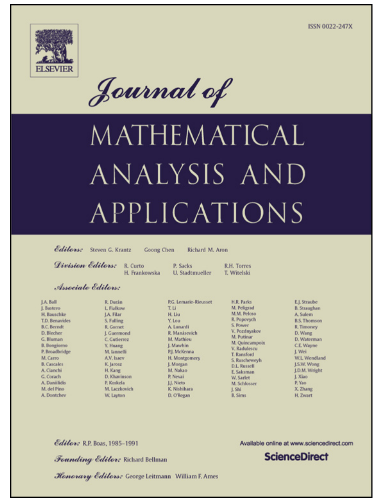
Journal Pre-proof



An approximate numerical method for solving Cauchy singular integral equations composed of multiple implicit parameter functions with unknown integral limits in contact mechanics

Qi Wen, Qungui Du

PII: S0022-247X(19)30798-X
DOI: https://doi.org/10.1016/j.jmaa.2019.123530
Reference: YJMAA 123530

To appear in: *Journal of Mathematical Analysis and Applications*

Received date: 3 March 2019

An approximate numerical method for solving Cauchy singular integral equations composed of multiple implicit parameter functions with unknown integral limits in contact mechanics

Qi Wen[a,b], Qungui Du[a,b,*]

a. School of Mechanical & Automotive Engineering, South China University of Technology, 381 WuShan Road, TianHe District, Guangzhou 510641, P.R. China

b. Guangdong Provincial Key Laboratory of Automotive Engineering, 381 WuShan Road, TianHe District, Guangzhou 510641, P.R. China

* Corresponding author: School of Mechanical & Automotive Engineering, South China University of Technology, 381 WuShan Road, TianHe District, Guangzhou 510641, P.R. China. E-mail address: ctqgdu@scut.edu.cn (Professor Qungui Du)

Abstract: Cauchy singular integral equations are widely used in physics and mathematics, especially in solid contact mechanics. The solution of Cauchy singular integral equations composed of explicit functions has already been achieved in existing research. However, when dealing with the contact problem between two solid bodies with irregular surfaces described by implicit parametric functions, difficulties arise when trying to solve the Cauchy singular integral because it is composed of multiple implicit parameter functions. Moreover, the integral limits are constrained by physical characteristics and the unknown. To solve this kind of problem, an approximate calculation method with high accuracy will be provided in this paper. Specifically, based on the quadrature method and taking the constraint function of the boundary as the convergence criterion, both the integral limits satisfying the physical characteristics condition and the solution of the Cauchy singular integral equations composed of multiple parameter functions can be derived by an iterative method. Finally, five different examples are calculated using the new method, and the absolute errors between the approximate values provided by the new method and the true values are analysed.

Keywords: Cauchy singular integral; Unknown integral limits; Implicit parameter function; Approximate numerical method

1. Introduction

There are two main methods for solving contact problems between solids. One is numerical method, including finite element method and boundary integral method. This kind of method is often used to solve dynamic contact problems such as the contact between spheres and general surfaces under the influence of impact[1], gear drives contact under impact condition[2], collision between ball-beam and thin walled tube-rigid wall[3], large deformation frictionless dynamic contact-impact[4], wheel-rail impact contact[5], contact between a rigid sphere pressing into an elastic half-space under cyclic loading[6], gear teeth contact under cyclic loading[7], contact between flat-on-flat and cylinder-on-flat under cyclic loading[8], rolling/sliding beam to beam contact without or with frictional conditions[9; 10; 11], two-dimensional large deformation frictional contact of large sliding contact[12; 13], sliding contact of rubber friction on rough rigid surfaces[14], contact and adhesion between a rough sphere and a flat surface[15].

Another method to solve the contact problem between solids is the analytical calculation method based on classical contact mechanics such as the sliding frictional contact between a rigid conducting cylindrical punch and a functionally graded piezoelectric coated half-plane[16], contact between an elastic coating and a moving punch[17], frictional sliding contact between a rigid circular conducting punch and a magneto-electro-elastic half-plane[18], dynamics with allowance for crack edge contact interaction[19], contact with friction for bodies with cracks[20], half-plane contact with initial stresses[21], rigid punch moving along the boundary of the elastic half-plane with initial stresses[22]. In these methods, different types of singular integral equations which will also appear in other application fields such as aeronautical technologies[23], queueing problems[24], etc. need to be solved. Besides, different solution methods are discussed due to the different characteristics of singular integral equations[25; 26]. Cauchy singular integral equation is most common in the classical contact problems, and the general form is usually defined as Eq.(1)[27; 28; 29; 30]. On this basis, other forms with some specific characteristics have also been discussed [31; 32; 33; 34; 35].

$$\int_{-1}^{1}\frac{P(s)}{x-s}ds+\int_{-1}^{1}K\left(x,s\right)P(s)ds=g(x)\qquad -1<x<1 \tag{1}$$

In Eq.(1), If $K(x, s)=0$, then the integral is reduced to the characteristic Cauchy singular integral equation in the form of Eq.(2), which is a type of Cauchy principal value integral. Here, $g(x)$ is determined by the explicit equation as follows:

$$\int_{-1}^{1}\frac{P(s)}{x-s}ds = g(x) \qquad -1<x<1 \tag{2}$$

In engineering applications, the function types of $g(x)$ are quite different. Therefore, their different solution methods should be chosen according to their characteristics. On the basis of different principles, these solution methods can be summarized as follows: the polynomial approximation method based on Chebyshev polynomials[36; 37; 38], Bernstein polynomials[39; 40], or Jacobi polynomials[41]; the differential transform method[42]; the Gaussian quadrature method[43]; the Nyström method for equations with negative index[44]; the Galerkin method[45; 46; 47]; the quadrature method[48; 49; 50]; the Adomian decomposition method[51]; the Sinc approximations method[52]; and the reproducing kernel Hilbert space method[53].

In the above methods, $g(x)$ is determined by the explicit equation, and the integral limits are known. When solving the problem of contact between two solid bodies with irregular profiles in the application of solid contact mechanics, the problem of solving Cauchy singular integral equations with the following characteristics will be faced:

(Ⅰ) $g(x)$ is a derivative function of solid profile, which is obtained by parametric functions $[X_1(k_1), Y_1(k_1)]$ and $[X_2(k_2), Y_2(k_2)]$. Moreover, the parameters k_1 and k_2 cannot be eliminated in $[X_1(k_1), Y_1(k_1)]$ and $[X_2(k_2), Y_2(k_2)]$.

(Ⅱ) $g(x)$ is a bivariate function and the integral limits a and b are unknown.

(Ⅲ) There is no primitive function for the integral function $P(s)/(x\text{-}s)$.

In accordance with the above characteristics, the integral equation Eq.(2) is converted to Eq.(3) as follows:

$$\int_{-b}^{a}\frac{P(s)}{x-s}ds = g(k_1, k_2) \tag{3}$$

Here, $P(s)$ is the pressure distribution function that needs to be solved for, and $g(k_1,k_2)$ could be defined as in Eq.(4) as follows:[30].

$$g\left(k_1,k_2\right)=Y_2^{'}\left(k_2\right)/X_2^{'}\left(k_2\right)-Y_1^{'}\left(k_1\right)/X_1^{'}\left(k_1\right) \tag{4}$$

In addition, Eq.(3) needs to satisfy the conditional equation as follows:

$$X_1\left(k_1\right)=X_2\left(k_2\right) \tag{5}$$

The upper and lower limits are related to the integral constant C, and the constraints can be written as follows:

$$\begin{aligned} P(a-\varepsilon_a) &= A+\varepsilon_1 \\ P(\varepsilon_b-b) &= B+\varepsilon_2 \end{aligned} \tag{6}$$

Here, ε_a, ε_b, ε_1, ε_2 are the extreme values, and A, B are true value at the boundaries.

In summary, Eq.(3) can be defined as the Cauchy singular integral equations composed of multiple parameter functions with unknown integral limits. However, there is no effective method to solve those equations in existing research. Therefore, an approximate numerical solution method based on the quadrature method is proposed in this paper.

2. The solution method

Because the method in this paper is mainly aimed at solving solid contact problems, and the parametric function groups [$X_1(k_1)$, $Y_1(k_1)$] and [$X_2(k_2)$, $Y_2(k_2)$] are used to describe the profiles of two solids in contact, respectively [54; 55]. Therefore, [$X_1(k_1)$, $Y_1(k_1)$] and [$X_2(k_2)$, $Y_2(k_2)$] are not only continuous, but also have and only have one intersection point on the contact region [$-b$, a][56]. In addition, $Y_2[X_2^{-1}(k_2)]>Y_1$, then the general solution of Eq.(3) can be expressed as Eq.(7) as below:

$$P(x)=\frac{1}{\pi^2\sqrt{(x+b)\cdot(a-x)}}\left(\int_{-b}^{a}\frac{\sqrt{(s+b)\cdot(a-s)}\cdot g(k_1,k_2)}{x-s}ds+C\right) \tag{7}$$

Here,

$$\begin{aligned} k_1 &= X_1^{-1}\left(s\right) \\ k_2 &= X_2^{-1}\left(s\right) \end{aligned} \tag{8}$$

Since the upper and lower integral limits a and b in Eq.(7) are unknown, the substitution should be carried out first in order to transform these into the integrand function. Let $s=(b+a)t/2+(a-b)/2$ and $z=(2x-a+b)/(a+b)$, then Eq.(7) can be turned into Eq.(9) as follows:

$$P(x)=\frac{a+b}{2\pi^2\sqrt{(x+b)\cdot(a-x)}}\left(\int_{-1}^{1}H(t)dt+C\right) \tag{9}$$

Rewrite it as:

$$P(x)=P_1(x)+P_2(x) \tag{10}$$

Here,

$$P_1(x)=\frac{a+b}{2\pi^2\sqrt{(x+b)\cdot(a-x)}}\int_{-1}^{1}H(t)dt \tag{11}$$

with

$$H(t)=\frac{h(t)}{z-t} \tag{12}$$

$$h(t)=\sqrt{1-t^2}Y_2^{'}\left\{X_2^{-1}\left(\frac{at+bt+a-b}{2}\right)\right\}/X_2^{'}\left\{X_2^{-1}\left(\frac{at+bt+a-b}{2}\right)\right\}$$
$$-\sqrt{1-t^2}Y_1^{'}\left\{X_1^{-1}\left(\frac{at+bt+a-b}{2}\right)\right\}/X_1^{'}\left\{X_1^{-1}\left(\frac{at+bt+a-b}{2}\right)\right\} \quad (13)$$

and

$$P_2(x)=\frac{(a+b)\cdot C}{2\pi^2\sqrt{(x+b)\cdot(a-x)}} \quad (14)$$

It can be determined from Eq.(11)-(14) that $P_2(x)$ is a known function; the calculation of $P_1(x)$ is mainly discussed below.

Theorem 1: If the parameter function groups $[X_1(k_1), Y_1(k_1)]$ and $[X_2(k_2), Y_2(k_2)]$ are not only continuous, but also have and only have one intersection point on $[-b, a]$. In addition, $Y_2[X_2^{-1}(k_2)]>Y_1\{X_1^{-1}[X_2(k_2)]\}$, and $h(t)$ is a $2m$ order differentiable on $[-1,1]$, then Eq.(15) can be obtained as follows:

$$\int_{-1}^{1}H(t)dt=w\sum_{\substack{j=0\\t_j\neq z}}^{2N}H(t_j)-w\sum_{\substack{i=0\\t_i\neq z}}^{N}H(t_i)+Re_1(w) \quad (15)$$

Here, $t_i=-1+iw$, $w=2/N$, N is the number of segments in the integral region, B_{2u} are the Bernoulli numbers, and $Re_1(w)$ is represented by Eq.(16) as follows:

$$Re_1(w)=\frac{B_2w^2}{4}[H^1(1)-H^1(-1)]+\sum_{u=2}^{m-1}\frac{B_{2u}w^{2u}}{(2u)!}[H^{2u-1}(-1)-H^{2u-1}(1)]\left(\frac{1}{2^{2u-1}}-1\right)+O(w^{2m}) \quad (16)$$

Proof.

First, when $w\to 0$, Eq.(17) can be obtained, based on the Euler-Maclaurin expansion theory provided by references[50; 57] as follows:

$$\int_{-1}^{1}H(t)dt-w\sum_{\substack{i=0\\t_i\neq z}}^{N}H(t_i)=wh^{'}(z)+\sum_{u=1}^{m-1}\frac{B_{2u}}{(2u)!}[H^{2u-1}(-1)-H^{2u-1}(1)]w^{2u}+O(w^{2m}) \quad (17)$$

To obtain a higher level of accuracy, Eq.(17) is made the first extrapolation. That is, the integral regions are divided into $2N$ equal parts, and the integral width of each segment is changed to $w/2$. Then Eq.(18) can be derived as follows:

$$\int_{-1}^{1}H(t)dt-\frac{w}{2}\sum_{\substack{j=0\\t_j\neq z}}^{2N}H(t_j)=\frac{w}{2}h^{'}(z)+\sum_{u=1}^{m-1}\frac{B_{2u}}{(2u)!}[H^{2u-1}(-1)-H^{2u-1}(1)]\left(\frac{w}{2}\right)^{2u}+O\left(\frac{w^{2m}}{2^{2m}}\right) \quad (18)$$

Here, $t_j=-1+jw/2$.

Carry out Eq.(18)×2−Eq.(17); then Eq.(19) can be obtained, which completes the proof of Theorem 1.

$$\int_{-1}^{1} H(t)dt - w\sum_{\substack{j=0\\ t_j\neq z}}^{2N} H(t_j) + w\sum_{\substack{i=0\\ t_i\neq z}}^{N} H(t_i)$$

$$= \frac{B_2 w^2}{4}[H^1(1) - H^1(-1)] + \sum_{u=2}^{m-1}\frac{B_{2u}w^{2u}}{(2u)!}[H^{2u-1}(-1) - H^{2u-1}(1)]\left(\frac{1}{2^{2u-1}} - 1\right) + O(w^{2m})$$

(19)

Remark. The integral accuracy of Eq.(19) has been raised to level $O(w^2)$ compared to that of Eq.(17). The analytic formula for $P_1(x)$ after the first extrapolation (ET=1) can be constructed as follows:

$$P_1(x)_{ET=1} \approx \frac{a+b}{2\pi^2\sqrt{(x+b)\cdot(a-x)}}\left[w\sum_{\substack{j=0\\ t_j\neq z}}^{2N} H(t_j) - w\sum_{\substack{i=0\\ t_i\neq z}}^{N} H(t_i)\right] \tag{20}$$

Corollary 1. If the parameter function groups $[X_1(k_1), Y_1(k_1)]$ and $[X_2(k_2), Y_2(k_2)]$ are not only continuous, but also have and only have one intersection point on $[-b, a]$. In addition, $Y_2\ [X_2^{-1}(k_2)] > Y_1\{X_1^{-1}[X_2(k_2)]\}$, and $h(t)$ is a $4m$ order differentiable on $[-1,1]$, then Eq.(21) can be obtained:

$$\int_{-1}^{1} H(t)\,dt = \frac{2w}{3}\sum_{\substack{p=0\\ t_{2p+1}\neq z}}^{2N-1} H\left(t_{2p+1}\right) - \frac{w}{3}\sum_{\substack{q=0\\ t_{2q+1}\neq z}}^{N-1} H\left(t_{2q+1}\right) + Re_2(w) \tag{21}$$

Here, $t_{2p+1}=(2p+1)w/4-1$, $t_{2q+1}=(2q+1)w/2-1$, and $Re_2\ (w)$ are represented by Eq.(22) as follows:

$$Re_2(w) = \frac{7B_4 w^4}{768}[H^3(-1) - H^3(1)] + \sum_{u=3}^{m-1}\frac{B_{2u}w^{2u}}{3(2u)!}[H^{2u-1}(-1) - H^{2u-1}(1)]\left(\frac{2^{2u}+2^{3-2u}-6}{2^{2u}}\right) + O(w^{2m}) \tag{22}$$

Proof.

The second extrapolation is done in the same way. The integral interval in Eq.(19) is halved and the integral width of each segment is changed to $w/4$.

Then, Eq.(23) can be derived as follows:

$$\int_{-1}^{1} H(t)dt - \frac{w}{2}\sum_{\substack{j=0\\ t_k\neq z}}^{4N} H(t_k) + \frac{w}{2}\sum_{\substack{i=0\\ t_j\neq z}}^{2N} H(t_j)$$

$$= \frac{B_2 w^2}{16}[H(1) - H(-1)] + \sum_{u=2}^{m-1}\frac{B_{2u}w^{2u}}{(2u)!}[H^{2u-1}(-1) - H^{2u-1}(1)]\left(\frac{2^{1-2u}-1}{2^{2u}}\right) + O\left(\frac{w^{2m}}{2^{2m}}\right) \tag{23}$$

Here, $t_k=-1+kw/4$.

Carry out Eq.(23)×4–Eq.(19); then Eq.(24) can be derived as follows:

$$
\begin{aligned}
&3\int_{-1}^{1}H(t)dt-2w\sum_{\substack{j=0\\t_k\neq z}}^{4N}H(t_k)+2w\sum_{\substack{i=0\\t_j\neq z}}^{2N}H(t_j)+w\sum_{\substack{j=0\\t_j\neq z}}^{2N}H(t_j)-w\sum_{\substack{i=0\\t_i\neq z}}^{N}H(t_i)\\
&=\frac{21B_4w^4}{768}[H^3(-1)-H^3(1)]+\sum_{u=3}^{m-1}\frac{B_{2u}w^{2u}}{(2u)!}[H^{2u-1}(-1)-H^{2u-1}(1)]\left(\frac{2^{2u}+2^{3-2u}-6}{2^{2u}}\right)+O(w^{2m})
\end{aligned}
\tag{24}
$$

Rewrite it as:

$$
\begin{aligned}
&\int_{-1}^{1}H(t)dt=\frac{2w}{3}\sum_{\substack{p=0\\t_{2p+1}\neq z}}^{2N-1}H(t_{2p+1})-\frac{w}{3}\sum_{\substack{q=0\\t_{2q+1}\neq z}}^{N-1}H(t_{2q+1})+\frac{7B_4w^4}{768}[H^3(-1)-H^3(1)]\\
&+\sum_{u=3}^{m-1}\frac{B_{2u}w^{2u}}{3(2u)!}[H^{2u-1}(-1)-H^{2u-1}(1)]\left(\frac{2^{2u}+2^{3-2u}-6}{2^{2u}}\right)+O(w^{2m})
\end{aligned}
\tag{25}
$$

This completes the proof of corollary 1.

Remark. Eq.(25) has the integral accuracy of level $O(w^4)$. Using the above processes, a high level of accuracy can be obtained by further extrapolation based on the same extrapolation principle. Therefore, further extrapolation calculations are not performed in this paper.

By substituting Eq.(21) into Eq.(11), the approximate numerical formula Eq.(26) of $P_1(x)$ double extrapolation can be obtained (ET=2) as follows:

$$
P_1(x)_{ET=2}\approx\frac{a+b}{2\pi^2\sqrt{(x+b)\cdot(a-x)}}\left[\frac{2w}{3}\sum_{\substack{p=0\\t_{2p+1}\neq z}}^{2N-1}H(t_{2p+1})-\frac{w}{3}\sum_{\substack{q=0\\t_{2q+1}\neq z}}^{N-1}H(t_{2q+1})\right]
\tag{26}
$$

In Eq.(20) and Eq.(26), both a and b are unknown and need to satisfy the conditional function shown in Eq.(6). Therefore, $P_1(x)$ cannot be calculated by the conventional method but can be calculated by the iterative method. The main steps are as follows:

(Ⅰ) Define the initial values of a and b and the number of segments N, then calculate the arrays t_i and t_j under the condition of ET=1 (the arrays t_{2p+1} and t_{2q+1} for ET=2).

(Ⅱ) In accordance with Eq.(12)-(13), calculate the arrays $H(t_i)$ and $H(t_j)$ under the condition of ET=1 (the arrays $H(t_{2p+1})$ and $H(t_{2q+1})$ for ET=2).

(Ⅲ) Substitute a, b, $H(t_i)$ and $H(t_j)$ into Eq.(20) and calculate $P_1(x)$ under the condition of ET=1 (Substitute $H(t_{2p+1})$ and $H(t_{2q+1})$ into Eq.(26) under the condition of ET=2).

(Ⅳ) Use Eq.(14) to compute $P_2(x)$ and use Eq.(10) to calculate $P(x)$.

(Ⅴ) According to engineering requirements, set appropriate constraint thresholds ε_a, ε_b, ε_1, ε_2, and calculate the boundary function values $P(a-\varepsilon_a)$ and $P(\varepsilon_b-b)$ based on the expression of $P(x)$ in step Ⅳ.

(Ⅵ) If the calculated values of $P(a-\varepsilon_a)$ and $P(\varepsilon_b-b)$ satisfy the condition in Eq.(6), then a and b are the boundaries. If not, the values of a and b need to be readjusted and steps Ⅰ-Ⅴ repeated until Eq.(6) is satisfied.

Take the problem of contact between two solid bodies as an example to explain the readjustment rule. According to reference[58], if the profiles defined by parametric equations $[X_1(k_1), Y_1(k_1)]$ and $[X_2(k_2), Y_2(k_2)]$ are smooth, then the solution of Eq.(9) must be continuously decreasing to the boundary values a and b and reaches zero on the boundary (A=0 and B=0 in Eq.(6)). Thus, the readjustment rule for a and b is obtained as follows:

If the boundary function values $P(a-\varepsilon_a)$ and $P(\varepsilon_b-b)$ are greater than the minimum value of ε_1 or ε_2 in the previous calculation, then the boundary a or b needs to be expanded, respectively. Otherwise, the boundary should be narrowed. Some methods, such as a binary search method, can certainly be used to speed up the iteration.

The flowchart of the above calculation processes is described in Fig. 1.

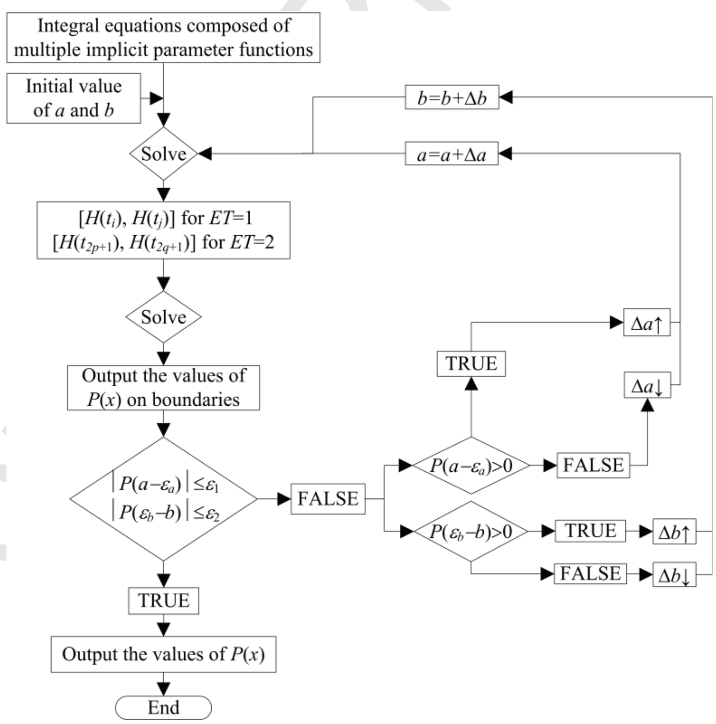


Fig. 1 Calculation flowchart

3 Numerical results

To analyse the effectiveness of the new method, five different examples are calculated in this section. In examples 1 and 2, the absolute errors between the true values of $P_1(x)$ and $P(x)$ and the numerical solutions calculate by the new method is analysed. Three singular integral equations without analytic solutions are calculated in examples 3-5.

3.1 Example 1

From the function expressions of $P_1(x)$ and $P_2(x)$, it can be seen that the absolute error for $P(x)$ is produced in the process of computing $P_1(x)$. Therefore, only the absolute errors of Eq.(20) (ET=1) and Eq.(26) (ET=2), which are the expansion of $P_1(x)$, are discussed in Example 1 (C=0). To avoid the error caused by the iteration process, suppose a=1 and b=1. The singular integral function composed of the following parameter functions is considered:

$$\begin{aligned} X_1(k_1) &= k_1 + 1 \\ Y_1(k_1) &= -0.5\operatorname{asin}(k_1+1) \end{aligned} \tag{27}$$

$$\begin{aligned} X_2(k_2) &= 3k_2 \\ Y_2(k_2) &= 1.5k_2\sqrt{1-9{k_2}^2} \end{aligned} \tag{28}$$

The expression of the true solution is:

$$P(x) = \frac{2x + \left(1-x^2\right)\ln\left(\dfrac{1+x}{1-x}\right)}{-\pi^2\sqrt{1-x^2}} \tag{29}$$

Following calculation steps Ⅰ-Ⅲ in section 2, the absolute error between the numerical solution of $P_1(x)$ and the true value is shown in Table. 1.

Table. 1 Absolute error of approximate solution compared with true value for example 1

x	$N=2^6$ Absolute error(ET=1)	$N=2^6$ Absolute error(ET=2)	$N=2^8$ Absolute error(ET=1)	$N=2^8$ Absolute error(ET=2)	$N=2^{10}$ Absolute error(ET=1)	$N=2^{10}$ Absolute error(ET=2)
-0.75	4.26E-05	2.95E-08	2.67E-06	1.16E-10	1.67E-07	4.55E-13
-0.50	1.27E-05	2.40E-09	7.93E-07	9.41E-12	4.96E-08	3.68E-14
-0.25	4.54E-06	4.69E-10	2.84E-07	1.83E-12	1.77E-08	7.17E-15
0.00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00	0.00E+00
0.25	4.54E-06	4.69E-10	2.84E-07	1.83E-12	1.77E-08	7.17E-15
0.50	1.27E-05	2.40E-09	7.93E-07	9.41E-12	4.96E-08	3.68E-14
0.75	4.26E-05	2.95E-08	2.67E-06	1.16E-10	1.67E-07	4.55E-13

3.2 Example 2

In this example, the approximate numerical solution of the function $P(x)$ given by Eqs.(30)-(31) is calculated with $C=128\pi^2/25$. The integral limits a and b are unknown but constrained by Eq.(6).

$$X_1(k_1) = 2k_1 + 0.2$$
$$Y_1(k_1) = -36k_1^2 \tag{30}$$

$$X_2(k_2) = 3k_2 + 0.2$$
$$Y_2(k_2) = 63k_2^2 \tag{31}$$

In engineering applications, integral principle values are usually used to represent solutions of singular integrals, which in this example can be expressed as in Eq.(32) as follows [30]:

$$P(x) = 16\sqrt{(a-x)(b+x)} \tag{32}$$

After calculation, the absolute error between the approximate numerical solution and the principal value solution is shown in Table. 2.

Table. 2 Absolute error of approximate solution compared with principal value for example 2

	N=2^8		N=2^{10}		N=2^{12}	
x	Absolute error(ET=1)	Absolute error(ET=2)	Absolute error(ET=1)	Absolute error(ET=2)	Absolute error(ET=1)	Absolute error(ET=2)
-0.5	4.33E-03	5.84E-04	5.36E-04	7.35E-05	6.68E-05	9.21E-06
-0.3	1.02E-03	1.40E-04	1.27E-04	1.76E-05	1.59E-05	2.20E-06
-0.1	6.08E-04	8.40E-05	7.60E-05	1.05E-05	9.50E-06	1.31E-06
0.1	4.96E-04	6.85E-05	6.20E-05	8.56E-06	7.75E-06	1.07E-06
0.2	4.84E-04	6.69E-05	6.06E-05	8.36E-06	7.57E-06	1.05E-06
0.3	4.96E-04	6.85E-05	6.20E-05	8.56E-06	7.75E-06	1.07E-06
0.5	6.08E-04	8.40E-05	7.60E-05	1.05E-05	9.50E-06	1.31E-06
0.7	1.02E-03	1.40E-04	1.27E-04	1.76E-05	1.59E-05	2.20E-06
0.9	4.33E-03	5.84E-04	5.36E-04	7.35E-05	6.68E-05	9.21E-06

3.3 Example 3

Examples 1 and 2 have accurate analytical solutions; therefore, the absolute error between the values calculated by the new method presented in this paper and the exact values is provided. However, the new method is mainly used to solve Cauchy singular integral equations composed of multiple implicit parameter functions with unknown integral limits but constrained by physical characteristics, and these equations are without analytical solutions. In Example 3, one of these types of equations is defined. Eq.(33) and Eq.(34) are the implicit parameter functions, with N=2^{10}, C=300.

$$X_1(k_1) = 2\cos(k_1) - 8k_1\cos(k_1) - 9\sin(k_1) + 7k_1\sin(k_1) - 2$$
$$Y_1(k_1) = \left[15\sin(k_1) + 9\cos(k_1) - 15k_1\cos(k_1) - 6k_1\sin(k_1) - 9\right]\times 10^5 \tag{33}$$

$$X_2(k_2) = 7\sin(k_2) - 8\cos(k_2) + 3k_2\cos(k_2) - 9k_2\sin(k_2) + 8$$
$$Y_2(k_2) = \left[35k_2\cos(k_2) - 35\sin(k_2) - 30\cos(k_2) + 10k_2\sin(k_2) + 30\right]\times 10^5 \tag{34}$$

The solid shapes represented by parameter functions [$X_1(k_1)$, $Y_1(k_1)$] and [$X_2(k_2)$, $Y_2(k_2)$] are shown in Fig. 2.

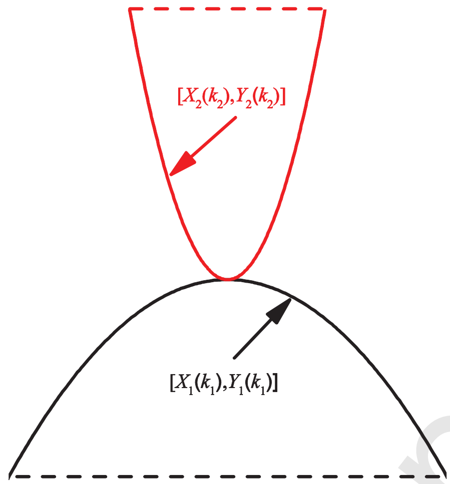


Fig. 2 The solid shapes represented by parameter functions in example 3

The calculation results are illustrated in Fig. 3.

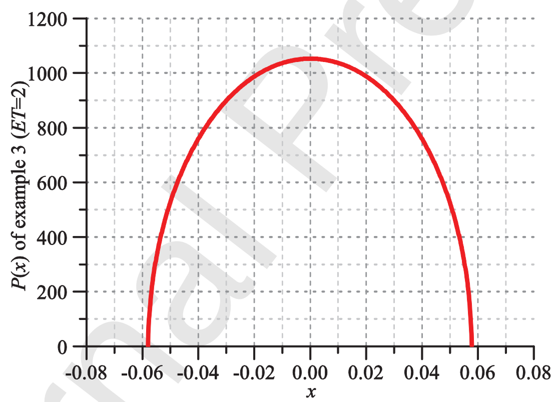


Fig. 3 $P(x)$ for example 3 (ET=2)

3.4 Example 4

The method presented in this paper can also be applied to contact problems of solid described by piecewise functions. In this example, parametric equations describing solid profiles are composed of a piecewise parameter function (Eq.(35)) and a standard parameter function (Eq.(36)) with N=2^9, C=20. The singular equation used for calculation has no analytic solution and the integral limits are unknown.

$$X_1(k_1)=\begin{cases}12\cos(k_1)-2\sin(k_1)-3k_1\cos(k_1)+2k_1\sin(k_1)-12, & X_1\le 0\\ 6\cos(k_1)-5\sin(k_1)-3k_1\cos(k_1)+9k_1\sin(k_1)-6, & X_1>0\end{cases}$$
$$Y_1(k_1)=\begin{cases}\left[7.11\cos(k_1)+0.63\sin(k_1)-0.63k_1\cos(k_1)-7.83k_1\sin(k_1)-7.11\right]\times 10^6, & X_1\le 0\\ \left[0.7k_1\sin(k_1)-5600k_1^2-3.5k_1^2\cos(k_1)\right]\times 10^5, & X_1>0\end{cases} \tag{35}$$

$$
\begin{aligned}
&X_2(k_2)=3\sin(k_2)+2k_2\cos(k_2)-k_2\sin(k_2)\\
&Y_2(k_2)=\left[0.027k_2\cos(k_2)-0.027\sin(k_2)-1.8\cos(k_2)+1.8\right]\times10^7
\end{aligned}
\tag{36}
$$

The solid shapes represented by parameter functions [$X_1(k_1)$, $Y_1(k_1)$] and [$X_2(k_2)$, $Y_2(k_2)$] are described in Fig. 4.

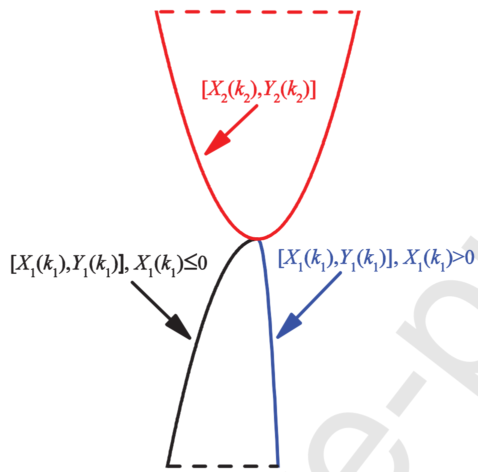


Fig. 4 The solid shapes represented by parameter functions in example 4

The results are shown in Fig. 5.

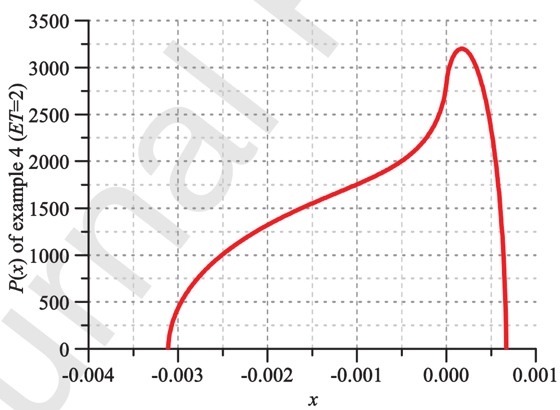


Fig. 5 $P(x)$ for example 4 (ET=2)

3.5 Example 5

In this example, the solid profile equation for contact is composed of two sets of piecewise functions (Eq.(37) and Eq.(38)) with N=2^8, C=36. The singular integral equation also has no analytic solution and the integral limits are unknown, just as in examples 3-4.

$$X_1(k_1)=\begin{cases}5k_1^2\tan^2(k_1)-2\tan(k_1)-3k_1\cos(k_1), & X_1\le 0\\ 7\cos(k_1)-2\tan(k_1)-5k_1\cos(k_1)+6k_1\tan(k_1)-7, & X_1>0\end{cases}$$
$$Y_1(k_1)=\begin{cases}\left[2.08\sin(k_1)+6.50\cos^2(k_1)-2.08k_1\cos(k_1)-6.50\right]\times 10^7, & X_1\le 0\\ 33600k_1\tan(k_1)-33600000\tan^2(k_1)+53200k_1^2\cos(k_1), & X_1>0\end{cases} \quad (37)$$

$$X_2(k_2)=\begin{cases}2\cos^2(k_2)-3\tan(k_2)+10k_2^2\tan^2(k_2)-5k_2\cos(k_2)-2, & X_2\le 0\\ 6\tan(k_2)+5k_2\cos(k_2)-7k_2\tan(k_2), & X_2>0\end{cases}$$
$$Y_2(k_2)=\begin{cases}9\times 10^6\tan^3(k_2)-200k_2\tan^2(k_2)-300k_2^2\cos(k_2), & X_2\le 0\\ \left[8.5k_2\cos(k_2)-8.5\sin(k_2)-7.5\cos(k_2)+7.5\right]\times 10^5, & X_2>0\end{cases} \quad (38)$$

The solid shapes represented by parameter functions [$X_1(k_1)$, $Y_1(k_1)$] and [$X_2(k_2)$, $Y_2(k_2)$] are shown in Fig. 6.

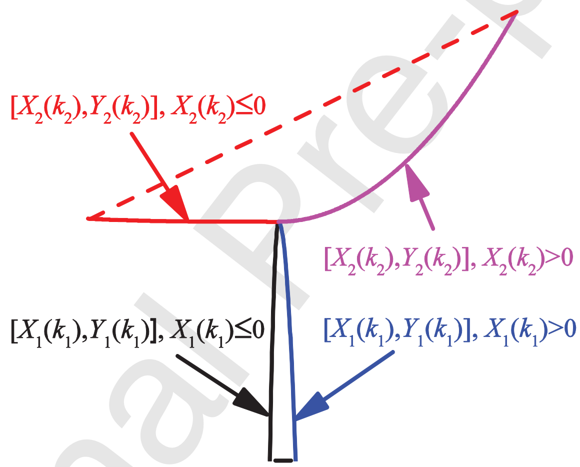


Fig. 6 The solid shapes represented by parameter functions in example 5

The calculation results are shown in Fig. 7.

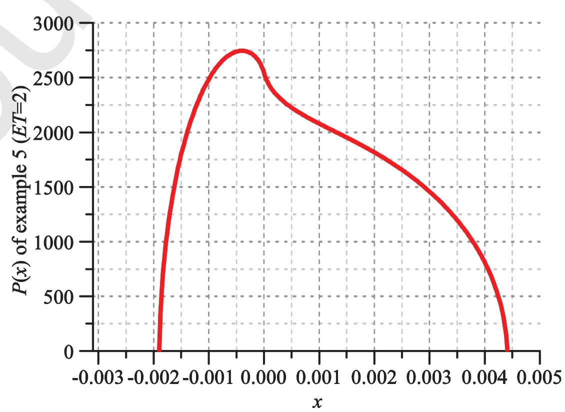


Fig. 7 $P(x)$ for example 5 (ET=2)

4. Conclusion

Cauchy singular integral equations that are composed of multiple implicit parameter functions with unknown integral limits constrained by physical characteristics appear in (but are not limited to) the problem of contact between two solid bodies with irregular profiles described by parametric functions. A numerical approximation method, which consists mainly of transforming integrals into numerical calculations, is found to be useful for solving those equations based on an iterative method. Higher accuracy can be obtained by making an extrapolation. In addition, this method can also directly solve the singular integral equations composed of multiple implicit parameter functions with known upper and lower limits. The numerical results show that the absolute error between the calculated value and the true value is very small and decreases as N increases. This indicates that the proposed method is accurate and reliable.

Acknowledgements

The authors are grateful for the financial support provided by the Research and Development Program in Key Areas (2019B090917004), Special Foundation for Science and Technology Development (2017B090910010), and Natural Science Foundation (2016A030313514) of Guangdong province in China, and the State Key Laboratory of Mechanical Transmission in China (SKLMT-KFKT-201503).

References

[1] A. Gay Neto, P.d.M. Pimenta, P. Wriggers, Contact between spheres and general surfaces. Computer Methods in Applied Mechanics and Engineering 328 (2018) 686-716.

[2] T. Lin, H. Ou, R. Li, A finite element method for 3D static and dynamic contact/impact analysis of gear drives. Computer Methods in Applied Mechanics and Engineering 196 (2007) 1716-1728.

[3] Y. Cai, X. Cui, G. Li, W. Liu, A parallel finite element procedure for contact-impact problems using edge-based smooth triangular element and GPU. Computer Physics Communications 225 (2018) 47-58.

[4] S.O.R. Biabanaki, A.R. Khoei, P. Wriggers, Polygonal finite element methods for contact-impact problems on non-conformal meshes. Computer Methods in Applied Mechanics and Engineering 269 (2014) 198-221.

[5] Z. Yang, A. Boogaard, Z. Wei, J. Liu, R. Dollevoet, Z. Li, Numerical study of wheel-rail impact contact solutions at an insulated rail joint. International Journal of Mechanical Sciences 138-139 (2018) 310-322.

[6] J.H. Kim, Y.H. Jang, Frictional Hertzian contact problems under cyclic loading using static reduction. International Journal of Solids and Structures 51 (2014) 252-258.

[7] G. Fajdiga, M. Sraml, Fatigue crack initiation and propagation under cyclic contact loading. Engineering Fracture Mechanics 76 (2009) 1320-1335.

[8] G.H. Majzoobi, F. Abbasi, On the Effect of Contact Geometry on Fretting Fatigue Life Under Cyclic Contact Loading. Tribology Letters 65 (2017) 125-121-117.

[9] A. Gay Neto, P.M. Pimenta, P. Wriggers, A master-surface to master-surface formulation for beam to beam contact. Part I: frictionless interaction. Computer Methods in Applied Mechanics and Engineering 303 (2016) 400-429.

[10] A. Gay Neto, P.M. Pimenta, P. Wriggers, A master-surface to master-surface formulation for beam to beam contact. Part II: Frictional interaction. Computer Methods in Applied Mechanics and Engineering 319 (2017) 146-174.

[11] A. Gay Neto, P. Wriggers, Computing pointwise contact between bodies: a class of formulations based on master–master approach. Computational Mechanics (2019).

[12] K.A. Fischer, P. Wriggers, Mortar based frictional contact formulation for higher order interpolations using the moving friction cone. Computer Methods in Applied Mechanics and Engineering 195 (2006) 5020-5036.

[13] R. Mlika, Y. Renard, F. Chouly, An unbiased Nitsche's formulation of large deformation frictional contact and self-contact. Computer Methods in Applied Mechanics and Engineering 325 (2017) 265-288.

[14] L. De Lorenzis, P. Wriggers, Computational homogenization of rubber friction on rough rigid surfaces. Computational Materials Science 77 (2013) 264-280.

[15] H. Eid, G.G. Adams, N.E. Mcgruer, A. Fortini, S. Buldyrev, D. Srolovitz, A Combined Molecular Dynamics and Finite Element Analysis of Contact and Adhesion of a Rough Sphere and a Flat Surface. Tribology Transactions 54 (2011) 920-928.

[16] J. Su, L. Ke, B. El, Sami, Y. Xiang, Y. Wang, An effective method for the sliding frictional contact of a conducting cylindrical punch on FGPMs. International Journal of Solids and Structures 141-142 (2018) 127-136.

[17] M.N. Balci, S. Dag, Dynamic frictional contact problems involving elastic coatings. Tribology International 124 (2018) 70-92.

[18] R. Elloumi, I. Kallel-Kamoun, S. El-Borgi, M.A. Guler, On the frictional sliding contact problem between a rigid circular conducting punch and a magneto-electro-elastic half-plane. International Journal of Mechanical Sciences 87 (2014) 1-17.

[19] A.N. Guz, V.V. Zozulya, Fracture Dynamics with Allowance for Crack Edge Contact Interaction. International Journal of Nonlinear Sciences & Numerical Simulation 2 (2001) 173---233.

[20] A.N. Guz, V.V. Zozulya, Elastodynamic Unilateral Contact Problems with Friction for Bodies with Cracks. International Applied Mechanics 38 (2002) 895-932.

[21] A.N. Guz, Contact problems of the theory of elasticity for a half-plane with

initial stresses. Soviet Applied Mechanics 16 (1980) 691-699.

[22] A.N. Guz, On dynamic contact problems for an elastic half-plane with initial stresses in the case of a moving rigid punch. International Applied Mechanics 35 (1999) 515-521.

[23] E. Ladopoulos, Singular integral equations: linear and non-linear theory and its applications in science and engineering, Springer Science & Business Media, 2013.

[24] J.W. Cohen, O.J. Boxma, Boundary Value Problems in Queueing System Analysis, Amsterdam ; New York : North-Holland Pub. Co., 1983.

[25] H. Du, J. Shen, Reproducing kernel method of solving singular integral equation with cosecant kernel. Journal of Mathematical Analysis and Applications 348 (2008) 308-314.

[26] H.A.H. Salem, On the quadratic integral equations and their applications. Computers & Mathematics with Applications 62 (2011) 2931-2943.

[27] Y. Alinia, M. Hosseini-nasab, M.A. Güler, The sliding contact problem for an orthotropic coating bonded to an isotropic substrate. European Journal of Mechanics - A/Solids 70 (2018) 156-171.

[28] İ. Çömez, Continuous and discontinuous contact problem of a functionally graded layer pressed by a rigid cylindrical punch. European Journal of Mechanics - A/Solids 73 (2019) 437-448.

[29] S.J. Chidlow, W.W.F. Chong, M. Teodorescu, On the two-dimensional solution of both adhesive and non-adhesive contact problems involving functionally graded materials. European Journal of Mechanics - A/Solids 39 (2013) 86-103.

[30] K.L. Johnson, Contact Mechanics, Cambridge University Press, Cambridge, 1985.

[31] L.P. Castro, E.M. Rojas, Explicit solutions of Cauchy singular integral equations with weighted Carleman shift. Journal of Mathematical Analysis and Applications 371 (2010) 128-133.

[32] M.C. De Bonis, C. Laurita, Numerical solution of systems of Cauchy singular integral equations with constant coefficients. Applied Mathematics and Computation 219 (2012) 1391-1410.

[33] C. Gu, I.S. Hwang, D.-O. Kang, W.Y. Lee, Normal singular Cauchy integral operators with operator-valued symbols. Journal of Mathematical Analysis and Applications 447 (2017) 289-308.

[34] V.V. Kravchenko, On a transplant operator and explicit construction of Cauchy-type integral representations for p-analytic functions. Journal of Mathematical Analysis and Applications 339 (2008) 1103-1111.

[35] M.M. Nayak, M. Acharya, B.P. Acharya, Numerical evaluation of two-dimensional Cauchy Principal Value integrals. Applied Mathematics and Computation 225 (2013) 807-810.

[36] A. Chakrabarti, G. Vanden Berghe, Approximate solution of singular integral equations. Applied Mathematics Letters 17 (2004) 553-559.

[37] Z.K. Eshkuvatov, N.M.A. Nik Long, M. Abdulkawi, Approximate solution of singular integral equations of the first kind with Cauchy kernel. Applied Mathematics Letters 22 (2009) 651-657.

[38] İ. Turhan, H. Oğuz, E. Yusufoğlu, Chebyshev polynomial solution of the system of Cauchy-type singular integral equations of the first kind. International Journal of Computer Mathematics 90 (2013) 944-954.

[39] R.K. Pandey, O.P. Singh, V.K. Singh, Numerical solution of system of Volterra integral equations using Bernstein polynomials. International Journal of Nonlinear Sciences & Numerical Simulation 10 (2009) 891-896.

[40] A. Setia, Numerical solution of various cases of Cauchy type singular integral equation. Applied Mathematics and Computation 230 (2014) 200-207.

[41] P. Karczmarek, D. Pylak, M.A. Sheshko, Application of Jacobi polynomials to approximate solution of a singular integral equation with Cauchy kernel. Applied Mathematics and Computation 181 (2006) 694-707.

[42] M. Abdulkawi, Solution of Cauchy type singular integral equations of the first kind by using differential transform method. Applied Mathematical Modelling 39 (2015) 2107-2118.

[43] S. Kim, Solving singular integral equations using Gaussian quadrature and overdetermined system. Computers & Mathematics with Applications 35 (1998) 63-71.

[44] M.C. De Bonis, C. Laurita, Nyström method for Cauchy singular integral equations with negative index. Journal of Computational and Applied Mathematics 232 (2009) 523-538.

[45] H. Cai, The numerical integration scheme for a fast Petrov–Galerkin method for solving the generalized airfoil equation. Calcolo 50 (2013) 141-162.

[46] M. Paripour, Note on numerical solution of the Fredholm singular integro-differential equation with Cauchy kernel by using Taylor-series expansion and Galerkin method. Applied Mathematics and Computation 250 (2015) 530-532.

[47] V. Sharma, A. Setia, R.P. Agarwal, Numerical solution for system of Cauchy type singular integral equations with its error analysis in complex plane. Applied Mathematics and Computation 328 (2018) 338-352.

[48] M.C.D. Bonis, C. Laurita, A quadrature method for systems of Cauchy singular integral equations. Journal of Integral Equations & Applications 24 (2012) 241-271.

[49] K.M. Shadimetov, A.R. Hayotov, D.M. Akhmedov, Optimal quadrature formulas for Cauchy type singular integrals in Sobolev space. Applied Mathematics and Computation 263 (2015) 302-314.

[50] A. Sidi, M. Israeli, Quadrature methods for periodic singular and weakly singular Fredholm integral equations. Journal of Scientific Computing 3 (1988) 201-231.

[51] L. Bougoffa, A. Mennouni, R.C. Rach, Solving Cauchy integral equations of the first kind by the Adomian decomposition method. Applied Mathematics and Computation 219 (2013) 4423-4433.

[52] M.S. Akel, H.S. Hussein, Numerical treatment of solving singular integral equations by using Sinc approximations. Applied Mathematics and Computation 218 (2011) 3565-3573.

[53] A. Dezhbord, T. Lotfi, K. Mahdiani, A new efficient method for cases of the

singular integral equation of the first kind. Journal of Computational and Applied Mathematics 296 (2016) 156-169.

[54] S.G. Mikhlin, Integral Equations (Second Revised Edition), in: S.G. MIKHLIN (Ed.), Singular integral equations, Elsevier Ltd, 1957, pp. 335-338.

[55] H. Söhngen, Zur Theorie der endlichen Hilbert-Transformation. Mathematische Zeitschrift 60 (1954) 31-51.

[56] M. Costabel, E. Darrigrand, H. Sakly, Volume integral equations for electromagnetic scattering in two dimensions. Computers & Mathematics with Applications 70 (2015) 2087-2101.

[57] G. Zeng, J. Huang, H. Jia, The High Accuracy Algorithm for Cauchy Singular Integral and Cauchy Singular Integral Equation, 2010 4th International Conference on Bioinformatics and Biomedical Engineering, IEEE, Chengdu, China 2010, pp. 1-4.

[58] J. Boussinesq, Application des potentiels à l'étude de l'équilibre et du mouvement des solides élastiques, Gauthier-Villars, Imprimeur-Libraire, 1885.