any initial data $u_0(x)$, $v_0(x)$ and $w_0(x)$ satisfying* (1.2)*, the initial-boundary value problem* (1.1) *has a unique local-in-time nonnegative classical solution (u, v, w), in the sense that*

$$\begin{aligned} &u \in C^0(\overline{\Omega} \times [0, T^*)) \cap C^{2,1}(\overline{\Omega} \times (0, T^*)),\\ &v \in C^0(\overline{\Omega} \times [0, T^*)) \cap C^{2,1}(\overline{\Omega} \times (0, T^*)),\\ &w \in C^0(\overline{\Omega} \times [0, T^*)) \cap C^{2,1}(\overline{\Omega} \times (0, T^*)) \cap L^\infty_{loc}([0, T^*); W^{1,q}(\Omega)). \end{aligned}$$

Here,T^* denotes the maximal existence time. Moreover, if $T^* < \infty$, then

$$\limsup_{t \to T^*} \{\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)}\} = \infty. \tag{2.1}$$

Next we recall some elementary estimates, see [1, Lemma 2.2], [11, Lemma 2.2].

Lemma 2.2 *Let the assumptions of Lemma 2.1 hold, then the solution component u of* (1.1) *satisfies*

$$\|u(\cdot, t)\|_{L^1(\Omega)} \leq m_1 := \max\left\{\|u_0(x)\|_{L^1(\Omega)}, |\Omega|\right\}, \qquad \text{for all } t \in [0, T^*) \tag{2.2}$$

and

$$\|v(\cdot, t)\|_{L^1(\Omega)} \leq m_2 := \max\left\{\|u_0(x)\|_{L^1(\Omega)}, |\Omega|\right\}, \qquad \text{for all } t \in [0, T^*) \tag{2.3}$$

as well as

$$\int_t^{t+\sigma} \int_\Omega u^2 dx ds \leq K_1 := m_1 + \frac{m_1}{\mu_1} \qquad \text{for all} \quad t \in (0, T^* - \sigma), \tag{2.4}$$

and

$$\int_t^{t+\sigma} \int_\Omega v^2 dx ds \leq K_2 := m_2 + \frac{m_2}{\mu_2} \qquad \text{for all} \quad t \in (0, T^* - \sigma), \tag{2.5}$$

where $\sigma := \min\{1, \frac{1}{2}T^*\}$.

Based on the spatio-temporal estimate (2.4) and (2.5), one can test the third equation of (1.1) against $-\Delta w$ to obtain the uniform bound of $\int_\Omega |\nabla w|^2 dx$, which is crucial for the L^2 estimates of u and v.

Lemma 2.3 *Let the assumptions of Lemma 2.1 hold. Then the solution component w of* (1.1) *satisfies*

$$\int_\Omega |\nabla w|^2 dx \leq \max\left\{\int_\Omega |\nabla w_0(x)| dx + 2b_1^2 K_1 + 2b_2^2 K_2, \ \frac{b_1^2 K_1 + b_2^2 K_2}{\lambda} + 4b_1^2 K_1 + 4b_2^2 K_2\right\} =: K_3. \tag{2.6}$$

3 Some a priori estimates

As we know, in view of the extensibility criterion (2.1), the crucial ingredient to obtain the global existence and boundedness of classical solutions is the uniform boundedness of $\|u(\cdot,t)\|_{L^p(\Omega)}$ and $\|v(\cdot,t)\|_{L^p(\Omega)}$ for some $p > \frac{n}{2}$. Especially, if $n = 3$, then the uniform bounds of $\|u\|_{L^2(\Omega)}$ and $\|v(\cdot,t)\|_{L^2(\Omega)}$ are sufficient to ensure the global existence and boundedness of classical solutions. To this end, we shall establish a series of estimates on $\int_\Omega (b_1u + b_2v)^2dx$, and $\int_\Omega (b_1u + b_2v)|\nabla w|^2dx$, as well as $\int_\Omega |\nabla w|^4dx$, in Lemma 3.1, Lemma 3.2, and Lemma 3.3, respectively; and then we combine these estimates to construct a differential inequality (3.18) with several parameters undetermined; after a series of delicate analysis on this differential inequality (3.18), we can establish a Gronwall type inequality, from which the uniform boundedness of $\|u(\cdot,t)\|_{L^2(\Omega)}$ and also $\|v(\cdot,t)\|_{L^2(\Omega)}$ will be gotten. The ideas used in this section mainly come from [25] and [23].

Lemma 3.1 *Let the assumptions of Lemma 2.1 hold. Then we have*

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (b_1u+b_2v)^2dx + 2(1-\epsilon_1)\int_\Omega |\nabla(b_1u+b_2v)|^2dx + 2\mu_1b_1^2\int_\Omega u^3dx + 2\mu_2b_2^2\int_\Omega v^3dx\\ &\le \frac{1}{\epsilon_1}\int_\Omega (\chi_1^2b_1^2u^2+\chi_2^2b_2^2v^2)|\nabla w|^2dx + 2\mu_1b_1^2\int_\Omega u^2dx + 2\mu_2b_2^2\int_\Omega v^2dx + c_1\end{aligned} \tag{3.1}$$

with $c_1 := \frac{\mu_1b_1b_2m_2}{2} + \frac{\mu_2b_1b_2m_1}{2}$, *for all* $t \in (0,T^*)$ *and for any* $\epsilon_1 \in (0,1)$.

Proof. In view of the first two equations in (1.1), we have

$$(b_1u+b_2v)_t = \Delta(b_1u+b_2v) - \nabla\cdot((\chi_1b_1u+\chi_2b_2v)\nabla w) + \mu_1b_1u(1-u-a_1v) + \mu_2b_2v(1-v-a_2u). \tag{3.2}$$

Upon testing against $b_1u + b_2v$ and applying the Young inequality, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega (b_1u+b_2v)^2dx + \int_\Omega |\nabla(b_1u+b_2v)|^2dx\\ &= \int_\Omega (\chi_1b_1u+\chi_2b_2v)\nabla(b_1u+b_2v)\cdot\nabla w dx\\ &\quad + \int_\Omega [\mu_1b_1u(1-u-a_1v)+\mu_2b_2v(1-v-a_2u)](b_1u+b_2v)dx\\ &\le \epsilon_1\int_\Omega |\nabla(b_1u+b_2v)|^2dx + \frac{1}{4\epsilon_1}\int_\Omega (\chi_1b_1u+\chi_2b_2v)^2|\nabla w|^2dx\\ &\quad + \int_\Omega [\mu_1b_1u(1-u)+\mu_2b_2v(1-v)](b_1u+b_2v)dx\\ &\le \epsilon_1\int_\Omega |\nabla(b_1u+b_2v)|^2dx + \frac{1}{2\epsilon_1}\int_\Omega (\chi_1^2b_1^2u^2+\chi_2^2b_2^2v^2)|\nabla w|^2dx\\ &\quad + \int_\Omega [\mu_1b_1u(1-u)+\mu_2b_2v(1-v)](b_1u+b_2v)dx\end{aligned} \tag{3.3}$$

for all $t \in (0,T^*)$. Applying Young's inequality to the last term, we have

$$\begin{aligned}&\int_\Omega [\mu_1b_1u(1-u)+\mu_2b_2v(1-v)](b_1u+b_2v)dx\\ &= \int_\Omega (\mu_1b_1^2u^2 - \mu_1b_1^2u^3 + \mu_1b_1b_2uv - \mu_1b_1b_2u^2v + \mu_2b_2^2v^2 + \mu_2b_2b_1uv - \mu_2b_2b_1uv^2 - \mu_2b_2^2v^3)dx\\ &\le \int_\Omega (\mu_1b_1^2u^2 - \mu_1b_1^2u^3 + \mu_2b_2^2v^2 - \mu_2b_2^2v^3 + \frac{\mu_1b_1b_2v}{4} + \frac{\mu_2b_1b_2u}{4})dx\end{aligned}$$

Upon combining (3.3), rearranging, and using (2.2) as well as (2.3), then we have (3.1). □

Lemma 3.2 *Let the assumptions of Lemma 2.1 hold. Then we have*

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega (b_1u+b_2v)|\nabla w|^2dx + 2\lambda\int_\Omega (b_1u+b_2v)|\nabla w|^2dx \\
&\quad + (\mu_1b_1 - \frac{\chi_1^2b_1^2}{2\epsilon_3})\int_\Omega u^2|\nabla w|^2dx + (\mu_2b_2 - \frac{\chi_2^2b_2^2}{2\epsilon_3})\int_\Omega v^2|\nabla w|^2dx \\
&\le \frac{1}{\epsilon_2}\int_\Omega |\nabla(b_1u+b_2v)|^2dx + (\epsilon_2+\epsilon_3)\int_\Omega |\nabla|\nabla w|^2|^2dx + \int_\Omega (\mu_1b_1u+\mu_2b_2v)|\nabla w|^2dx \\
&\quad + \frac{3n}{8}\int_\Omega (b_1^3u^3+b_2^3v^3)dx + \int_{\partial\Omega}(b_1u+b_2v)\partial_\nu|\nabla w|^2dS
\end{aligned}
\tag{3.4}
$$

for all $t \in (0, T^*)$, and for any $\epsilon_2 > 0$ and $\epsilon_3 > 0$.

Proof. By simple computation, we have

$$
\frac{d}{dt}\int_\Omega (b_1u+b_2v)|\nabla w|^2dx = \int_\Omega (b_1u+b_2v)_t|\nabla w|^2dx + \int_\Omega (b_1u+b_2v)(|\nabla w|^2)_tdx. \tag{3.5}
$$

Using (3.2), the first term on the right hand side of (3.5) can be estimated as

$$
\begin{aligned}
\int_\Omega (b_1u+b_2v)_t|\nabla w|^2dx &= \int_\Omega [\Delta(b_1u+b_2v) - \nabla\cdot((\chi_1b_1u+\chi_2b_2v)\nabla w)]|\nabla w|^2dx \\
&\quad + \int_\Omega [\mu_1b_1u(1-u-a_1v) + \mu_2b_2v(1-v-a_2u)]|\nabla w|^2dx \\
&= -\int_\Omega \nabla(b_1u+b_2v)\cdot\nabla|\nabla w|^2dx + \int_\Omega (\chi_1b_1u+\chi_2b_2v)\nabla w\cdot\nabla|\nabla w|^2dx \\
&\quad + \int_\Omega [\mu_1b_1u(1-u-a_1v) + \mu_2b_2v(1-v-a_2u)]|\nabla w|^2dx.
\end{aligned}
\tag{3.6}
$$

Applying Young's inequality with ϵ to the first two terms on the right hand side of (3.6), we then have

$$
\begin{aligned}
\int_\Omega (b_1u+b_2v)_t|\nabla w|^2dx &\le \frac{1}{2\epsilon_2}\int_\Omega |\nabla(b_1u+b_2v)|^2dx + (\frac{\epsilon_2}{2}+\epsilon_3)\int_\Omega |\nabla|\nabla w|^2|^2dx \\
&\quad + \frac{1}{4\epsilon_3}\int_\Omega (\chi_1b_1u+\chi_2b_2v)^2|\nabla w|^2dx \\
&\quad + \int_\Omega [\mu_1b_1u(1-u-a_1v) + \mu_2b_2v(1-v-a_2u)]|\nabla w|^2dx \\
&\le \frac{1}{2\epsilon_2}\int_\Omega |\nabla(b_1u+b_2v)|^2dx + (\frac{\epsilon_2}{2}+\epsilon_3)\int_\Omega |\nabla|\nabla w|^2|^2dx \\
&\quad + \frac{1}{2\epsilon_3}\int_\Omega (\chi_1^2b_1^2u^2+\chi_2^2b_2^2v^2)|\nabla w|^2dx \\
&\quad + \int_\Omega [\mu_1b_1u(1-u) + \mu_2b_2v(1-v)]|\nabla w|^2dx.
\end{aligned}
\tag{3.7}
$$

For the last term on the right hand side of (3.5), we first note that

$$
\begin{aligned}
\frac{d}{dt}|\nabla w|^2 &= 2\nabla w\cdot\nabla(\Delta w - \lambda w + b_1u + b_2v) \\
&= \Delta|\nabla w|^2 - 2|D^2w|^2 - 2\lambda|\nabla w|^2 + 2\nabla w\cdot\nabla(b_1u+b_2v),
\end{aligned}
\tag{3.8}
$$

where we have used the point-wise identity $2\nabla w \cdot \nabla(\Delta w) = \Delta|\nabla w|^2 - 2|D^2 w|^2$. Thus, we have

$$\begin{aligned}
&\int_\Omega (b_1 u + b_2 v)(|\nabla w|^2)_t dx \\
&= \int_\Omega (b_1 u + b_2 v)[\Delta|\nabla w|^2 - 2|D^2 w|^2 - 2\lambda|\nabla w|^2 + 2\nabla w \cdot \nabla(b_1 u + b_2 v)]dx \\
&= -\int_\Omega \nabla(b_1 u + b_2 v) \cdot \nabla|\nabla w|^2 dx + \int_{\partial\Omega} (b_1 u + b_2 v)\partial_\nu |\nabla w|^2 dS \\
&\quad - 2\int_\Omega (b_1 u + b_2 v)(|D^2 w|^2 + \lambda|\nabla w|^2)dx - \int_\Omega (b_1 u + b_2 v)^2 \Delta w dx.
\end{aligned} \tag{3.9}$$

Applying Young's inequality with ϵ to the first and the last terms, and using the fact that $|\Delta v|^2 \leq n|D^2 w|^2$, we have

$$-\int_\Omega \nabla(b_1 u + b_2 v) \cdot \nabla|\nabla w|^2 dx \leq \frac{1}{2\epsilon_2}\int_\Omega |\nabla(b_1 u + b_2 v)|^2 dx + \frac{\epsilon_2}{2}\int_\Omega |\nabla|\nabla w|^2|^2 dx, \tag{3.10}$$

where ϵ_2 is the same as in (3.7), and

$$\begin{aligned}
\int_\Omega (b_1 u + b_2 v)^2 \Delta w dx &\leq \frac{2}{n}\int_\Omega (b_1 u + b_2 v)|\Delta w|^2 dx + \frac{n}{8}\int_\Omega (b_1 u + b_2 v)^3 dx \\
&\leq 2\int_\Omega (b_1 u + b_2 v)|D^2 w|^2 dx + \frac{3n}{8}\int_\Omega (b_1^3 u^3 + b_2^3 v^3)dx.
\end{aligned} \tag{3.11}$$

Upon combining (3.9), then yields

$$\begin{aligned}
\int_\Omega (b_1 u + b_2 v)(|\nabla w|^2)_t dx \leq &\frac{1}{2\epsilon_2}\int_\Omega |\nabla(b_1 u + b_2 v)|^2 dx + \frac{\epsilon_2}{2}\int_\Omega |\nabla|\nabla w|^2|^2 dx \\
&+ \frac{3n}{8}\int_\Omega (b_1^3 u^3 + b_2^3 v^3)dx + \int_{\partial\Omega} (b_1 u + b_2 v)\partial_\nu |\nabla w|^2 dS \\
&- 2\lambda\int_\Omega (b_1 u + b_2 v)|\nabla w|^2 dx.
\end{aligned} \tag{3.12}$$

Substituting (3.12) and (3.7) into (3.5), then we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (b_1 u + b_2 v)|\nabla w|^2 dx \leq &\frac{1}{\epsilon_2}\int_\Omega |\nabla(b_1 u + b_2 v)|^2 dx + (\epsilon_2 + \epsilon_3)\int_\Omega |\nabla|\nabla w|^2|^2 dx \\
&+ \frac{1}{2\epsilon_3}\int_\Omega (\chi_1^2 b_1^2 u^2 + \chi_2^2 b_2^2 v^2)|\nabla w|^2 dx - 2\lambda\int_\Omega (b_1 u + b_2 v)|\nabla w|^2 dx \\
&+ \int_\Omega [\mu_1 b_1 u(1-u) + \mu_2 b_2 v(1-v)]|\nabla w|^2 dx \\
&+ \frac{3n}{8}\int_\Omega (b_1^3 u^3 + b_2^3 v^3)dx + \int_{\partial\Omega} (b_1 u + b_2 v)\partial_\nu |\nabla w|^2 dS,
\end{aligned} \tag{3.13}$$

which immediately implies (3.4). □

Similar to [25, Lemma 3.2], see also [9, Lemma 4.2], we establish the following estimate to control $\int_\Omega |\nabla|\nabla w|^2|^2 dx$ in (3.4).

Lemma 3.3 *Let the assumptions of Lemma 2.1 hold. Then we have*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla w|^4 dx + 2(1-\epsilon_4)\int_\Omega |\nabla|\nabla w|^2|^2 dx + 4\lambda\int_\Omega |\nabla w|^4 dx \\
&\leq (\frac{4}{\epsilon_4} + 2n)\int_\Omega (b_1^2 u^2 + b_2^2 v^2)|\nabla w|^2 dx + 2\int_{\partial\Omega} |\nabla w|^2 \partial_\nu |\nabla w|^2 dS
\end{aligned} \tag{3.14}$$

for all $t \in (0, T^*)$ and any $\epsilon_4 \in (0, 1)$.

Proof. Using (3.8), simple computation shows that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w|^4 dx &= \int_\Omega |\nabla w|^2(|\nabla w|^2)_t \\ &= \int_\Omega |\nabla w|^2[\Delta|\nabla w|^2 - 2|D^2w|^2 - 2\lambda|\nabla w|^2 + 2\nabla w\cdot\nabla(b_1u+b_2v)],\end{aligned} \tag{3.15}$$

for all $t \in (0, T^*)$. By an integration by parts, we further obtain that

$$\int_\Omega |\nabla w|^2\Delta|\nabla w|^2 dx = -\int_\Omega |\nabla|\nabla w|^2|^2 dx + \int_{\partial\Omega} |\nabla w|^2\partial_\nu|\nabla w|^2 dS \tag{3.16}$$

and

$$\begin{aligned}2\int_\Omega |\nabla w|^2\nabla w\cdot\nabla(b_1u+b_2v)dx &= -2\int_\Omega (b_1u+b_2v)\nabla|\nabla w|^2\cdot\nabla w dx - 2\int_\Omega |\nabla w|^2\Delta w(b_1u+b_2v)dx \\ &\le \epsilon_4\int_\Omega |\nabla|\nabla w|^2|^2 dx + \frac{1}{\epsilon_4}\int_\Omega (b_1u+b_2v)^2|\nabla w|^2 dx \\ &\quad + \frac{2}{n}\int_\Omega |\nabla w|^2|\Delta w|^2 dx + \frac{n}{2}\int_\Omega (b_1u+b_2v)^2|\nabla w|^2 dx \\ &\le \epsilon_4\int_\Omega |\nabla|\nabla w|^2|^2 dx + (\frac{2}{\epsilon_4}+n)\int_\Omega (b_1^2u^2+b_2^2v^2)|\nabla w|^2 dx \\ &\quad + 2\int_\Omega |\nabla w|^2|D^2w|^2 dx\end{aligned} \tag{3.17}$$

where ϵ_4 is an arbitrary positive constant. Combining (3.16) and (3.17) with (3.15), and rearranging the terms then yields (3.14). □

A linear combination $\delta_1 \times (3.1) + \delta_2 \times (3.4) + \delta_3 \times (3.14)$ with some positive constants δ_1, δ_2, δ_3, then yields the following inequality.

Corollary 3.1 *Let the assumptions of Lemma 2.1 be satisfied, then there holds:*

$$\begin{aligned}&\frac{d}{dt}\left\{\delta_1\int_\Omega (b_1u+b_2v)^2dx + \delta_2\int_\Omega (b_1u+b_2v)|\nabla w|^2dx + \delta_3\int_\Omega |\nabla w|^4dx\right\} + 4\lambda\delta_3\int_\Omega |\nabla w|^4dx \\ &\quad + 2\lambda\delta_2\int_\Omega (b_1u+b_2v)|\nabla w|^2dx + A_1\int_\Omega |\nabla(b_1u+b_2v)|^2dx + A_2\int_\Omega u^3dx \\ &\quad + A_3\int_\Omega v^3dx + A_4\int_\Omega |\nabla|\nabla w|^2|^2dx + A_5\int_\Omega u^2|\nabla w|^2dx + A_6\int_\Omega v^2|\nabla w|^2dx \\ &\le 2\mu_1b_1^2\delta_1\int_\Omega u^2dx + 2\mu_2b_2^2\delta_1\int_\Omega v^2dx + \delta_2\int_\Omega (\mu_1b_1u+\mu_2b_2v)|\nabla w|^2dx + \delta_1c_1 \\ &\quad + \delta_2\int_{\partial\Omega} (b_1u+b_2v)\partial_\nu|\nabla w|^2dS + 2\delta_3\int_{\partial\Omega} |\nabla w|^2\partial_\nu|\nabla w|^2dS\end{aligned} \tag{3.18}$$

for all $t \in (0, T^*)$, where

$$\begin{aligned}
&A_1 := 2\delta_1(1-\epsilon_1) - \frac{\delta_2}{\epsilon_2}; \quad A_2 := 2\mu_1 b_1^2\delta_1 - \frac{3nb_1^3\delta_2}{8};\\
&A_3 := 2\mu_2 b_2^2\delta_1 - \frac{3nb_2^3\delta_2}{8}; \quad A_4 := 2(1-\epsilon_4)\delta_3 - (\epsilon_2+\epsilon_3)\delta_2;\\
&A_5 := \mu_1 b_1\delta_2 - \frac{\chi_1^2 b_1^2\delta_1}{\epsilon_1} - \frac{\chi_1^2 b_1^2\delta_2}{2\epsilon_3} - (\frac{4}{\epsilon_4}+2n)b_1^2\delta_3;\\
&A_6 := \mu_2 b_2\delta_2 - \frac{\chi_2^2 b_2^2\delta_1}{\epsilon_1} - \frac{\chi_2^2 b_2^2\delta_2}{2\epsilon_3} - (\frac{4}{\epsilon_4}+2n)b_2^2\delta_3.
\end{aligned}$$

To obtain the boundedeness of $\|(b_1u+b_2v)\|_{L^2(\Omega)}$, we need to select parameters appropriately such that $z(t) := \delta_1\int_\Omega(b_1u+b_2v)^2dx+\delta_2\int_\Omega(b_1u+b_2v)|\nabla w|^2dx+\delta_3\int_\Omega|\nabla w|^4dx$ satisfies a delicate Gronwall's inequality. To this end, we need to control the terms on the right of inequality (3.18) by the dissipative terms on the left. We first note that, using the Young inequality and the uniform boundedness of $\|\nabla w\|^2_{L^2(\Omega)}$, the headmost three terms can be handled. For the remaining two boundary integrals, if Ω is convex, then $\frac{\partial|\nabla w|^2}{\partial\nu} \le 0$ ([16]), i.e, the two boundary integrals are nonpositive. In this case, we only need to choose parameters such that $A_1 = 0$, $A_2 > 0$, $A_3 > 0$, $A_4 = 0$, $A_5 > 0$, $A_6 > 0$, that is to say:

$$\begin{cases}
2\delta_1(1-\epsilon_1) - \frac{\delta_2}{\epsilon_2} = 0,\\
2\mu_1 b_1^2\delta_1 - \frac{3nb_1^3\delta_2}{8} > 0,\\
2\mu_2 b_2^2\delta_1 - \frac{3nb_2^3\delta_2}{8} > 0,\\
2(1-\epsilon_4)\delta_3 - (\epsilon_2+\epsilon_3)\delta_2 = 0,\\
\mu_1 b_1\delta_2 - \frac{\chi_1^2 b_1^2\delta_1}{\epsilon_1} - \frac{\chi_1^2 b_1^2\delta_2}{2\epsilon_3} - (\frac{4}{\epsilon_4}+2n)b_1^2\delta_3 > 0,\\
\mu_2 b_2\delta_2 - \frac{\chi_2^2 b_2^2\delta_1}{\epsilon_1} - \frac{\chi_2^2 b_2^2\delta_2}{2\epsilon_3} - (\frac{4}{\epsilon_4}+2n)b_2^2\delta_3 > 0,
\end{cases} \tag{3.19}$$

which is equivalent to

$$\begin{cases}
\frac{\delta_1}{\delta_2} = \frac{1}{2(1-\epsilon_1)\epsilon_2},\\
\frac{\mu_1}{b_1} > \frac{3n}{16}\frac{\delta_2}{\delta_1},\\
\frac{\mu_2}{b_2} > \frac{3n}{16}\frac{\delta_2}{\delta_1},\\
\frac{\delta_3}{\delta_2} = \frac{\epsilon_2+\epsilon_3}{2(1-\epsilon_4)},\\
\frac{\mu_1}{b_1} > \frac{\chi_1^2}{\epsilon_1}\frac{\delta_1}{\delta_2} + \frac{\chi_1^2}{2\epsilon_3} + (\frac{4}{\epsilon_4}+2n)\frac{\delta_3}{\delta_2},\\
\frac{\mu_2}{b_2} > \frac{\chi_2^2}{\epsilon_1}\frac{\delta_1}{\delta_2} + \frac{\chi_2^2}{2\epsilon_3} + (\frac{4}{\epsilon_4}+2n)\frac{\delta_3}{\delta_2}.
\end{cases} \tag{3.20}$$

Whereupon, we have

$$\frac{\mu_1}{b_1} > \frac{\chi_1^2}{2\epsilon_1(1-\epsilon_1)\epsilon_2} + \frac{\chi_1^2}{2\epsilon_3} + (\frac{2}{\epsilon_4}+n)\frac{\epsilon_2+\epsilon_3}{1-\epsilon_4}, \tag{3.21}$$

and

$$\frac{\mu_2}{b_2} > \frac{\chi_2^2}{2\epsilon_1(1-\epsilon_1)\epsilon_2} + \frac{\chi_2^2}{2\epsilon_3} + (\frac{2}{\epsilon_4} + n)\frac{\epsilon_2+\epsilon_3}{1-\epsilon_4}. \tag{3.22}$$

To minimize the right hands of (3.21) and (3.22), we first find that

$$\frac{1}{2\epsilon_1}\frac{1}{(1-\epsilon_1)} \geq 2, \tag{3.23}$$

the equality holds if and only if $\epsilon_1 = \frac{1}{2}$, and

$$(\frac{2}{\epsilon_4} + n)\frac{1}{(1-\epsilon_4)} \geq [\frac{n}{\sqrt{n+2}-\sqrt{2}}]^2, \tag{3.24}$$

where the equality holds if and only if $\epsilon_4 = \frac{-2+\sqrt{2n+4}}{n}$. Upon combining (3.21), (3.22), respectively, we have

$$\frac{\mu_1}{b_1} > \frac{\chi_1^2}{2\epsilon_3} + \frac{2\chi_1^2}{\epsilon_2} + (\frac{n}{\sqrt{n+2}-\sqrt{2}})^2(\epsilon_2+\epsilon_3), \tag{3.25}$$

and

$$\frac{\mu_2}{b_2} > \frac{\chi_2^2}{2\epsilon_3} + \frac{2\chi_2^2}{\epsilon_2} + (\frac{n}{\sqrt{n+2}-\sqrt{2}})^2(\epsilon_2+\epsilon_3). \tag{3.26}$$

Adding (3.25) and (3.26), we have

$$\begin{aligned}\frac{\mu_1}{b_1} + \frac{\mu_2}{b_2} &> \frac{\chi_1^2+\chi_2^2}{2\epsilon_3} + \frac{2\chi_1^2+2\chi_2^2}{\epsilon_2} + 2(\frac{n}{\sqrt{n+2}-\sqrt{2}})^2(\epsilon_2+\epsilon_3)\\ &\geq 6\sqrt{\chi_1^2+\chi_2^2}(\sqrt{n+2}+\sqrt{2}),\end{aligned} \tag{3.27}$$

where the equality holds if and only if $\epsilon_3 = \frac{\sqrt{\chi_1^2+\chi_2^2}(\sqrt{n+2}-\sqrt{2})}{2n}$ and $\epsilon_2 = \frac{\sqrt{\chi_1^2+\chi_2^2}(\sqrt{n+2}-\sqrt{2})}{n}$. Substituting them into (3.25) and (3.26), we then obtain

$$\frac{\mu_1}{b_1} > \frac{(9\chi_1^2+3\chi_2^2)(\sqrt{n+2}+\sqrt{2})}{2\sqrt{\chi_1^2+\chi_2^2}} \tag{3.28}$$

and

$$\frac{\mu_2}{b_2} > \frac{(9\chi_2^2+3\chi_1^2)(\sqrt{n+2}+\sqrt{2})}{2\sqrt{\chi_1^2+\chi_2^2}}. \tag{3.29}$$

All in all, for any μ_1, μ_2, b_1, b_2 satisfying (3.28) and (3.29), in order to achieve (3.20), we can first take

$$\begin{cases}\epsilon_1 = \frac{1}{2},\\ \epsilon_2 = \frac{\sqrt{\chi_1^2+\chi_2^2}(\sqrt{n+2}-\sqrt{2})}{n},\\ \epsilon_3 = \frac{\sqrt{\chi_1^2+\chi_2^2}(\sqrt{n+2}-\sqrt{2})}{2n},\\ \epsilon_4 = \frac{-2+\sqrt{2n+4}}{n},\end{cases} \tag{3.30}$$

and then we can choose δ_i as follows:

$$\begin{cases} \delta_1 = \dfrac{\sqrt{n+2}+\sqrt{2}}{\sqrt{\chi_1^2+\chi_2^2}}, \\ \delta_2 = 1, \\ \delta_3 = \dfrac{3\sqrt{\chi_1^2+\chi_2^2}}{4\sqrt{n+2}}. \end{cases} \tag{3.31}$$

In fact, for such ϵ_i and δ_i, the previous discussions have shown that, except for the second and the third inequalities, the inequalities in (3.20) are valid, which implies that $A_1 = A_4 = 0, A_5 > 0, A_6 > 0$. By simple computations, one can further find that

$$\frac{3n}{16}\frac{\delta_2}{\delta_1} = \frac{3n\sqrt{\chi_1^2+\chi_2^2}}{16(\sqrt{n+2}+\sqrt{2})} < \frac{(9\chi_1^2+3\chi_2^2)(\sqrt{n+2}+\sqrt{2})}{2\sqrt{\chi_1^2+\chi_2^2}} < \frac{\mu_1}{b_1}. \tag{3.32}$$

Therefore, the second inequality in (3.20) is valid, and similarly, the third inequality in (3.20) holds too, i.e., A_2 and A_3 are also positive. Fix the aforementioned ϵ_i and δ_i, then now we can establish the uniform boundedness of $\|u\|_{L^2(\Omega)}$ and $\|v\|_{L^2(\Omega)}$.

Lemma 3.4 *Let $\Omega \subset \mathbb{R}^n (n \geq 1)$ be a bounded convex domain with smooth boundary, and assume that μ_1, b_1, μ_2, b_2 satisfy* (3.28) *and* (3.29)*, respectively. Then there exist a positive constant C such that*

$$\|u\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)} + \|\nabla w\|_{L^4(\Omega)} \leq C \qquad \text{for all } t \in [0, T^*). \tag{3.33}$$

Proof. In fact, from (3.18) and (3.20), we have

$$\begin{aligned} &\frac{d}{dt}\left\{\delta_1\int_\Omega (b_1u+b_2v)^2dx + \delta_2\int_\Omega (b_1u+b_2v)|\nabla w|^2dx + \delta_3\int_\Omega |\nabla w|^4dx\right\} \\ &\quad + 4\lambda\delta_3\int_\Omega |\nabla w|^4dx + 2\lambda\delta_2\int_\Omega (b_1u+b_2v)|\nabla w|^2dx + \lambda\delta_1\int_\Omega (b_1u+b_2v)^2dx \\ &\quad + A_2\int_\Omega u^3dx + A_3\int_\Omega v^3dx + A_5\int_\Omega u^2|\nabla w|^2dx + A_6\int_\Omega v^2|\nabla w|^2dx \\ &\leq 2\mu_1b_1^2\delta_1\int_\Omega u^2dx + 2\mu_2b_2^2\delta_1\int_\Omega v^2dx + \lambda\delta_1\int_\Omega (b_1u+b_2v)^2dx \\ &\quad + \delta_2\int_\Omega (\mu_1b_1u+\mu_2b_2v)|\nabla w|^2dx + \delta_1c_1 \\ &\leq 2(\mu_1+\lambda)b_1^2\delta_1\int_\Omega u^2dx + 2(\mu_2+\lambda)b_2^2\delta_1\int_\Omega v^2dx \\ &\quad + \delta_2\int_\Omega (\mu_1b_1u+\mu_2b_2v)|\nabla w|^2dx + \delta_1c_1 \qquad \text{for all } t \in [0, T^*), \end{aligned} \tag{3.34}$$

with some positive constants A_2, A_3, A_5, A_6 as those in Corollary 3.1, and δ_1, δ_2, δ_3 are defined in (3.31).

By Young's inequality, we have

$$2(\mu_1+\lambda)b_1^2\delta_1\int_\Omega u^2dx \leq \frac{2}{3}A_2\int_\Omega u^3dx + \frac{8}{3}(\mu_1+\lambda)^3b_1^6\delta_1^3A_2^{-2}|\Omega|, \tag{3.35}$$

and

$$2(\mu_2+\lambda)b_2^2\delta_1\int_\Omega v^2dx \le \frac{2}{3}A_3\int_\Omega v^3dx+\frac{8}{3}(\mu_2+\lambda)^3b_2^6\delta_1^3A_3^{-2}|\Omega|, \tag{3.36}$$

as well as

$$\begin{aligned}\delta_2\int_\Omega(\mu_1b_1u+\mu_2b_2v)|\nabla w|^2dx &\le \frac{1}{2}A_5\int_\Omega u^2|\nabla w|^2dx+\frac{1}{2A_5}\delta_2^2\mu_1^2b_1^2\int_\Omega|\nabla w|^2dx\\ &\quad+\frac{1}{2}A_6\int_\Omega v^2|\nabla w|^2dx+\frac{1}{2A_6}\delta_2^2\mu_2^2b_2^2\int_\Omega|\nabla w|^2dx\\ &\le \frac{1}{2}A_5\int_\Omega u^2|\nabla w|^2dx+\frac{1}{2}A_6\int_\Omega v^2|\nabla w|^2dx\\ &\quad+(\frac{1}{2A_6}\delta_2^2\mu_2^2b_2^2+\frac{1}{2A_5}\delta_2^2\mu_1^2b_1^2)K_3\end{aligned} \tag{3.37}$$

with K_3 as in (2.6). Upon substituting into (3.34), then we have the following Gronwall type inequality:

$$\begin{aligned}&\frac{d}{dt}\left\{\delta_1\int_\Omega(b_1u+b_2v)^2dx+\delta_2\int_\Omega(b_1u+b_2v)|\nabla w|^2dx+\delta_3\int_\Omega|\nabla w|^4dx\right\}\\ &\quad+\lambda\left\{\delta_1\int_\Omega(b_1u+b_2v)^2dx+\delta_2\int_\Omega(b_1u+b_2v)|\nabla w|^2dx+\delta_3\int_\Omega|\nabla w|^4dx\right\}\\ &\le(\frac{1}{2A_6}\delta_2^2\mu_2^2b_2^2+\frac{1}{2A_5}\delta_2^2\mu_1^2b_1^2)K_3+\delta_1c_1 \qquad \text{for all } t\in[0,T^*),\end{aligned} \tag{3.38}$$

which implies (3.3). □

If Ω is non-convex domain, then the two boundary integrals in (3.18) may be positive. However, similar to [25, 3.22], see also [8, 9],etc., we can control them as follows:

Lemma 3.5 *Let $\Omega\subset\mathbb{R}^n(n\ge1)$ be a bounded domain with smooth boundary. Then the solution of* (1.1) *satisfies:*

$$\begin{aligned}&\delta_2\int_{\partial\Omega}(b_1u+b_2v)\partial_\nu|\nabla w|^2dS+2\delta_3\int_{\partial\Omega}|\nabla w|^2\partial_\nu|\nabla w|^2dS\\ &\le\epsilon\int_\Omega|\nabla|\nabla w|^2|^2dx+\epsilon\int_\Omega|\nabla(b_1u+b_2v)|^2dx\\ &\quad+C(\epsilon)\left(\int_\Omega|\nabla w|^2dx\right)^2+C(\epsilon)\left(\int_\Omega(b_1u+b_2v)dx\right)^2\end{aligned} \tag{3.39}$$

for all $t\in(0,T^*)$, where $\epsilon>0$ is arbitrary, and $C(\epsilon)$ is some positive constant only depending on ϵ and Ω.

In this case, to establish a Gronwall's inequality through (3.18), we need to take parameters such that A_1, A_2, A_3, A_4, A_5, A_6 are all positive. In fact, it's obviously that both the left parts in the fifth and the sixth inequalities in (3.20) are continuously dependent on δ_1 and δ_3; moreover, they are equal to the left sides of (3.28) and (3.29), respectively, once δ_i satisfy (3.31) and ϵ_i satisfy (3.30). Therefore, for any μ_1, b_1, μ_2, b_2 satisfying (3.28) and (3.29), we can also take ϵ_i as (3.30), and just select δ_1, δ_2, δ_3 satisfying

$$\begin{cases}\delta_1>\dfrac{\sqrt{n+2}+\sqrt{2}}{\sqrt{\chi_1^2+\chi_2^2}},\\ \delta_2=1,\\ \delta_3>\dfrac{3\sqrt{\chi_1^2+\chi_2^2}}{4\sqrt{n+2}}\end{cases} \tag{3.40}$$

such that A_1, A_2, A_3, A_4, A_5, A_6 are all positive. With such δ_i and ϵ_i, we can take $\epsilon = \min\{A_1, A_4\}$ in (3.39), and then combine (3.18), (3.39), (3.35)-(3.37) to deduce that

$$\begin{aligned}
&\frac{d}{dt}\left\{\delta_1 \int_\Omega (b_1u + b_2v)^2 dx + \delta_2 \int_\Omega (b_1u + b_2v)|\nabla w|^2 dx + \delta_3 \int_\Omega |\nabla w|^4 dx\right\} \\
&\quad + \lambda\left\{\delta_1 \int_\Omega (b_1u + b_2v)^2 dx + \delta_2 \int_\Omega (b_1u + b_2v)|\nabla w|^2 dx + \delta_3 \int_\Omega |\nabla w|^4 dx\right\} \\
&\quad \le (\frac{1}{2A_6}\delta_2^2\mu_2^2 b_2^2 + \frac{1}{2A_5}\delta_2^2\mu_1^2 b_1^2)K_3 + \delta_1 c_1 + C(\epsilon)K_3^2 + C(\epsilon)(b_1m_1 + b_2m_2)^2 \qquad \text{for all } t \in [0, T^*),
\end{aligned} \tag{3.41}$$

from which one can also obtain (3.33). □

4 Proof of Theorem 1.1

Proof of Theorem 1.1. In view of [1, Lemma 2.6], the global existence and boundedness of solutions to system (1.1) can be established once the uniform bound of $\|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}$ for some $p > \frac{n}{2}$ is obtained. In the case of $n = 3$, the uniform bound of $\|u\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}$ obtained in (3.3) then implies our statements in Theorem 1.1. □

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