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A Note on the LaSalle-Type Theorems for Stochastic Differential Delay Equations[1]

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The main aim of this note is to improve some results obtained in the author's earlier paper (1999, *J. Math. Anal. Appl.* **236**, 350–369). From the improved result follow some useful criteria on the stochastic asymptotic stability and boundedness.

Key Words: LaSalle's theorem; nonnegative semimartingale convergence theorem; Itô's formula.

1. INTRODUCTION

In his earlier paper [5] the author considered the n-dimensional stochastic differential delay equation

$$dx(t) = f(x(t), x(t-\tau), t)\,dt + g(x(t), x(t-\tau), t)\,dB(t) \qquad (1.1)$$

on $t \geq 0$ with initial data $\{x(\theta) : -\tau \leq \theta \leq 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0];\ R^n)$. Here $B(t) = (B_1(t), \ldots, B_m(t))^T$ was an m-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., right continuous and $\mathcal{F}_0$ containing all $\mathbb{P}$-null sets). The author in [5] imposed the following hypothesis:

(H1) *Both* f: $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ *and* g: $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}$ *are Borel-measurable functions. They satisfy the local Lipschitz condition and*

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the linear growth condition. That is, for each $k = 1, 2, \ldots,$ *there is a* $c_k > 0$ *such that*

$$|f(x, y, t) - f(\bar{x}, \bar{y}, t)| \vee |g(x, y, t) - g(\bar{x}, \bar{y}, t)| \le c_k(|x - \bar{x}| + |y - \bar{y}|),$$

for all $t \ge 0$ *and those* $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$ *with* $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \le k$, *and there is moreover a* $c > 0$ *such that*

$$|f(x, y, t)| \vee |g(x, y, t)| \le c(1 + |x| + |y|),$$

for all $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$.

Under the hypothesis (H1), it is well-known (cf. Mao [4]) that for any initial data $\{x(\theta) : -\tau \le \theta \le 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ (which will be defined later), Eq. (1.1) has a unique solution that is denoted by $x(t; \xi)$ on $t \ge -\tau$.

One of the main results established in Mao [5] is the following (the notations used will be explained later):

THEOREM 1.1 (Mao [5, Theorem 2.4]). *Let* (H1) *hold. Assume that there are functions* $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, $\gamma \in L^1(\mathbb{R}_+; \mathbb{R}_+)$, *and* $w_1, w_2 \in C(\mathbb{R}^n; \mathbb{R}_+)$ *such that*

$$\begin{aligned} \mathcal{L}V(x, y, t) &:= V_t(x, t) + V_x(x, t)f(x, y, t) \\ &\quad + \frac{1}{2}\operatorname{trace}\big[g^T(x, y, t)V_{xx}(x, t)g(x, y, t)\big] \\ &\le \gamma(t) - w_1(x) + w_2(y), \qquad (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+, \end{aligned} \tag{1.2}$$

$$w_1(x) \ge w_2(x), \qquad x \in \mathbb{R}^n, \tag{1.3}$$

and

$$\lim_{|x|\to\infty} \inf_{0\le t<\infty} V(x, t) = \infty. \tag{1.4}$$

Assume also that for each initial data $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ *there is a* $p > 2$ *such that*

$$\sup_{-\tau\le t<\infty} E|x(t; \xi)|^p < \infty. \tag{1.5}$$

Then, for every $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$,

$$\lim_{t\to\infty} \big[w_1(x(t; \xi)) - w_2(x(t; \xi))\big] = 0 \ a.s. \tag{1.6}$$

This is a stochastic version of the well-known LaSalle theorem (cf. Hale and Lunel [1] or LaSalle [2]). The main aim of this note is to show that this theorem still holds without condition (1.5) while Hypothesis (H1) can also be replaced with a weaker one to cover much more general stochastic differential delay equations.

2. GENERALIZED RESULT

The notations used in this paper are the same as in Mao [5], but for the convenience of the reader we explain here. Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$. Let $\tau > 0$ and $C([-\tau, 0]; \mathbb{R}^n)$ denote the family of all continuous $\mathbb{R}^n$-valued functions on $[-\tau, 0]$. Let $C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$ be the family of all $\mathcal{F}_0$-measurable bounded $C([-\tau, 0]; \mathbb{R}^n)$-valued random variables $\xi = \{\xi(\theta) : -\tau \le \theta \le 0\}$. Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ denote the family of all nonnegative functions $V(x, t)$ on $\mathbb{R}^n \times \mathbb{R}_+$ which are continuously twice differentiable in x and once differentiable in t. For each $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, define an operator $\mathcal{L}V$ from $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$ to $\mathbb{R}$ by

$$\mathcal{L}V(x, y, t) = V_t(x, t) + V_x(x, t) f(x, y, t) + \frac{1}{2}\text{trace}\big[g^T(x, y, t) V_{xx}(x, t) g(x, y, t)\big],$$

where

$$V_t(x, t) = \frac{\partial V(x, t)}{\partial t}, \qquad V_x(x, t) = \left(\frac{\partial V(x, t)}{\partial x_1}, \ldots, \frac{\partial V(x, t)}{\partial x_n}\right),$$

$$V_{xx}(x, t) = \left(\frac{\partial^2 V(x, t)}{\partial x_i \partial x_j}\right)_{n\times n}.$$

Moreover, let $C(\mathbb{R}^n; \mathbb{R}_+)$ denote the family of continuous functions from $\mathbb{R}^n$ to $\mathbb{R}_+$. If K is a subset of $\mathbb{R}^n$, denote by $d(x, K)$ the Haussdorf semi-distance between $x \in \mathbb{R}^n$ and the set K, that is, $d(x, K) = \inf_{y\in K} |x - y|$. If w is a real-valued function defined on $\mathbb{R}^n$, then its kernel is denoted by $Ker(w)$, namely $Ker(w) = \{x \in \mathbb{R}^n : w(x) = 0\}$. We also denote by $L^1(\mathbb{R}_+; \mathbb{R}_+)$ the family of all functions $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^\infty \gamma(t)\, dt < \infty$.

Instead of Hypothesis (H1) we shall impose the following weaker one:

(H2) *Given any initial data* $\{x(\theta) : -\tau \le \theta \le 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, *Eq.* (1.1) *has a unique solution denoted by* $x(t; \xi)$ *on* $t \ge 0$. *Moreover, both* $f(x, y, t)$ *and* $g(x, y, t)$ *are locally bounded in* (x, y) *while uniformly bounded in* t. *That is, for any* $h > 0$ *there is a* $K_h > 0$ *such that*

$$|f(x, y, t)| \vee |g(x, y, t)| \le K_h,$$

for all $t \ge 0$ *and* $x, y \in \mathbb{R}^n$ *with* $|x| \vee |y| \le h$.

Hypothesis (H2) covers much more stochastic differential delay equations than does (H1). For example, consider a one-dimensional stochastic differential delay equation

$$dx(t) = \big[-x^3(t) + x(t - \tau)\big]\, dt + \sin(x(t - \tau))\, dB(t),$$

where $B(t)$ is a one-dimensional Brownian motion. This equation does not satisfy Hypothesis (H1) but we shall show that it satisfies Hypothesis (H2) in the Appendix. Let us now establish an improved result of Theorem 1.1.

THEOREM 2.1. *Let* (H2) *hold. Assume that there are functions* $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, $\gamma \in L^1(\mathbb{R}_+; \mathbb{R}_+)$, *and* $w_1, w_2 \in C(\mathbb{R}^n; \mathbb{R}_+)$ *such that*

$$\mathcal{L}V(x, y, t) \le \gamma(t) - w_1(x) + w_2(y), \qquad (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+, \tag{2.1}$$

$$w_1(x) \ge w_2(x), \qquad x \in \mathbb{R}^n, \tag{2.2}$$

and

$$\lim_{|x|\to\infty} \inf_{0\le t<\infty} V(x, t) = \infty. \tag{2.3}$$

Then $Ker(w_1 - w_2) \neq \emptyset$ *and*

$$\lim_{t\to\infty} d(x(t;\xi), Ker(w_1 - w_2)) = 0 \ a.s. \tag{2.4}$$

for every $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$.

Proof. The proof is rather technical so we divide it into three steps.

Step 1. Fix any ξ and write $x(t;\xi) = x(t)$ for simplicity. By Itô's formula,

$$V(x(t), t) = V(x(0), 0) + \int_0^t \mathcal{L}V(x(s), x(s-\tau), s)\, ds + \int_0^t V_x(x(s), s) g(x(s), x(s-\tau), s)\, dB(s).$$

Using the conditions (2.1) and (2.2) we compute

$$\begin{aligned}
&\int_0^t \mathcal{L}V(x(s), x(s-\tau), s)\, ds \\
&\quad \le \int_0^t \big[\gamma(s) - w_1(x(s)) + w_2(x(s-\tau))\big]\, ds \\
&\quad = \int_0^t \gamma(s)\, ds - \int_0^t w_1(x(s))\, ds + \int_{-\tau}^{t-\tau} w_2(x(s))\, ds \\
&\quad \le \int_0^t \gamma(s)\, ds + \int_{-\tau}^0 w_2(x(s))\, ds - \int_0^t \big[w_1(x(s)) - w_2(x(s))\big]\, ds.
\end{aligned}$$

So

$$\begin{aligned}
V(x(t), t) \le V(\xi(0), 0) &+ \int_{-\tau}^0 w_2(\xi(s))\, ds + \int_0^t \gamma(s)\, ds \\
&- \int_0^t \big[w_1(x(s)) - w_2(x(s))\big]\, ds \\
&+ \int_0^t V_x(x(s), s) g(x(s), x(s-\tau), s)\, dB(s).
\end{aligned} \tag{2.5}$$

Applying the nonnegative semimartingale convergence theorem (cf. Liptser and Shiryayev [3, Theorem 7 on page 139] or Mao [5, Lemma 2.2]) we immediately obtain

$$\limsup_{t\to\infty} V(x(t), t) < \infty \text{ a.s.} \tag{2.6}$$

Moreover, taking the expectations on both sides of (2.5) yields

$$\begin{aligned}\mathbb{E}\int_0^t \big[w_1(x(s)) - w_2(x(s))\big]\,ds \le E\bigg[V(\xi(0), 0) + \int_{-\tau}^0 w_2(\xi(s))\,ds\bigg]\\ + \int_0^\infty \gamma(s)\,ds < \infty.\end{aligned}$$

Letting $t \to \infty$ we obtain that

$$\mathbb{E}\int_0^\infty \big[w_1(x(s)) - w_2(x(s))\big]\,ds < \infty. \tag{2.7}$$

This implies

$$\int_0^\infty \big[w_1(x(s)) - w_2(x(s))\big]\,ds < \infty \text{ a.s.} \tag{2.8}$$

Step 2. Set $w = w_1 - w_2$. Clearly, $w \in C(\mathbb{R}^n; \mathbb{R}_+)$. It is straightforward to see from (2.8) that

$$\liminf_{t\to\infty} w(x(t)) = 0 \text{ a.s.} \tag{2.9}$$

We now claim that

$$\lim_{t\to\infty} w(x(t)) = 0 \text{ a.s.} \tag{2.10}$$

If this is false, then

$$\mathbb{P}\Big\{\limsup_{t\to\infty} w(x(t)) > 0\Big\} > 0.$$

Hence there is a (fixed) number $\varepsilon > 0$ such that

$$\mathbb{P}(\Omega_1) \ge 3\varepsilon, \tag{2.11}$$

where

$$\Omega_1 = \Big\{\limsup_{t\to\infty} w(x(t)) > 2\varepsilon\Big\}.$$

It is easy to observe from (2.6) and the continuity of both the solution $x(t)$ and the function $V(x, t)$ that

$$\sup_{-\tau\le t<\infty} V(x(t), t) < \infty \text{ a.s.}$$

Define μ: $\mathbb{R}_+ \to \mathbb{R}_+$ by

$$\mu(r) = \inf_{|x| \geq r,\, 0 \leq t \leq \infty} V(x, t) \qquad \text{for } r \geq 0.$$

Clearly, $\mu(|x(t)|) \leq V(x(t), t)$ so

$$\sup_{-\tau \leq t < \infty} \mu(|x(t)|) \leq \sup_{-\tau \leq t < \infty} V(x(t), t) < \infty \qquad \text{a.s.}$$

On the other hand, it follows from condition (2.3) that

$$\lim_{r \to \infty} \mu(r) = \infty.$$

We therefore must have

$$\sup_{-\tau \leq t < \infty} |x(t)| < \infty \qquad \text{a.s.} \tag{2.12}$$

Recalling the boundedness of the initial data we can then find a positive number h, which depends on ε, sufficiently large for $|\xi(\theta)| < h$ for all $-\tau \leq \theta \leq 0$ almost surely while

$$\mathbb{P}(\Omega_2) \geq 1 - \varepsilon, \tag{2.13}$$

where

$$\Omega_2 = \left\{ \sup_{-\tau \leq t < \infty} |x(t)| < h \right\}.$$

It is easy to see from (2.11) and (2.13) that

$$\mathbb{P}(\Omega_1 \cap \Omega_2) \geq 2\varepsilon. \tag{2.14}$$

Let us now define a sequence of stopping times,

$$\begin{aligned}
\tau_h &= \inf\{t \geq 0 : |x(t)| \geq h\}, \\
\sigma_1 &= \inf\{t \geq 0 : w(x(t)) \geq 2\varepsilon\}, \\
\sigma_{2i} &= \inf\{t \geq \sigma_{2i-1} : w(x(t)) \leq \varepsilon\}, \qquad i = 1, 2, \ldots, \\
\sigma_{2i+1} &= \inf\{t \geq \sigma_{2i} : w(x(t)) \geq 2\varepsilon\}, \qquad i = 1, 2, \ldots,
\end{aligned}$$

where throughout this paper we set $\inf \varnothing = \infty$. Note from (2.9) and the definitions of Ω_1 and Ω_2 that

$$\tau_h(\omega) = \infty \text{ and } \sigma_i(\omega) < \infty \qquad \text{for } \forall\, i \geq 1 \text{ whenever } \omega \in \Omega_1 \cap \Omega_2. \tag{2.15}$$

By (2.7), we compute

$$
\begin{aligned}
\infty &> \mathbb{E}\int_0^\infty w(x(t))\,dt \\
&\ge \sum_{i=1}^\infty \mathbb{E}\bigg[I_{\{\sigma_{2i-1}<\infty,\ \sigma_{2i}<\infty,\ \tau_h=\infty\}}\int_{\sigma_{2i-1}}^{\sigma_{2i}} w(x(t))\,dt\bigg] \\
&\ge \varepsilon\sum_{i=1}^\infty \mathbb{E}\big[I_{\{\sigma_{2i-1}<\infty,\ \tau_h=\infty\}}(\sigma_{2i}-\sigma_{2i-1})\big],
\end{aligned}
\tag{2.16}
$$

where I_A is the indicator function of set A and we have noted from (2.9) that $\sigma_{2i} < \infty$ whenever $\sigma_{2i-1} < \infty$. On the other hand, by Hypothesis (H2), Hölder's inequality, and Doob's martingale inequality, we compute

$$
\begin{aligned}
&\mathbb{E}\Big[I_{\{\tau_h\wedge\sigma_{2i-1}<\infty\}}\sup_{0\le t\le T}|x(\tau_h\wedge(\sigma_{2i-1}+t))-x(\tau_h\wedge\sigma_{2i-1})|^2\Big] \\
&\quad\le 2\mathbb{E}\bigg[I_{\{\tau_h\wedge\sigma_{2i-1}<\infty\}}\sup_{0\le t\le T}\bigg|\int_{\tau_h\wedge\sigma_{2i-1}}^{\tau_h\wedge(\sigma_{2i-1}+t)} f(x(s),x(s-\tau),s)\,ds\bigg|^2\bigg] \\
&\qquad+2\mathbb{E}\bigg[I_{\{\tau_h\wedge\sigma_{2i-1}<\infty\}}\sup_{0\le t\le T}\bigg|\int_{\tau_h\wedge\sigma_{2i-1}}^{\tau_h\wedge(\sigma_{2i-1}+t)} g(x(s),x(s-\tau),s)\,dB(s)\bigg|^2\bigg] \\
&\quad\le 2T\mathbb{E}\bigg[I_{\{\tau_h\wedge\sigma_{2i-1}<\infty\}}\int_{\tau_h\wedge\sigma_{2i-1}}^{\tau_h\wedge(\sigma_{2i-1}+T)} |f(x(s),x(s-\tau),s)|^2 ds\bigg] \\
&\qquad+8\mathbb{E}\bigg[I_{\{\tau_h\wedge\sigma_{2i-1}<\infty\}}\int_{\tau_h\wedge\sigma_{2i-1}}^{\tau_h\wedge(\sigma_{2i-1}+T)} |g(x(s),x(s-\tau),s)|^2 ds\bigg] \\
&\quad\le 2K_h^2(T+4)T.
\end{aligned}
\tag{2.17}
$$

Since $w(\cdot)$ is continuous in $\mathbb{R}^n$, it must be uniformly continuous in the closed ball $\overline{S}_h = \{x \in \mathbb{R}^n : |x| \le h\}$. We can therefore choose $\delta = \delta(\varepsilon) > 0$ so small that

$$
|w(x)-w(y)| < \varepsilon \qquad \text{whenever } |x-y|<\delta,\ x,y\in\overline{S}_h. \tag{2.18}
$$

We furthermore choose $T = T(\varepsilon, \delta, h) > 0$ sufficiently small for

$$
\frac{2K_h^2(T+4)T}{\delta^2} < \varepsilon.
$$

It then follows from (2.17) that

$$
\begin{aligned}
&\mathbb{P}\Big(\{\tau_h\wedge\sigma_{2i-1}<\infty\}\cap\Big\{\sup_{0\le t\le T}|x(\tau_h\wedge(\sigma_{2i-1}+t))-x(\tau_h\wedge\sigma_{2i-1})|\ge\delta\Big\}\Big) \\
&\quad\le \frac{2K_h^2(T+4)T}{\delta^2} < \varepsilon.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
\mathbb{P}\bigg(\{\sigma_{2i-1}<\infty,\ \tau_h=\infty\}\cap\Big\{\sup_{0\le t\le T}|x(\sigma_{2i-1}+t)-x(\sigma_{2i-1})|\ge\delta\Big\}\bigg)
&=\mathbb{P}\bigg(\{\tau_h\wedge\sigma_{2i-1}<\infty,\ \tau_h=\infty\}\cap\Big\{\sup_{0\le t\le T}|x(\tau_h\wedge(\sigma_{2i-1}+t))\\
&\qquad -x(\tau_h\wedge\sigma_{2i-1})|\ge\delta\Big\}\bigg)\\
&\le\mathbb{P}\bigg(\{\tau_h\wedge\sigma_{2i-1}<\infty\}\cap\Big\{\sup_{0\le t\le T}|x(\tau_h\wedge(\sigma_{2i-1}+t))\\
&\qquad -x(\tau_h\wedge\sigma_{2i-1})|\ge\delta\Big\}\bigg)\\
&\le\varepsilon.
\end{aligned}
$$

Recalling (2.14) and (2.15), we further compute

$$
\begin{aligned}
&\mathbb{P}\bigg(\{\sigma_{2i-1}<\infty,\ \tau_h=\infty\}\cap\Big\{\sup_{0\le t\le T}|x(\sigma_{2i-1}+t)-x(\sigma_{2i-1})|<\delta\Big\}\bigg)\\
&\quad=\mathbb{P}(\{\sigma_{2i-1}<\infty,\ \tau_h=\infty\})\\
&\qquad-\mathbb{P}\bigg(\{\sigma_{2i-1}<\infty,\ \tau_h=\infty\}\cap\Big\{\sup_{0\le t\le T}|x(\sigma_{2i-1}+t)-x(\sigma_{2i-1})|\ge\delta\Big\}\bigg)\\
&\quad\ge 2\varepsilon-\varepsilon=\varepsilon.
\end{aligned}
$$

Using (2.18), we derive that

$$
\begin{aligned}
&\mathbb{P}\bigg(\{\sigma_{2i-1}<\infty,\ \tau_h=\infty\}\cap\Big\{\sup_{0\le t\le T}|w(x(\sigma_{2i-1}+t))-w(x(\sigma_{2i-1}))|<\varepsilon\Big\}\bigg)\\
&\quad\ge\mathbb{P}\bigg(\{\sigma_{2i-1}<\infty,\ \tau_h=\infty\}\cap\Big\{\sup_{0\le t\le T}|x(\sigma_{2i-1}+t)-x(\sigma_{2i-1})|<\delta\Big\}\bigg)\\
&\quad\ge\varepsilon.
\end{aligned}
\tag{2.19}
$$

Set

$$
\overline{\Omega}_i=\Big\{\sup_{0\le t\le T}|w(x(\sigma_{2i-1}+t))-w(x(\sigma_{2i-1}))|<\varepsilon\Big\}.
$$

Noting that

$$
\sigma_{2i}(\omega)-\sigma_{2i-1}(\omega)\ge T \quad \text{if } \omega\in\{\sigma_{2i-1}<\infty,\ \tau_h=\infty\}\cap\overline{\Omega}_i,
$$

we derive from (2.16) and (2.19) that

$$\begin{aligned}\infty &> \varepsilon \sum_{i=1}^{\infty} \mathbb{E}\Big[I_{\{\sigma_{2i-1}<\infty, \tau_h=\infty\}}(\sigma_{2i}-\sigma_{2i-1})\Big] \\ &\geq \varepsilon \sum_{i=1}^{\infty} \mathbb{E}\Big[I_{\{\sigma_{2i-1}<\infty, \tau_h=\infty\}\cap\overline{\Omega}_i}(\sigma_{2i}-\sigma_{2i-1})\Big] \\ &\geq \varepsilon T \sum_{i=1}^{\infty} \mathbb{P}\Big(\{\sigma_{2i-1}<\infty, \tau_h=\infty\}\cap\overline{\Omega}_i\Big) \\ &\geq \varepsilon T \sum_{i=1}^{\infty} \varepsilon = \infty,\end{aligned}$$

which is a contradiction. So (2.10) must hold.

Step 3. Let us now show that $Ker(w_1 - w_2) = Ker(w) \neq \emptyset$. Observe from (2.10) and (2.12) that there is an $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that

$$\lim_{t\to\infty} w(x(t,\omega)) = 0 \text{ and } \sup_{0\leq t<\infty} |x(t,\omega)| < \infty \quad \text{for all } \omega \in \Omega_0. \tag{2.20}$$

Fix any $\omega \in \Omega_0$. Then $\{x(t,\omega)\}_{t\geq 0}$ is bounded in $\mathbb{R}^n$ so there must be an increasing sequence $\{t_i\}_{i\geq 1}$ such that $\{x(t_i,\omega)\}_{i\geq 1}$ converges to some $y \in \mathbb{R}^n$. Hence

$$w(y) = \lim_{i\to\infty} w(x(t_i,\omega)) = 0,$$

which implies $y \in Ker(w)$ so $Ker(w) \neq \emptyset$. We shall now show that

$$\lim_{t\to\infty} d(x(t,\omega), Ker(w)) = 0 \quad \text{for all } \omega \in \Omega_0. \tag{2.21}$$

If this is false, then there is some $\bar{\omega} \in \Omega_0$ such that

$$\limsup_{t\to\infty} d(x(t,\bar{\omega}), Ker(w)) > 0,$$

whence there is a subsequence $\{x(t_i,\bar{\omega})\}_{i\geq 1}$ of $\{x(t,\bar{\omega})\}_{t\geq 0}$ such that

$$d(x(t_i,\bar{\omega}), Ker(w)) \geq \varepsilon, \quad \forall\, i \geq 1,$$

for some $\varepsilon > 0$. Since $\{x(t_i,\bar{\omega})\}_{i\geq 1}$ is bounded, we can find a subsequence $\{x(\bar{t}_i,\bar{\omega})\}_{i\geq 1}$ which converges to z. Clearly, $z \notin Ker(w)$ so $w(z) > 0$. However, by (2.20),

$$w(z) = \lim_{i\to\infty} w(x(\bar{t}_i,\bar{\omega})) = 0,$$

which contradicts $w(z) > 0$. Hence (2.21) must hold and the required assertion (2.4) follows since $\mathbb{P}(\Omega_0) = 1$. The proof is therefore complete.

From Steps 1 and 2 of the proof above we see clearly that Theorem 1.1 still holds without condition (1.5). In other words, Theorem 2.1 improves one of the main results of Mao [5].

3. USEFUL COROLLARIES

Theorem 2.1 reveals the limit set $Ker(w_1 - w_2)$ of the solutions to the stochastic differential delay equation. Should we know more about the set $Ker(w_1 - w_2)$, we can then show the asymptotic properties of the solutions more precisely. For example, if we know that the set $Ker(w_1 - w_2)$ contains only the origin of $\mathbb{R}^n$, that is, $Ker(w_1 - w_2) = \{0\}$, then the solutions will all tend to the origin asymptotically with probability one. This leads to the following useful criterion on the asymptotic stability.

COROLLARY 3.1. *Let* (H2) *hold. Assume that there are functions* $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, $\gamma \in L^1(\mathbb{R}_+; \mathbb{R}_+)$, *and* $w_1, w_2 \in C(\mathbb{R}^n; \mathbb{R}_+)$ *such that*

$$\mathscr{L}V(x, y, t) \le \gamma(t) - w_1(x) + w_2(y), \quad (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+,$$
$$w_1(x) > w_2(x), \quad \forall\, x \neq 0, \tag{3.1}$$

and

$$\lim_{|x|\to\infty} \inf_{0 \le t < \infty} V(x, t) = \infty.$$

Then

$$\lim_{t\to\infty} x(t; \xi) = 0 \ \ a.s. \tag{3.2}$$

for every $\xi \in C^b_{\mathscr{F}_0}([-\tau, 0]; \mathbb{R}^n)$.

Proof. By the condition (3.1), $x \notin Ker(w_1 - w_2)$ if $x \neq 0$. On the other hand, Theorem 2.1 shows that $Ker(w_1 - w_2) \neq \emptyset$. We therefore must have $Ker(w_1 - w_2) = \{0\}$ and the desired assertion (3.2) follows from Theorem 2.1 immediately.

Theorem 2.1 can also be used to establish the criteria on the partially asymptotic stability. Let $1 \le \hat{n} < n$ and $1 \le i_1 < i_2 < \cdots < i_{\hat{n}} \le n$ be integers. Let $\hat{x} = (x_{i_1}, x_{i_2}, \ldots, x_{i_{\hat{n}}})$ be the partial coordinates of x, which can be regarded as in $\mathbb{R}^{\hat{n}}$ with the norm $|\hat{x}| = \sqrt{x_{i_1}^2 + x_{i_2}^2 + \cdots + x_{i_{\hat{n}}}^2}$. Moreover, let $\mathscr{K}$ denote the class of continuous (strictly) increasing functions μ from $\mathbb{R}_+$ to itself with $\mu(0) = 0$. The following corollary is an improvement of Corollary 3.4 of Mao [5].

COROLLARY 3.2. *Let* (H2) *hold. Assume that there are functions* $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, $\gamma \in L^1(\mathbb{R}_+; \mathbb{R}_+)$, $w_1, w_2 \in C(\mathbb{R}^n; \mathbb{R}_+)$, *and* $\mu \in \mathscr{K}$ *such that*

$$\mathscr{L}V(x, y, t) \le \gamma(t) - w_1(x) + w_2(y), \quad (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+,$$
$$w_1(x) - w_2(x) \ge \mu(|\hat{x}|), \quad x \in \mathbb{R}^n, \tag{3.3}$$

and

$$\lim_{|x|\to\infty} \inf_{0\le t<\infty} V(x,t) = \infty.$$

Then

$$\lim_{t\to\infty} \hat{x}(t;\xi) = 0 \quad a.s. \tag{3.4}$$

for every $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$.

Proof. It is straightforward to observe from the assumption (3.3) that if $x \in Ker(w_1 - w_2)$ then $\hat{x} = 0$. The conclusion then follows from Theorem 2.1.

Theorem 2.1 can also be applied to discuss the asymptotic boundedness of the solutions. The following is a straightforward corollary from the theorem.

COROLLARY 3.3. *Let all the assumptions of Theorem* 2.1 *hold. If* $Ker(w_1 - w_2)$ *is bounded, then for every* $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$,

$$\lim_{t\to\infty} |x(t;\xi)| \le C \quad a.s. \tag{3.5}$$

where $C = \sup\{|x| : x \in Ker(w_1 - w_2)\}$.

4. LINEAR STOCHASTIC DIFFERENTIAL DELAY EQUATIONS

Let us now employ our new result to discuss the linear stochastic differential delay equation

$$dx(t) = [A_0x(t) + D_0x(t-\tau)]\,dt + \sum_{i=1}^m [A_ix(t) + D_ix(t-\tau)]\,dB_i(t), \tag{4.1}$$

where A_i's and D_i's are all $n \times n$ matrices. Let Q be a symmetric positive-definite $n \times n$ matrix and let $V(x) = x^TQx$. Then the operator $\mathscr{L}V: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ has the form

$$\mathscr{L}V(x,y,t) = 2x^TQ(A_0x + D_0y) + \sum_{i=1}^m (A_ix + D_iy)^TQ(A_ix + D_iy).$$

Note that

$$\begin{aligned} 2x^TQA_0x &= x^T(QA_0 + A_0^TQ)x, \\ 2x^TQD_0y &\le 2|x^TQ^{\frac{1}{2}}\|Q^{\frac{1}{2}}D_0y| \le |x^TQ^{\frac{1}{2}}|^2 + |Q^{\frac{1}{2}}D_0y|^2 \\ &= x^TQx + y^TD_0^TQD_0y, \end{aligned}$$

and

$$
\begin{aligned}
&(A_i x + D_i y)^T Q (A_i x + D_i y) \\
&\quad = x^T A_i^T Q A_i x + x^T A_i^T Q D_i y + y^T D_i^T Q A_i x + y^T D_i^T Q D_i y \\
&\quad = x^T A_i^T Q A_i x + x^T A_i^T Q^{\frac{1}{2}} Q^{\frac{1}{2}} D_i y + y^T D_i^T Q^{\frac{1}{2}} Q^{\frac{1}{2}} A_i x + y^T D_i^T Q D_i y \\
&\quad \le x^T A_i^T Q A_i x + 2|x^T A_i^T Q^{\frac{1}{2}}||Q^{\frac{1}{2}} D_i y| + y^T D_i^T Q D_i y \\
&\quad \le x^T A_i^T Q A_i x + |x^T A_i^T Q^{\frac{1}{2}}|^2 + |Q^{\frac{1}{2}} D_i y|^2 + y^T D_i^T Q D_i y \\
&\quad \le 2x^T A_i^T Q A_i x + 2y^T D_i^T Q D_i y.
\end{aligned}
$$

So

$$
\mathscr{L}V(x, y, t) \le -w_1(x) + w_2(y),
$$

where

$$
w_1(x) = x^T \left(-QA_0 - A_0^T Q - Q - 2\sum_{i=1}^m A_i^T Q A_i \right) x
$$

and

$$
w_2(x) = x^T \left(D_0^T Q D_0 + 2\sum_{i=1}^m D_i^T Q D_i \right) x.
$$

Clearly, $D_0^T Q D_0 + 2\sum_{i=1}^m D_i^T Q D_i$ is nonnegative-definite so $w_2(x) \ge 0$. If we impose the condition that

$$
M := -QA_0 - A_0^T Q - Q - 2\sum_{i=1}^m A_i^T Q A_i - D_0^T Q D_0 - 2\sum_{i=1}^m D_i^T Q D_i \quad (4.2)
$$

is positive-definite, then $w_1(x) \ge w_2(x)$ and $Ker(w_1 - w_2) = \{0\}$. In this case, Theorem 2.1 shows that the solutions of Eq. (4.1) will tend to zero asymptotically with probability one. However, if the matrix M is only nonnegative-definite (but not positive-definite), then we still have that $w_1(x) \ge w_2(x)$. In this case, we can conclude by Theorem 2.1 that the solutions of Eq. (4.1) will approach $Ker(w_1 - w_2)$ asymptotically with probability one. If we let $v_1, \dots, v_k$ be the all-orthogonal eigenvectors of M corresponding to the eigenvalue 0, then $Ker(w_1 - w_2)$ is the linear span of $v_1, \dots, v_k$; that is,

$$
Ker(w_1 - w_2) = \{\alpha_1 v_1 + \cdots + \alpha_k v_k : (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k\}.
$$

Summarizing the above, we obtain the following useful corollary.

COROLLARY 4.1. *If there is a symmetric positive-definite $n \times n$ matrix Q such that the matrix M defined by* (4.2) *is positive-definite, then the solutions of Eq.* (4.1) *will tend to zero asymptotically with probability one. However, if the matrix M is only nonnegative-definite, then the solutions of Eq.* (4.1) *will approach the linear span of $v_1, \dots, v_k$ asymptotically with probability one, where $v_1, \dots, v_k$ are the all-orthogonal eigenvectors of M corresponding to the eigenvalue* 0.

5. NONLINEAR EXAMPLES

In the previous section we have shown the efficiency of our new result in the study of asymptotic properties for stochastic differential delay equations. Let us now discuss a couple of nonlinear examples to illustrate our new results furthermore. In this section we will let $B(t)$ be a one-dimensional Brownian motion while we omit mentioning the initial data and simply write the solutions as $x(t)$. It is also easy to verify by the existence-and-uniqueness theorems of solutions established in Mao [4] that all the equations discussed in this section satisfy Hypothesis (H2).

EXAMPLE 5.1. Let α and β be two constants such that $2\alpha > \beta^2$. Consider the one-dimensional stochastic differential delay equation

$$dx(t) = -\alpha x(t) \sin^2(x(t))\, dt + \beta x(t-\tau) \sin(x(t-\tau))\, dB(t). \tag{5.1}$$

Let $V(x) = x^2$. Then the operator

$$\mathscr{L}V(x, y, t) = -w_1(x) + w_2(y),$$

where

$$w_1(x) = 2\alpha x^2 \sin^2(x) \quad \text{and} \quad w_2(x) = \beta^2 x^2 \sin^2(x).$$

Theorem 2.1 shows that the solution of Eq. (5.1) has the property

$$\lim_{t\to\infty} d(x(t), Ker(w_1 - w_2)) = 0 \quad \text{a.s.}$$

Noting that $Ker(w_1 - w_2) = \{\pm k\pi : k = 0, 1, 2, \dots\}$ while the solution $x(t)$ is continuous, we can then conclude that

$$\lim_{t\to\infty} x(t) = \kappa\pi \quad \text{a.s.},$$

where κ is a random integer which may depend on the initial data.

EXAMPLE 5.2. Consider a stochastic delay oscillator

$$\ddot{z}(t) + 2\dot{z}(t) + 2z(t) = e^{-t} + [\sin(z(t-\tau)) + \dot{z}(t-\tau)]\dot{B}(t). \tag{5.2}$$

Introducing a new variable $x = (x_1, x_2)^T = (z, \dot{z})^T$, we can write this oscillator as a stochastic differential delay equation

$$dx(t) = \begin{bmatrix} x_2(t) \\ e^{-t} - 2x_1(t) - 2x_2(t) \end{bmatrix} dt + \begin{bmatrix} 0 \\ \sin(x_1(t-\tau)) + x_2(t-\tau) \end{bmatrix} dB(t). \tag{5.3}$$

Let $V(x) = \alpha x_1^2 + 2\beta x_1 x_2 + x_2^2$, where α and β are two positive constants to be determined and we require $\alpha > \beta^2$ in order for V to be positive-definite. Then the operator $\mathscr{L}V\colon \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}$ has the form

$$\begin{aligned} \mathscr{L}V(x, y, t) &= (2\alpha x_1 + 2\beta x_2)x_2 + (2\beta x_1 + 2x_2)(e^{-t} - 2x_1 - 2x_2) \\ &\quad + (\sin y_1 + y_2)^2 \\ &\le -4\beta x_1^2 - (4 - 2\beta)x_2^2 + (2\alpha - 4 - 4\beta)x_1 x_2 \\ &\quad + e^{-t}(2\beta x_1 + 2x_2) + 2(y_1^2 + y_2^2). \end{aligned}$$

Choosing α and β for

$$4\beta = 4 - 2\beta \quad \text{and} \quad 2\alpha - 4 - 4\beta = 0,$$

namely, $\alpha = 10/3$ and $\beta = 2/3$ which satisfy $\alpha > \beta^2$, we find that

$$\mathscr{L}V(x, y, t) \le -\frac{8}{3}|x|^2 + e^{-t}\left(\frac{4}{3}x_1 + 2x_2\right) + 2|y|^2.$$

Noting

$$\frac{4}{3}x_1 e^{-t} \le \frac{1}{3}x_1^2 + \frac{4}{3}e^{-2t} \qquad 2x_2 e^{-t} \le \frac{1}{3}x_2^2 + 3e^{-2t},$$

we see that

$$\mathscr{L}V(x, y, t) \le \frac{13}{3}e^{-2t} - \frac{7}{3}|x|^2 + 2|y|^2.$$

By Corollary 3.1, we can then conclude that the stochastic oscillator (5.2) has the property that

$$\lim_{t\to\infty} [|z(t)| + |\dot{z}(t)|] = 0 \quad \text{a.s.}$$

6. EQUATIONS WITH MULTIPLE DELAYS

The theory developed in Section 2 can be generalized to cope with equations with multiple delays of the form

$$\begin{aligned} dx(t) = f(x(t), x(t-\tau_1), \dots, x(t-\tau_k), t)\, dt \\ + g(x(t), x(t-\tau_1), \dots, x(t-\tau_k), t)\, dB(t) \end{aligned} \tag{6.1}$$

on $t \geq 0$, where $0 < \tau_1 < \tau_2 < \cdots < \tau_k = \tau$, $f\colon \mathbb{R}^n \times \mathbb{R}^{n\times k} \times \mathbb{R}_+ \to \mathbb{R}^n$, and $g\colon \mathbb{R}^n \times \mathbb{R}^{n\times k} \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}$. For this equation, we impose the following hypothesis:

(H3) *Given any initial data* $\{x(\theta) : -\tau \leq \theta \leq 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; R^n)$, *Eq.* (6.1) *has a unique solution denoted by* $x(t;\xi)$ *on* $t \geq 0$. *Moreover, both* $f(x, y_1, \dots, y_k, t)$ *and* $g(x, y_1, \dots, y_k, t)$ *are locally bounded in* $(x, y_1, \dots, y_k)$ *while uniformly bounded in* t. *That is, for any* $h > 0$ *there is a* $K_h > 0$ *such that*

$$|f(x, y_1, \dots, y_k, t)| \vee |g(x, y_1, \dots, y_k, t)| \leq K_h,$$

for all $t \geq 0$ *and* $x, y_1, \dots, y_k \in \mathbb{R}^n$ *with* $|x| \vee |y_1| \vee \cdots \vee |y_k| \leq h$.

The corresponding operator $\mathcal{L}V$ from $\mathbb{R}^n \times \mathbb{R}^{n\times k} \times \mathbb{R}_+$ to $\mathbb{R}$ is now defined as

$$\begin{aligned} \mathcal{L}V(x, y_1, \dots, y_k, t) = V_t(x, t) + V_x(x, t) f(x, y_1, \dots, y_k, t) \\ + \frac{1}{2}\operatorname{trace}\Big[g^T(x, y_1, \dots, y_k, t) V_{xx}(x, t) \\ \times g(x, y_1, \dots, y_k, t)\Big]. \end{aligned}$$

To close this paper let us state the following more general result, which can be proved in the same way as in the proof of Theorem 2.1.

THEOREM 6.1. *Let* (H3) *hold. Assume that there are functions* $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$, $\gamma \in L^1(\mathbb{R}_+; \mathbb{R}_+)$, *and* $w, w_1, \dots, w_k \in C(\mathbb{R}^n; \mathbb{R}_+)$ *such that*

$$\begin{aligned} \mathcal{L}V(x, y_1, \dots, y_k, t) \leq \gamma(t) - w(x) + \sum_{i=1}^{k} w_i(y_i), \\ (x, y_1, \dots, y_k, t) \in \mathbb{R}^n \times \mathbb{R}^{n\times k} \times \mathbb{R}_+, \end{aligned} \tag{6.2}$$

$$w(x) \geq \sum_{i=1}^{k} w_i(x), \qquad x \in \mathbb{R}^n, \tag{6.3}$$

and

$$\lim_{|x|\to\infty} \inf_{0\le t<\infty} V(x,t) = \infty. \tag{6.4}$$

Then $Ker(w - w_1 - \cdots - w_k) \neq \emptyset$ *and the solutions of Eq.* (6.1) *have the property that*

$$\lim_{t\to\infty} d(x(t;\xi), Ker(w - w_1 - \cdots - w_k)) = 0 \quad a.s. \tag{6.5}$$

for every $\xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$.

A. APPENDIX

In this appendix we show that equation

$$dx(t) = [-x^3(t) + x(t-\tau)]\,dt + \sin(x(t-\tau))\,dB(t) \tag{A.1}$$

satisfies Hypothesis (H2). In fact, the corresponding coefficients $f(x, y, t) = -x^3 + y$ and $g(x, y, t) = \sin(y)$ are locally bounded in (x, y). Hence, to show that this equation satisfies Hypothesis (H2), we need only to show that this equation has a unique solution for any initial data.

To the best knowledge of the author, there is no existing result that can be applied to this nonlinear equation to guarantee the existence and uniqueness of its (global) solution. We therefore first establish a very useful new criterion on the existence and uniqueness and then apply it to Eq. (A.1).

THEOREM A.1. *Assume that both coefficients* $f(x, y, t)$ *and* $g(x, y, t)$ *of Eq.* (1.1) *are locally Lipschitz in* (x, y). *Assume also that there is a function* $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ *and a positive constant K such that*

$$\mathcal{L}V(x, y, t) \le K(1 + V(x, t) + V(y, t-\tau)), \qquad (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+, \tag{A.2}$$

and

$$\lim_{|x|\to\infty} \inf_{0\le t\le\infty} V(x,t) = \infty. \tag{A.3}$$

Then Eq. (1.1) *has a unique* (*global*) *solution for any initial data* $\{x(\theta) : -\tau \le \theta \le 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$.

Proof. Since the coefficients are locally Lipschitz in (x, y), Eq. (1.1) has a unique maximal local solution $x(t)$ on $t \in [[-\tau, \sigma_\infty[[$ for any initial data $\{x(\theta) : -\tau \le \theta \le 0\} = \xi \in C^b_{\mathcal{F}_0}([-\tau, 0]; \mathbb{R}^n)$, where σ_∞ is the explosion time (cf. Mao [6, Theorem 3.2.2 on p. 95]). We therefore need only to show that $\sigma_\infty = \infty$ a.s. For any integer $k \ge 1$, define the stopping time

$$\tau_k = \sigma_\infty \wedge \inf\{t \in [[0, \sigma_\infty[[: |x(t)| \ge k\},$$

where, as usual, we set $\inf \varnothing = \infty$. Clearly, τ_k's are increasing so they have the limit $\tau_\infty = \lim_{k\to\infty} \tau_k$. Obviously, $\tau_\infty \le \sigma_\infty$ a.s. For any $k \ge 1$ and $t \ge 0$, the Itô formula shows that

$$\mathbb{E}V(x(t\wedge\tau_k), t\wedge\tau_k) = \mathbb{E}V(x(0),0) + \mathbb{E}\int_0^{t\wedge\tau_k} \mathscr{L}V(x(s), x(s-\tau), s)\, ds.$$

By the condition (A.2), we compute that

$$\begin{aligned}
&\mathbb{E}V(x(t\wedge\tau_k), t\wedge\tau_k)\\
&\quad \le \mathbb{E}V(x(0),0) + \mathbb{E}\int_0^{t\wedge\tau_k} K\big[1 + V(x(s),s) + V(x(s-\tau), s-\tau)\big]\, ds\\
&\quad \le \mathbb{E}V(x(0),0) + Kt + K\mathbb{E}\int_0^t \big[V(x(s\wedge\tau_k), s\wedge\tau_k)\\
&\qquad\qquad + V(x((s-\tau)\wedge\tau_k), (s-\tau)\wedge\tau_k)\big]\, ds\\
&\quad = \mathbb{E}V(x(0),0) + Kt + K\int_0^t \big[\mathbb{E}V(x(s\wedge\tau_k), s\wedge\tau_k)\\
&\qquad\qquad + \mathbb{E}V(x((s-\tau)\wedge\tau_k), (s-\tau)\wedge\tau_k)\big]\, ds\\
&\quad \le C + Kt + 2K\int_0^t \Big[\sup_{0\le r\le s} \mathbb{E}V(x(r\wedge\tau_k), r\wedge\tau_k)\Big]\, ds,
\end{aligned} \tag{A.4}$$

where

$$C = \mathbb{E}V(x(0),0) + K\int_0^\tau \mathbb{E}V(x(s-\tau), s-\tau)]\, ds < \infty.$$

Since the right-hand side of (A.4) is increasing in t, we must have

$$\sup_{0\le r\le t} \mathbb{E}V(x(r\wedge\tau_k), r\wedge\tau_k) \le C + Kt + 2K\int_0^t \Big[\sup_{0\le r\le s} \mathbb{E}V(x(r\wedge\tau_k), r\wedge\tau_k)\Big]\, ds.$$

The well-known Gronwall inequality yields

$$\sup_{0\le r\le t} \mathbb{E}V(x(r\wedge\tau_k), r\wedge\tau_k) \le (C+Kt)e^{2Kt}, \qquad \forall\, t\ge 0.$$

Hence

$$\mathbb{E}V(x(t\wedge\tau_k), t\wedge\tau_k) \le (C+Kt)e^{2Kt}, \qquad \forall\, t\ge 0. \tag{A.5}$$

On the other hand, define μ: $\mathbb{R}_+ \to \mathbb{R}_+$ by

$$\mu(r) = \inf_{|x|\ge r,\, 0\le t\le\infty} V(x,t) \qquad \text{for } r\ge 0.$$

Clearly, $\mu(|x(t)|) \le V(x(t),t)$ and, by the condition (A.3),

$$\lim_{r\to\infty} \mu(r) = \infty.$$

It therefore follows from (A.5) that

$$(C + Kt)e^{2Kt} \geq \mathbb{E}\mu(|x(t \wedge \tau_k)|) \geq \mu(k)\mathbb{P}(\tau_k \leq t).$$

Letting $k \to \infty$ and then $t \to \infty$ we obtain that

$$\mathbb{P}(\tau_\infty < \infty) = 0.$$

That is, $\tau_\infty = \infty$ a.s. We therefore must have that $\sigma_\infty = \infty$ a.s. This completes the proof.

Let us now return to Eq. (A.1). Let $V(x, t) = V(x) = x^2$ and compute, with the coefficients $f(x, y, t) = -x^3 + y$ and $g(x, y, t) = \sin(y)$, that

$$\mathscr{L}V(x, y, t) = 2x(-x^3 + y) + \sin^2(y) \leq x^2 + 2y^2 \leq 2(1 + V(x) + V(y)).$$

Applying Theorem A.1 we can see that Eq. (A.1) does have a unique solution for any initial data.

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