J. Math. Anal. Appl. 471 (2019) 448–480

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Analytic semigroups generated by the dispersal process in two habitats incorporating individual behavior at the interface

Angelo Favini [a], Rabah Labbas [b], Ahmed Medeghri [c], Abdallah Menad [c]

[a] *Università degli Studi di Bologna, Dipartimento di Matematica, Piazza di Porta S. Donato, 5, 40126 Bologna, Italy*
[b] *Normandie Univ., UNIHAVRE, LMAH, FR-CNRS-3335, 76600 Le Havre, France*
[c] *Laboratoire de Mathématiques Pures et Appliquées, Université Abdelhamid Ibn Badis, 27000 Mostaganem, Algeria*

ARTICLE INFO

Article history:
Received 30 March 2017
Available online 30 October 2018
Submitted by A. Lunardi

Keywords:
Population dynamics
Dispersal process
Diffusion equation
Buffer zones
Semigroups

ABSTRACT

In this work we study an elliptic differential equation set in two habitats which models a linear stationary case of dispersal problems of population dynamics incorporating responses at interfaces between the habitats. We prove that this operator generates an analytic semigroup in an adapted space of Hölder-continuous functions.

1. Introduction

In the paper of Cantrell, R.S., and Cosner, C. [2], we find an interesting study of a diffusion model for population dynamics with dispersal incorporating individual behavior at boundaries. This study was detailed in one space dimension.

In [7], we have studied a similar problem, but without a spectral parameter and with a different transmission condition. This work was done in three habitats.

Our goal in this work is to analyze the situation in two dimension space and more precisely, we will be concerned with the study of the analyticity of the C_0-semigroup generated by the dispersal process in two habitats under some skewness condition and continuous dispersal condition at the interface which represent the behavior of the individuals at boundaries.

These problems are based on partial differential equations of parabolic type set in the following landscape constituted by two different habitats:

E-mail addresses: angelo.favini@unibo.it (A. Favini), rabah.labbas@univ-lehavre.fr (R. Labbas), ahmed.medeghri@univ-mosta.dz (A. Medeghri), abellah.menad@univ-mosta.dz (A. Menad).

https://doi.org/10.1016/j.jmaa.2018.10.085
0022-247X/

$$\Omega = \Omega_- \cup \Omega_+,$$

where

$$\begin{cases} \Omega_- =]-l, 0[\times]0,1[\\ \Omega_+ =]0, L[\times]0,1[\,, \end{cases} \tag{1.1}$$

with l, $L > 0$. The diffusion equation is

$$\frac{\partial u}{\partial t}(t,x,y) = \begin{cases} d_-.\Delta u_-(t,x,y) + F_-(u(t,x,y)) & \text{in }]0,T[\times \Omega_- \\ d_+.\Delta u_+(t,x,y) + F_+(u(t,x,y)) & \text{in }]0,T[\times \Omega_+, \end{cases} \tag{1.2}$$

under the initial data

$$u(0,x,y) = \begin{cases} \varphi_-(x,y) & \text{in } \Omega_- \\ \varphi_+(x,y) & \text{in } \Omega_+, \end{cases} \tag{1.3}$$

the boundary conditions

$$\begin{cases} u_-(t,-l,y) = f_-(t,y), \quad y \in]0,1[\\ u_-(t,x,0) = u_-(t,x,1) = 0, \quad x \in]-l,0[\\ u_+(t,x,0) = u_+(t,x,1) = 0, \quad x \in]0,L[\\ \dfrac{\partial u_+}{\partial x}(t,L,y) = f_+(t,y), \quad y \in]0,1[\,, \end{cases} \tag{1.4}$$

the continuity of the dispersal at the interface $\Gamma_0 = \{0\} \times]0,1[$,

$$\begin{cases} d_-.\Delta u_-(t,0,y) + F_-(u_-(t,0,y)) = d_+.\Delta u_+(t,0,y) + F_+(u_+(t,0,y)), \\ y \in]0,1[\,, \end{cases} \tag{1.5}$$

and the non-continuity of the flux at Γ_0

$$(1-p)d_- \frac{\partial u_-}{\partial x}(t,0,y) = p d_+ \frac{\partial u_+}{\partial x}(t,0,y), \;\; y \in]0,1[\,. \tag{1.6}$$

We have used above the natural notations

$$u_- = u_{|\Omega_-}, \quad u_+ = u_{|\Omega_+}.$$

Note that condition (1.5) is different of the one considered in [7].

Here, $u(t,x,y)$ represents a population density, Ω_+ is the refuge while Ω_- is the buffer zone with their corresponding diffusion coefficients d_+ and d_-. Equations (1.2) describe the different diffusion in the habitats with their growth or decline logistic functions F_- and F_+. The boundary conditions (1.4) simply mean that the individuals die when they reach on the other parts of the boundaries $]-l, L[\times \{0\}$ and $]-l, L[\times \{1\}$ (which mean that our bounded domain is surrounded by an hostile habitat); the population density is given on $\{-l\} \times]0,1[$ for instance and its flux also on $\{L\} \times]0,1[$.

As it was specified in [2], conditions (1.5) are essential in this work and express the fact that the dispersal process does not allow individuals to become stuck at any fixed location and does not allow any impenetrable barriers to dispersal, so, it is necessary to assume its continuity in all the domain.

Finally, the interface conditions in (1.6) are based on a skew Brownian motion characterized by the parameter $p \in]0, 1[$ which is the probability that an individual on the interface will move into the refuge Ω_+. Of course, when $p = 1/2$ the conditions signify the continuity of the flux.

The justification of conditions (1.6) is due to the fact that the so called local time of the process with $p \neq 1/2$ is discontinuous, see for instance Otso Ovaskainen and Stephen J. Cornell in [9].

Note also that, when we consider different types of habitats, the response of individuals at the interface is important for the overall movement behavior.

We will focus ourselves to study the linear stationary problem by taking:

$$\begin{cases} F_-(u_-) = -r_- u_- & \text{on }]-l, 0[\times]0, 1[\\ F_+(u_+) = r_+ u_+ & \text{on }]0, L[\times]0, 1[\,, \end{cases}$$

here r_+ is the growth rate inside the refuge Ω_+ and r_- is the death rate in the buffer zone Ω_-.

The organization of the paper is the following.

In Section 2 we recall some preliminary results on sectorial operators and their H^∞-calculus.

Sections 3 and 4 are devoted to the spectral equation of the dispersal operator $\mathcal{L}$ (see section 3) and its operational formulation. Our main result is the following

Theorem 1.1. *The dispersal operator $\mathcal{L}$ defined in the two habitats described in (3.1) below generates an analytic semigroup (not necessarily continuous at 0) in the Banach space $\mathcal{E}_*$ introduced in (4.3).*

For semigroup theory we refer to monograph [10].

All next sections concern the proof of this theorem.

Section 5 contains the complete study of the resolvent equation in an adapted space of continuous functions.

Section 6 is devoted to the invertibility of the determinant of the system induced by the spectral equation; here we use the H^∞-calculus for sectorial operators.

In Sections 7, 8 and 9 we give the complete resolution of the spectral equation, in section 10 we give the estimate of the resolvent operator which concludes the proof of our main result concerning the analyticity of the semigroup generated by $\mathcal{L}$ in some natural space $\mathcal{E}_*$ described in (4.3). Finally in section 11, we will specify the nature of the closure of $D(\mathcal{L})$.

2. Some results on sectorial operators

Consider $\omega \in [0, \pi]$ and define

$$S_\omega := \begin{cases} \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \omega\} & \text{if } \omega \in]0, \pi] \\ ]0, +\infty[\quad \text{if } \omega = 0. \end{cases}$$

We will use the following lemmas.

Lemma 2.1. *Let $\eta \in]0, \pi/2[$. For any $z \in S_\eta$, we have*

(1) $|\arg(1 - e^{-z}) - \arg(1 + e^{-z})| < \eta$
(2) $|1 + e^{-z}| \geq C_\eta = 1 - e^{-\pi/2 \tan \eta} > 0$
(3) $\dfrac{|z| \cos \eta}{1 + |z| \cos \eta} \leqslant |1 - e^{-z}| \leqslant \dfrac{2\,|z|}{1 + |z| \cos \eta}$.

The complete proof is in [4], Proposition 4.10, p. 1880.

Lemma 2.2. *Let $w, z \in \mathbb{C}\setminus\{0\}$. We have*

$$|w+z| \geq (|w|+|z|)\left|\cos\frac{\arg w - \arg z}{2}\right|.$$

See Proposition 4.9, p. 1879 in [4].

Now, let us recall some results from Haase [6]. Assume that $\omega \in [0,\pi[$. A linear operator Λ on a complex Banach space E is called sectorial of angle ω if

(1) $\sigma(\Lambda) \subset \overline{S_\omega}$ and
(2) $M(\Lambda,\omega') := \sup\limits_{\lambda\in\mathbb{C}\overline{S_{\omega'}}} \left\|\lambda(\Lambda-\lambda I)^{-1}\right\| < \infty$ for all $\omega' \in]\omega,\pi[$.

We then write that $\Lambda \in Sect(\omega)$.

The following angle

$$\omega_A := \min\{\omega \in [0,\pi[: \Lambda \in Sect(\omega)\},$$

is called the spectral angle of A. We recall the following properties of the set $Sect(\omega)$. It is clear that Statement (2) implies necessarily that Λ is closed.

Proposition 2.3. *If $]-\infty,0[\subset \rho(\Lambda)$ and*

$$M(\Lambda) := M(\Lambda,\pi) := \sup_{t>0}\left\|t(\Lambda+tI)^{-1}\right\| < \infty,$$

then $M(\Lambda) \geqslant 1$ and

$$\Lambda \in Sect(\pi - \arcsin(1/M(\Lambda)).$$

Let Λ be sectorial operator and let $\nu \in]0,1/2]$. Then $\Lambda^\nu \in Sect(\nu\omega_A)$, thus $-\Lambda^\nu$ generates an analytic semigroup. See Haase [6] pp. 80–81.

Now, consider the following space

$$H^\infty(S_\omega) = \{f : f \text{ is an holomorphic and bounded function on } S_\omega\},$$

with $\omega \in]0,\pi[$; then we recall that if $f \in H^\infty(S_\omega)$ is such that $1/f \in H^\infty(S_\omega)$ and

$$(1/f)(\Lambda) \in \mathcal{L}(X),$$

then $f(\Lambda)$ is invertible with a bounded inverse and

$$[f(\Lambda)]^{-1} = (1/f)(\Lambda), \tag{2.1}$$

see, for instance [3].

3. Operator $\mathcal{L}$ and its resolvent equation

We will be concerned with the study of the following operator

$$\mathcal{L}u = \begin{cases} d_-\Delta u_- - r_-u_- & \text{on }]-l,0[\times]0,1[\\ d_+\Delta u_+ + r_+u_+ & \text{on }]0,L[\times]0,1[, \end{cases}$$

where u satisfies

$$\begin{cases} (\text{Bound.C.}) \begin{cases} u = 0 \text{ in } \ ]-l, L[\times \{0\} \\ u = 0 \text{ in } \ ]-l, L[\times \{1\} \\ u_- = 0 \text{ in } \ \{-l\} \times]0,1[\\ \dfrac{\partial u_+}{\partial x} = 0 \text{ in } \ \{L\} \times]0,1[\end{cases} \\ (\text{Interf.C.}) \quad d_-.\Delta u_-(0,y) - r_- u_- = d_+.\Delta u_+(0,y) + r_+ u_+, \ y \in]0,1[\\ (\text{Skew.C.}) \quad (1-p) d_- \dfrac{\partial u_-}{\partial x}(0,y) = p d_+ \dfrac{\partial u_+}{\partial x}(0,y), \ y \in]0,1[\, . \end{cases}$$

We must study the spectral equation

$$\mathcal{L}u - \lambda u = g, \tag{3.1}$$

in an adequate space, where u verifies (Bound.C.), (Interf.C.) and (Skew.C.).

We recall that all the constants r_-, r_+, d_-, d_+ are strictly positive, $p \in]0,1[$ and λ is in some sector described in Section 5.

4. Operational formulation of (3.1) and the main result

Consider the Banach space

$$E = C_0([0,1]) = \{\psi \in C([0,1]) : \psi(0) = \psi(1) = 0\},$$

endowed with the sup-norm of $C([0,1])$. Let us define the following operator A in E by

$$\begin{cases} D(A) = \left\{\psi \in C^2([0,1]) : \psi(0) = \psi(1) = 0 \text{ and } \psi'' \in C_0([0,1])\right\} \\ (A\psi)(y) = \psi''(y). \end{cases}$$

Then we know that this closed linear operator verifies

$$\begin{cases} \overline{D(A)} = E; \text{for any } \eta \in]0,\pi[\, , \rho(A) \supset S_{\pi-\eta} \cup \{0\} \text{ and} \\ \exists C > 0 : \forall z \in S_{\pi-\eta} \cup \{0\}, \left\|(zI - A)^{-1}\right\|_{L(E)} \leq \dfrac{C}{1+|z|}. \end{cases} \tag{4.1}$$

We also know that there exists a ball $\mathcal{B}(0,\delta)$, $\delta > 0$, such that $\rho(A) \supset \overline{\mathcal{B}(0,\delta)}$ and the estimate in (4.1) is still true in $S_{\pi-\eta} \cup \overline{\mathcal{B}(0,\delta)}$. Here $\rho(A)$ denotes the resolvent set of A.

Remark 4.1. The conditions that ψ'' vanishes in 0 and 1 in the definition of $D(A)$ is necessary to define correctly operator A in E. This is not a restriction since we assume naturally that the spatial dispersal in direction of the variable y outside the domain is null.

Remark 4.2.

(1) We can consider the more natural spaces

$$E = C_0([0,h]) = \{\psi \in C([0,h]) : \psi(0) = \psi(h) = 0\}, \ h > 0,$$

or

$$E = C_0(\overline{U}) = \left\{ \psi \in C(\overline{U}) : \psi = 0 \text{ on } \partial U \right\},$$

where U is an open bounded set of $\mathbb{R}^2$.

(2) Note that the dimensions l, L and h of the two habitats are important for the analysis of the spectrum of $\mathcal{L}$ as it was proved in [2] in one dimension. We will study the similar situation in greater dimension for our problem in a future work.

The following well known vector-valued notations of functions

$$u_\pm(x)(y) := u_\pm(x, y),$$

lead us to write problem (3.1) in the space E, as:

$$\begin{cases} \begin{cases} u_-''(x) + Au_-(x) - \dfrac{r_-}{d_-}u_-(x) - \dfrac{\lambda}{d_-}u_-(x) = \dfrac{g_-(x)}{d_-} & \text{on }]-l, 0[\\ u_+''(x) + Au_+(x) + \dfrac{r_+}{d_+}u_+(x) - \dfrac{\lambda}{d_+}u_+(x) = \dfrac{g_+(x)}{d_+} & \text{on }]0, L[\end{cases} \\ \begin{cases} u_-(-l) = 0 \\ u_+'(L) = 0 \end{cases} \\ \begin{cases} \quad d_- \left[u_-''(0^-) + Au_-(0^-)\right] - r_-u_-(0^-) \\ = d_+ \left[u_+''(0^+) + Au_+(0^+)\right] + r_+u_+(0^+) \end{cases} \\ (1-p)d_-u_-'(0^-) = pd_+u_+'(0^+). \end{cases}$$

Note that there is no reason that the condition

$$d_- \left[u_-''(0^-) + Au_-(0^-)\right] - r_-u_-(0^-) = d_+ \left[u_+''(0^+) + Au_+(0^+)\right] + r_+u_+(0^+),$$

implies the continuity of the density itself across the interface. Taking account of our boundary and transmission conditions, it is clear that the continuity of the spatial dispersal in direction of the variable x is in some sense more "nuanced" than the ones in direction of the variable y. This is why we will consider our work in the following space

$$\begin{aligned} \mathcal{E} = \Big\{ & u \in C\left([-l, L] \setminus \{0\}; E\right) : \exists\ u(0^-) \in E, \exists u(0^+) \in E, \\ & \text{and } u_{|[-l,0[} \in C^\theta\left([-l, 0[; E\right), u_{|]0,L]} \in C^\theta\left(]0, L]; E\right) \Big\}. \end{aligned}$$

Now, we will use the following result.

Lemma 4.3. *Let U an open bounded set of $\mathbb{R}^n$. Then any function in $C^\theta(U; E)$ can be extended as a function in $C^\theta\left(\overline{U}; E\right)$. In particular $C^\theta(U; E) \subset C\left(\overline{U}; E\right)$.*

The proof is in C.L. Zuily and H. Queffélec [12]. A consequence of this Lemma is that

$$\begin{cases} C^\theta\left([-l, 0[; E\right) = C^\theta\left([-l, 0]; E\right) \\ C^\theta\left(]0, L]; E\right) = C^\theta\left([0, L]; E\right), \end{cases}$$

which imply that the spaces $C^\theta\left([-l, 0[; E\right)$ and $C^\theta\left(]0, L]; E\right)$ are equipped by the same norms that $C^\theta\left([-l, 0]; E\right)$ and $C^\theta\left([0, L]; E\right)$.

Due to this Lemma we can write:

$$\mathcal{E} = \left\{u \in C\left([-l,L]\setminus\{0\};E\right) : u_{|[-l,0[} \in C^\theta\left([-l,0];E\right) \text{ and } u_{|]0,L]} \in C^\theta\left([0,L];E\right)\right\}.$$

Then $\mathcal{E}$ is a Banach space endowed with the norm

$$\begin{aligned}\|u\|_{\mathcal{E}} &= \max\left(\|u_-\|_{C^\theta([-l,0[;E)}, \|u_+\|_{C^\theta(]0,L];E)}\right)\\ &= \max\left(\|u_-\|_{C^\theta([-l,0];E)}, \|u_+\|_{C^\theta([0,L];E)}\right).\end{aligned}$$

The function $g \in \mathcal{E}$ is such that

$$\begin{cases} g_- = g|_{[-l,0]} \in C^\theta\left([-l,0];E\right) \\ g_+ = g|_{[0,L]} \in C^\theta\left([0,L];E\right), \end{cases}$$

(with $0 < \theta < 1$). It is not difficult to prove that the global hölderianity of g on $[-l, L]$ is true if and only if

$$g_-(0) = g_+(0).$$

We do not assume this condition.

Let us specify the domain of $\mathcal{L}$:

$$D(\mathcal{L}) = \left\{\begin{array}{l} u \in \mathcal{E} : \forall x \in [-l,L]\setminus\{0\} \;\; u(x) \in D(A), \\ u_- \in C^2\left(]-l,0[;E\right), u_+ \in C^2\left(]0,L[;E\right), \\ x \mapsto \left[u_-''(x) + Au_-(x)\right] \in C^\theta\left([-l,0[;E\right), \\ x \mapsto \left[u_+''(x) + Au_+(x)\right] \in C^\theta\left(]0,L];E\right), \\ \text{and (Bound.C.), (Interf.C.), (Skew.C.).} \end{array}\right\}. \tag{4.2}$$

Our analysis leads us to consider the following natural space

$$\begin{aligned}\mathcal{E}_* = \big\{&u \in C\left([-l,L]\setminus\{0\};E\right) : \exists\, u(0^-) \in E_\theta, \exists u(0^+) \in E_\theta, \\ &u(-l) = 0, u(L) \in E_\theta,\, u_{|[-l,0[} \in C^\theta\left([-l,0[;E\right) \text{ and} \\ &u_{|]0,L]} \in C^\theta\left(]0,L];E\right)\big\},\end{aligned} \tag{4.3}$$

where

$$E_\theta = D_A\left(\theta/2, +\infty\right) = \left\{\varphi \in C^\theta\left([0,1]\right) : \varphi(0) = \varphi(1) = 0\right\};$$

then, we can prove that $\mathcal{E}_*$ is a Banach space endowed with the norm

$$\begin{aligned}\|u\|_{\mathcal{E}_*} &= \max\left(\|u_-\|_{C^\theta([-l,0[;E)}, \|u_+\|_{C^\theta(]0,L];E)}\right) \\ &\quad + \max\left(\left\|u_-(0^-)\right\|_{C^\theta([0,1])}, \left\|u_+(0^+)\right\|_{C^\theta([0,1])}, \|u_+(L)\|_{C^\theta([0,1])}\right).\end{aligned}$$

Remark 4.4. Note that the condition that the population density vanishes on the exterior boundary of the buffer zone Ω_- is natural and the fact that on $\{L\}\times]0,1[$ and on the interface $\{0\}\times]0,1[$ we have a Hölder-continuous condition is in some sense realistic.

All the following sections are devoted to the proof of Theorem 1.1.

5. Study of the resolvent equation

Fix a small number ε_0. Let us assume that the complex λ verifies

$$|\arg(\lambda)| < \pi - \varepsilon_0. \tag{5.1}$$

Our spectral equation for $\mathcal{L}$ writes:

$$(P_A)\begin{cases}\begin{cases} u_-''(x) + Au_-(x) - \dfrac{r_-}{d_-}u_-(x) - \dfrac{\lambda}{d_-}u_-(x) = \dfrac{g_-(x)}{d_-} = G_-(x) \text{ on }]-l,0[\\ u_+''(x) + Au_+(x) + \dfrac{r_+}{d_+}u_+(x) - \dfrac{\lambda}{d_+}u_+(x) = \dfrac{g_+(x)}{d_+} = G_+(x) \text{ on }]0,L[\end{cases} \\ \begin{cases} u_-(-l) = 0 \\ u_+'(L) = 0 \end{cases} \\ \begin{cases} d_-\left[u_-''(0^-) + Au_-(0^-)\right] - r_-u_-(0^-) \\ = d_+\left[u_+''(0^+) + Au_+(0^+)\right] + r_+u_+(0^+) \end{cases} \\ (1-p)d_-u_-'(0^-) = pd_+u_+'(0^+). \end{cases}$$

Put

$$A_- = A - \frac{r_-}{d_-}I - \frac{\lambda}{d_-}I, \qquad A_+ = A + \frac{r_+}{d_+}I - \frac{\lambda}{d_+}I,$$

which have the same domain

$$D(A_-) = D(A_+) = D(A).$$

As for operator $-A$, we can verify that operator

$$-A_- = -A + \frac{r_-}{d_-} + \frac{\lambda}{d_-},$$

is sectorial in E. In fact, we can easily prove that $]-\infty, 0] \subset \rho(-A_-)$ and, if we put

$$M(-A_-) := M(-A_-, \pi) := \sup_{t>0} \left\| t(-A_- + tI)^{-1} \right\|,$$

then, by an explicit calculus we obtain for all λ verifying

$$|\arg(\lambda)| < \pi - \varepsilon_0,$$

$$M(-A_-) \leqslant \sup_{t>0} \left(\frac{t}{\cos\left[\frac{1}{2}\arg\left(\frac{r_-}{d_-} + \frac{\lambda}{d_-} + t\right)\right]} \frac{1}{\left|\frac{r_-}{d_-} + \frac{\lambda}{d_-} + t\right|} \right).$$

We have two cases:

(1) if $|\arg(\lambda)| < \pi/2$ then obviously for $t > 0$

$$\left|\frac{r_-}{d_-} + \frac{\lambda}{d_-} + t\right| = d\left(\frac{\lambda}{d_-}, -t - \frac{r_-}{d_-}\right) \geqslant \frac{r_-}{d_-} + t,$$

where $d(,)$ is the Euclidean distance.

(2) If $\pi/2 \leqslant |\arg(\lambda)| < \pi - \varepsilon_0$

$$\left|\frac{r_-}{d_-} + \frac{\lambda}{d_-} + t\right| = d\left(\frac{\lambda}{d_-}, -t - \frac{r_-}{d_-}\right) \geqslant \left(\frac{r_-}{d_-} + t\right)\sin\alpha,$$

with $\alpha \in]\varepsilon_0, \pi/2]$ and then

$$\left|\frac{r_-}{d_-} + \frac{\lambda}{d_-} + t\right| = d\left(\frac{\lambda}{d_-}, -t - \frac{r_-}{d_-}\right) \geqslant \left(\frac{r_-}{d_-} + t\right)\sin\varepsilon_0,$$

therefore there exists a constant C independent of λ such that

$$\begin{aligned} M(-A_-) &\leqslant \sup_{t>0}\left(\frac{t}{\cos\left[\frac{1}{2}\arg\left(\frac{r_-}{d_-} + \frac{\lambda}{d_-} + t\right)\right]}\frac{1}{\left|\frac{r_-}{d_-} + \frac{\lambda}{d_-} + t\right|}\right) \\ &\leqslant \frac{C}{\cos\frac{\pi-\varepsilon_0}{2}}\sup_{t>0}\left(\frac{t}{\left|\frac{r_-}{d_-} + t\right|}\right) \\ &= \frac{C}{\sin(\varepsilon_0/2)}\sup_{t>0}\left(\frac{t}{\left|\frac{r_-}{d_-} + t\right|}\right) < \infty, \end{aligned}$$

thus

$$-A_- \in \mathit{Sect}\left[\pi - \arcsin\left(1/M(-A_-)\right)\right],$$

with $M(-A_-)$ independent of λ, see Proposition 2.1.1 in [6].

Now, since

$$|\arg(\lambda)| < \pi - \varepsilon_0,$$

then there exists a small $\varepsilon_0(r_+, d_+) > 0$ such that

$$\varepsilon_0(r_+, d_+) < \varepsilon_0 \quad \text{and} \quad \left|\arg\left(\frac{\lambda}{d_+} - \frac{r_+}{d_+}\right)\right| < \pi - \varepsilon_0(r_+, d_+);$$

we then obtain similarly, but differently

$$-A_+ \in \mathit{Sect}\left(\pi - \arcsin\left[1/M(-A_+)\right]\right),$$

with $M(-A_+)$ independent of λ.

We also deduce that the following operators

$$B_- = -\left[-\left(A - \frac{r_-}{d_-}I - \frac{\lambda}{d_-}I\right)\right]^{1/2}, \quad B_+ = -\left[-\left(A + \frac{r_+}{d_+}I - \frac{\lambda}{d_+}I\right)\right]^{1/2},$$

of the same domain

$$D(B_-) = D(B_+) = D(B),$$

are well defined and generate analytic semigroups on E, see [6], p. 81 and also [1].

On the other hand, using the scaling property (see Proposition 3.1.2 in [6]), we deduce the fact that

$$\begin{cases} \text{for all } z \in \left\{z \in \mathbb{C}\setminus\{0\} : |\arg(z)| \leqslant \frac{\pi}{2} + \epsilon(B_-)\right\} \\ \left\|(B_- - zI)^{-1}\right\| \leqslant \dfrac{C_-}{|z|}, \\ \text{and for all } z \in \left\{z \in \mathbb{C}\setminus\{0\} : |\arg(z)| \leqslant \frac{\pi}{2} + \epsilon(B_+)\right\}, \\ \left\|(B_+ - zI)^{-1}\right\| \leqslant \dfrac{C_+}{|z|}, \end{cases} \tag{5.2}$$

where the constants

$$C_-, \quad C_+, \quad \epsilon(B_-) \quad \text{and} \quad \epsilon(B_+),$$

are independent of λ.

Set

$$p_- = (1-p)d_-, \qquad p_+ = pd_+.$$

Our spectral equation

$$\mathcal{L}u - \lambda u = g \in \mathcal{E},$$

can be written in the following way

$$\begin{cases} \begin{cases} u_-''(x) - B_-^2 u_-(x) = G_-(x) & \text{on } \left]-l, 0\right[\\ u_+''(x) - B_+^2 u_+(x) = G_+(x) & \text{on } \left]0, L\right[\end{cases} \\ \begin{cases} u_-(-l) = 0 \\ u_+'(L) = 0 \end{cases} \\ \begin{cases} \quad d_-\left[u_-''(0^-) + Au_-(0^-)\right] - r_-u_-(0^-) \\ = d_+\left[u_+''(0^+) + Au_+(0^+)\right] + r_+u_+(0^+) \end{cases} \\ (1-p)d_-u_-'(0^-) = pd_+u_+'(0^+). \end{cases}$$

Then

$$u_\pm(x) = e^{(x-a_\pm)B_\pm}\alpha_\pm + e^{(b_\pm - x)B_\pm}\beta_\pm + v_\pm(G)(x),$$

with $\alpha_\pm$, $\beta_\pm \in E$; $a_- = -l, b_- = 0 : a_+ = 0, b_+ = L$ and

$$v_\pm(G_\pm)(x) = \frac{1}{2}\int_{a_\pm}^{x} e^{(x-t)B_\pm}B_\pm^{-1}G_\pm(t)dt + \frac{1}{2}\int_{x}^{b_\pm} e^{(t-x)B_\pm}B_\pm^{-1}G_\pm(t)dt.$$

Therefore

$$\begin{aligned} u_-(x) &= e^{(x+l)B_-}\alpha_- + e^{-xB_-}\beta_- + v_-(G_-)(x), \ \ x \in \left]-l, 0\right[\\ u_+(x) &= e^{xB_+}\alpha_+ + e^{(L-x)B_+}\beta_+ + v_+(G_+)(x), \ \ x \in \left]0, L\right[, \end{aligned}$$

and

$$v_-(G_-)(x) = \frac{1}{2}\int_{-l}^{x} e^{(x-t)B_-}B_-^{-1}G_-(t)dt + \frac{1}{2}\int_{x}^{0} e^{(t-x)B_-}B_-^{-1}G_-(t)dt$$
$$v_+(G_+)(x) = \frac{1}{2}\int_{0}^{x} e^{(x-t)B_+}B_+^{-1}G_+(t)dt + \frac{1}{2}\int_{x}^{L} e^{(t-x)B_+}B_+^{-1}G_+(t)dt,$$

which give

$$\begin{aligned} u_-'(x) &= B_-e^{(x+l)B_-}\alpha_- - B_-e^{-xB_-}\beta_- + v_-'(G_-)(x), \ \ x \in]-l,0[\\ u_+'(x) &= B_+e^{xB_+}\alpha_+ - B_+e^{(L-x)B_+}\beta_+ + v_+'(G_+)(x), \ \ x \in]0,L[\,, \\ u_-'(0) &= B_-e^{lB_-}\alpha_- - B_-\beta_- + v_-'(G_-)(0) \\ u_+'(0) &= B_+\alpha_+ - B_+e^{LB_+}\beta_+ + v_+'(G_+)(0). \end{aligned}$$

The boundary conditions

$$\begin{cases} u_-(-l) = \alpha_- + e^{lB_-}\beta_- + v_-(G_-)(-l) = 0 \\ u_+'(L) = B_+e^{LB_+}\alpha_+ - B_+\beta_+ + v_+'(G_+)(L) = 0, \end{cases}$$

give

$$\begin{cases} \alpha_- = -e^{lB_-}\beta_- - v_-(G_-)(-l) \\ \beta_+ = e^{LB_+}\alpha_+ + B_+^{-1}v_+'(G_+)(L). \end{cases} \tag{5.3}$$

The interface dispersal condition

$$d_-\left[u_-''(0) - B_-^2u_-(0)\right] - r_-u_-(0) = d_+\left[u_+''(0) - B_+^2u_+(0)\right] + r_+u_+(0),$$

becomes

$$d_-G_-(0) + \lambda u_-(0) = d_+G_+(0) + \lambda u_+(0),$$

or

$$\begin{aligned} &d_-G_-(0) + \lambda\left[e^{lB_-}\alpha_- + \beta_- + v_-(G_-)(0)\right] \\ &\quad = d_+G_+(0) + \lambda\left[\alpha_+ + e^{LB_+}\beta_+ + v_+(G_+)(0)\right], \end{aligned}$$

and since

$$\begin{aligned} u_-'(0) &= B_-e^{lB_-}\alpha_- - B_-\beta_- + v_-'(G_-)(0) \\ u_+'(0) &= B_+\alpha_+ - B_+e^{LB_+}\beta_+ + v_+'(G_+)(0), \end{aligned}$$

the skewness condition

$$p_-(u_-)'(0) - p_+(u_+)'(0) = 0,$$

gives

$$\begin{aligned}
&p_- \left(B_- e^{lB_-}\alpha_- - B_-\beta_-\right) - p_+ \left(B_+\alpha_+ - B_+ e^{LB_+}\beta_+\right) \\
&\quad = -p_- v_-'(G_-)(0) + p_+ v_+'(G_+)(0).
\end{aligned}$$

We obtain the system

$$\begin{cases}
\alpha_- = -e^{lB_-}\beta_- - v_-(G_-)(-l) \\
\beta_+ = e^{LB_+}\alpha_+ + B_+^{-1} v_+'(G_+)(L) \\
p_- \left(B_- e^{lB_-}\alpha_- - B_-\beta_-\right) - p_+ \left(B_+\alpha_+ - B_+ e^{LB_+}\beta_+\right) \\
= -p_- v_-'(G_-)(0) + p_+ v_+'(G_+)(0) \\
\lambda \left[e^{lB_-}\alpha_- + \beta_-\right] - \lambda \left[\alpha_+ + e^{LB_+}\beta_+\right] \\
= d_+ G_+(0) - d_- G_-(0) + \lambda v_+(G_+)(0) - \lambda v_-(G_-)(0),
\end{cases}$$

which is equivalent to

$$\begin{cases}
\alpha_- = -e^{lB_-}\beta_- - v_-(G_-)(-l) \\
\beta_+ = e^{LB_+}\alpha_+ + B_+^{-1} v_+'(G_+)(L) \\
p_- \left(e^{lB_-}\alpha_- - \beta_-\right) - p_+ \left(B_-^{-1} B_+\alpha_+ - B_-^{-1} B_+ e^{LB_+}\beta_+\right) \\
= B_-^{-1}\left[-p_- v_-'(G_-)(0) + p_+ v_+'(G_+)(0)\right] := (I) \\
\left[e^{lB_-}\alpha_- + \beta_-\right] - \left[\alpha_+ + e^{LB_+}\beta_+\right] \\
= \frac{1}{\lambda}\left[d_+ G_+(0) - d_- G_-(0)\right] + v_+(G_+)(0) - v_-(G_-)(0) := (II),
\end{cases}$$

thus

$$\begin{cases}
\alpha_- = -e^{lB_-}\beta_- - v_-(G_-)(-l) \\
\beta_+ = e^{LB_+}\alpha_+ + B_+^{-1} v_+'(G_+)(L) \\
p_- \left(e^{lB_-}\alpha_- - \beta_-\right) - p_+ \left(B_-^{-1} B_+\alpha_+ - B_-^{-1} B_+ e^{LB_+}\beta_+\right) = (I) \\
\left[e^{lB_-}\alpha_- + \beta_-\right] - \left[\alpha_+ + e^{LB_+}\beta_+\right] = (II);
\end{cases}$$

using the two first equations, we get

$$\begin{aligned}
&p_- \left(e^{lB_-}\alpha_- - \beta_-\right) - p_+ \left(B_-^{-1} B_+\alpha_+ - B_-^{-1} B_+ e^{LB_+}\beta_+\right) \\
&\quad = p_- \left(e^{lB_-}\left[-e^{lB_-}\beta_- - v_-(G_-)(-l)\right] - \beta_-\right) \\
&\qquad - p_+ \left(B_-^{-1} B_+\alpha_+ - B_-^{-1} B_+ e^{LB_+}\left[e^{LB_+}\alpha_+ + B_+^{-1} v_+'(G_+)(L)\right]\right) \\
&\quad = -p_- \left[e^{2lB_-}\beta_- + \beta_-\right] - p_- e^{lB_-} v_-(G_-)(-l) - p_+ B_-^{-1} B_+\alpha_+ \\
&\qquad + p_+ B_-^{-1} B_+ e^{2LB_+}\alpha_+ + p_+ B_-^{-1} B_+ e^{LB_+} B_+^{-1} v_+'(G_+)(L) \\
&\quad = -p_- \left[I + e^{2lB_-}\right]\beta_- - p_+ B_-^{-1} B_+ \left[I - e^{2LB_+}\right]\alpha_+ \\
&\qquad + p_+ B_-^{-1} e^{LB_+} v_+'(G_+)(L) - p_- e^{lB_-} v_-(G_-)(-l),
\end{aligned}$$

and

$$\begin{aligned}
&\left[e^{lB_-}\alpha_- + \beta_-\right] - \left[\alpha_+ + e^{LB_+}\beta_+\right] \\
&\quad = e^{lB_-}\left[-e^{lB_-}\beta_- - v_-(G_-)(-l)\right] + \beta_- \\
&\qquad - \left[\alpha_+ + e^{LB_+}\left[e^{LB_+}\alpha_+ + B_+^{-1} v_+'(G_+)(L)\right]\right] \\
&\quad = \left(I - e^{2lB_-}\right)\beta_- - \left(I + e^{2LB_+}\right)\alpha_+ \\
&\qquad - e^{lB_-} v_-(G_-)(-l) - e^{LB_+} B_+^{-1} v_+'(G_+)(L),
\end{aligned}$$

then the system above becomes

$$\begin{cases} -p_- \left[I + e^{2lB_-}\right] \beta_- - p_+ B_-^{-1} B_+ \left[I - e^{2LB_+}\right] \alpha_+ \\ = (I) - p_+ B_-^{-1} e^{LB_+} v'_+(G_+)(L) + p_- e^{lB_-} v_-(G_-)(-l) := (I') \\ \left(I - e^{2lB_-}\right) \beta_- - \left(I + e^{2LB_+}\right) \alpha_+ \\ = (II) + e^{lB_-} v_-(G_-)(-l) + e^{LB_+} B_+^{-1} v'_+(G_+)(L) := (II'), \end{cases}$$

or

$$\begin{cases} -p_- \left[I + e^{2lB_-}\right] \beta_- - p_+ B_-^{-1} B_+ \left[I - e^{2LB_+}\right] \alpha_+ = (I') \\ \left(I - e^{2lB_-}\right) \beta_- - \left(I + e^{2LB_+}\right) \alpha_+ = (II'). \end{cases}$$

The abstract determinant of this system is

$$\begin{aligned} \Delta_{\lambda,p_-,p_+} = {} & p_- \left(I + e^{2lB_-}\right)\left(I + e^{2LB_+}\right) \\ & + p_+ B_-^{-1} B_+ \left(I - e^{2LB_+}\right)\left(I - e^{2lB_-}\right). \end{aligned} \tag{5.4}$$

For the invertibility of Δ_{λ,p_-,p_+}, we will use the H^∞-calculus for sectorial operators.

Note that all the operators $\left(I + e^{2lB_-}\right), \left(I + e^{2LB_+}\right), \left(I - e^{2LB_+}\right)$ and $\left(I - e^{2lB_-}\right)$ are boundedly invertible by applying Proposition 2.3.6, page 60 in A. Lunardi [8].

6. Invertibility of Δ

Let us set

$$\frac{r_-}{d_-} = \rho_-, \qquad \frac{\lambda}{d_-} = \lambda_-, \quad \frac{r_+}{d_+} = \rho_+, \qquad \frac{\lambda}{d_+} = \lambda_+,$$

consider $\omega \in]0, \pi[$ and recall

$$S_\omega = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \omega \}.$$

Consider the following function

$$e_{\lambda,p_-,p_+} : S_\omega \ni z \longmapsto e_{\lambda,p_-,p_+}(z),$$

defined by

$$\begin{aligned} & e_{\lambda,p_-,p_+}(z) \\ & \quad = p_- \left(1 + e^{-2l(z+\lambda_-+\rho_-)^{1/2}}\right)\left(1 + e^{-2L(z+\lambda_+-\rho_+)^{1/2}}\right) \\ & \qquad + p_+ \frac{(z+\lambda_+-\rho_+)^{1/2}}{(z+\lambda_-+\rho_-)^{1/2}} \left(1 - e^{-2L(z+\lambda_+-\rho_+)^{1/2}}\right)\left(1 - e^{-2l(z+\lambda_-+\rho_-)^{1/2}}\right); \end{aligned}$$

this function is clearly holomorphic and bounded since

$$Re\,(z + \lambda_- + \rho_-)^{1/2} > 0 \quad \text{and} \quad Re\,(z + \lambda_+ - \rho_+)^{1/2} > 0,$$

thus $e_{\lambda,p_-,p_+} \in H^\infty(S_\omega)$. On the other hand we know that the operator $-A$ has bounded $H^\infty(S_\omega)$ functional calculus; hence we have

$$\Delta_{\lambda,p_-,p_+} = e_{\lambda,p_-,p_+}(-A).$$

We have

$$\begin{aligned}
&\left|e_{\lambda,p_-,p_+}(z)\right| \\
&\geqslant \left(p_+\left|\frac{(z+\lambda_+-\rho_+\ )^{1/2}}{(z+\lambda_-+\rho_-)^{1/2}}\right|\left|1-e^{-2L(z+\lambda_+-\rho_+\ )^{1/2}}\right|\left|1-e^{-2l(z+\lambda_-+\rho_-)^{1/2}}\right|\right. \\
&\quad \left.+\,p_-\left|1+e^{-2l(z+\lambda_-+\rho_-)^{1/2}}\right|\left|1+e^{-2L(z+\lambda_+-\rho_+\ )^{1/2}}\right|\right)\cdot\left|\cos\left(\frac{\Phi}{2}\right)\right|,
\end{aligned}$$

where

$$\begin{aligned}
\Phi = {}& \arg\left(\frac{(z+\lambda_+-\rho_+\ )^{1/2}}{(z+\lambda_-+\rho_-)^{1/2}}\right)+\arg\left(1-e^{-2L(z+\lambda_+-\rho_+\ )^{1/2}}\right) \\
&+\arg\left(1-e^{-2l(z+\lambda_-+\rho_-)^{1/2}}\right)-\arg\left(1+e^{-2l(z+\lambda_-+\rho_-)^{1/2}}\right) \\
&-\arg\left(1+e^{-2L(z+\lambda_+-\rho_+\ )^{1/2}}\right),
\end{aligned}$$

then

$$\begin{aligned}
&|\Phi| \\
&\leqslant \left|\arg\left(z+\lambda_+-\rho_+\ \right)^{1/2}-\arg\left(z+\lambda_-+\rho_-\right)^{1/2}\right| \\
&\quad+\left|\arg\left(1-e^{-2l(z+\lambda_-+\rho_-)^{1/2}}\right)-\arg\left(1+e^{-2l(z+\lambda_-+\rho_-)^{1/2}}\right)\right| \\
&\quad+\left|\arg\left(1-e^{-2l(z+\lambda_-+\rho_-)^{1/2}}\right)-\arg\left(1+e^{-2L(z+\lambda_+-\rho_+\ )^{1/2}}\right)\right|.
\end{aligned}$$

Now, since

$$z \in S_\omega = \left\{z \in \mathbb{C}\backslash\left\{0\right\} : |\arg z| < \omega\right\},$$

then, we can see that there exist some $\omega_{\lambda_-,\rho_-} \in]0,\omega/2[$ and some $\omega_{\lambda_+,\rho_+} \in]0,\omega/2[$, (with $\omega_{\lambda_-,\rho_-} < \omega_{\lambda_+,\rho_+}$) such that

$$\begin{cases} 2l\left(z+\lambda_-+\rho_-\right)^{1/2} \in S_{\omega_{\lambda_-,\rho_-}} \\ 2L\left(z+\lambda_+-\rho_+\ \right)^{1/2} \in S_{\omega_{\lambda_+,\rho_+}}; \end{cases}$$

by using the lemma in Section 2, we have

$$\begin{cases} \left|\arg\left(1-e^{-2l(z+\lambda_-+\rho_-)^{1/2}}\right)-\arg\left(1+e^{-2l(z+\lambda_-+\rho_-)^{1/2}}\right)\right| < \omega_{\lambda_-,\rho_-} \\ \left|\arg\left(z+\lambda_+-\rho_+\ \right)^{1/2}-\arg\left(z+\lambda_-+\rho_-\right)^{1/2}\right| \leqslant \omega_{\lambda_+,\rho_+}, \end{cases}$$

from which we deduce

$$|\Phi| < \omega_{\lambda_+,\rho_+} - \omega_{\lambda_-,\rho_-} + \omega_{\lambda_-,\rho_-} + \omega_{\lambda_+,\rho_+} < 2\omega_{\lambda_+,\rho_+} < \omega,$$

and

$$
\begin{aligned}
&\left|e_{\lambda,p_-,p_+}(z)\right| \\
&\quad \geqslant \left(p_+\left|\frac{(z+\lambda_+-\rho_+\ )^{1/2}}{(z+\lambda_-+\rho_-)^{1/2}}\right|\left|1-e^{-2L(z+\lambda_+-\rho_+\ )^{1/2}}\right|\left|1-e^{-2l(z+\lambda_-+\rho_-)^{1/2}}\right|\right. \\
&\qquad \left. + p_-\left|1+e^{-2l(z+\lambda_-+\rho_-)^{1/2}}\right|\left|1+e^{-2L(z+\lambda_+-\rho_+\ )^{1/2}}\right|\right)\cdot\cos\left(\frac{\omega}{2}\right),
\end{aligned}
$$

or

$$
\begin{aligned}
\left|e_{\lambda,p_-,p_+}(z)\right| &\geqslant p_-\left|1+e^{-2l(z+\lambda_-+\rho_-)^{1/2}}\right|\left|1+e^{-2L(z+\lambda_+-\rho_+\ )^{1/2}}\right|\cos\left(\frac{\omega}{2}\right) \\
&\geqslant p_-\left[1-e^{-\pi/(2\tan(\omega_{\lambda_-,\rho_-})}\right]\left[1-e^{-\pi/(2\tan(\omega_{\lambda_+,\rho_+})}\right]\cos\left(\frac{\omega}{2}\right) \\
&\geqslant p_-\left(1-e^{-\pi/(2\tan(\omega/2)}\right)^2\cos\left(\frac{\omega}{2}\right) > 0;
\end{aligned}
$$

we have used the fact that, for any $z \in S_\omega$, we have

$$
\tan(\omega_{\lambda_-,\rho_-}) \leqslant \tan(\omega/2) \quad \text{and} \quad \tan(\omega_{\lambda_+,\rho_+}) \leqslant \tan(\omega/2).
$$

Therefore the function $e_{\lambda,p_-,p_+}(z)$ does not vanish on S_ω and the function $1/e_{\lambda,p_-,p_+}(z)$ is bounded, hence it belongs to $H^\infty(S_\omega)$. Moreover

$$
\left\|1/e_{\lambda,p_-,p_+}\right\|_\infty \leqslant \frac{C}{p_-} = \frac{C}{(1-p)d_-}.
$$

We then conclude that Δ_{λ,p_-,p_+} is boundedly invertible and

$$
\begin{cases}
\Delta^{-1}_{\lambda,p_-,p_+} = \left(1/e_{\lambda,p_-,p_+}\right)(-A) \\
\left\|\Delta^{-1}_{\lambda,p_-,p_+}\right\|_{\mathcal{L}(X)} \leqslant \dfrac{C}{(1-p)d_-}.
\end{cases}
$$

Note that the constant C is independent of the habitats and parameter λ.

Now, from equality

$$
\Delta_{\lambda,p_-,p_+}A^{-1} = A^{-1}\Delta_{\lambda,p_-,p_+},
$$

it follows that

$$
\Delta^{-1}_{\lambda,p_-,p_+}A = A\Delta^{-1}_{\lambda,p_-,p_+},
$$

on $D(A)$, hence $\Delta^{-1}_{\lambda,p_-,p_+}$ is a bounded operator from $D(A)$ into itself. Therefore, by interpolation $\Delta^{-1}_{\lambda,p_-,p_+}$ is bounded from any interpolation space $(D(A), X)_{\alpha,q}$ (see the definition in [5]) into itself and clearly we have also the same estimate

$$
\left\|\Delta^{-1}_{\lambda,p_-,p_+}\right\|_{\mathcal{L}((D(A),X)_{\alpha,q})} \leqslant \frac{C}{(1-p)d_-}.
$$

7. Resolution of the spectral equation

Recall that if

$$
\begin{cases}
(I) = B_-^{-1}\left[-p_-v_-'(G_-)(0) + p_+v_+'(G_+)(0)\right] \\
(II) = \frac{1}{\lambda}\left[d_+G_+(0) - d_-G_-(0)\right] + v_+(G_+)(0) - v_-(G_-)(0),
\end{cases}
$$

and

$$\begin{cases} (I') = (I) - p_+ B_-^{-1} e^{LB_+} v_+'(G_+)(L) + p_- e^{lB_-} v_-(G_-)(-l) \\ (II') = (II) + e^{lB_-} v_-(G_-)(-l) + e^{LB_+} B_+^{-1} v_+'(G_+)(L); \end{cases}$$

then

$$\begin{aligned} (I') &= (I) - p_+ B_-^{-1} e^{LB_+} v_+'(G_+)(L) + p_- e^{lB_-} v_-(G_-)(-l) \\ &= B_-^{-1} \left[-p_- v_-'(G_-)(0) + p_+ v_+'(G_+)(0) \right] \\ &\quad - p_+ B_-^{-1} e^{LB_+} v_+'(G_+)(L) + p_- e^{lB_-} v_-(G_-)(-l), \\ (II') &= (II) + e^{lB_-} v_-(G_-)(-l) + e^{LB_+} B_+^{-1} v_+'(G_+)(L) \\ &= \frac{1}{\lambda} \left[d_+ G_+(0) - d_- G_-(0) \right] + v_+(G_+)(0) - v_-(G_-)(0) \\ &\quad + e^{lB_-} v_-(G_-)(-l) + e^{LB_+} B_+^{-1} v_+'(G_+)(L). \end{aligned}$$

From the system

$$\begin{cases} -p_- \left(I + e^{2lB_-} \right) \beta_- - p_+ B_-^{-1} B_+ \left(I - e^{2LB_+} \right) \alpha_+ = (I') \\ \left(I - e^{2lB_-} \right) \beta_- - \left(I + e^{2LB_+} \right) \alpha_+ = (II'), \end{cases}$$

we have

$$\begin{cases} \beta_- = \Delta_{\lambda,p_-,p_+}^{-1} \left[p_+ B_-^{-1} B_+ \left(I - e^{2LB_+} \right) (II') - \left(I + e^{2LB_+} \right) (I') \right] \\ \alpha_+ = \Delta_{\lambda,p_-,p_+}^{-1} \left[-p_- \left(I + e^{2lB_-} \right) (II') - \left(I - e^{2lB_-} \right) (I') \right], \end{cases}$$

and

$$\begin{aligned} \alpha_- &= -e^{lB_-} \beta_- - v_-(G_-)(-l) \\ &= -\Delta_{\lambda,p_-,p_+}^{-1} e^{lB_-} \left(\left[p_+ B_-^{-1} B_+ \left(I - e^{2LB_+} \right) (II') - \left(I + e^{2LB_+} \right) (I') \right] \right) \\ &\quad - v_-(G_-)(-l), \end{aligned}$$

from which we deduce that

$$\begin{aligned} u_-(x) &= e^{(x+l)B_-} \alpha_- + e^{-xB_-} \beta_- + v_-(G_-)(x) \\ &= -\Delta_{\lambda,p_-,p_+}^{-1} e^{lB_-} e^{(x+l)B_-} \left[p_+ B_-^{-1} B_+ \left(I - e^{2LB_+} \right) (II') - \left(I + e^{2LB_+} \right) (I') \right] \\ &\quad - e^{(x+l)B_-} \left[v_-(G_-)(-l) \right] \\ &\quad + \Delta_{\lambda,p_-,p_+}^{-1} e^{-xB_-} \left[p_+ B_-^{-1} B_+ \left(I - e^{2LB_+} \right) (II') - \left(I + e^{2LB_+} \right) (I') \right] \\ &\quad + v_-(G_-)(x); \end{aligned}$$

now, using

$$\begin{aligned} &p_+ B_-^{-1} B_+ \left(I - e^{2LB_+} \right) (II') - \left(I + e^{2LB_+} \right) (I') \\ &\quad = p_+ B_-^{-1} B_+ \left(I - e^{2LB_+} \right) \left[\frac{1}{\lambda} \left[d_+ G_+(0) - d_- G_-(0) \right] + \right] \\ &\qquad + p_+ B_-^{-1} B_+ \left(I - e^{2LB_+} \right) \left[v_+(G_+)(0) - v_-(G_-)(0) \right] \end{aligned}$$

$$
\begin{aligned}
&+ p_+ B_-^{-1} B_+ \left(I - e^{2LB_+}\right) \left[e^{lB_-} v_-(G_-)(-l) + e^{LB_+} B_+^{-1} v_+'(G_+)(L)\right] \\
&- B_-^{-1} \left(I + e^{2LB_+}\right) \left[-p_- v_-'(G_-)(0) + p_+ v_+'(G_+)(0)\right] \\
&- \left(I + e^{2LB_+}\right) \left[p_- e^{lB_-} v_-(G_-)(-l) - p_+ B_-^{-1} e^{LB_+} v_+'(G_+)(L)\right],
\end{aligned}
$$

we get for $x \in \,]-l, 0[$,

$$
\begin{aligned}
&u_- (x) \\
&= e^{(x+l)B_-} \alpha_- + e^{-xB_-} \beta_- + v_-(G_-)(x) \\
&= -\Delta_{\lambda,p_-,p_+}^{-1} e^{lB_-} e^{(x+l)B_-} \left[p_+ B_-^{-1} B_+ \left(I - e^{2LB_+}\right) (II') - \left(I + e^{2LB_+}\right) (I')\right] \\
&\quad - e^{(x+l)B_-} \left[v_-(G_-)(-l)\right] \\
&\quad + \Delta_{\lambda,p_-,p_+}^{-1} e^{-xB_-} \left[p_+ B_-^{-1} B_+ \left(I - e^{2LB_+}\right) (II') - \left(I + e^{2LB_+}\right) (I')\right] \\
&\quad + v_-(G_-)(x),
\end{aligned}
$$

or

$$
\begin{aligned}
&u_- (x) \\
&= \Delta_{\lambda,p_-,p_+}^{-1} \left(e^{-xB_-} - e^{(x+2l)B_-}\right) \left[p_+ B_-^{-1} B_+ \left(I - e^{2LB_+}\right) (II')\right] \\
&\quad - \Delta_{\lambda,p_-,p_+}^{-1} \left(e^{-xB_-} - e^{(x+2l)B_-}\right) \left[\left(I + e^{2LB_+}\right) (I')\right] \\
&\quad - e^{(x+l)B_-} \left[v_-(G_-)(-l)\right] \\
&\quad + v_-(G_-)(x).
\end{aligned}
\tag{7.1}
$$

In the same way, we have

$$
\begin{aligned}
\beta_+ &= e^{LB_+} \alpha_+ + B_+^{-1} v_+'(G_+)(L) \\
&= \Delta_{\lambda,p_-,p_+}^{-1} e^{LB_+} \left(\left[-p_- \left(I + e^{2lB_-}\right) (II') - \left(I - e^{2lB_-}\right) (I')\right]\right) \\
&\quad + B_+^{-1} v_+'(G_+)(L),
\end{aligned}
$$

and for all $x \in \,]0, L[$,

$$
\begin{aligned}
&u_+ (x) \\
&= \Delta_{\lambda,p_-,p_+}^{-1} \left(e^{xB_+} + e^{(2L-x)B_+}\right) \left[-p_- \left(I + e^{2lB_-}\right) (II') - \left(I - e^{2lB_-}\right) (I')\right] \\
&\quad + B_+^{-1} e^{(L-x)B_+} v_+'(G_+)(L) \\
&\quad + v_+(G_+)(x).
\end{aligned}
$$

8. The non-continuity of the density

As we have mentioned, the population density is not continuous across the interface. In fact we have

$$
\begin{aligned}
&u_- (0) \\
&= \Delta_{\lambda,p_-,p_+}^{-1} \left(I - e^{2lB_-}\right) \left[p_+ B_-^{-1} B_+ \left(I - e^{2LB_+}\right) (II') - \left(I + e^{2LB_+}\right) (I')\right] \\
&\quad - e^{lB_-} \left[v_-(G_-)(-l)\right] + v_-(G_-)(0),
\end{aligned}
\tag{8.1}
$$

and

$$\begin{aligned} u_+(0) &= \Delta^{-1}_{\lambda,p_-,p_+}\left(I+e^{2LB_+}\right)\left[-p_-\left(I+e^{2lB_-}\right)(II') - \left(I-e^{2lB_-}\right)(I')\right] \\ &\quad + B_+^{-1}e^{LB_+}v'_+(G_+)(L) + v_+(G_+)(0), \end{aligned}$$

then

$$\begin{aligned} & u_-(0) - u_+(0) \\ &\quad = \Delta^{-1}_{\lambda,p_-,p_+}\left(I-e^{2lB_-}\right)\left[p_+B_-^{-1}B_+\left(I-e^{2LB_+}\right)(II') - \left(I+e^{2LB_+}\right)(I')\right] \\ &\qquad + \Delta^{-1}_{\lambda,p_-,p_+}\left(I+e^{2LB_+}\right)\left[p_-\left(I+e^{2lB_-}\right)(II') + \left(I-e^{2lB_-}\right)(I')\right] \\ &\qquad - e^{lB_-}\left[v_-(G_-)(-l)\right] - B_+^{-1}e^{LB_+}v'_+(G_+)(L) \\ &\qquad + v_-(G_-)(0) - v_+(G_+)(0); \end{aligned}$$

but

$$\Delta_{\lambda,p_-,p_+} = p_-\left(I+e^{2lB_-}\right)\left(I+e^{2LB_+}\right) + p_+B_-^{-1}B_+\left(I-e^{2LB_+}\right)\left(I-e^{2lB_-}\right),$$

so

$$\begin{aligned} & u_-(0) - u_+(0) \\ &\quad = \Delta^{-1}_{\lambda,p_-,p_+}\left[p_+B_-^{-1}B_+\left(I-e^{2LB_+}\right)\left(I-e^{2lB_-}\right)\right. \\ &\qquad \left. + p_-\left(I+e^{2LB_+}\right)\left(I+e^{2lB_-}\right)\right](II') \\ &\qquad + \Delta^{-1}_{\lambda,p_-,p_+}\left[\left(I+e^{2LB_+}\right)\left(I-e^{2lB_-}\right) - \left(I-e^{2lB_-}\right)\left(I+e^{2LB_+}\right)\right](I') \\ &\qquad - e^{lB_-}\left[v_-(G_-)(-l)\right] - B_+^{-1}e^{LB_+}v'_+(G_+)(L) \\ &\qquad + v_-(G_-)(0) - v_+(G_+)(0) \\ &\quad = (II') - e^{lB_-}\left[v_-(G_-)(-l)\right] - B_+^{-1}e^{LB_+}v'_+(G_+)(L) \\ &\qquad + v_-(G_-)(0) - v_+(G_+)(0); \end{aligned}$$

now, recall that

$$\begin{aligned} (II') &= (II) + e^{lB_-}v_-(G_-)(-l) + e^{LB_+}B_+^{-1}v'_+(G_+)(L) \\ &= \frac{1}{\lambda}\left[d_+G_+(0) - d_-G_-(0)\right] + v_+(G_+)(0) - v_-(G_-)(0) \\ &\quad + e^{lB_-}v_-(G_-)(-l) + e^{LB_+}B_+^{-1}v'_+(G_+)(L). \end{aligned}$$

Therefore

$$\begin{aligned} & u_-(0) - u_+(0) \\ &\quad = \frac{1}{\lambda}\left[d_+G_+(0) - d_-G_-(0)\right] + v_+(G_+)(0) - v_-(G_-)(0) \\ &\qquad + e^{lB_-}v_-(G_-)(-l) + e^{LB_+}B_+^{-1}v'_+(G_+)(L) \\ &\qquad - e^{lB_-}\left[v_-(G_-)(-l)\right] - B_+^{-1}e^{LB_+}v'_+(G_+)(L) \\ &\qquad + v_-(G_-)(0) - v_+(G_+)(0) \\ &\quad = \frac{1}{\lambda}\left[d_+G_+(0) - d_-G_-(0)\right] \neq 0, \end{aligned}$$

since

$$d_+G_+(0) - d_-G_-(0) = g_+(0) - g_-(0) \neq 0.$$

9. $u \in D(\mathcal{L})$

We must verify that $u \in D(\mathcal{L})$.

Recall that for all $x \in]-l, 0[$

$$\begin{aligned} &u_-(x) \\ &= \Delta^{-1}_{\lambda,p_-,p_+}\left(e^{-xB_-} - e^{(x+2l)B_-}\right)\left[p_+B_-^{-1}B_+\left(I - e^{2LB_+}\right)(II')\right] \\ &\quad - \Delta^{-1}_{\lambda,p_-,p_+}\left(e^{-xB_-} - e^{(x+2l)B_-}\right)\left[\left(I + e^{2LB_+}\right)(I')\right] \\ &\quad - e^{(x+l)B_-}\left[v_-(G_-)(-l)\right] + v_-(G_-)(x), \end{aligned}$$

and for all $x \in]0, L[$,

$$\begin{aligned} &u_+(x) \\ &= \Delta^{-1}_{\lambda,p_-,p_+}\left(e^{xB_+} + e^{(2L-x)B_+}\right)\left[-p_-\left(I + e^{2lB_-}\right)(II') - \left(I - e^{2lB_-}\right)(I')\right] \\ &\quad + B_+^{-1}e^{(L-x)B_+}v_+'(G_+)(L) + v_+(G_+)(x). \end{aligned}$$

We must prove that $u \in D(\mathcal{L})$, that is

$$\begin{cases} i)\ u_- \in C^\theta\left([-l, 0[; C_0([0,1])\right) \\ ii)\ u_-', u_-'' \in C\left([-l, 0[; C_0([0,1])\right) \\ iii)\ x \mapsto \left[u_-''(x) - B_-^2u_-(x)\right] \in C^\theta\left([-l, 0[; C_0([0,1])\right), \end{cases}$$

$$\begin{cases} iv)\ u_+ \in C^\theta\left(]0, L]; C_0([0,1])\right) \\ v)\ u_+', u_+'' \in C\left(]0, L]; C_0([0,1])\right) \\ vi)\ x \mapsto \left[u_+''(x) - B_+^2u_+(x)\right] \in C^\theta\left(]0, L]; C_0([0,1])\right), \end{cases}$$

and all the boundary, skewness and dispersal continuous conditions are verified.

Note that *iii*) and *vi*) imply that

$$\left[u_-''(0^-) - B_-^2u_-(0^-)\right] \text{ and } \left[u_+''(0^+) - B_+^2u_+(0^+)\right],$$

exist.

Let us for instance prove *iii*). We have, for all $x \in]-l, 0[$

$$\begin{aligned} &B_-^2u_-(x) \\ &= \Delta^{-1}_{\lambda,p_-,p_+}B_-^2\left(e^{-xB_-} - e^{(x+2l)B_-}\right)\left[p_+B_-^{-1}B_+\left(I - e^{2LB_+}\right)(II')\right] \\ &\quad - \Delta^{-1}_{\lambda,p_-,p_+}B_-^2\left(e^{-xB_-} - e^{(x+2l)B_-}\right)\left(I + e^{2LB_+}\right)(I') \\ &\quad - B_-^2e^{(x+l)B_-}\left[v_-(G_-)(-l)\right] + B_-^2v_-(G_-)(x), \end{aligned}$$

and

$$
\begin{aligned}
&u_-''(x)\\
&= \Delta_{\lambda,p_-,p_+}^{-1} B_-^2 \left(e^{-xB_-} - e^{(x+2l)B_-}\right) \left[p_+ B_-^{-1} B_+ \left(I - e^{2LB_+}\right)(II')\right]\\
&\quad - \Delta_{\lambda,p_-,p_+}^{-1} B_-^2 \left(e^{-xB_-} - e^{(x+2l)B_-}\right)\left(I + e^{2LB_+}\right)(I')\\
&\quad - B_-^2 e^{(x+l)B_-}\left[v_-(G_-)(-l)\right] + v_-''(G_-)(x).
\end{aligned}
$$

We have

$$
v_-(G_-)(x) = \frac{1}{2}\int_{-l}^{x} e^{(x-t)B_-} B_-^{-1} G_-(t)dt + \frac{1}{2}\int_{x}^{0} e^{(t-x)B_-} B_-^{-1} G_-(t)dt,
$$

$$
\begin{aligned}
v_-'(G_-)(x) &= \frac{1}{2}\int_{-l}^{x} e^{(x-t)B_-} G_-(t)dt - \frac{1}{2}\int_{x}^{0} e^{(t-x)B_-} G_-(t)dt\\
&= w_1(x) + w_2(x).
\end{aligned}
$$

Set for $\varepsilon > 0$ small enough

$$
w_{1\varepsilon}(x) = \frac{1}{2}\int_{-l}^{x-\varepsilon} e^{(x-t)B_-} G_-(t)dt, \quad w_{2\varepsilon}(x) = -\frac{1}{2}\int_{x+\varepsilon}^{0} e^{(t-x)B_-} G_-(t)dt,
$$

then

$$
\begin{aligned}
w_{1\varepsilon}'(x) &= \left(\frac{1}{2}\int_{-l}^{x-\varepsilon} e^{(x-t)B_-} G_-(t)dt\right)'\\
&= \frac{1}{2} e^{\varepsilon B_-} G_-(x-\varepsilon) + \frac{1}{2}\int_{-l}^{x-\varepsilon} B_- e^{(x-t)B_-} G_-(t)dt,\\
&= \frac{1}{2} e^{\varepsilon B_-} G_-(x-\varepsilon) + \frac{1}{2}\int_{-l}^{x-\varepsilon} B_- e^{(x-t)B_-} \left(G_-(t) - G_-(x)\right) dt\\
&\quad + \frac{1}{2}\int_{-l}^{x-\varepsilon} B_- e^{(x-t)B_-} G_-(x)dt,\\
w_{1\varepsilon}'(x) &= \frac{1}{2} e^{\varepsilon B_-}\left[G_-(x-\varepsilon) - G_-(x)\right] + \frac{1}{2} e^{(x+l)B_-} G_-(x)\\
&\quad + \frac{1}{2}\int_{-l}^{x-\varepsilon} B_- e^{(x-t)B_-}\left(G_-(t) - G_-(x)\right) dt.
\end{aligned}
$$

Similarly, we obtain

$$
w_{2\varepsilon}'(x) = \left(-\frac{1}{2}\int_{x+\varepsilon}^{0} e^{(t-x)B_-} G_-(t)dt\right)'
$$

$$
\begin{aligned}
&= \frac{1}{2} e^{\varepsilon B_-} G_-(x+\varepsilon) + \frac{1}{2} \int_{x+\varepsilon}^{0} B_- e^{(t-x)B_-} G_-(t) dt \\
&= \frac{1}{2} e^{\varepsilon B_-} G_-(x+\varepsilon) + \frac{1}{2} \int_{x+\varepsilon}^{0} B_- e^{(t-x)B_-} \left(G_-(t) - G_-(x)\right) dt \\
&\quad + \frac{1}{2} \int_{x+\varepsilon}^{0} B_- e^{(t-x)B_-} G_-(x) dt \\
&= \frac{1}{2} e^{\varepsilon B_-} \left(G_-(x+\varepsilon) - G_-(x)\right) + \frac{1}{2} e^{-xB_-} G_-(x) \\
&\quad + \frac{1}{2} \int_{x+\varepsilon}^{0} B_- e^{(t-x)B_-} \left(G_-(t) - G_-(x)\right) dt.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&-B_-^2 v_-(G_-)(x) \\
&= -\frac{1}{2} \int_{-l}^{x} B_- e^{(x-t)B_-} \left(G_-(t) - G_-(x)\right) dt - \frac{1}{2} \int_{-l}^{x} B_- e^{(x-t)B_-} G_-(x) dt \\
&\quad - \frac{1}{2} \int_{x}^{0} B_- e^{(t-x)B_-} \left(G_-(t) - G_-(x)\right) dt - \frac{1}{2} \int_{x}^{0} B_- e^{(t-x)B_-} G_-(x) dt \\
&= -\frac{1}{2} \int_{-l}^{x} B_- e^{(x-t)B_-} \left(G_-(t) - G_-(x)\right) dt - \frac{1}{2} \int_{x}^{0} B_- e^{(t-x)B_-} \left(G_-(t) - G_-(x)\right) dt \\
&\quad + \frac{1}{2} G_-(x) - \frac{1}{2} e^{(x+l)B_-} G_-(x) - \frac{1}{2} e^{-xB_-} G_-(x) + \frac{1}{2} G_-(x);
\end{aligned}
$$

we deduce

$$
\begin{aligned}
&[w'_{1\varepsilon}(x) + w'_{2\varepsilon}(x)] - B_-^2 v_-(G_-)(x) \\
&\quad = \frac{1}{2} e^{\varepsilon B_-} \left[G_-(x-\varepsilon) - G_-(x)\right] + \frac{1}{2} e^{(x+l)B_-} G_-(x) \\
&\qquad + \frac{1}{2} e^{\varepsilon B_-} \left[G_-(x+\varepsilon) - G_-(x)\right] + \frac{1}{2} e^{-xB_-} G_-(x) \\
&\qquad + \frac{1}{2} G_-(x) - \frac{1}{2} e^{(x+l)B_-} G_-(x) - \frac{1}{2} e^{-xB_-} G_-(x) + \frac{1}{2} G_-(x) \\
&\qquad + \frac{1}{2} \int_{-l}^{x-\varepsilon} B_- e^{(x-t)B_-} \left(G_-(t) - G_-(x)\right) dt \\
&\qquad + \frac{1}{2} \int_{x+\varepsilon}^{0} B_- e^{(t-x)B_-} \left(G_-(t) - G_-(x)\right) dt \\
&\qquad - \frac{1}{2} \int_{-l}^{x} B_- e^{(x-t)B_-} \left(G_-(t) - G_-(x)\right) dt
\end{aligned}
$$

$$-\frac{1}{2}\int_{x}^{0} B_-e^{(t-x)B_-}\left(G_-(t)-G_-(x)\right)dt,$$

and

$$\begin{aligned}
&\left\|[w'_{1\varepsilon}(x)+w'_{2\varepsilon}(x)]-B_-^2v_-(G_-)(x)-G_-(x)\right\| \\
&\leqslant \|[G_-(x-\varepsilon)-G_-(x)]\|+\|[G_-(x+\varepsilon)-G_-(x)]\| \\
&\quad+\frac{1}{2}\left\|\int_{x-\varepsilon}^{x} B_-e^{(x-t)B_-}\left(G_-(t)-G_-(x)\right)dt\right\| \\
&\quad+\frac{1}{2}\left\|\int_{x}^{x+\varepsilon} B_-e^{(t-x)B_-}\left(G_-(t)-G_-(x)\right)dt\right\| \\
&\leqslant C\left(\varepsilon^\theta+\int_{x-\varepsilon}^{x}(x-t)^{\theta-1}\,dt+\int_{x}^{x+\varepsilon}(t-x)^{\theta-1}\,dt\right)\|G_-\|_{C^\theta([-l,0];E)},
\end{aligned}$$

and this last term tends to 0 as $\varepsilon\to 0$. It follows that

$$v''_-(G_-)(x)-B_-^2v_-(G_-)(x)=G_-(x);$$

therefore

$$\begin{aligned}
&u''_-(x)-B_-^2u_-(x) \\
&\quad=v''_-(G_-)(x)-B_-^2v_-(G_-)(x)=G_-(x),
\end{aligned}$$

from which we deduce that

$$x\mapsto\left[u''_-(x)-B_-^2u_-(x)\right]\in C^\theta\left([-l,0[;C_0([0,1])\right),$$

and

$$d_-\left[u''_-(x)+Au_-(x)\right]-r_-u_-(x)=\lambda u_-(x)+d_-G_-(x).$$

Since we have computed $u_-(0^-)$, we then deduce that the limit

$$\lim_{x\to 0^-}\left[d_-\left[u''_-(x)+Au_-(x)\right]-r_-u_-(x)\right]$$

exists and

$$\left[d_-\left[u''_-(0^-)+Au_-(0^-)\right]-r_-u_-(0^-)\right]=\lambda u_-(0^-)+d_-G_-(0).$$

Similarly, we obtain, for all $x\in]0,L[$

$$d_+\left[u''_+(x)+Au_+(x)\right]+r_+u_+(x)=\lambda u_+(x)+d_+G_+(x),$$

which implies

$$d_+\left[u''_+\left(0^+\right)+Au_+\left(0^+\right)\right]+r_+u_+\left(0^+\right)=\lambda u_+(0^+)+d_+G_+(0).$$

10. Estimate of the resolvent operator

10.1. Estimate of the norm $\|u_-\|_\infty$

Here $x \in [-l, 0]$. All the constants C in this subsection are independent of λ in virtue of (5.2). Let us first estimate

$$
\begin{aligned}
(a) &= \frac{1}{2}\int_{-l}^{x} e^{(x-t)B_-} B_-^{-1} G_-(t)dt + \frac{1}{2}\int_{x}^{0} e^{(t-x)B_-} B_-^{-1} G_-(t)dt \\
&= \frac{1}{2}\int_{-l}^{x} e^{(x-t)B_-} B_-^{-1} \left(G_-(t) - G_-(x)\right) dt \\
&\quad + \frac{1}{2}\int_{x}^{0} e^{(t-x)B_-} B_-^{-1} \left(G_-(t) - G_-(x)\right) dt \\
&\quad + \frac{1}{2}\int_{-l}^{x} e^{(x-t)B_-} B_-^{-1} G_-(x)dt + \frac{1}{2}\int_{x}^{0} e^{(t-x)B_-} B_-^{-1} G_-(x)dt,
\end{aligned}
$$

or

$$
\begin{aligned}
(a) &= \frac{1}{2}B_-^{-2}\int_{-l}^{x} B_- e^{(x-t)B_-} \left(G_-(t) - G_-(x)\right) dt \\
&\quad + \frac{1}{2}B_-^{-2}\int_{x}^{0} B_- e^{(t-x)B_-} \left(G_-(t) - G_-(x)\right) dt \\
&\quad + \frac{1}{2}B_-^{-2} e^{(x+l)B_-} G_-(x) + \frac{1}{2}B_-^{-2} e^{-xB_-} G_-(x) - B_-^{-2} G_-(x).
\end{aligned}
$$

We have

$$
\begin{aligned}
\|(a)\|_E &\leqslant C \left\|B_-^{-2}\right\| \|G_-\|_{C^\theta([-l,0];E)} \\
&\leqslant C \left\| \left(A - \frac{r_-}{d_-}I - \frac{\lambda}{d_-}I\right)^{-1} \right\| \|G_-\|_{C^\theta([-l,0];E)} \\
&\leqslant \frac{C}{\left|\dfrac{\lambda}{d_-} + \dfrac{r_-}{d_-}\right|} \|G_-\|_{C^\theta([-l,0];E)} \,.
\end{aligned}
$$

Now, we estimate

$$
\begin{aligned}
(b) = v_-(G_-)(-l) &= \frac{1}{2}\int_{-l}^{0} e^{(t+l)B_-} B_-^{-1} G_-(t)dt \\
= \frac{1}{2}B_-^{-2}\int_{-l}^{0} & B_- e^{(t+l)B_-} \left(G_-(t) - G_-(-l)\right) dt + \frac{1}{2}B_-^{-1}\int_{-l}^{0} e^{(t+l)B_-} G_-(-l)dt
\end{aligned}
$$

$$= \frac{1}{2} B_-^{-2} \int_{-l}^{0} B_- e^{(t+l)B_-} \left(G_-(t) - G_-(-l)\right) dt + \frac{1}{2} B_-^{-2} \left(e^{lB_-} - I\right) G_-(-l)$$

as above. We obtain

$$\begin{aligned} \|(b)\|_E &\leqslant C \left\|B_-^{-2}\right\| \|G_-\|_{C^\theta([-l,0];E)} \\ &\leqslant \frac{C}{\left|\dfrac{\lambda}{d_-} + \dfrac{r_-}{d_-}\right|} \|G_-\|_{C^\theta([-l,0];E)} . \end{aligned}$$

For the third following term

$$\begin{aligned} (c) &= v_+(G_+)(0) - v_-(G_-)(0) \\ &= \frac{1}{2} \int_0^L e^{tB_+} B_+^{-1} G_+(t) dt - \frac{1}{2} \int_{-l}^{0} e^{-tB_-} B_-^{-1} G_-(t) dt, \end{aligned}$$

we write

$$\begin{aligned} (c) &= \frac{1}{2} \int_0^L e^{tB_+} B_+^{-1} G_+(t) dt - \frac{1}{2} \int_{-l}^{0} e^{-tB_-} B_-^{-1} G_-(t) dt \\ &= \frac{1}{2} \int_0^L B_+^{-1} e^{tB_+} \left(G_+(t) - G_+(0)\right) dt + \frac{1}{2} \int_0^L B_+^{-1} e^{tB_+} G_+(0) dt \\ &\quad - \frac{1}{2} \int_{-l}^{0} B_-^{-1} e^{-tB_-} \left(G_-(t) - G_-(0)\right) dt - \frac{1}{2} \int_{-l}^{0} B_-^{-1} e^{-tB_-} G_-(0) dt, \end{aligned}$$

so that

$$\begin{aligned} (c) = \frac{B_+^{-2}}{2} &\int_0^L B_+ e^{tB_+} \left(G_+(t) - G_+(0)\right) dt \\ &- \frac{B_-^{-2}}{2} \int_{-l}^{0} B_- e^{-tB_-} \left(G_-(t) - G_-(0)\right) dt \\ &+ \frac{1}{2} B_+^{-2} \left(e^{LB_+} - I\right) G_+(0) + \frac{1}{2} B_-^{-2} \left(I - e^{lB_+}\right) G_-(0), \end{aligned}$$

from which it follows

$$\begin{aligned} \|(c)\|_E &\leqslant C \left\|B_+^{-2}\right\| \|G_+\|_{C^\theta([0,L];E)} + C \left\|B_-^{-2}\right\| \|G_-\|_{C^\theta([-l,0];E)} \\ &\leqslant \frac{C}{\left|\dfrac{\lambda}{d_+} - \dfrac{r_+}{d_+}\right|} \|G_+\|_{C^\theta([0,L];E)} + \frac{C}{\left|\dfrac{\lambda}{d_-} + \dfrac{r_-}{d_-}\right|} \|G_-\|_{C^\theta([-l,0];E)} ; \end{aligned}$$

the similar estimate is obtained for all the terms in $u_-(x)$. Summarizing we have, for all complex λ such that

$$\begin{cases} |\lambda| > r_+ \\ |\arg(\lambda)| < \pi - \varepsilon_0 \end{cases}$$

the estimate

$$\|u_-\|_{C([-l,0];E)} \leqslant \frac{C}{\left|\dfrac{\lambda}{d_+} - \dfrac{r_+}{d_+}\right|} \|G_+\|_{C^\theta([0,L];E)} + \frac{C}{\left|\dfrac{\lambda}{d_-} + \dfrac{r_-}{d_-}\right|} \|G_-\|_{C^\theta([-l,0];E)}.$$

In the same way, we obtain

$$\|u_+\|_{C([0,L];E)} \leqslant \frac{C}{\left|\dfrac{\lambda}{d_+} - \dfrac{r_+}{d_+}\right|} \|G_+\|_{C^\theta([0,L];E)} + \frac{C}{\left|\dfrac{\lambda}{d_-} + \dfrac{r_-}{d_-}\right|} \|G_-\|_{C^\theta([-l,0];E)};$$

all these denominators do not vanish.

10.2. Estimate of the semi-norm $[u_-]_\theta$

As above, all the constants C in this subsection are independent of λ in virtue of (5.2).

We must now estimate the semi-norm

$$[u_-]_\theta = \sup_{\substack{x_1,x_2\in[-l,0] \\ x_1\neq x_2}} \frac{\|u_-(x_1) - u_-(x_2)\|_E}{|x_1 - x_2|^\theta};$$

recall that

$$\begin{aligned} &u_-(x) \\ &= \Delta^{-1}_{\lambda,p_-,p_+}\left(e^{-xB_-} - e^{(x+2l)B_-}\right)\left[p_+B_-^{-1}B_+\left(I - e^{2LB_+}\right)(II') - \left(I + e^{2LB_+}\right)(I')\right] \\ &\quad - e^{(x+l)B_-}\left[v_-(G_-)(-l)\right] \\ &\quad + v_-(G_-)(x), \end{aligned}$$

where

$$\begin{aligned} &p_+B_-^{-1}B_+\left(I - e^{2LB_+}\right)(II') - \left(I + e^{2LB_+}\right)(I') \\ &\quad = p_+B_-^{-1}B_+\left(I - e^{2LB_+}\right)\left[\frac{1}{\lambda}\left[d_+G_+(0) - d_-G_-(0)\right] + v_+(G_+)(0) - v_-(G_-)(0)\right] \\ &\qquad + p_+B_-^{-1}B_+\left(I - e^{2LB_+}\right)\left[e^{lB_-}v_-(G_-)(-l) + e^{LB_+}B_+^{-1}v_+'(G_+)(L)\right] \\ &\qquad - B_-^{-1}\left(I + e^{2LB_+}\right)\left[-p_-v_-'(G_-)(0) + p_+v_+'(G_+)(0)\right] \\ &\qquad - \left(I + e^{2LB_+}\right)\left[p_-e^{lB_-}v_-(G_-)(-l) - p_+B_-^{-1}e^{LB_+}v_+'(G_+)(L)\right]; \end{aligned}$$

or

$$\begin{aligned} &p_+B_-^{-1}B_+\left(I - e^{2LB_+}\right)(II') - \left(I + e^{2LB_+}\right)(I') \\ &\quad = p_+B_-^{-1}B_+\left[\frac{1}{\lambda}\left[d_+G_+(0) - d_-G_-(0)\right]\right] \\ &\qquad + p_+B_-^{-1}B_+\left[v_+(G_+)(0) - v_-(G_-)(0)\right] \end{aligned}$$

$$
\begin{aligned}
&- p_+ B_-^{-1} B_+ e^{2LB_+} \left[\frac{1}{\lambda} \left[d_+ G_+(0) - d_- G_-(0) \right] + v_+(G_+)(0) - v_-(G_-)(0) \right] \\
&+ p_+ B_-^{-1} B_+ \left(I - e^{2LB_+} \right) \left[e^{lB_-} v_-(G_-)(-l) + e^{LB_+} B_+^{-1} v_+'(G_+)(L) \right] \\
&- B_-^{-1} \left(I + e^{2LB_+} \right) \left[-p_- v_-'(G_-)(0) + p_+ v_+'(G_+)(0) \right] \\
&- \left(I + e^{2LB_+} \right) \left[p_- e^{lB_-} v_-(G_-)(-l) - p_+ B_-^{-1} e^{LB_+} v_+'(G_+)(L) \right];
\end{aligned}
$$

therefore we can write

$$
\begin{aligned}
&p_+ B_-^{-1} B_+ \left(I - e^{2LB_+} \right) (II') - \left(I + e^{2LB_+} \right) (I') \\
&\quad = \frac{1}{\lambda} p_+ B_-^{-1} B_+ \left[d_+ G_+(0) - d_- G_-(0) \right] + (III),
\end{aligned}
$$

where the term (III) is regular, that is, at least in the domain of B_-. We obtain for the representation of u_-

$$
\begin{aligned}
u_-(x) = {} & \frac{1}{\lambda} p_+ B_-^{-1} B_+ \Delta_{\lambda,p_-,p_+}^{-1} e^{-xB_-} \left[d_+ G_+(0) - d_- G_-(0) \right] \\
& + \Delta_{\lambda,p_-,p_+}^{-1} e^{(x+2l)B_-} \left[p_+ B_-^{-1} B_+ \left(I - e^{2LB_+} \right) (II') - \left(I + e^{2LB_+} \right) (I') \right] \\
& + \Delta_{\lambda,p_-,p_+}^{-1} \left(e^{-xB_-} - e^{(x+2l)B_-} \right) \left[(III) \right] \\
& - e^{(x+l)B_-} \left[v_-(G_-)(-l) \right] \\
& + v_-(G_-)(x).
\end{aligned}
$$

We have

$$
\begin{aligned}
e^{(x+l)B_-} \left[v_-(G_-)(-l) \right] &= \frac{1}{2} e^{(x+l)B_-} \left(\int_{-l}^{0} e^{(t+l)B_-} B_-^{-1} G_-(t) dt \right) \\
&= \frac{1}{2} e^{(x+l)B_-} \left(\int_{-l}^{0} e^{(t+l)B_-} B_-^{-1} \left(G_-(t) - G_-(-l) \right) dt \right) \\
&\quad + \frac{1}{2} e^{(x+l)B_-} \left(\int_{-l}^{0} e^{(t-x)B_-} B_-^{-1} G_-(-l) dt \right) \\
&= \frac{1}{2} B_-^{-2} e^{(x+l)B_-} \left(\int_{-l}^{0} B_- e^{(t+l)B_-} \left(G_-(t) - G_-(-l) \right) dt \right) \\
&\quad + \frac{1}{2} B_-^{-2} e^{(x+l)B_-} \left(e^{lB_-} G_-(-l) - G_-(-l) \right),
\end{aligned}
$$

and

$$
\begin{aligned}
&v_-(G_-)(x) \\
&\quad = \frac{1}{2} \int_{-l}^{x} e^{(x-t)B_-} B_-^{-1} G_-(t) dt + \frac{1}{2} \int_{x}^{0} e^{(t-x)B_-} B_-^{-1} G_-(t) dt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2} B_-^{-2} \int_{-l}^{x} B_- e^{(x-t)B_-} \left(G_-(t) - G_-(x)\right) dt \\
&\quad + \frac{1}{2} B_-^{-2} \int_{x}^{0} B_- e^{(t-x)B_-} \left(G_-(t) - G_-(x)\right) dt \\
&\quad + \frac{1}{2} B_-^{-2} e^{(x+l)B_-} G_-(x) + \frac{1}{2} B_-^{-2} e^{-xB_-} G_-(x) - B_-^{-2} G_-(x).
\end{aligned}
$$

Summarizing, we get

$$
\begin{aligned}
&u_-(x) \\
&= \frac{1}{\lambda} p_+ B_-^{-1} B_+ \Delta_{\lambda,p_-,p_+}^{-1} e^{-xB_-} \left[d_+ G_+(0) - d_- G_-(0)\right] + (IV) \\
&\quad - \frac{1}{2} B_-^{-2} e^{(x+l)B_-} \left(\int_{-l}^{0} B_- e^{(t+l)B_-} \left(G_-(t) - G_-(-l)\right) dt \right) \\
&\quad - \frac{1}{2} B_-^{-2} e^{(x+l)B_-} \left(e^{lB_-} G_-(-l) - G_-(-l) \right) \\
&\quad + \frac{1}{2} B_-^{-2} \int_{-l}^{x} B_- e^{(x-t)B_-} \left(G_-(t) - G_-(x)\right) dt \\
&\quad + \frac{1}{2} B_-^{-2} \int_{x}^{0} B_- e^{(t-x)B_-} \left(G_-(t) - G_-(x)\right) dt \\
&\quad + \frac{1}{2} B_-^{-2} e^{(x+l)B_-} G_-(x) + \frac{1}{2} B_-^{-2} e^{-xB_-} G_-(x) - B_-^{-2} G_-(x);
\end{aligned}
$$

or

$$
\begin{aligned}
&u_-(x) \\
&= \frac{1}{\lambda} p_+ B_-^{-1} B_+ \Delta_{\lambda,p_-,p_+}^{-1} e^{-xB_-} \left[d_+ G_+(0) - d_- G_-(0)\right] + (IV) \\
&\quad - \frac{1}{2} B_-^{-2} e^{(x+l)B_-} \left(\int_{-l}^{0} B_- e^{(t+l)B_-} \left(G_-(t) - G_-(-l)\right) dt \right) \\
&\quad - \frac{1}{2} B_-^{-2} e^{(x+l)B_-} e^{lB_-} G_-(-l) + \frac{1}{2} B_-^{-2} e^{(x+l)B_-} G_-(-l) \\
&\quad + \frac{1}{2} B_-^{-2} \int_{-l}^{x} B_- e^{(x-t)B_-} \left(G_-(t) - G_-(x)\right) dt \\
&\quad + \frac{1}{2} B_-^{-2} \int_{x}^{0} B_- e^{(t-x)B_-} \left(G_-(t) - G_-(x)\right) dt \\
&\quad + \frac{1}{2} B_-^{-2} e^{(x+l)B_-} \left(G_-(x) - G_-(-l)\right) + \frac{1}{2} B_-^{-2} e^{(x+l)B_-} G_-(-l) \\
&\quad + \frac{1}{2} B_-^{-2} e^{-xB_-} \left(G_-(x) - G_-(0)\right) + \frac{1}{2} B_-^{-2} e^{-xB_-} G_-(0)
\end{aligned}
$$

$$- B_-^{-2}G_-(x);$$

notice that the term (IV) is regular. Set

$$\begin{cases} w_1(x) = \dfrac{1}{2}\int_{-l}^{x} B_- e^{(x-t)B_-}\left(G_-(t) - G_-(x)\right) dt \\ w_2(x) = \dfrac{1}{2}\int_{x}^{0} B_- e^{(t-x)B_-}\left(G_-(t) - G_-(x)\right) dt \\ \Psi = \int_{-l}^{0} B_- e^{(t+l)B_-}\left(G_-(t) - G_-(-l)\right) dt. \end{cases}$$

Then we know that

$$[w_1]_\theta \leqslant C\,\|G_-\|_{C^\theta([-l,0];E)}\,, \qquad [w_2]_\theta \leqslant C\,\|G_-\|_{C^\theta([-l,0];E)}\,,$$

by applying exactly the same techniques in [11], p. 46. On the other hand we have

$$\Psi \in D_{B_-}(\theta, +\infty) = D_A\left(\theta/2, +\infty\right),$$

which implies that

$$x \mapsto e^{(x+l)B_-}\left(\int_{-l}^{0} B_- e^{(t+l)B_-}\left(G_-(t) - G_-(-l)\right) dt\right) \in C^\theta\left([-l,0]\,; E\right).$$

Now, write

$$w_3(x) = e^{-xB_-}\left(G_-(x) - G_-(0)\right) + e^{-xB_-}G_-(0) := w_{31}(x) + w_{32}(x),$$

and observe that

$$\lim_{x\to 0^-} w_3(x) = G_-(0),$$

since $G_-(0) \in E = \overline{D(A)} = \overline{D(B_-)}$. Thus $w_3 \in C([-l,0]\,;E)$ and we know that

$$w_{31} \in C^\theta\left([-l,0]\,; E\right),$$

in virtue of [11] (see p. 47 the hölderianity of the function in (4.19)). On the other hand

$$w_{32} \in C^\theta\left([-l,0]\,; E\right),$$

iff

$$G_-(0) \in D_{B_-}(\theta, +\infty) = D_A\left(\theta/2, +\infty\right).$$

Similarly we get

$$x \mapsto e^{(x+l)B_-}G_-(-l) \in C^\theta\left([-l,0]\,; E\right),$$

iff

$$G_-(-l) \in D_{B_-}(\theta, +\infty) = D_A\left(\theta/2, +\infty\right).$$

Summarizing, if we assume that

$$g_+(0),\ g_-(0),\ g_-(-l) \in D_A(\theta/2,+\infty),$$

we have

$$\begin{aligned}[u_-]_\theta \leqslant{} & \frac{C}{|\lambda|}\|g_+(0)-g_-(0)\|_{D_A(\theta/2,+\infty)} \\ & + C\left\|B_-^{-2}\right\|\|g_-\|_{C^\theta([-l,0];E)} \\ & + C\left\|B_-^{-2}\right\|\left[\left\|e^{lB_-}G_-(-l)\right\|_{D_A(\theta/2,+\infty)} + \|G_-(-l)\|_{D_A(\theta/2,+\infty)}\right] \\ & + C\left\|B_-^{-2}\right\|\|g_-(0)\|_{D_A(\theta/2,+\infty)}.\end{aligned}$$

But

$$\left\|B_-^{-2}\right\| = \left\|-\left[-\left(A-\frac{r_-}{d_-}I-\frac{\lambda}{d_-}I\right)\right]^{-1}\right\| \leqslant \frac{C}{\left|\dfrac{\lambda}{d_-}+\dfrac{r_-}{d_-}\right|},$$

thus

$$\begin{aligned}[u_-]_\theta \leqslant{} & \frac{C}{|\lambda|}\|g_+(0)-g_-(0)\|_{D_A(\theta/2,+\infty)} + \frac{C}{\left|\dfrac{\lambda}{d_-}+\dfrac{r_-}{d_-}\right|}\|g_-\|_{C^\theta([-l,0];E)} \\ & + \frac{C}{\left|\dfrac{\lambda}{d_-}+\dfrac{r_-}{d_-}\right|}\left[\left\|e^{lB_-}G_-(-l)\right\|_{D_A(\theta/2,+\infty)} + \|G_-(-l)\|_{D_A(\theta/2,+\infty)}\right] \\ & + \frac{C}{\left|\dfrac{\lambda}{d_-}+\dfrac{r_-}{d_-}\right|}\|g_-(0)\|_{D_A(\theta/2,+\infty)}.\end{aligned}$$

10.3. Estimate of the semi-norm $[u_+]_\theta$

As in the two subsections above, all the constants C here are independent of λ in virtue of (5.2). Recall that

$$\begin{aligned}& u_+(x) \\ & = \Delta_{\lambda,p_-,p_+}^{-1}\left(e^{xB_+}+e^{(2L-x)B_+}\right)\left[-p_-\left(I+e^{2lB_-}\right)(II')-\left(I-e^{2lB_-}\right)(I')\right] \\ & \quad + B_+^{-1}e^{(L-x)B_+}v_+'(G_+)(L) \\ & \quad + v_+(G_+)(x),\end{aligned}$$

where

$$\begin{cases}(I') = (I) - p_+B_-^{-1}e^{LB_+}v_+'(G_+)(L) + p_-e^{lB_-}v_-(G_-)(-l) \\ (II') = (II) + e^{lB_-}v_-(G_-)(-l) + e^{LB_+}B_+^{-1}v_+'(G_+)(L),\end{cases}$$

$$\begin{cases}(I) = B_-^{-1}\left[-p_-v_-'(G_-)(0)+p_+v_+'(G_+)(0)\right] \\ (II) = \frac{1}{\lambda}\left[d_+G_+(0)-d_-G_-(0)\right] + v_+(G_+)(0) - v_-(G_-)(0).\end{cases}$$

Then

$$
\begin{aligned}
&-p_-\left(I+e^{2lB_-}\right)(II')-\left(I-e^{2lB_-}\right)(I') \\
&\quad=-p_-\left(I+e^{2lB_-}\right)\left[(II)+e^{lB_-}v_-(G_-)(-l)+e^{LB_+}B_+^{-1}v_+'(G_+)(L)\right] \\
&\qquad-\left(I-e^{2lB_-}\right)\left[(I)-p_+B_-^{-1}e^{LB_+}v_+'(G_+)(L)+p_-e^{lB_-}v_-(G_-)(-l)\right] \\
&\quad=-p_-\left(I+e^{2lB_-}\right)\left[\frac{1}{\lambda}\left[d_+G_+(0)-d_-G_-(0)\right]+v_+(G_+)(0)-v_-(G_-)(0)\right] \\
&\qquad-p_-\left(I+e^{2lB_-}\right)\left[e^{lB_-}v_-(G_-)(-l)+e^{LB_+}B_+^{-1}v_+'(G_+)(L)\right] \\
&\qquad-\left(I-e^{2lB_-}\right)\left[B_-^{-1}\left[-p_-v_-'(G_-)(0)+p_+v_+'(G_+)(0)\right]\right] \\
&\qquad-\left(I-e^{2lB_-}\right)\left[-p_+B_-^{-1}e^{LB_+}v_+'(G_+)(L)+p_-e^{lB_-}v_-(G_-)(-l)\right].
\end{aligned}
$$

We have

$$
\begin{aligned}
&v_+(G_+)(x)+B_+^{-1}e^{(L-x)B_+}v_+'(G_+)(L) \\
&\quad=\frac{1}{2}\int_0^x e^{(x-t)B_+}B_+^{-1}G_+(t)dt+\frac{1}{2}\int_x^L e^{(t-x)B_+}B_+^{-1}G_+(t)dt \\
&\qquad+\frac{1}{2}B_+^{-1}e^{(L-x)B_+}\int_0^L e^{(L-t)B_+}G_+(t)dt.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{1}{2}B_+^{-1}e^{(L-x)B_+}\int_0^L e^{(L-t)B_+}G_+(t)dt \\
&\quad=\frac{1}{2}B_+^{-2}e^{(L-x)B_+}\int_0^L B_+e^{(L-t)B_+}\left[G_+(t)-G_+(L)\right]dt \\
&\qquad+\frac{1}{2}B_+^{-1}e^{(L-x)B_+}\int_0^L e^{(L-t)B_+}G_+(L)dt \\
&\quad=\frac{1}{2}B_+^{-2}e^{(L-x)B_+}\int_0^L B_+e^{(L-t)B_+}\left[G_+(t)-G_+(L)\right]dt \\
&\qquad-\frac{1}{2}B_+^{-2}e^{(L-x)B_+}G_+(L)+\frac{1}{2}B_+^{-2}e^{(L-x)B_+}e^{LB_+}G_+(L),
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2}\int_0^x e^{(x-t)B_+}B_+^{-1}G_+(t)dt+\frac{1}{2}\int_x^L e^{(t-x)B_+}B_+^{-1}G_+(t)dt \\
&\quad=\frac{1}{2}B_+^{-2}\int_0^x B_+e^{(x-t)B_+}\left(G_+(t)-G_+(x)\right)dt
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{2} B_+^{-2} \int_x^L B_+ e^{(t-x)B_+} \left(G_+(t) - G_+(x)\right) dt \\
&+ \frac{1}{2} B_+^{-1} \int_0^x e^{(x-t)B_+} G_+(x) dt + \frac{1}{2} B_+^{-1} \int_x^L e^{(t-x)B_+} G_+(x) dt \\
= &\frac{1}{2} B_+^{-2} \int_0^x B_+ e^{(x-t)B_+} \left(G_+(t) - G_+(x)\right) dt \\
&+ \frac{1}{2} B_+^{-2} \int_x^L B_+ e^{(t-x)B_+} \left(G_+(t) - G_+(x)\right) dt \\
&+ \frac{1}{2} B_+^{-2} e^{xB_+} \left[G_+(x) - G_+(0)\right] + \frac{1}{2} B_+^{-2} e^{xB_+} G_+(0) \\
&+ \frac{1}{2} B_+^{-2} e^{(L-x)B_+} \left[G_+(x) - G_+(L)\right] + \frac{1}{2} B_+^{-2} e^{(L-x)B_+} G_+(L) \\
&- B_+^{-2} G_+(x).
\end{aligned}
$$

Now, since

$$
\left\| B_+^{-2} \right\| = \left\| - \left[- \left(A + \frac{r_+}{d_+} I - \frac{\lambda}{d_+} I \right) \right]^{-1} \right\| \leqslant \frac{C}{\left| \frac{\lambda}{d_+} - \frac{r_+}{d_+} \right|},
$$

the $[u_+]_\theta$-estimates hold as for $[u_-]_\theta$ if we assume

$$
g_+(0),\ g_+(L) \in D_A\left(\theta/2, +\infty\right),
$$

and we obtain

$$
\begin{aligned}
[u_+]_\theta \leqslant &\frac{C}{|\lambda|} \left\| g_+(0) - g_-(0) \right\|_{D_A(\theta/2,+\infty)} + \frac{C}{\left| \frac{\lambda}{d_+} - \frac{r_+}{d_+} \right|} \left\| g_+ \right\|_{C^\theta([0,L];E)} \\
&+ \frac{C}{\left| \frac{\lambda}{d_+} - \frac{r_+}{d_+} \right|} \left\| g_+(L) \right\|_{D_A(\theta/2,+\infty)} \\
&+ \frac{C}{\left| \frac{\lambda}{d_+} - \frac{r_+}{d_+} \right|} \left\| g_+(0) \right\|_{D_A(\theta/2,+\infty)}.
\end{aligned}
$$

This completes the proof of our main result Theorem 1.1.

Remark 10.1. We can specify the closure of the domain $D(\mathcal{L})$ by using the famous little Hölder continuous Banach spaces.

11. On the closure of $D(\mathcal{L})$ in $\mathcal{E}_*$

Let $I \subset \mathbb{R}$ be an interval. For $0 < \theta < 1$, the spaces of little-Hölder continuous functions are defined by

$$h^\theta(I;E) = \left\{ f \in C^\theta(I;E) : \lim_{\delta \to 0} \sup_{t,s\in I, |t-s|<\delta} \frac{\|f(t)-f(s)\|_E}{|t-s|^\theta} = 0 \right\};$$

and we have the following result: $h^\theta(I;E)$ is the closure of $C^{k+\theta}(I;E)$ in $C^\theta(I;E)$ for any $k \in \mathbb{N}$, see [8], p. 4, Proposition 0.2.1.

If we consider uniquely the function u_-, we have

$$\begin{cases} u_- \in C^2([-l,0[;E), u(-l) = 0, \quad u(0^-) \in E_\theta \\ \forall x \in [-l,L] \setminus \{0\}, \ u(x) \in D(A), \\ x \mapsto [u''_-(x) + Au_-(x)] \in C^\theta([-l,0[;E) \\ \text{and (Bound.C.), (Interf.C.), (Skew.C.).} \end{cases}$$

We know, from [8], p. 4, Proposition 0.2.1 that the closure of

$$\left\{ u_- \in C^2([-l,0[;E), u(-l) = 0, \quad u(0^-) \in E_\theta \right\},$$

in the norm of the space

$$\mathcal{E}_- = \left\{ u_- \in C^\theta([-l,0[;E) : u_-(-l) = 0, u_-(0^-) \in E_\theta \right\},$$

is exactly the following little-Hölder space

$$\begin{aligned} &\left\{ f \in h^\theta([-l,0[;E) : \ f(-l) = 0, \quad f(0^-) \in E_\theta \right\} \\ &\quad = \left\{ f \in h^\theta([-l,0];E) : \ f(-l) = 0, \quad f(0^-) \in E_\theta \right\}. \end{aligned}$$

Similarly the same arguments can be applied for u_+.

Therefore the closure of $D(\mathcal{L})$ in the norm of $\mathcal{E}_*$ is built on the little-Hölder spaces.

Acknowledgments

We would like to thank the Referees for their valuable remarks and comments on this article.

References

[1] A.V. Balakrishnan, Fractional powers of closed operators and the semigroups generated by them, Pacific J. Math. 10 (1960) 419–437.
[2] R.S. Cantrell, C. Cosner, Diffusion models for population dynamics incorporating individual behavior at boundaries: applications to refuge design, Theor. Popul. Biol. 55 (1999) 189–207.
[3] M. Cowling, I. Doust, A. McIntosh, A. Yagi, Banach space operator with a bounded H^∞ functional calculus, J. Aust. Math. Soc. A 60 (1996) 51–89.
[4] G. Dore, A. Favini, R. Labbas, K. Lemrabet, An abstract transmission problem in a thin layer, I: sharp estimates, J. Funct. Anal. 261 (2011) 1865–1922.
[5] P. Grisvard, Spazi di Tracce e Applicazioni, Rend. Mat. (4) 5 (VI) (1972) 657–729.
[6] M. Haase, The Functional Calculus for Sectorial Operators, Oper. Theory Adv. Appl., vol. 169, 2006.
[7] R. Labbas, A. Medeghri, A. Mened, Solvability of elliptic differential equation set in three habitats with skewness boundary conditions at the interfaces, Mediterr. J. Math. (2018) 1–23.
[8] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, 1995.
[9] O. Ovaskainen, S.J. Cornell, Biased movement at a boundary and conditional occupancy times for diffusion process, J. Appl. Probab. 40 (2003) 557–580.

[10] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Applied Mathematical Sciences, Springer-Verlag, New York, 1983, p. 119.

[11] E. Sinestrari, On the abstract Cauchy problem of parabolic type in space of continuous functions, J. Math. Anal. Appl. 66 (1985) 16–66.

[12] C.L. Zuily, H. Queffélec, Eléments d'analyse pour l'agrégation, Masson, Paris, 1995.