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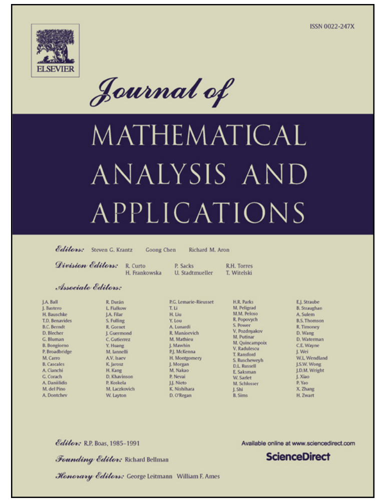

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Extremal problems of Hardy-Littlewood-Sobolev inequalities on compact Riemannian manifolds *

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Abstract

In this paper we study the existence of extremal problems for the Hardy-Littlewood-Sobolev inequalities on compact Riemannian manifolds without boundary via Concentration-Compactness principle.

keywords: Hardy-Littlewood-Sobolev inequalities, Existence of extremals, Concentration-Compactness principle, Compact Riemannian manifold.

1 Introduction

It is well known that classical Sobolev inequalities and Hardy-Littlewood-Sobolev (HLS) inequalities are basic tools in analysis and geometry, and their sharp constants play essential role on certain geometric and probabilistic information. In fact, in past decades, these sharp inequalities were applied extensively in the study of curvature equations, see, e.g. [1, 3, 4, 14–16, 26] and references therein. Recently, there have been some interesting results concerning the globally defined fractional operators such as fractional Yamabe problems, fractional prescribing curvature problems, fractional Paneitz operators, etc. (see, e.g. [11–13, 20–23] and references therein), which are closely related to singular integral operators. In particular, sharp HLS inequalities and reversed HLS inequalities are immediately applied to discuss a class of prescribing integral curvature problems by Zhu [30] and integral equations on bounded domain in [6, 7]. So, HLS inequalities play essential role in the global analysis of some operators of geometric interest.

Motivated by these studies, there are some extensions of classical HLS inequalities, such as reversed HLS inequalities on $\mathbb{R}^n$ [9, 28], HLS inequalities and reversed HLS inequalities on the upper half space [5, 8, 29], HLS inequalities on compact Riemannian manifolds [17], or HLS inequalities on the Heisenberg group [10]. This paper is mainly devoted to discuss the sharp HLS inequalities on compact manifolds without boundary.

Let (M^n, g) be a given compact Riemmanian manifold without boundary, $\alpha \in (0, n)$ be a parameter and $|x - y|_g$ represent the distance from x to y on

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M^n under metric g. In [17], Han and Zhu have introduced the following integral operator

$$I_\alpha f(x) = \int_{M^n} \frac{f(y)}{|x-y|_g^{n-\alpha}} dV_y \tag{1.1}$$

and studied the following Hardy-Littlewood-Sobolev inequalities, which are well known in the community.

Proposition 1.1 (Proposition 1.1. in [17])**.** *Assume that $\alpha \in (0, n)$, $1 < p < \frac{n}{\alpha}$ and q is given by*

$$\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}. \tag{1.2}$$

Then there is a positive constant $C(\alpha, p, M^n, g)$, such that

$$||I_\alpha f||_{L^q(M^n)} \leq C(\alpha, p, M^n, g)||f||_{L^p(M^n)} \tag{1.3}$$

holds for all $f \in L^p(M^n)$. Moreover, for $1 \leq r < q$, the operator $I_\alpha : L^p(M^n) \to L^r(M^n)$ is a compact embedding.

As everyone knowns, it is important to study the extremal problems of (1.3), which can be stated as follows:

$$\begin{aligned} N_{p,\alpha,M} :=& \sup\{\|I_\alpha f\|_{L^q(M^n)} : \ \|f\|_{L^p(M^n)} = 1\} \\ :=& \sup\Big\{\frac{\|I_\alpha f\|_{L^q(M^n)}}{\|f\|_{L^p(M^n)}} : \ f \in L^p(M^n)\backslash\{0\}\Big\}. \end{aligned} \tag{1.4}$$

By duality, $N_{p,\alpha,M}$ is also defined equivalently as

$$N_{p,\alpha,M} := \sup\Big\{\Big|\int_{M^n}\int_{M^n} f(x)g(y)|x-y|_g^{\alpha-n} dV_x dV_y\Big| : \ \|f\|_p = \|g\|_t = 1\Big\} \tag{1.5}$$

$$:= \sup_{\|f\|_p>0, \|g\|_t>0} \frac{\Big|\int_{M^n}\int_{M^n} f(x)g(y)|x-y|_g^{\alpha-n} dV_x dV_y\Big|}{\|f\|_p\|g\|_t}, \tag{1.6}$$

where $t = \frac{q}{q-1}$. In particular, we denote $N_{p,\alpha,\mathbb{R}^n}$ as $N_{p,\alpha}$.

In [17], Han and Zhu have discussed the extremal problems (1.4) for the conformal case, i.e. the case $p = t$ and $f = g$. Then as an application, they studied a class of integral curvature problems. Particularly, they gave a new proof for the Yamabe problems on compact locally conformally flat manifold.

This paper will deal with the remaining cases. Firstly, we will get the following estimate for the sharp constant.

Proposition 1.2 (Estimate)**.** $N_{p,\alpha,M} \geq N_{p,\alpha}$.

Then, similar to the existence criteria for the classical Yamabe problems, we will give the following existence criteria of the extremal problems (1.4) by the Concentration-Compactness principle introduced by Lions (see [24, 25]).

Theorem 1.3 (Criteria of Existence)**.** *Under the assumption of Proposition 1.1 and if $N_{p,\alpha,M} > N_{p,\alpha}$, then the supremum is attained, i.e., there exists some function $f(x) \in L^p(M^n)$ such that $N_{p,\alpha,M} = \frac{\|I_\alpha f\|_{L^q(M^n)}}{\|f\|_{L^p(M^n)}}$.*

Remark 1.4. Let $G^g_x(y) = n(n-2)\omega_n \Gamma^g_x(y)$, where $\Gamma^g_x(y)$ is the Green's function with pole at x for the conformal Laplacian operator $-\Delta_g + \frac{n-2}{4(n-1)} R_g$ and ω_n is the volume of the unit ball. Arguing as in [17], for the operator

$$I_{M^n,g,\alpha} = \int_{M^n} \left[G^g_x(y)\right]^{\frac{\alpha-n}{2-n}} g(y) dV_y,$$

we can also get the similar results of estimate (Proposition 1.2) and existence criteria (Theorem 1.3). Since the details of the proof is similar, we omit it for conciseness.

The plan of the paper is following. In Section 2, we introduce some known facts and give a new proof of compactness of the operator (1.1) for convenience. Then, we present our Concentration-Compactness Lemma in Section 3. Finally, Section 4 is devoted to get the estimate (Proposition 1.2) and prove the existence for the extremal problems (Theorem 1.3).

2 Preliminary

Firstly, we recall the existence of the extremal problems of classical Hardy-Littlewood-Sobolev inequalities on $\mathbb{R}^n$.

Theorem 2.1 (Theorem 2.3 of [27] & Theorem 2.1 of [25])**.** *There exist a pair of nonnegative functions $f \in L^p(\mathbb{R}^n)$ and $g \in L^t(\mathbb{R}^n)$ such that*

$$\begin{cases} \int_{\mathbb{R}^n} |f|^p dx = \int_{\mathbb{R}^n} |g|^t dy = 1 \\ N_{p,\alpha} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x) g(y) |x-y|^{\alpha-n} dx dy. \end{cases} \tag{2.1}$$

Hence, the extremal pair satisfies the Euler-Lagrange equations

$$\begin{cases} |x|^{\alpha-n} * g = N_{p,\alpha} f^{p-1}(x), \\ |x|^{\alpha-n} * f = N_{p,\alpha} g^{t-1}(x). \end{cases} \tag{2.2}$$

Furthermore, by scaling, we know that function pairs

$$f_\lambda(x) = \lambda^{-p/n} f(x/\lambda), \quad g_\lambda(y) = \lambda^{-t/n} g(y/\lambda), \quad \forall \lambda > 0 \tag{2.3}$$

also satisfy (2.1) *and* (2.2).

For convenience, we introduce the following Young's inequality.

Lemma 2.2 (Young's inequality, Lemma 2.1 of [17])**.** *For a given compact manifold (M^n, g), define*

$$g * h(x) = \int_{M^n} g(y) h(|y-x|_g) dV_y.$$

Then, there is a constant $C > 0$, such that

$$||g * h||_{L^r} \le C ||g||_{L^q} \cdot ||h||_{L^p},$$

where $p, q, r \in (1, \infty)$ and satisfy $1 + \frac{1}{r} = \frac{1}{q} + \frac{1}{p}$.

Following, we give a new proof of compactness for the operator (1.1).

Proposition 2.3 (Compactness)**.** *For all* $r \in [1, q)$*, where* q *is defined as* (1.2)*, the operator* $I_\alpha : L^p(M^n) \to L^r(M^n)$ *is compact.*

Proof. Take any bounded sequence $\{f_m\}$ in $L^p(M^n)$. Then, there exists a subsequence (still denoted by $\{f_m\}$) and some function $f \in L^p(M)$ such that

$$f_m \rightharpoonup f \quad \text{weakly in} \quad L^p(M^n). \tag{2.4}$$

It is known that the proof will be completed if it holds that

$$I_\alpha f_m \to I_\alpha f \quad \text{strongly in} \quad L^r(M^n).$$

Denoted by $K^\rho(t) = t^{\alpha-n}\chi_{\{t>\rho\}}$ and $K_\rho(t) = t^{\alpha-n} - K^\rho(t)$ for $t > 0$, where $\rho > 0$ is a parameter to be chosen later. Then, we decompose the integral operator as

$$I_\alpha f_m(x) = K^\rho * f_m(x) + K_\rho * f_m(x) \triangleq I^1_\alpha f_m(x) + I^2_\alpha f_m(x).$$

Since, for any fixed $x \in M^n$, $K^\rho(|x-y|_g) \in L^{p'}(M^n)$ with respect to y, then weak convergence implies that $K^\rho * f_m \to K^\rho * f$ pointwisely. Notice also that

$$\left|K^\rho * f_m(x)\right| \le \|K^\rho\|_{P'}\|f_m\|_p \le C(\rho),$$

where $C(\rho)$ is independent of x and m. By dominated convergence theorem, we have

$$K^\rho * f_m \to K^\rho f \quad \text{strongly in} \quad L^r(M^n).$$

Since

$$\int_{M^n} K_\rho(|x-y|_g)^s dV_x \le C\rho^{(\alpha-n)s+n},$$

where $0 < s < \frac{n}{n-\alpha}$, we take parameter $s > 1$ satisfying $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{s}$ and get from the Young's inequality (see Lemma 2.2) that

$$\|K_\rho * (f_m - f)\|_r \le C\rho^{(\alpha-n)+n/s}\|f_m - f\|_p \le C\rho^{(\alpha-n)+n/s}.$$

By now, through choosing first ρ small and then m large, we deduce the claimed convergence in $L^r(M^n)$. □

Based on the Proposition 2.3, we have the following conclusions.

Remark 2.4. For any bounded sequence $\{f_m\} \subset L^p(M^n)$, there exists a subsequence (still denoted by $\{f_m\}$) and some function $f \in L^p(M^n)$ such that

$$\begin{gathered} f_m \rightharpoonup f \quad \text{weakly in} \quad L^p(M^n), \\ I_\alpha f_m \rightharpoonup I_\alpha f \quad \text{weakly in} \quad L^q(M^n), \\ I_\alpha f_m \to I_\alpha f \quad \text{strongly in} \quad L^r(M^n) \end{gathered}$$

for all $r \in [1, q)$. Furthermore, $I_\alpha f_m \to I_\alpha f$ pointwisely a.e. in M^n.

3 Concentration-Compactness Lemma

Lemma 3.1. *Let $\{f_m\} \subset L^p(M^n)$ be a bounded nonnegative sequence and suppose that there exists some function $f \in L^p(M^n)$ such that*

$$f_m \rightharpoonup f \quad \text{weakly in} \quad L^p(M^n).$$

After passing to a subsequence, assume that $|I_\alpha f_m|^q dV_x$, $|f_m|^p dV_x$ converge weakly in the sense of measure to some bounded nonnegative measures ν, μ on M^n, respectively. Then we have:

i) There exist some countable set J, a family $\{P_j : j \in J\}$ of distinct points in M^n, and a family $\{\nu_j : j \in J\}$ of nonnegative numbers such that

$$\nu = |I_\alpha f|^q dV_x + \sum_{j\in J} \nu_j \delta_{P_j}, \tag{3.1}$$

where δ_{P_j} is the Dirac-mass of mass 1 concentrated at $P_j \in M^n$;

ii) In addition we have

$$\mu \geq |f|^p dV_x + \sum_{j\in J} \mu_j \delta_{P_j} \tag{3.2}$$

for some family $\{\mu_j > 0 : j \in J\}$, where μ_j satisfy

$$\nu_j^{1/q} \leq N_{p,\alpha} \mu_j^{1/p} \quad \text{for all} \quad j \in J. \tag{3.3}$$

In particular, $\sum_{j\in J} \nu_j^{p/q} < +\infty$.

Proof of i). By the conditions of the sequence $\{f_m\} \subset L^p(M^n)$, we know from the Remark 2.4 that

$$\begin{aligned} I_\alpha f_m \rightharpoonup I_\alpha f \quad &\text{weakly in} \quad L^q(M^n), \\ I_\alpha f_m \to I_\alpha f \quad &\text{strongly in} \quad L^r(M^n) \\ I_\alpha f_m \to I_\alpha f \quad &\text{pointwisely a.e. in} \quad M^n, \end{aligned}$$

where $r \in [1, q)$. Then, Brézis-Lieb Lemma leads that

$$\begin{aligned} 0 &= \lim_{m\to+\infty} \int_{M^n} \left(|I_\alpha f_m|^q - |I_\alpha (f_m - f)|^q - |I_\alpha f|^q\right) dV_x \\ &= \int_{M^n} d\nu - \int_{M^n} |I_\alpha f|^q dV_x - \lim_{m\to+\infty} |I_\alpha (f_m - f)|^q dV_x. \end{aligned}$$

So, it is sufficient to discuss the case $f \equiv 0$. By the classical argument of Lions (see [24, 25]), it is sufficient to prove

$$\left(\int_{M^n} |\varphi|^q d\nu\right)^{1/q} \leq N_{p,\alpha,M} \left(\int_{M^n} |\varphi|^p d\mu\right)^{1/p}, \quad \forall \varphi \in C_0^\infty(M^n). \tag{3.4}$$

In fact, (3.4) is the main assumption of Lemma 1.2 in [24] and is the key in the proof of the second Concentration-Compactness Lemma given by Lions, see the proof of Lemma 1.1 in [24].

Since, for any $\varphi(x) \in C_0^\infty(M^n)$,

$$\begin{aligned}
&\left(\int_{M^n} |\varphi(x) I_\alpha f_m|^q dV_x\right)^{1/q} \\
\le &\left(\int_{M^n} |I_\alpha(\varphi f_m)|^q dV_x\right)^{1/q} + \left(\int_{M^n} |\varphi(x) I_\alpha f_m - I_\alpha(\varphi f_m)|^q dV_x\right)^{1/q} \\
\le &N_{p,\alpha,M}\left(\int_{M^n} |\varphi f_m|^p dV_x\right)^{1/p} + \left(\int_{M^n} |\varphi(x) I_\alpha f_m - I_\alpha(\varphi f_m)|^q dV_x\right)^{1/q},
\end{aligned}$$

we get as $m \to +\infty$ that

$$\begin{aligned}
\left(\int_{M^n} |\varphi|^q d\nu\right)^{1/q} \le &N_{p,\alpha,M}\left(\int_{M^n} |\varphi|^p d\mu\right)^{1/p} \\
&+ \lim_{m\to+\infty}\left(\int_{M^n} |\varphi(x) I_\alpha f_m - I_\alpha(\varphi f_m)|^q dV_x\right)^{1/q}.
\end{aligned}$$

So, we can obtain (3.4) if

$$\lim_{m\to+\infty}\left(\int_{M^n} |\varphi(x) I_\alpha f_m - I_\alpha(\varphi f_m)|^q dV_x\right)^{1/q} = 0. \tag{3.5}$$

Notice that

$$\begin{aligned}
|\varphi(x) I_\alpha f_m - I_\alpha(\varphi f_m)| = &\left|\int_{M^n} (\varphi(x) - \varphi(y))|x-y|_g^{\alpha-n} f_m(y) dV_y\right| \\
\le &C \int_{M^n} |x-y|_g^{\alpha+1-n} |f_m(y)| dV_y
\end{aligned}$$

and

$$R(x,y) := (\varphi(x) - \varphi(y))|x-y|_g^{\alpha-n} \in L^r(M^n),$$

where $r \le +\infty$ if $\alpha+1-n \ge 0$ and $r < \frac{n}{n-\alpha-1}$ if $\alpha+1-n < 0$. If $\alpha+1-n \ge 0$, we can prove (3.5) by dominated convergence theorem. While for the case $\alpha+1-n < 0$, we obtain through the Hardy-Littlewood-Sobolev inequalities (1.3) that

$$\int_{M^n} R(x,y) f_m(y) dV_y \in L^s(M^n),$$

where $s = (\frac{1}{p} - \frac{\alpha+1}{n})^{-1} > q$. Furthermore, repeating the proof of Proposition 2.3, we have

$$\int_{M^n} R(x,y) f_m(y) dV_y \to \int_{M^n} R(x,y) f(y) dV_y = 0 \quad \text{strongly in} \quad L^q(M^n).$$

So, we get (3.5) and complete the proof of i). $\square$

Proof of ii). Since

$$f_m \rightharpoonup f \quad \text{weakly in} \quad L^p(M^n),$$

then, $\mu \ge |f|^p dV_x$. So, we just have to show that for each fixed $j \in J$,

$$\nu_j^{1/q} = \nu(\{P_j\})^{1/q} \le N_{p,\alpha} \mu(\{P_j\})^{1/p} = N_{p,\alpha} \mu_j^{1/p}.$$

For point $P_j \in M^n$, choose a neighbourhood $\Omega_{P_j} \subset M^n$ so that for $\delta > 0$ small enough, in normal coordinates, $\exp(B_\delta) \subset \Omega_{P_j}$ and

$$(1-\epsilon)I \leq g(x) \leq (1+\epsilon)I, \quad \forall x \in B_\delta.$$

Take $\varphi_\lambda(x) = \varphi(\frac{x}{\lambda})$, where $\varphi(x) \in C_0^\infty(\mathbb{R}^n)$ satisfies $0 \leq \varphi(x) \leq 1$, $\varphi(0) = 1$, $\operatorname{supp} \varphi \subset B_1$ and $\lambda \in (0, \delta)$. Then,

$$\begin{aligned}
I_\alpha((\varphi_\lambda \circ \exp^{-1}) \cdot f_m) &= \int_{M^n} (\varphi_\lambda \circ \exp^{-1})(y) f_m(y) |x-y|_g^{\alpha-n} dV_g(y) \\
&= \int_{B_\delta} \varphi_\lambda(y)(f_m \circ \exp)(y)|x-y|_g^{\alpha-n}\sqrt{\det g(y)}dy \\
&\leq \frac{(1+\epsilon)^{n/2}}{(1-\epsilon)^{n-\alpha}} \int_{B_\delta} \varphi_\lambda(y)(f_m \circ \exp)(y)|x-y|^{\alpha-n}dy
\end{aligned}$$

and

$$\begin{aligned}
&\left(\int_{\exp(B_\delta)} |I_\alpha((\varphi_\lambda \circ \exp^{-1}) \cdot f_m)|^q dV_x\right)^{1/q} \\
\leq &(1+\epsilon)^{n/(2q)} \left(\int_{B_\delta} |I_\alpha((\varphi_\lambda \circ \exp^{-1}) \cdot f_m)|^q dx\right)^{1/q} \\
\leq &\frac{(1+\epsilon)^{\frac{n}{2}(1+\frac{1}{q})}}{(1-\epsilon)^{n-\alpha}} \left(\int_{B_\delta} \left|\int_{B_\delta} \varphi_\lambda(y)(f_m \circ \exp)(y)|x-y|^{\alpha-n}dy\right|^q dx\right)^{1/q} \\
\leq &\frac{(1+\epsilon)^{\frac{n}{2}(1+\frac{1}{q})}}{(1-\epsilon)^{n-\alpha}} N_{p,\alpha} \left(\int_{B_\delta} |\varphi_\lambda(y)(f_m \circ \exp)(y)|^p dy\right)^{1/p} \\
\leq &\frac{(1+\epsilon)^{\frac{n}{2}(1+\frac{1}{q})}}{(1-\epsilon)^{\frac{n}{2p}+n-\alpha}} N_{p,\alpha} \left(\int_{\exp(B_\delta)} |(\varphi_\lambda \circ \exp^{-1}) \cdot f_m|^p dV_y\right)^{1/p}.
\end{aligned}$$

So,

$$\begin{aligned}
&\left(\int_{M^n} |(\varphi_\lambda \circ \exp^{-1}) \cdot I_\alpha f_m|^q dV_x\right)^{1/q} \\
\leq &\left(\int_{\exp(B_\delta)} |I_\alpha((\varphi_\lambda \circ \exp^{-1}) \cdot f_m)|^q dV_x\right)^{1/q} \\
&+ \left(\int_{\exp(B_\delta)} |(\varphi_\lambda \circ \exp^{-1}) \cdot I_\alpha f_m - I_\alpha((\varphi_\lambda \circ \exp^{-1}) \cdot f_m)|^q dV_x\right)^{1/q} \\
\leq &\frac{(1+\epsilon)^{\frac{n}{2}(1+\frac{1}{q})}}{(1-\epsilon)^{\frac{n}{2p}+n-\alpha}} N_{p,\alpha} \left(\int_{\exp(B_\delta)} |(\varphi_\lambda \circ \exp^{-1}) \cdot f_m|^p dV_y\right)^{1/p} + \mathbf{I}, \qquad (3.6)
\end{aligned}$$

where

$$\mathbf{I} := \left(\int_{\exp(B_\delta)} |(\varphi_\lambda \circ \exp^{-1}) \cdot I_\alpha f_m - I_\alpha((\varphi_\lambda \circ \exp^{-1}) \cdot f_m)|^q dV_x\right)^{1/q}.$$

Repeating the argument of (3.5), we have, as $m \to +\infty$,

$$\mathbf{I} \to \left(\int_{\exp(B_\delta)} |(\varphi_\lambda \circ \exp^{-1}) \cdot I_\alpha f - I_\alpha((\varphi_\lambda \circ \exp^{-1}) \cdot f)|^q dV_x \right)^{1/q}.$$

So, letting $m \to +\infty$ leads

$$\begin{aligned} &\left(\int_{M^n} |\varphi_\lambda \circ \exp^{-1}|^q d\nu \right)^{1/q} \\ \le & \frac{(1+\epsilon)^{\frac{n}{2}(1+\frac{1}{q})}}{(1-\epsilon)^{\frac{n}{2p}+n-\alpha}} N_{p,\alpha} \left(\int_{M^n} |(\varphi_\lambda \circ \exp^{-1})|^p d\mu \right)^{1/p} \\ & + \left(\int_{\exp(B_\delta)} |(\varphi_\lambda \circ \exp^{-1}) \cdot I_\alpha f - I_\alpha((\varphi_\lambda \circ \exp^{-1}) \cdot f)|^q dV_x \right)^{1/q}. \end{aligned} \tag{3.7}$$

Since

$$\int_{\exp(B_\delta)} |(\varphi_\lambda \circ \exp^{-1}) \cdot I_\alpha f|^q dV_x \to 0 \quad \text{as} \quad \lambda \to 0^+$$

and

$$\begin{aligned} &\left(\int_{\exp(B_\delta)} |I_\alpha((\varphi_\lambda \circ \exp^{-1}) \cdot f)|^q dV_x \right)^{1/q} \\ \le & C \left(\int_{B_\delta} |(\varphi_\lambda \circ \exp^{-1}) \cdot f|^p dV_y \right)^{1/p} \to 0 \quad \text{as} \quad \lambda \to 0^+, \end{aligned}$$

we can complete the proof by letting $\lambda \to 0^+$ and $\epsilon \to 0^+$. □

4 Estimate and criteria of existence

Proof of Proposition 1.2. For small positive constant $\lambda > 0$, recall that $f_\lambda(x)$ and $g_\lambda(y)$ are given in (2.3). Take

$$\tilde{f}(x) = \begin{cases} f_\lambda(x), & \text{in } B_\delta(0) \\ 0, & \text{in } \mathbb{R}^n \setminus B_\delta(0) \end{cases} \quad \text{and} \quad \tilde{g}(y) = \begin{cases} g_\lambda(y), & \text{in } B_\delta(0) \\ 0, & \text{in } \mathbb{R}^n \setminus B_\delta(0) \end{cases}$$

where $\delta > 0$ is a fixed constant to be determined later. Then, for small enough λ and by (2.2),

$$\begin{aligned} & \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \tilde{f}(x) \tilde{g}(y) |x-y|^{\alpha-n} dx dy \\ = & \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f_\lambda(x) g_\lambda(y) |x-y|^{\alpha-n} dx dy \\ & - \int_{|x|>\delta} \int_{\mathbb{R}^n} f_\lambda(x) g_\lambda(y) |x-y|^{\alpha-n} dx dy \\ & - \int_{\mathbb{R}^n} \int_{|y|>\delta} f_\lambda(x) g_\lambda(y) |x-y|^{\alpha-n} dx dy \end{aligned}$$

$$
\begin{aligned}
&+\int_{|x|>\delta}\int_{|y|>\delta} f_\lambda(x)g_\lambda(y)|x-y|^{\alpha-n}dxdy\\
=&N_{p,\alpha}-N_{p,\alpha}\int_{|x|>\delta} f_\lambda^p(x)dx-N_{p,\alpha}\int_{|y|>\delta} g_\lambda^t(y)dy\\
&+\int_{|x|>\delta}\int_{|y|>\delta} f_\lambda(x)g_\lambda(y)|x-y|^{\alpha-n}dxdy\\
:=&N_{p,\alpha}-\mathbf{I}-\mathbf{II}+\mathbf{III}, \qquad (4.1)
\end{aligned}
$$

where, for fixed $\delta>0$ and as $\lambda\to 0^+$,

$$
\begin{aligned}
\mathbf{I} &:= N_{p,\alpha}\int_{|x|>\delta} f_\lambda^p(x)dx = N_{p,\alpha}\int_{|x|>\delta/\lambda} f^p(x)dx \to 0,\\
\mathbf{II} &:= N_{p,\alpha}\int_{|y|>\delta} g_\lambda^t(y)dy \to 0,\\
\mathbf{III} &:= \int_{|x|>\delta}\int_{|y|>\delta} f_\lambda(x)g_\lambda(y)|x-y|^{\alpha-n}dxdy\\
&\le C\left(\int_{|x|>\delta} f_\lambda^p dx\right)^{1/p}\left(\int_{|y|>\delta} g_\lambda^t dx\right)^{1/t}\to 0.
\end{aligned}
$$

So, for small enough λ,

$$
\begin{aligned}
&\frac{\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\tilde{f}(x)\tilde{g}(y)|x-y|^{\alpha-n}dxdy}{\|\tilde{f}\|_{L^p(\mathbb{R}^n)}\|\tilde{g}\|_{L^t(\mathbb{R}^n)}}\\
\ge&\frac{N_{p,\alpha}-\mathbf{I}-\mathbf{II}}{\|f_\lambda\|_{L^p(\mathbb{R}^n)}\|g_\lambda\|_{L^t(\mathbb{R}^n)}}=N_{p,\alpha}-\mathbf{I}-\mathbf{II}. \qquad (4.2)
\end{aligned}
$$

For any given point $P\in M^n$, choose a neighbourhood $\Omega_P\subset M^n$ so that for $\delta>0$ small enough, in normal coordinates, $\exp(B_\delta)\subset\Omega_P$ and

$$(1-\epsilon)I\le g(x)\le(1+\epsilon)I,\quad \forall x\in B_\delta.$$

Thus,

$$(1-\epsilon)|x-y|\le|x-y|_g\le(1+\epsilon)|x-y|,\qquad \forall x,\ y\in B_\delta.$$

In normal coordinates with respect to the center $P\in M^n$, let

$$u(x)=\begin{cases} f_\lambda(\exp^{-1}(x)), & \text{in}\quad \exp(B_\delta)\\ 0, & \text{in}\quad M^n\setminus\exp(B_\delta)\end{cases}$$

and

$$v(y)=\begin{cases} g_\lambda(\exp^{-1}(y)), & \text{in}\quad \exp(B_\delta)\\ 0, & \text{in}\quad M^n\setminus\exp(B_\delta).\end{cases}$$

Then

$$\int_{M^n}|u|^p dV_x\le(1+\epsilon)^{\frac{n}{2}}\int_{B_\delta(0)}|f_\lambda(x)|^p dx,$$

$$\begin{aligned}
&\int_{M^n} |v|^t dV_y \le (1+\epsilon)^{\frac{n}{2}} \int_{B_\delta(0)} |g_\lambda(y)|^t dy,\\
&\int_{M^n}\int_{M^n} u(x)v(y)|x-y|_g^{\alpha-n} dV_x dV_y\\
&\quad = \int_{B_\delta(0)}\int_{B_\delta(0)} \frac{u(x)v(y)}{|x-y|_g^{n-\alpha}} \sqrt{\det g(x)}\sqrt{\det g(y)}dxdy\\
&\quad \ge \int_{B_\delta(0)}\int_{B_\delta(0)} \frac{f_\lambda(x)g_\lambda(y)}{(1+\epsilon)^{n-\alpha}|x-y|^{n-\alpha}}(1-\epsilon)^n dxdy\\
&\quad = \frac{(1-\epsilon)^n}{(1+\epsilon)^{n-\alpha}}\int_{B_\delta(0)}\int_{B_\delta(0)} \frac{f_\lambda(x)g_\lambda(y)}{|x-y|^{n-\alpha}}dxdy. \qquad (4.3)
\end{aligned}$$

Thus

$$\begin{aligned}
N_{p,\alpha,M} &\ge \frac{\int_{M^n}\int_{M^n} u(x)v(y)|x-y|_g^{\alpha-n} dV_x dV_y}{\|u\|_{L^p(M^n)}\|v\|_{L^t(M^n)}}\\
&\ge \frac{\frac{(1-\epsilon)^n}{(1+\epsilon)^{n-\alpha}}\int_{B_\delta(0)}\int_{B_\delta(0)} f_\lambda(x)g_\lambda(y)|x-y|^{\alpha-n}dxdy}{(1+\epsilon)^{\frac{n}{2}(\frac{1}{p}+\frac{1}{t})}\|f_\lambda\|_{L^p(B_\delta(0))}\|g_\lambda\|_{L^t(B_\delta(0))}}\\
&\ge \frac{(1-\epsilon)^n}{(1+\epsilon)^{\frac{n}{2}(\frac{1}{p}+\frac{1}{t})+n-\alpha}}\left(N_{p,\alpha} - \mathbf{I} - \mathbf{II}\right).
\end{aligned}$$

Sending ϵ and λ to 0, we obtain the estimate. □

Prof of Theorem 1.3. Take a maximizing nonnegative sequence $\{f_m(x)\} \subset L^p(M^n)$ satisfying $\int_{M^n} f_m^p dV_x = 1$ and

$$\|I_\alpha f_m\|_{L^q(M^n)} \to N_{p,\alpha,M}, \quad \text{as} \quad m \to +\infty. \qquad (4.4)$$

Then, there exist a subsequence of $\{f_m\}$ (still denoted by $\{f_m\}$) and some function $f \in L^p(M^n)$ such that

$$f_m \rightharpoonup f \quad \text{weakly in} \quad L^p(M^n).$$

By Hardy-Littlewood-Sobolev inequalities (1.3), we know that

$$\mu_m = |f_m|^p dV_x, \quad \nu_m = |I_\alpha f_m|^q dV_x \qquad (4.5)$$

are two families of bounded measures. So, there exist two nonnegative bounded measures μ and ν on M^n such that

$$\mu_m \rightharpoonup \mu, \quad \nu_m \rightharpoonup \nu$$

weakly in the sense of measures.

Applying the Concentration-Compactness Lemma (see Lemma 3.1), we have

$$\nu = |I_\alpha f|^q dV_x + \sum_{j\in J}\nu_j\delta_{P_j}, \quad \mu \ge |f|^p dV_x + \sum_{j\in J}\mu_j\delta_{P_j}, \qquad (4.6)$$

and $\nu_j^{1/q} \le N_{p,\alpha}\mu_j^{1/p}$ for all $j \in J$. Since $\int_{M^n} d\mu = \lim_{m\to+\infty}\int_{M^n}|f_m|^p dV_x = 1$, then $\int_{M^n}|f|^p dV_x \le 1$ and $\mu_j \le 1,\ j \in J$.

We claim that $\mu_j = 0,\ j \in J$, which implies that $\nu_j = 0,\ j \in J$.
In fact, otherwise, combining (4.6) and the fact $\frac{q}{p} > 1$, we have

$$\begin{aligned}
N^q_{p,\alpha,M} &= \lim_{m\to+\infty} \int_{M^n} |I_\alpha f_m|^q dV_x = \int_{M^n} d\nu \\
&= \int_{M^n} |I_\alpha f|^q dV_x + \sum_{j\in J} \nu_j \\
&\le N^q_{p,\alpha,M} \|f\|^q_{L^p(M^n)} + \sum_{j\in J} N^q_{p,\alpha} \mu_j^{q/p} \\
&< N^q_{p,\alpha,M} \left(\int_{M^n} |f|^p dV_x \right)^{q/p} + \sum_{j\in J} N^q_{p,\alpha,M} \mu_j^{q/p} \\
&\le N^q_{p,\alpha,M} \left(\int_{M^n} |f|^p dV_x + \sum_{j\in J} \mu_j \right)^{q/p} \\
&= N^q_{p,\alpha,M} \left(\int_{M^n} d\mu \right)^{q/p} = N^q_{p,\alpha,M}, \qquad (4.7)
\end{aligned}$$

which is a contradiction.

Repeating the process of (4.7), we have that

$$N^q_{p,\alpha,M} = \int_{M^n} |I_\alpha f|^q dV_x \quad \text{and} \quad \int_{M^n} |f|^p dV_x = 1,$$

i.e., f is a maximizer. □

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