

Strong Comparison Principle for Radial Solutions of Quasi-Linear Equations

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Let Ω be either a ball or an annulus centered about the origin in \mathbb{R}^N and Δ_p the usual p -Laplace operator in \mathbb{R}^N . Let $f_1, f_2 \in L^1_{loc}(\Omega)$ be two radial functions on Ω with $f_1 \leq f_2, f_1 \not\equiv f_2$. Let $b : \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing continuous function. Let $u_1, u_2 \in C^{1,\beta}(\Omega)$, $\beta \in (0, 1]$ be any two radial weak solutions of $-\Delta_p u_i = b(u_i) + f_i$ in Ω . We then show that $u_1 \leq u_2$ in Ω implies $u_1 < u_2$ in Ω and also that appropriate versions of Hopf boundary point principle hold. © 2001 Academic Press

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0. INTRODUCTION

Let $p \in (1, \infty)$ and let $-\Delta_p$ denote the usual p -Laplace operator in \mathbb{R}^N defined by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$. Let $\Omega \subset \mathbb{R}^N, N \geq 1$, be a bounded connected domain whose boundary $\partial\Omega$ is a C^2 -manifold if $N \geq 2$. Let $b : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function which is monotone in the second variable. We are interested in the following:

Strong Comparison Principle (SCP). Take $f_1, f_2 \in L^1_{loc}(\Omega), f_1 \leq f_2$, and $f_1 \not\equiv f_2$. Let $u_1, u_2 \in C^{1,\beta}(\Omega)$, $\beta \in (0, 1]$, be any two weak solutions of

$$(P) \quad -\Delta_p u_i = b(x, u_i) + f_i(x) \text{ in } \Omega, i = 1, 2.$$

We say that the strong comparison principle (SCP) holds for the problem (P) if

$$u_2 \geq u_1 \quad \text{in } \Omega \quad \Rightarrow \quad u_2 > u_1 \quad \text{in } \Omega.$$



We now give a quick review of known results concerning (SCP) for the problem (P). We refer to [1] for more details and references to related works in the area. In this interesting reference, the following results are shown (see Theorem 2.1 in [1]):

(i) Let $b(x, u)$ be nondecreasing in the second variable. Suppose the following problem admits a unique nonnegative solution for any $f \in L^\infty(\Omega)$, $f \geq 0$ in Ω :

$$\begin{aligned} -\Delta_p u &= b(x, u) + f(x) \text{ in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Then SCP holds if $\partial\Omega$ is a connected manifold when $N \geq 2$ and Ω is an interval when $N = 1$ with additional assumptions: $f_2 \geq f_1 \geq 0$, $u_1 = u_2 = 0$ on $\partial\Omega$, $u_1 \geq 0$, and $u_2 \geq 0$ in Ω .

(ii) Let $b(x, u)$ be a locally Lipschitz function, nonincreasing in the second variable, with the following behaviour when $|u| \rightarrow 0$:

$$\limsup_{|u| \rightarrow 0} \sup_{x \in \bar{\Omega}} \left| \frac{\partial b}{\partial u}(x, u) \right| \leq c \begin{cases} |u|^{p-2} & \text{if } 1 < p < 2 \\ 1 & \text{if } p \geq 2 \end{cases}$$

for some $c > 0$. Then SCP holds in any dimension $N \geq 1$ provided that $1 < p \leq 2$, Ω is a ball (or an interval in \mathbb{R}) in \mathbb{R}^N centered about the origin, $b(x, u)$, $f_1(x)$, $f_2(x)$, $u_1(x)$, and $u_2(x)$ are all radial functions of the variable x on Ω , and u_1 , and u_2 are nonnegative on Ω with zero boundary values. Also, if $p > 2$ and $\frac{\partial b}{\partial u}$ is a large enough negative number, then a counterexample to SCP is shown.

In this paper we concentrate only on the case when $b(x, u)$ is a non-decreasing function in the second variable and radial in the first variable. We show that by restricting ourselves to radial solutions on a ball or an annulus, SCP holds without any additional assumptions.

1. THE MAIN LEMMA

Fix $0 \leq r < R \leq \infty$, $c \geq 0$. Let $\alpha, \beta : [r, R] \times \mathbb{R} \rightarrow \mathbb{R}$ be continuous functions, with α strictly increasing and β nondecreasing in the second variable. We then have the following.

LEMMA 1. Let $f_1, f_2 \in L^1_{loc}(r, R)$, $f_2 \geq f_1$, and $f_1 \not\equiv f_2$. Let $u_1, u_2 \in C^{1,\beta}[r, R]$ solve, in the weak sense,

$$\begin{cases} -(\alpha(x, u'_i))' - (\frac{c}{x})\alpha(x, u'_i) = \beta(x, u_i) + f_i(x), & x \in (r, R), i = 1, 2 \\ u'_i(0) = 0, & i = 1, 2 \text{ if } c > 0 \text{ and } r = 0. \end{cases}$$

Then SCP holds; that is,

$$u_2 \geq u_1 \text{ in } (r, R) \Rightarrow u_2 > u_1 \text{ in } (r, R).$$

Furthermore, the following boundary Hopf lemma holds: When $0 < r < R \leq \infty$ and $c \geq 0$, $u_1(r) = u_2(r) \Rightarrow u'_1(r) < u'_2(r)$, and if $R < \infty$, then $u_1(R) = u_2(R) \Rightarrow u'_1(R) > u'_2(R)$. When $r = 0$ and $c > 0$, and if $R < \infty$, then $u_1(R) = u_2(R) \Rightarrow u'_1(R) > u'_2(R)$.

When $r = 0$ and $c = 0$, $u_1(0) = u_2(0) \Rightarrow u'_1(0) < u'_2(0)$ and if $R < \infty$, then $u_1(R) = u_2(R) \Rightarrow u'_1(R) > u'_2(R)$.

Proof. By definition, $\forall \phi \in C_0^1(r, R)$, we have

$$(I_i) \quad \int_r^R \alpha(x, u'_i) \phi' = c \int_r^R \frac{\alpha(x, u'_i)}{x} \phi + \int_r^R \beta(x, u_i) \phi + \int_r^R f_i \phi, \quad i = 1, 2.$$

Now $(I_2) - (I_1)$ gives

$$\begin{aligned} \int_r^R [\alpha(x, u'_2) - \alpha(x, u'_1)] \phi' &= c \int_r^R \frac{[\alpha(x, u'_2) - \alpha(x, u'_1)]}{x} \phi \\ &\quad + \int_r^R [\beta(x, u_2) - \beta(x, u_1)] \phi + \int_r^R (f_2 - f_1) \phi. \end{aligned}$$

Define the functions $G, f^*: [r, R] \rightarrow \mathbb{R}$ by $G(x) = \alpha(x, u'_2(x)) - \alpha(x, u'_1(x))$, $f^*(x) = \beta(x, u_2(x)) + f_2(x) - \beta(x, u_1(x)) - f_1(x)$. Note that the hypotheses $u_2 \geq u_1$, $\beta(x, \cdot)$ is a nondecreasing function, and $f_2 \geq f_1$, together imply $f^* \geq 0$. We can now rewrite the above equation as

$$(*) \quad \int_r^R G \phi' = c \int_r^R \frac{G}{x} \phi + \int_r^R f^* \phi, \quad \phi \in C_0^1(r, R).$$

Claim. $(*)$ implies that the function $x^c G(x)$ is nonincreasing in $[r, R]$.

Proof. Let $r < x_0 < y_0 < R$. Define for $\varepsilon > 0$, a piecewise smooth ϕ_ε in (r, R) by

$$\phi_\varepsilon(x) = \begin{cases} 0, & x \in (r, x_0 - \varepsilon) \cup (y_0 + \varepsilon, R) \\ \varepsilon, & x \in [x_0, y_0] \\ \text{"linear"} & x \in [x_0 - \varepsilon, x_0) \cup (y_0, y_0 + \varepsilon]. \end{cases}$$

Plugging in such a ϕ_ε into $(*)$, we get

$$\int_{x_0 - \varepsilon}^{x_0} G - \int_{y_0}^{y_0 + \varepsilon} G = c \int_{x_0 - \varepsilon}^{y_0 + \varepsilon} \frac{G}{x} \phi_\varepsilon + \int_{x_0 - \varepsilon}^{y_0 + \varepsilon} f^* \phi_\varepsilon.$$

Dividing by ε on both sides and letting $\varepsilon \rightarrow 0$, we get

$$(**) \quad G(x_0) - G(y_0) = c \int_{x_0}^{y_0} \frac{G}{x} + \int_{x_0}^{y_0} f^*.$$

Now fixing $y_0 \in (r, R)$ and thinking of x_0 as varying in any compact interval $I \subset (r, R)$, we obtain from (**) that G is absolutely continuous on I . Hence G is differentiable a.e. on (r, R) . Choose $h > 0$ and x any point of differentiability for G . Set $x_0 = x$, $y_0 = x + h$ in (**), divide by h , and pass to the limit $h \rightarrow 0$ to obtain

$$-G'(x) = \left(\frac{c}{x}\right)G(x) + f^*(x) \quad \text{a.e. } x \in (r, R).$$

Multiplying by the integrating factor x^c on either side, for a.e. $x \in (r, R)$ we obtain

$$[x^c G(x)]' = -x^c f^*(x) \leq 0.$$

The claim now follows. ■

Define $w(x) = (u_2 - u_1)(x)$. Then, by hypotheses, $w \geq 0$ in (r, R) . We wish to show $w > 0$ in (r, R) . Suppose $w(x_0) = 0$ for some $x_0 \in (r, R)$. We show that this leads to a contradiction. Clearly, x_0 is a point of global minimum for w in (r, R) , and hence $w'(x_0) = 0$. Hence at x_0 : $u_1(x_0) = u_2(x_0)$, $u_1'(x_0) = u_2'(x_0)$. This means $G(x_0) = 0$. From the claim, since $x^c G(x)$ is nonincreasing on (r, R) , it follows that $G(x) \geq 0$ in (r, x_0) and $G(x) \leq 0$ in (x_0, R) . Since α is a strictly increasing function in the second variable, it follows $u_2' \geq u_1'$ in (r, x_0) , and $u_2' \leq u_1'$ in (x_0, R) . Thus $w' \geq 0$ in (r, x_0) and $w' \leq 0$ in (x_0, R) . This means that w is a nonnegative function on (r, R) which vanishes at x_0 and which is nondecreasing to the left of x_0 and nonincreasing to the right of x_0 . This forces $w \equiv 0$ in (r, R) , which implies $u_1 \equiv u_2$ and hence $f_1 \equiv f_2$ in (r, R) , which is a contradiction. This contradiction shows that $u_2 > u_1$ in (r, R) .

We now show the Hopf boundary point principle. We suppose that $u_1(r) = u_2(r)$ and $u_1'(r) = u_2'(r)$, and show that this leads to a contradiction. We now have $G(r) = 0$, and hence $G \leq 0$ in (r, R) by the claim. This means $w(r) = 0$ and w is nonincreasing to the right of r and hence $w \equiv 0$ in (r, R) . As before, we obtain a contradiction to the assumption $f_1 \not\equiv f_2$. Hence $u_1(r) = u_2(r) \Rightarrow u_1'(r) < u_2'(r)$. Similarly, if $R < \infty$, $r \geq 0$, and $c \geq 0$, then we obtain $u_1(R) = u_2(R)$ and $u_1'(R) = u_2'(R) \Rightarrow G(R) = 0$ and $G \geq 0$ in (r, R) . Hence as before, $w \equiv 0$ in (r, R) , which again leads to the same contradiction. Therefore, if $R < \infty$, then $u_1(R) = u_2(R) \Rightarrow u_1'(R) > u_2'(R)$. This proves the lemma. ■

2. STRONG COMPARISON PRINCIPLE FOR QUASI-LINEAR EQUATIONS IN A BALL OR AN ANNULUS

In this section we assume that the domain Ω is either a ball centered at the origin or an annulus symmetric about the origin in \mathbb{R}^N . We also allow

the limiting cases when $\Omega = \mathbb{R}^N$ (a ball of infinite radius) and the exterior of a ball in \mathbb{R}^N (an annulus with an infinite outer radius). Accordingly, we may take $\Omega = B(0, R)$ or $\Omega = B(0, R) \setminus \overline{B(0, r)}$ for $0 \leq r < R \leq \infty$. Let $J : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a continuous function with the following structure:

$$J(x, \nabla u(x)) = \alpha(|x|, u'(|x|)) \frac{x}{|x|}, \quad \forall u \in C^1(\Omega), \quad u \text{ radial},$$

where $\alpha : [r, R] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function that is strictly increasing in the second variable. Let $b : [r, R] \times \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function that is nondecreasing in the second variable. Then, as a direct application of Lemma 1, we have the following SCP results.

THEOREM 1. *Let $\Omega = B(0, R)$, $0 < R \leq \infty$. Let $f_1, f_2 \in L^1_{loc}(\Omega)$ be two radial functions on Ω with $f_2 \geq f_1$, and $f_1 \not\equiv f_2$ in Ω . Let u_1 , and u_2 be any two $C^{1,\beta}(\Omega)$, $\beta \in (0, 1]$, radial weak solutions of*

$$-\operatorname{div}(J(x, \nabla u_i(x))) = b(|x|, u_i) + f_i(x) \text{ in } \Omega, i = 1, 2.$$

If $u_2 \geq u_1$ in Ω , then $u_2 > u_1$ in Ω . Furthermore, the following Hopf lemma holds: If $R < \infty$ and $u_1(R) = u_2(R)$, then $u'_1(R) > u'_2(R)$.

Proof. A short computation gives, for $0 < |x| < R$, $-\operatorname{div}(J(x, \nabla u_i(x))) = -(\alpha(|x|, u'_i(|x|)))' - \frac{(N-1)}{|x|} \alpha(|x|, u'_i(|x|))$. Also, since u_i are C^1 radial functions, $u'_1(0) = u'_2(0) = 0$. Hence we may appeal to Lemma 1 with $c = N - 1$ to conclude the theorem.

THEOREM 2. *Let $\Omega = B(0, R) \setminus \overline{B(0, r)}$, $0 < r < R \leq \infty$. Let f_1, f_2, u_1 , and u_2 be as in the statement of Theorem 1. If $u_2 \geq u_1$ in Ω , then $u_2 > u_1$ in Ω . Furthermore, the following Hopf lemma holds: $u_1(r) = u_2(r) \Rightarrow u'_1(r) < u'_2(r)$; if $R < \infty$, $u_1(R) = u_2(R) \Rightarrow u'_1(R) > u'_2(R)$.*

Proof. Follows similarly by appealing to Lemma 1.

REFERENCE

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