



Some remarks on the Fuglede p -modulus

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ARTICLE INFO

Article history:

Received 10 December 2018

Available online xxxx

Submitted by A. Cianchi

Keywords:

p -modulus

Extremal function

Optimal plan

Double fibration

ABSTRACT

We study the properties of extremal function for the Fuglede's p -modulus of a measure family Σ . Based on the recent result concerning p -modulus showing the existence of certain Borel measure \mathfrak{n}_Σ on Σ , we give an interpretation of \mathfrak{n}_Σ in some simple cases. In particular, we consider the case of disjoint supports and a natural family of measures associated with a double fibration.

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1. Introduction

The p -modulus of a family of measures, introduced by Fuglede in [8], plays important role in many aspects of analysis on metric measure spaces. The significance of this notion has been recently recognized by many authors. The crucial fact noticed by Fuglede is that families of p -modulus equal to zero, called p -exceptional, may play the same role as equivalence of functions on sets of measure zero in L^p spaces. The second remarkable observation is, so called, Fuglede lemma, which states that convergence in L^p space implies convergence in L^1 with respect to all measures in considered family except for measures in an p -exceptional family.

Above mentioned results led Shumunghamam [13] to introduce the notion of the first Sobolev spaces $N^{1,p}(X)$ on the metric measure space X . Hence, p -harmonicity may be studied on such spaces [14]. In this case the measures considered are the arc length measures on curves. Considering measures associated with hypersurfaces, we get the equivalent notion of p -capacity of a condenser [15]. Capacity is an important tool in potential theory, partial differential equations and in differential geometry, by its conformal invariance for certain choice of the coefficient $p > 1$ (see [11,4,7] for more details).

Recently, Ambrosio, Di Marino and Savare [1] associated to any family Σ of Borel measures on the Polish space X with the reference measure \mathfrak{m} the unique optimal measure \mathfrak{n}_Σ on Σ (see Proposition 2 in Section 3 for more details). This measure is absolutely continuous with respect to the p -modulus, thus provides weaker

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condition for negligible sets and considered on the family of measures associated to parametric curves allows to provide alternative definition of p -weak upper gradient (on metric measure spaces).

In this article we describe the measure \mathbf{n}_Σ in some simple cases. We consider a family of measures with disjoint supports and a family naturally associated with a double fibration. The common feature of these cases is existence of a Borel map onto Σ . The push-forward of \mathbf{n}_Σ with respect to this map defines a measure which may be described with a given data, more precisely, by the extremal function f_Σ for the p -modulus of Σ . The main ingredient, is the integral formula derived in [1]

$$\int_{\Sigma} \hat{f} d\mathbf{n}_\Sigma = \text{mod}_p(\Sigma)^{-1} \int_X f_\Sigma^{p-1} f d\mathbf{m},$$

where $\text{mod}_p(\Sigma)$ is the Fuglede p -modulus of Σ with respect to a measure \mathbf{m} and $\hat{f} : \Sigma \rightarrow \mathbb{R}$ is given by $\hat{f}(\mu) = \int_X f d\mu$.

2. Fuglede's p -modulus and its properties

In this section we review the notion and properties of the Fuglede's p -modulus. For more details see [8,2]. Let X be a metric space and denote by \mathcal{M} the σ -algebra of Borel sets. Fix a reference Borel measure \mathbf{m} and let Σ be a set of Borel measures on X . Denote by $\mathcal{L}_+^p(X, \mathbf{m})$ the space of all Borel functions $f : X \rightarrow [0, \infty]$ such that $\int_X f^p d\mathbf{m} < \infty$ and by $\mathcal{L}_{[-\infty, \infty]}^p(X, \mathbf{m})$ the space of all Borel measurable functions taking values in $[-\infty, \infty]$ and such that $\int_X f^p d\mathbf{m} < \infty$. We define the p -modulus of Σ by

$$\text{mod}_p(\Sigma) = \inf \left\{ \int_X f_X^p d\mathbf{m} \mid f \in \mathcal{L}_+^p(X, \mathbf{m}), \int_X f d\mu \geq 1 \text{ for } \mu \in \Sigma \right\}.$$

A function $f \in \mathcal{L}_+^p(X, \mathbf{m})$ such that $\int_X f d\mu \geq 1$ for all $\mu \in \Sigma$ is called p -admissible or admissible for the p -modulus of Σ . p -modulus has the following properties [8, Theorem 1]:

1. if $T \subset \Sigma$, then $\text{mod}_p(T) \leq \text{mod}_p(\Sigma)$,
2. if $T \subset \bigcup_i \Sigma_i$, then $\text{mod}_p(T) \leq \sum_i \text{mod}_p(\Sigma_i)$.

In other words, p -modulus is an outer measure on the space of measures. Moreover, one can easily show that

$$\text{mod}_p(T \cup \Sigma) = \text{mod}_p(T) + \text{mod}_p(\Sigma),$$

for families T and Σ such that $\text{supp} T \cap \text{supp} \Sigma = \emptyset$, where

$$\text{supp} \Sigma = \bigcup_{\mu \in \Sigma} \text{supp} \mu.$$

We say that a family Σ is p -exceptional if its p -modulus is equal to zero. Moreover, we say that a certain property (P) holds p -almost everywhere (p -a.e., for short) with respect to Σ , if there is a subfamily $T \subset \Sigma$ such that (P) holds for every measure $\mu \in \Sigma \setminus T$ and T is p -exceptional. A p -admissible function f_Σ which realizes the infimum for the p -modulus of Σ is called *extremal* for the p -modulus of Σ . One can show, that up to a subfamily of p -modulus zero, there is unique extremal function [8]. It can be characterized by the following Badger's criterion [2].

Theorem 1. Let X be a metric space and \mathbf{m} a Borel measure on X . Assume $\text{mod}_p(\Sigma) < \infty$. A p -admissible function $f_\Sigma \in \mathcal{L}_+^p(X, \mathbf{m})$ is extremal for the p -modulus of the family Σ if and only if there is a family Σ_0 such that $\text{mod}_p(\Sigma \cup \Sigma_0) = \text{mod}_p(\Sigma)$ and the following two conditions hold:

- (1) $\int_X f_\Sigma d\nu = 1$ for $\nu \in \Sigma_0$,
- (2) if $\int_X f d\nu \geq 0$ for all $\nu \in \Sigma_0$, where $f \in \mathcal{L}_{[-\infty, \infty]}^p(X, \mathbf{m})$, then $\int_X f_\Sigma^{p-1} f d\mathbf{m} \geq 0$.

3. Measures associated with families of Borel measures

All facts in this section are taken from [1]. Let X be a Polish space. Denote by $\mathcal{M}(X)$ the set of all Borel measures on X and fix a measure $\mathbf{m} \in \mathcal{M}(X)$. We can endow $\mathcal{M}(X)$ with the topology of $*$ -weak convergence, i.e., $\mu_n \rightarrow \mu$ if

$$\int_X f d\mu_n \rightarrow \int_X f d\mu$$

for all $f \in C_b(X)$ (here, $C_b(X)$ denotes the set of all continuous and bounded functions on X). To simplify the notation, for any Borel function f put

$$\hat{f} : \mathcal{M}(X) \rightarrow \mathbb{R}, \quad \hat{f}(\mu) = \int_X f d\mu.$$

Notice that if $f \in C_b(X)$ then, by the definition, \hat{f} is continuous, hence Borel. If, more generally, $f \in \mathcal{L}^p(X, \mathbf{m})$, then \hat{f} is again Borel. This follows by the use of monotone class theorem. In fact, we can find a sequence (f_n) in $C_b(X)$ converging in $L^1(X, \mu)$ to f for any $\mu \in \mathcal{M}(X)$. Since \hat{f}_n are Borel, the limit \hat{f} is Borel.

Let $\Sigma \subset \mathcal{M}(X)$. In [1] the authors show, assuming Σ is Suslin, existence of a Borel probability measure $\mathbf{n} = \mathbf{n}_\Sigma$ on $\mathcal{M}(X)$ concentrated on Σ such that

$$\int_{\mathcal{M}(X)} \hat{f} d\mathbf{n} \leq c(\mathbf{n}) \|f\|_p, \quad f \in \mathcal{L}^p(X, \mathbf{m}), \quad (1)$$

for some nonnegative constant $c(\mathbf{n})$, which we choose smallest possible. Any measure \mathbf{n} on $\mathcal{M}(X)$ satisfying (1) is called a *plan with barycenter in $L^q(X, \mathbf{m})$* , where q is conjugate to p . Assume now that $\text{mod}_p(\Sigma) > 0$ and $\sup_\Sigma \mu(X) < \infty$ and let Σ be a Suslin set in $\mathcal{M}(X)$. Recall, that a set S is *Suslin* if it is a image of a Polish space under continuous map. Among all plans with barycenter (not necessary probability measures) in $L^q(X, \mathbf{m})$ there is one *optimal*, what is explained in a proposition below.

Proposition 2. [1, Lemma 4.4, Theorem 5.1, Corollary 5.2] Let Σ be a Suslin set such that $\text{mod}_p(\Sigma) > 0$ and $\sup_\Sigma \mu(X) < \infty$. Put

$$C_{p, \mathbf{m}} := \sup_{c(\mathbf{n}) > 0} \frac{\mathbf{n}(\Sigma)}{c(\mathbf{n})}. \quad (2)$$

Then there exists an optimal plan \mathbf{n}_Σ with barycenter in $L^q(X, \mathbf{m})$ in (2). Moreover, $c(\mathbf{n}_\Sigma) = C_{p, \mathbf{m}}^{-1} = \text{mod}_p(\Sigma)^{-\frac{1}{p}}$. In addition, there is an extremal function f_Σ for the p -modulus of Σ . It satisfies

$$\hat{f}_\Sigma = 1 \quad \mathbf{n}_\Sigma\text{-a.e. on } \Sigma. \quad (3)$$

Let us recall the alternative definition of a plan with barycenter in $L^q(X, \mathbf{m})$ [1]. Let \mathbf{n} be a plan with barycenter in $L^q(X, \mathbf{m})$. Define a Borel measure $\underline{\mathbf{n}}$ on X by

$$\underline{\mathbf{n}}(A) = \int_{\mathcal{M}(X)} \mu(A) d\mathbf{n}(\mu) = \int_{\mathcal{M}(X)} \widehat{\chi_A} d\mathbf{n}, \quad A \in \mathcal{B}(X).$$

Thus by (1)

$$\underline{\mathbf{n}}(A) \leq c \mathbf{m}(A)^{\frac{1}{p}}, \quad A \in \mathcal{B}(X).$$

By Radon–Nikodym theorem there is a density $\rho \in L^1(X, \mathbf{m})$ such that $\underline{\mathbf{n}} = \rho \mathbf{m}$. By argument of approximation by simple functions we get that

$$\int_X f d\underline{\mathbf{n}} = \int_{\mathcal{M}(X)} \hat{f} d\mathbf{n}, \quad f \in \mathcal{L}^p(X, \mathbf{m}). \quad (4)$$

Hence,

$$\int_X f \rho d\mathbf{m} \leq c \|f\|_p, \quad f \in \mathcal{L}^p(X, \mathbf{m}).$$

By duality of Lebesgue spaces $L^p(X, \mathbf{m})$ and $L^q(X, \mathbf{m})^*$ it follows that $\rho \in L^q(X, \mathbf{m})$ and the smallest possible constant c equals $\|\rho\|_q$.

Notice, that for a plan \mathbf{n}_Σ we have

$$\rho = \text{mod}_p(\Sigma)^{-1} f_\Sigma^{p-1}.$$

In other words, the following formula holds

$$\int_\Sigma \hat{f} d\mathbf{n}_\Sigma = \text{mod}_p(\Sigma)^{-1} \int_X f_\Sigma^{p-1} f d\mathbf{m}, \quad f \in \mathcal{L}^p(X, \mathbf{m}). \quad (5)$$

4. Some results

In this section we will provide a formula for a plan \mathbf{n}_Σ assuming there is a Borel surjective map $\pi : X \rightarrow \Sigma$.

Let us start with some preliminary observations. Let $\Sigma \subset \mathcal{M}(X)$ be Suslin and assume $\text{mod}_p(\Sigma) > 0$. Then there is an optimal plan \mathbf{n}_Σ with barycenter in L^q . Assume there is a Borel surjective map $\pi : X \rightarrow \Sigma$. Put

$$X_\mu = \pi^{-1}(\mu). \quad (6)$$

We may push-forward a measure $\text{mod}_p(\Sigma)^{-1} f_\Sigma^p \mathbf{m}$ with respect to π to obtain a measure \mathbf{n}'_Σ on Σ ,

$$\mathbf{n}'_\Sigma = \pi_\#(\text{mod}_p(\Sigma)^{-1} f_\Sigma^p \mathbf{m}). \quad (7)$$

Moreover, denote by \bar{f} a composition $\hat{f} \circ \pi$ for $f \in \mathcal{L}^p(X, \mathbf{m})$. Then \bar{f} is a Borel function on Σ . By the definition we have

$$\int_\Sigma \hat{f} d\mathbf{n}'_\Sigma = \text{mod}_p(\Sigma')^{-1} \int_X f_\Sigma^p \bar{f} d\mathbf{m}, \quad f \in \mathcal{L}^p(X, \mathbf{m}). \quad (8)$$

4.1. Case of disjoint supports

Assume that support of μ is contained in X_μ for any $\mu \in \Sigma$. We say in this case that a family Σ is *separate*.

Lemma 3. *For the p -modulus of separate family Σ we have $\hat{f}_\Sigma = 1$ p -a.e. on Σ .*

Proof. Suppose $\hat{f}_\Sigma(\mu) > 1$. Then we could replace f_Σ by $\frac{1}{\hat{f}_\Sigma(\mu)} f_\Sigma$ on X_μ to get a function with the smaller $L^p(X, \mathbf{m})$ -norm. This contradicts the minimality of f_Σ . \square

Lemma 4. *For any Suslin subset Π of a separate family Σ , f_Σ restricted to $\bigcup_{\mu \in \Pi} X_\mu$ and zero elsewhere is extremal for the p -modulus of family Π .*

Proof. Suppose there exist f_Π , which is admissible for Π and such that $\|f_\Pi\|_p < \|f_\Sigma\|_p$ on $\bigcup_{\mu \in \Pi} X_\mu$. Put

$$f(x) = \begin{cases} f_\Pi(x) & \text{for } x \in \bigcup_{\mu \in \Pi} X_\mu \\ f_\Sigma(x) & \text{for remaining } x \end{cases}$$

Then f is p -admissible for Σ and $\|f\|_p < \|f_\Sigma\|_p$, which contradicts extremality of f_Σ . \square

For any $f \in \mathcal{L}^p(X, \mathbf{m})$ by Lemma 3 we have

$$\widehat{f_\Sigma f}(\mu) = \int_{X_\mu} f_\Sigma \bar{f} d\mu = \hat{f}(\mu) \hat{f}_\Sigma(\mu) = \hat{f}(\mu), \quad \mu \in \Sigma.$$

Hence, by (8) and (5) we get

$$\int_{\Sigma} \hat{f} d\mathbf{n}'_{\Sigma} = \int_{\Sigma} \widehat{f_\Sigma f} d\mathbf{n}_{\Sigma} = \int_{\Sigma} \hat{f} d\mathbf{n}_{\Sigma}.$$

Choose a Suslin subset $\Pi \subset \Sigma$. By Lemmas 3 and 4 we see that $\mathbf{n}'_{\Sigma}(\Pi) = \mathbf{n}_{\Sigma}(\Pi)$, hence these two measures agree on Suslin sets. Moreover, taking $f = f_\Pi$ in (5) we get

$$\mathbf{n}_{\Sigma}(\Pi) = \text{mod}_p(\Sigma)^{-1} \int_X f_\Sigma^{p-1} f_\Pi d\mathbf{m} = \text{mod}_p(\Sigma)^{-1} \int_X f_\Pi^p d\mathbf{m} = \frac{\text{mod}_p(\Pi)}{\text{mod}_p(\Sigma)}.$$

Concluding, we have the following proposition.

Proposition 5. *The plan \mathbf{n}_{Σ} on separate family Σ is equal to the conditional probability with respect to p -modulus for Suslin subsets. Moreover,*

$$\mathbf{n}_{\Sigma} = \mathbf{n}'_{\Sigma} = \pi_{\#}(\text{mod}_p(\Sigma) f_{\Sigma}^p \mathbf{m}).$$

By above proposition we have the following integral formula

$$\int_X f_{\Sigma}^p \bar{f} d\mathbf{m} = \int_X f_{\Sigma}^{p-1} f d\mathbf{m}, \quad f \in \mathcal{L}^p(X, \mathbf{m}). \quad (9)$$

Remark 1. Notice that the integral formula (9) was obtained by the author in the case of Lebesgue measures associated with a foliation on a Riemannian manifold [5,6].

4.2. Connections with Disintegration Theorem

Let Σ be a Suslin family of Borel measures on a space X such that $\text{mod}_p(\Sigma) > 0$ with respect to a fixed Borel measure \mathbf{m} is positive. Assume there is a Borel map $\pi : X \rightarrow \Sigma$. Then $\mathbf{n}_{\Sigma'}$, defined by (7), is a Borel measure on Σ . By Disintegration Theorem (see [3, Proposition 10.4.12]) applied to a map π and a measure \mathbf{n}'_{Σ} , there is a family of Borel measures $\Sigma' = \{\nu_{\mu}\}_{\mu \in \Sigma}$ on X such that

$$\text{mod}_p(\Sigma)^{-1} \int_X f_{\Sigma}^{p-1} f \, d\mathbf{m} = \int_{\Sigma} \int_{\pi^{-1}(\mu)} f \, d\nu_{\mu} \, d\mathbf{n}_{\Sigma'}(\mu). \quad (10)$$

Let us compute the p -modulus of Σ' . If f is admissible for the p -modulus of Σ' then by (10) it follows that

$$1 \leq \int_{\Sigma} 1 \, d\mathbf{n}_{\Sigma'} \leq \int_{\Sigma} \int_{\pi^{-1}(\mu)} f \, d\nu_{\mu} \, d\mathbf{n}_{\Sigma'}(\mu) \leq \text{mod}_p(\Sigma)^{-\frac{1}{p}} \|f\|_p.$$

Thus

$$\text{mod}_p(\Sigma') \geq \text{mod}_p(\Sigma). \quad (11)$$

Moreover, above last inequality can be stated as

$$\int_{\Sigma} \hat{f} \, d\mathbf{n}_{\Sigma'} \leq \text{mod}_p(\Sigma)^{-\frac{1}{p}} \|f\|_p,$$

where \hat{f} is taken with respect to Σ' . This shows that $\mathbf{n}_{\Sigma'}$ is a plan with barycenter in $L^q(X, \mathbf{m})$ associated with Σ' and that $c(\mathbf{n}_{\Sigma'}) \leq \text{mod}_p(\Sigma)$. Notice that measures ν_{μ} , $\mu \in \Sigma$, have disjoint supports.

It would be interesting to check when there is equality in (11). In the example below, we show that it may happen.

Example 1. Let $X = [0, 1]$ and consider a Lebesgue measure \mathbf{m} on X . Let μ_t , $t \in (0, 1]$, be a normalized Lebesgue measure on $[0, t]$, i.e. $\mu_t = \frac{1}{t} \mathbf{m}|_{[0, t]}$ and let $\mu_0 = \delta_0$ be the Dirac measure at 0. Put $\Sigma = \{\mu_t\}_{t \in [0, 1]}$. A map $\pi : X \rightarrow \Sigma$ given by $\pi(t) = \mu_t$ is continuous. An admissible function f_{Σ} is extremal for the p -modulus of Σ if

$$\frac{1}{t} \int_0^t f_{\Sigma}(s) \, ds = 1$$

for all t . Differentiating with respect to t , we get that $f_{\Sigma} = 1$. Thus $\text{mod}_p(\Sigma) = 1$. We will derive the formula for the p -modulus of a family Σ' of measures ν_t on X , which existence follows by Disintegration Theorem. We know that ν_t is concentrated on a set $X_t = \pi^{-1}(\mu_t) = \{t\}$. Hence $\nu_t = c_t \delta_t$ for some positive c_t . Thus, by Disintegration Theorem

$$\int_0^1 f(t) \, dt = \int_0^1 c_t f(t) \, dt,$$

for any f . Hence $c_t = t$ and $\nu_t = \delta_t$ is a Dirac measure. We thus have $\text{mod}_p(\Sigma') = 1 = \text{mod}_p(\Sigma)$.

Moreover, let us determine the plan \mathfrak{n}_Σ . By a definition we have

$$\int_{\Sigma} \frac{1}{t} \int_0^t f(s) ds d\mathfrak{n}_\Sigma(\mu_t) = \int_0^1 f(t) dt$$

for any continuous f . Taking $f = F'$ such that $F(0) = 0$ we obtain

$$\int_{\Sigma} \frac{F(t)}{t} d\mathfrak{n}_\Sigma(\mu_t) = F(1).$$

Now it easily follows that \mathfrak{n}_Σ is a Dirac measure at μ_1 .

5. Double fibration

Let G be a locally compact group, H_X and H_A closed subgroups of G . Define two left coset spaces

$$X = G/H_X, \quad \mathcal{A} = G/H_A.$$

Moreover, consider the following assumptions:

1. the groups G, H_X, H_A and $H = H_X \cap H_A$ are unimodular, i.e., left-invariant Haar measure are also right-invariant,
2. the set $H_X H_A$ is closed in G ,
3. if $hH_A \subset H_A H_X$, then $h \in H_X$. If $hH_X \subset H_X H_A$, then $h \in H_A$.

We say that two elements $x \in X$ and $\xi \in \mathcal{A}$ are *incident* if as cosets in G they intersect. The set of elements in \mathcal{A} which are incident with $x \in X$ is denoted by \check{x} . Analogously, the set of points in X which are incident with $\xi \in \mathcal{A}$ is denoted by $\hat{\xi}$. Put

$$F = \{(x, \xi) \in X \times \mathcal{A} \mid x \text{ and } \xi \text{ are incident}\}.$$

Then we have two fibrations $F \mapsto X$ and $F \mapsto \mathcal{A}$, this we speak about a *double fibration*.

Let us recall important results concerning existence of invariant measures for a double fibration [9]. Let $x_0 = \{H_X\}$ and $\xi_0 = \{H_A\}$. There is a unique H_X -invariant measure μ_0 on \check{x}_0 and a unique H_A -invariant measure \mathfrak{n}_0 on $\hat{\xi}_0$. Moreover, there exists a nonzero measure on each \check{x} and on $\hat{\xi}$, such that they coincide with μ_0 on \check{x}_0 and \mathfrak{n}_0 , respectively. For these measures, if \check{x}_1 corresponds to \check{x}_2 by $g \in G$, then μ_{x_1} corresponds to μ_{x_2} by g . Analogous correspondence holds for elements in \mathcal{A} . Denote by \mathfrak{m} and \mathfrak{n} the (normalized) G -invariant measures on X and \mathcal{A} , respectively, induced from Haar measure on G . Fix a smooth positive and bounded function ρ on F and consider the following Radon and its dual transformations

$$\begin{aligned} \hat{f}(\xi) &= \int_{\check{\xi}} f(x) \rho(x, \xi) d\mu_{\xi}(x), \\ \check{\varphi}(x) &= \int_{\hat{x}} \varphi(\xi) \rho(x, \xi) d\mathfrak{n}_x(\xi) \end{aligned}$$

for Borel and non-negative functions f and φ . Then it is not hard to see that (compare [12,9] in the continuous or L^p category; see also [8] for similar and general approach)

$$\int_X f(x) \check{\varphi}(x) d\mathbf{m}(x) = \int_{\mathcal{A}} \hat{f}(\xi) \varphi(\xi) d\mathbf{n}(\xi) \quad (12)$$

for any non-negative Borel functions f and φ .

Put

$$\Sigma = \{\rho(\cdot, \xi) \mu_\xi \mid \xi \in \mathcal{A}\}.$$

Each μ_ξ is supported on $\hat{\xi} \subset X$, whereas each \mathbf{n}_x is supported on $\check{x} \subset \mathcal{A}$. We seek for the extremal function f_Σ for the family Σ .

Proposition 6. Assume that there is an admissible function f_Σ for Σ such that the following conditions hold:

1. $\hat{f}_\Sigma = 1$,
2. there exists Borel positive function φ_Σ on \mathcal{A} such that $f_\Sigma^{p-1} = \check{\varphi}_\Sigma$.

Then f_Σ is extremal for the p -modulus of Σ .

Proof. The proof follows by using standard methods. Let f be admissible for Σ . By (12) and Hölder inequality,

$$\begin{aligned} 0 &\leq \int_{\mathcal{A}} \widehat{f - f_\Sigma} \varphi_\Sigma d\mathbf{m} = \int_X f_\Sigma^{p-1} f d\mathbf{m} - \int_X f_\Sigma^p d\mathbf{m} \\ &\leq \left(\int_X f_\Sigma^p d\mathbf{m} \right)^{\frac{1}{q}} \left(\int_X f^p d\mathbf{m} \right)^{\frac{1}{p}} - \int_X f_\Sigma^p d\mathbf{m}, \end{aligned}$$

where q is a coefficient conjugate to p . Thus

$$\int_X f_\Sigma^p d\mathbf{m} \leq \int_X f^p d\mathbf{m}.$$

Hence f_Σ is extremal. \square

Remark 2. Let us remark on the similarity of above theorem to the Badger's criterion (Theorem 1). In this case, Σ_0 is just Σ . Thus, condition (1) of Proposition 6 and Theorem 1 coincide. By the 'Plancherel' formula (12) for any positive Borel function f on X such that $\hat{f} \geq 0$ on Σ we have

$$\int_X f_\Sigma^{p-1} f d\mathbf{m} = \int_{\mathcal{A}} \varphi \hat{f} d\mathbf{n} \geq 0$$

assuming (2) of Proposition 6.

Remark 3. If the Radon transform is injective, i.e. the map $f \mapsto \hat{f}$ is on the suitable class injective with the constant functions in the range of this correspondence, then the condition (1) of Proposition 6 characterizes f_Σ completely. In such case, the second assumption is useless.

We will justify above theorem on the concrete examples. The classical example in this theory is the Radon transform on $X = \mathbb{R}^n$ with the dual homogeneous space \mathcal{A} being the manifold of all hyperplanes

in \mathbb{R}^n . By well-known results extremal function should be harmonic and bounded, hence constant. Since it integrates to 1 on each element of \mathcal{A} , we conclude that there is no extremal function in this case and, consequently, p -modulus of considered family Σ is zero for any $p > 1$. Hence, Proposition 6 does not apply in this situation.

Let us begin with the ‘elementary’ example.

Example 2. Consider a $(n-1)$ -dimensional unit sphere S^{n-1} centered at the origin in \mathbb{R}^n . Then $S^{n-1} = SO(n)/SO(n-1)$, where $SO(n-1)$ is considered as a subgroup fixing the north pole x_0 . Consider a double fibration with fibrations $SO(n) \mapsto S^{n-1}$ and a trivial one $SO(n) \mapsto SO(n)$, where we consider $SO(n)$ as a trivial quotient $SO(n) = SO(n)/\{e\}$. Then a coset $gSO(n-1)$ intersects coset $\tilde{g}\{e\}$ if and only if $g\tilde{g}^{-1} \in SO(n-1)$, hence $x \in M$ is incident to a transformation $g \in SO(n)$ if and only if $gx_0 = x$. In other words

$$\check{x} = gSO(n-1) \subset SO(n), \quad \hat{g} = \{x\} \subset M,$$

where $x = gx_0$. Then μ_g is a Dirac measure δ_x at x and \mathbf{n}_x is a Haar measure on $gSO(n-1)$. Hence, for any Borel non-negative functions f and φ we have (we choose $\rho = 1$)

$$\hat{f}(g) = f(x), \quad \check{\varphi}(x) = \int_{gSO(n-1)} \varphi(h) d\mathbf{n}_x(h), \quad gx_0 = x.$$

Thus, any admissible function f for $\Sigma = \{\mu_x \mid x \in S^{n-1}\}$ satisfies $f \geq 1$. Now, it is obvious that the extremal function f_Σ is identically 1. Let us apply Proposition 6. Clearly, $f_\Sigma = 1$ satisfies condition (1) of Proposition 6. Taking $\varphi_\Sigma = 1$ also the second condition holds. Notice, that formula (12) may be obtained straightforward by the coarea formula ($gx_0 = x$)

$$\begin{aligned} \int_{SO(n)} \hat{f}(g) \varphi(g) d\mathbf{n}(g) &= \int_{SO(n)} f(gx_0) \varphi(g) d\mathbf{n}(g) \\ &= \int_M \int_{SO(n-1)} f(hx_0) \varphi(h) d\mathbf{n}_{x_0}(h) d\mathbf{m}(x) \\ &= \int_M f(x) \int_{SO(n-1)} \varphi(h) d\mathbf{n}_{x_0}(h) d\mathbf{m}(x) \\ &= \int_M f(x) \int_{gSO(n-1)} \varphi(g^{-1}h) d\mathbf{n}_{x_0}(h) d\mathbf{m}(x) \\ &= \int_M f(x) \check{\varphi}(x) d\mathbf{m}(x), \end{aligned}$$

since $hx_0 = x_0$ for any $h \in SO(n-1)$ and by invariance of \mathbf{n}_x .

The second example concerns the Funk transform on a sphere.

Example 3. Let $X = S^2$ be the unit 2-dimensional sphere in \mathbb{R}^3 , $S^2 = O(3)/O(2)$. Let \mathcal{A} be a set of all great circles in S^2 , i.e., $\mathcal{A} = O(3)/O(2) \times \mathbb{Z}_2$, where $O(2) \times \mathbb{Z}_2$ is a subgroup fixing a line through a north pole. Then, for $x \in S^2$, \hat{x} is a set of all great circles passing through x and for $\xi \in \mathcal{A}$, $\check{\xi}$ is just ξ considered as a great circle in S^2 . Consider a family Σ of all Lebesgue measures on great circles. By an inversion formula for

a Funk transform [10] it follows that the extremal function for the p -modulus of Σ equals $f_\Sigma = \frac{1}{2\pi}$. Notice that in this case we can apply Proposition 6. In fact, f_Σ satisfies (1) and putting φ_Σ to be a constant such that $\check{\varphi} = (2\pi)^{1-p}$ it satisfies (2) of Proposition 6.

To finish this section, we show the relation with the plan \mathbf{n}_Σ on Σ . Firstly, we have a bijective map $\Phi : \mathcal{A} \rightarrow \Sigma$, $\Phi(\xi) = \rho(\cdot, \xi)\mu_\xi$. This map is continuous, since it is equivalent to the continuity of \hat{f} for any $f \in C_b(X)$. Hence, is Borel. Consider a measure \mathbf{n}'_Σ given by the relation

$$\mathbf{n}'_\Sigma = \Phi_\#(\varphi_\Sigma \mathbf{n}).$$

Then \mathbf{n}'_Σ is a Borel measure on Σ . Denote also by \hat{f} a function on Σ corresponding by Φ to \hat{f} , i.e., $\hat{f}(\rho(\cdot, \xi)\mu_\xi) \equiv \hat{f}(\xi)$. By (5) and (12),

$$\int_\Sigma \hat{f} d\mathbf{n}'_\Sigma = \int_{\mathcal{A}} \hat{f} \varphi_\Sigma d\mathbf{n} = \int_X f \check{\varphi}_\Sigma d\mathbf{m} = \int_X f_\Sigma^{p-1} f d\mathbf{m} = \text{mod}_p(\Sigma)^{-1} \int_\Sigma \hat{f} d\mathbf{n}_\Sigma$$

for any $f \in \mathcal{L}^p(X, \mathbf{m})$. Thus

$$\text{mod}_p(\Sigma) \mathbf{n}'_\Sigma = \mathbf{n}_\Sigma.$$

Acknowledgment

The author wishes to thank anonymous referee for useful comments that led to the improvement of the paper.

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