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Random iterations, fixed points and invariant CRF-horospheres in complex Banach spaces

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Abstract

For holomorphic noncontractive maps on (not necessarily bounded) domains in complex Banach spaces, we establish the conditions guaranteeing locally uniform convergence of random iterations and study the existence of fixed points and boundary behaviour of iterations. In particular, we show that the problem, concerning the existence of the horospheres determined by Carathéodory–Reiffen–Finsler pseudometrics defined on unbounded domains, has the solution and we prove new results of type of Julia's lemma and Wolff's theorem.

Keywords: Holomorphic noncontractive map; Unbounded domain; Complex Banach space; Random iteration; Fixed point; Invariant CRF-horosphere; Julia's lemma; Wolff's theorem

1. Introduction

The past few years have seen several significant developments in the fields of random iteration theory, iteration theory and fixed point theory for holomorphic self-maps on bounded convex domains G in complex Banach spaces. The important questions concern the behaviour of the random iterations $f_1 \circ f_2 \circ \cdots \circ f_n$ and $f_n \circ f_{n-1} \circ \cdots \circ f_1$, where $f_n \in \mathrm{Hol}(G, G)$ and $n \in \mathbb{N}$ [11,12,18–21,31,35,44,47,53], the iterative behaviour of an individual map $f^{[n]} = f \circ \cdots \circ f$ (n times) [1–3,5–8,13,14,26–30,33,34,36,37,41,42,45,49] and the existence of fixed points for $f \in \mathrm{Hol}(G, G)$ [13,24,40,43,48] and the existence of

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invariant horocycles, ellipsoids or horospheres in G determined by Kobayashi metrics k_G [1–8,15–17,26–30,34,42,49,53]. Many researchers (see, e.g., [2,6,7,14,22–24,29,30,34,43, 45,46,48]) have made essential contributions to this areas, starting with the celebrated fixed point theorem of Earle and Hamilton [13]. The source of those studies were fundamental results of Julia [28], Fatou [16], Wolff [50], Denjoy [8] and Valiron [39] in $\mathbb{C}$.

If we assume that G is unbounded domain in infinite dimensional complex spaces, then the problems concerning the methods of investigations of the behaviour of random iterations and iterations of holomorphic maps, the existence of fixed points of holomorphic maps and the existence of invariant horospheres determined by pseudometrics on G, are quite different.

Let G be a nonempty (not necessarily bounded) domain in a complex Banach space E and let ρ_G denote the CRF-pseudometric on G (abbreviation for the term "Carathéodory–Reiffen–Finsler pseudometric"). We say that the bounded subset D of G lies strictly inside G if there exists a neighbourhood W of the origin in E such that $D + W \subset G$. In [44,47] we characterize ρ_G-bounded subsets D of G, give conditions when pseudometric ρ_G is metric and prove uniform convergence to constant maps of random iterations $f_1 \circ f_2 \circ \cdots \circ f_n$, $n \in \mathbb{N}$, of holomorphic contractions, i.e., such maps that, for each $n \in \mathbb{N}$, $f_n \in \mathrm{Hol}(G, E)$ and $f_n(G) \subset D$, where subset D of G is bounded, ρ_G-bounded and lies strictly inside G. In particular, Theorem 2.1 of paper [47, Section 4] includes the fixed point theorem of C.J. Earle and R.S. Hamilton [13] and the result of W.J. Zhang and F.J. Ren [53, Theorem 2.1].

The present paper is a continuation of our investigations [47] and has two purposes. The first purpose is to give conditions guaranteeing locally uniform convergence of random iterations $f_n \circ f_{n-1} \circ \cdots \circ f_1$, $n \in \mathbb{N}$, for $\{f_n\}$ in $\mathrm{Hol}(G, G)$ and establish the fixed point theorems for $f \in \mathrm{Hol}(G, G)$ when G are not necessarily bounded domains in complex Banach spaces E and maps f_n, f, $n \in \mathbb{N}$, are not necessarily contractive. The second purpose is to construct the CRF-horospheres (i.e., the horospheres determined by CRF-pseudometrics ρ_G) and establish the results of type of Julia's lemma and Wolff's theorem for maps $f \in \mathrm{Hol}(G, G)$ when E are separable.

We use the terminology from [3,9,10,13,23,25,32,38,40,51,52].

Remark 1.1. It is worth noticing that the problems concerning the convergence of two kinds of random iterations $f_1 \circ f_2 \circ \cdots \circ f_n$ and $f_n \circ f_{n-1} \circ \cdots \circ f_1$, $n \in \mathbb{N}$, differ even for unit ball G in l^2; for details see Example 2.1(a).

2. Locally uniform convergence of random iterations of holomorphic noncontractive maps

Let E be a complex Banach space, let $G \subset E$ be a nonempty (not necessarily bounded) domain and let D be a subset of G.The sequence $\{F_n\}$ in $\mathrm{Hol}(G, D)$ converges to $F \in \mathrm{Hol}(G, \bar{D})$ locally uniformly on G iff, for every $x \in G$, $\{F_n\}$ converges uniformly to F on some ball $B(x_0, r) = \{y \in E\colon \|y - x_0\| < r\}$, $r > 0$, containing x and such that $B(x_0, r)$ lies strictly inside G.

We show

Theorem 2.1. *Let E be a complex Banach space, let $G \subset E$ be a nonempty (not necessarily bounded) domain and let D be an open bounded convex subset of G. Assume K is a subset of D such that $\bar{K}$ is compact in E, let $\{f_n\}$ be a sequence in $\{f \in \mathrm{Hol}(G,E)\colon\ f(G) \subset K\}$, and set $F_n = f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$ for all $n \in \mathbb{N}$.*

(i) *The sequence $\{F_n\}$ has a subsequence converging locally uniformly on G to a map $F \in \mathrm{Hol}(G, \bar{D})$.*
(ii) *In addition, if E is strictly convex and D is open unit ball, then either $F(G) \subset D$ or there exists $w \in \partial D$ such that F is a constant map of the form $x \mapsto w$, $x \in G$.*

Proof. (i) Let $a \in D$ be arbitrary and fixed and let $m_0 \in \mathbb{N}$ be such that $K \cap D_1 \neq \emptyset$, where $D_n = a + [1 - 1/(m_0 + n - 1)](D - a)$ for $n \in \mathbb{N}$. Denote $d_n = \mathrm{dist}(D_n, E \setminus D)$ and $K_n = K \cap D_n$, $n \in \mathbb{N}$. Obviously, $K = \bigcup_{n=1}^{\infty} K_n$.

Since the set $\bar{K}_1$ is compact, thus there exist a countable set $\{y_{n,1}\}$ in $\bar{K}_1$ which is dense in $\bar{K}_1$.

We consider the sequence $\{T_n\}$, $T_n = f_n \circ f_{n-1} \circ \cdots \circ f_3 \circ f_2$, $n \geqslant 2$. Since $\{T_n(y_{1,1})\}$ is a subset of a compact set $\bar{K}$, then there exists a subsequence $\{T_{n,1}\}$ of $\{T_n\}$ such that $\{T_{n,1}(y_{1,1})\}$ converges. By using analogous considerations, by induction, for each $k \in \mathbb{N}$, we find subsequence $\{T_{n,k}\}$ of $\{T_{n,k-1}\}$ which converges at $y_{1,1}, y_{2,1}, \ldots, y_{k,1}$. Consequently, the diagonal sequence $\{T_{n,n}\}$ converges in the points of sequence $\{y_{n,1}\}$.

Let $\varepsilon > 0$ be arbitrary and fixed. Let $M = \sup\{\|y\|\colon\ y \in D\}$. Since the set $\{y_{n,1}\}$ is dense in $\bar{K}_1$, thus $\bar{K}_1 \subset \bigcup_{j=1}^{\infty} B(y_{n,1}, \varepsilon d_1/(3M))$. By compactness, there exists a finite subsequence $y_{n_1,1}, y_{n_2,1}, \ldots, y_{n_m,1}$ of $\{y_{n,1}\}$ such that

$$\bar{K}_1 \subset \bigcup_{j=1}^{m} B\bigl(y_{n_j,1}, \varepsilon d_1/(3M)\bigr).$$

Obviously, the sequences $\{T_{n,n}(y_{n_j,1})\}$, $j = 1, 2, \ldots, m$, are Cauchy's sequences and, consequently, there exists $n_0 \in \mathbb{N}$ such that if $p, q > n_0$, then

$$\bigl\|T_{p,p}(y_{n_j,1}) - T_{q,q}(y_{n_j,1})\bigr\| < \varepsilon/3 \quad \text{for } j = 1, 2, \ldots, m.$$

Let now $y \in K_1$. Then $y \in B(y_{n_j,1}, \varepsilon d_1/(3M))$ for some $j = 1, 2, \ldots, m$ and, using the Hahn–Banach theorem, we define the maps $g_k = (1/M)(L_k \circ T_{k,k})$, where

$$L_k \in E^*, \quad \bigl|L_k(v)\bigr| \leqslant \|v\| \quad \text{for each } v \in E,$$

and

$$L_k\bigl[T_{k,k}(y) - T_{k,k}(y_{n_j,1})\bigr] = \bigl\|T_{k,k}(y) - T_{k,k}(y_{n_j,1})\bigr\|, \quad k \in \mathbb{N}.$$

Then, using Cauchy's integral formula, we get $|Dg_k(u)(v)| \leqslant \|v\|/d_1$ for $u \in B(y_{n_j,1}, \varepsilon d_1/(3M))$ and $v \in E$. Consequently,

$$\bigl\|T_{k,k}(y) - T_{k,k}(y_{n_j,1})\bigr\| \leqslant (M/d_1)\|y - y_{n_j,1}\| < \varepsilon/3, \quad k \in \mathbb{N}.$$

By the above two estimates we obtain

$$\begin{aligned}\|T_{p,p}(y) - T_{q,q}(y)\| &\leqslant \|T_{p,p}(y) - T_{p,p}(y_{n_j,1})\| + \|T_{p,p}(y_{n_j,1}) - T_{q,q}(y_{n_j,1})\| \\ &\quad + \|T_{q,q}(y_{n_j,1}) - T_{q,q}(y)\| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon\end{aligned}$$

whenever $p, q > n_0$. This shows that $\{T_{n,n}\}$ is uniformly convergent on K_1.

We denote $T_{n_1} = T_{n,n}$, $n \in \mathbb{N}$, and, using analogous consideration as the above, let $\{T_{n_2}\}$ denote the subsequence of $\{T_{n_1}\}$ which converges uniformly on K_2. Generally, let $\{T_{n_{p+1}}\}$denote, converging uniformly on K_{p+1}, $p = 2, 3, \ldots,$ the subsequence of $\{T_{n_p}\}$. Let us observe that then the diagonal sequence $\{T_{n_n}\}$ converges uniformly on K_p for each $p \in \mathbb{N}$.

Finally, let $x \in G$ be arbitrary and fixed. Denote $\lambda = \operatorname{dist}(x, E \setminus G)$ and $\mu = \operatorname{dist}(f_1(x), E \setminus D)$ and let $r > 0$ be such that $\lambda = r + \gamma$ and $rM/\gamma < \mu$ for some $\gamma > 0$. Obviously, then $B(x, r) \subset G$ and $B(f_1(x), rM/\gamma) \subset D$. Moreover, since $\operatorname{dist}(B(x, r), E \setminus G) \geqslant \gamma > 0$ thus

$$\|f_1(x) - f_1(u)\| = \left\| \int_0^1 Df_1\big[tx + (1-t)u\big](x-u)\,dt \right\| \leqslant M\|x-u\| < Mr/\gamma$$

for each $u \in B(x, r)$. The above implies that

$$f_1\big(B(x, r)\big) \subset B\big(f_1(x), Mr/\gamma\big) \cap K \subset K \cap D_s = K_s$$

for some $s \in \mathbb{N}$. Consequently $\{T_{n_n}\}$ converges uniformly on $f_1(B(x, r))$ and thus $\{F_{n_n}\}$, where $F_{n_n} = T_{n_n} \circ f_1$, $n \in \mathbb{N}$, converges uniformly on $B(x, r)$. This proves the assertion.

(ii) Applying E. Thorp and R. Whitley theorem [38] for F we obtain the assertion. □

Remarks 2.1. (a) The set K does not necessarily lie strictly inside G.

(b) Theorem 2.1 is new even when G is bounded and E is finite dimensional (compare with results of J. Gill [18–21] and L. Lorentzen [31] in $\mathbb{C}$ and results of Z. Zhang and F. Ren [53, Theorem 4.2] in $\mathbb{C}^n$).

Examples 2.1. Let $E = l^2$ and $G = \{x \in l^2\colon \|x\| = (\sum_{k=1}^{\infty} |x_k|^2)^{1/2} < 1\}$. Here l^2 denotes the complex Hilbert space of all sequences $x = \{x_k\}$ of complex numbers.

(a) For each $n \in \mathbb{N}$, let $f_n \in \operatorname{Hol}(G, G)$ be defined by the formulae

$$\begin{aligned}f_{2n-1}(x) &= \big\{3^{-1}(x_1 - 2/3), 3^{-2/2}x_2, 3^{-3/2}x_3, 3^{-4/2}x_4, \ldots\big\},\\ f_{2n}(x) &= \{3^{-1}x_1, 3^{-2/2}x_2, 3^{-3/2}x_3, 3^{-4/2}x_4, \ldots\}.\end{aligned}$$

Thus the maps f_1 and f_2 are compact [43, Criterion, pp. 156–157] and we may assume that $K = f_1(G) \cup f_2(G)$. Of course, $\bar{K}$ lies strictly inside G and $\{F_n\}$ cannot converge at any point of G. Moreover, for each $n \in \mathbb{N}$, $F_{2n-1}(G) \subset K$, $F_{2n}(G) \subset K$ and the sequences $\{F_{2n-1}\}$ and $\{F_{2n}\}$ converges to constant maps $x \mapsto -4^{-1}$ and $x \mapsto -12^{-1}$, respectively. However, the sequence $\{f_1 \circ f_2 \circ \cdots \circ f_n\}$ converges to map $x \mapsto -4^{-1}$ on G.

(b) Let $f_n \in \operatorname{Hol}(G, G)$, $n \in \mathbb{N}$, be defined by the formulae

$$\begin{aligned}f_1(x) &= 2^{-1}\{x_1, 2^{-2/2}x_2, 2^{-3/2}x_3, 2^{-4/2}x_4, \ldots\},\\ f_n(x) &= 2^{-1}\{A_n x_1, 2^{-2/2}x_2, 2^{-3/2}x_3, 2^{-4/2}x_4, \ldots\},\end{aligned}$$

where $A_n = 1 - n^{-2}$, $n \geqslant 2$, and $\lim_{m\to\infty} \prod_{n=2}^{m} A_m = 2^{-1}$. Then f_1 is compact, $f_n(G) \subset K = f_1(G)$, $n \in \mathbb{N}$, $\bar{K}$ lies strictly inside G and $\{F_n\}$ converges to a constant map F such that $F(x) = \{0, 0, 0, \ldots\}$, $x \in G$.

(c) Define $f_n \in \mathrm{Hol}(G, G)$, $n \in \mathbb{N}$, by the formulae

$$f_n(x) = \left\{2^{-1} + 4^{-1}\left(x_1 + x_1^2\right), 2^{-2n/2}x_2, 2^{-3n/2}x_3, 2^{-4n/2}x_4, \ldots\right\}.$$

Then f_1 is compact, $f_n(G) \subset K = f_1(G)$, $n \in \mathbb{N}$, K does not lie strictly inside G and $\{F_n\}$ converges to a constant map F such that $F(x) = \{1, 0, 0, \ldots\}$, $x \in G$.

3. Fixed points and boundary behaviour of holomorphic maps on unbounded domains

The following generalization of C.J. Earle and R.S. Hamilton result [13] is established in [47, Section 4 and inequalities (3.19) and (3.20)].

Theorem 3.1. *Let G be a nonempty domain* (*not necessarily bounded*) *in a complex Banach space E and suppose that a subset D of G is bounded, ρ_G-bounded and lies strictly inside G. Each map $f \in \mathrm{Hol}(G, E)$ such that $f(G) \subset D$ has a unique fixed point $w \in G$ and the sequence $\{f^{[n]}\}$ converges to the constant map $x \mapsto w$ uniformly on G.*

In this section we study the existence of fixed points and boundary behaviour of maps $f \in \mathrm{Hol}(G, G)$ when G is unbounded, $f(G)$ is bounded and $f(G)$ not lies strictly inside G. As a consequence of Theorem 3.1, the proof of Theorem 2.1 in [47] and ideas from [43] we obtain the following three facts.

Theorem 3.2. *Let E be a complex Banach space, let $G \subset E$ be a nonempty* (*not necessarily bounded*) *domain, let $f \in \mathrm{Hol}(G, E)$ and let $f(G)$ be contained in some open bounded convex subset D of G.*

(i) *Assume*

$$h_{t,a}(x) = tf(x) + (1-t)a \quad \text{for } x \in G, \tag{3.1}$$

where $a \in D$ and $t \in [0, 1)$ are arbitrary and fixed. Then there exists a unique fixed point $w(t, a)$ of the map $h_{t,a} : G \to D$.

(ii) *Let, additionally, D lie strictly inside G. Then, for each $t \in [0; 1)$, the map $w(t, \cdot) : D \to D$ is holomorphic in D.*

We call the set $\mathrm{Appr}(f) = \{w(t, a)\colon (t, a) \in [0; 1) \times D\}$ *an approximative set for f.*

Proof. (i) We may assume without loss of generality that $0 \in D$. We have

$$tD + (1-t)D \subset D \quad \text{and} \quad tf(G) \subset tD. \tag{3.2}$$

Further, a is an interior point of D. Consequently, there exists $r = r(a) > 0$ such that $y \in B(0, r)$ implies $a + y \in D$. So,

$$(1-t)a + (1-t)y \in (1-t)D \quad \text{for all } y \in B(0, r). \tag{3.3}$$

Thus (3.2) and (3.3) imply $tf(x) + (1-t)a + (1-t)y \in D$ for all $x \in G$ and $y \in B(0, r)$, or, equivalently, $tf(x) + (1-t)a + w \in D$ for all $x \in G$ and $w \in B[0, (1-t)r]$. Thus the set $h_{t,a}(G)$ lies strictly inside D. Consequently, the set $h_{t,a}(G)$ is bounded and lies strictly inside G and, by Theorem 3.1, there exists a unique fixed point $w(t, a)$ of the map $h_{t,a} : G \to D$ defined by (3.1), which show the first assertion.

(ii) Let $t \in [0; 1)$ be arbitrary and fixed. Let $w(t, a) = h_{t,a}(w(t, a))$ for $a \in D$. By Theorem 3.1, we get $w(t, a) = \lim_{n\to\infty}(h_{t,a})^{[n]}(0)$.

Note $w_n(t, a) = (h_{t,a})^{[n]}(0)$. Obviously, $\{w_n(t, \cdot)\}$ is locally uniformly bounded on D and converges uniformly to $w(t, \cdot)$ on D. Indeed, there exists $0 < \lambda < 1$ such that

$$\begin{aligned}\left\| w_n(t, a) - w_{n+1}(t, a)\right\| &= \left\|(h_{t,a})^{[n]}(0) - (h_{t,a})^{[n+1]}(0)\right\| \\ &\leqslant \operatorname{diam}\bigl(h_{t,a}(G)\bigr) \cdot \lambda^n \cdot \rho_G\bigl(0, h_{t,a}(0)\bigr) \\ &\leqslant \operatorname{diam}(D) \cdot \lambda^n \cdot M_D,\end{aligned}$$

where $M_D = \sup\{\rho_G(x, y)\colon x, y \in D\} < +\infty$ (see [47, Theorem 2.1, formula (3.14) and inequalities (3.19) and (3.4)]). Hence the sequence of maps $\{w_n(t, \cdot)\}$ is uniformly Cauchy on D. Consequently, the map $w(t, \cdot)$ is holomorphic in D. $\square$

Theorem 3.3. *Let E be a complex Banach space, $G \subset E$ an nonempty (not necessarily bounded) domain and D an open bounded convex subset of G. Assume that $f \in \mathrm{Hol}(G, D)$ is a compact map having no fixed points in G. Then there exist $w \in \partial G \cap \overline{f(G)}$ and sequence $\{w(t_n, a_n)\} \subset \mathrm{Appr}(f)$ such that $\lim_{n\to\infty} t_n = 1$ and*

$$\lim_{n\to\infty} w(t_n, a_n) = \lim_{n\to\infty} f\bigl[w(t_n, a_n)\bigr] = w. \tag{3.4}$$

In addition, if f is continuous in w, then $f(w) = w$.

Proof. The set $f(G)$ is contained in a compact subset of E and f having no fixed points in G. By Theorem 3.2(i), there exists $w \in D \cup (\partial G \cap \partial D)$ and sequence $\{w(t_n, a_n)\} \subset \mathrm{Appr}(f)$ such that $\lim_{n\to\infty} t_n = 1$ and $\lim_{n\to\infty} \|w(t_n, a_n) - w\| = 0$.

If $w \in G$, then

$$f(w) = \lim_{n\to\infty} h_{t_n,a_n}\bigl(w(t_n, a_n)\bigr) = \lim_{n\to\infty} w(t_n, a_n) = w,$$

which is impossible. Therefore $w \in \partial G \cap \overline{f(G)}$.

Furthermore, $\lim_{n\to\infty} \| f[w(t_n, a_n) - w\| = 0$ since

$$\begin{aligned}&f\bigl[w(t_n, a_n)\bigr] = w(t_n, a_n)/t_n - (1-t_n)a_n/t_n, \\ &\left\| f\bigl[w(t_n, a_n)\bigr] - w\right\| \leqslant \left\|\bigl[w(t_n, a_n) - t_n w\bigr]/t_n\right\| + (1-t_n)\|a_n\|/t_n \\ &\qquad\qquad \leqslant \left\| w(t_n, a_n) - w\right\|/t_n + (1-t_n)\bigl(\|w\| + \|a_n\|\bigr)/t_n, \quad n \in \mathbb{N},\end{aligned}$$

and D is bounded.

If f is continuous in w, then we may assume that $f(w) = w$. $\square$

Theorem 3.4. *Let E be a complex Banach space, G an nonempty (not necessarily bounded) domain in E and D an open bounded convex subset of G. Let $f \in \mathrm{Hol}(G, D)$ and let $f : \overline{f(G)} \to E$ be continuous. If $(I_E - f)(\overline{f(G)})$ is closed, then f has a fixed point in $\overline{f(G)}$.*

Proof. Let $a \in D$ be arbitrary and fixed. By Theorem 3.1, for each $t \in (0,1)$, the map $h_{t,a}$ defined by (3.1), has a unique fixed point $w(t,a) \in D$. But $\|w(t,a) - f(w(t,a))\| \leqslant |1-t|(\|f(w(t,a))\| + \|a\|)$ and D is bounded. This means that $\lim_{t\to 1}[w(t,a) - f(w(t,a))] = 0$. Hence we conclude that $0 \in (I_E - f)(\overline{f(G)})$. □

Examples 3.1. Let $E = \mathbb{C}$,

$$G = \{x \in E\colon\ 0 < |x|,\ 0 < \operatorname{Arg}(x) < \pi/2\},$$
$$D = \{y \in E\colon\ |y-1| < 1,\ \operatorname{Im}(y) > 0\}$$

and let $f\colon G \to D$ be of the form $f(x) = 1 + (x-1)/(x+1)$. Thus f is a biholomorphic compact map of unbounded domain G onto bounded convex domain $D \subset G$, continuous on $G \cup (\partial G \cap \partial D)$, having no fixed point in G and $\operatorname{Fix}(f) = \{0, 1\} \subset \partial G \cap \partial D$.

4. Construction of invariant CRF-horospheres, Julia's lemma and Wolff's theorem

The different constructions of horospheres and results of type of Julia's lemma and Wolff's theorem in bounded domains were given by several authors [1–4,7,15,17,26,29, 30,34,36,42,45,46,49].

For our further study of the boundary behaviour of holomorphic maps and iterations of holomorphic maps on unbounded domains G, we need a new tool: the horospheres in G. In this section we show that the question concerning the existence of horospheres in G determined by CRF-pseudometrics ρ_G has positive answer and as applications of those we give new results of type of Julia's lemma and Wolff's theorem. For details concerning CRF-pseudometrics, see [10,13,23,40,44,47].

First, we obtain

Lemma 4.1. *Let G be a nonempty* (*not necessarily bounded*) *domain contained in a complex Banach space E. Let a subset D of G be bounded and convex, and suppose that D lies strictly inside G. Then there exists $\varepsilon > 0$ such that*

$$\rho_G(x,y) \leqslant (1/\varepsilon)\|x-y\| \quad \textit{for all } x, y \in D.$$

Proof. We proceed as in the proof of [47, Theorem 2.1]. Suppose that $H^\infty(G)$ is a normed vector space of bounded maps $g \in \operatorname{Hol}(G, \mathbb{C})$ with the norm $\|g\| = \sup\{|g(x)|\colon x \in G\}$. For $(x,g) \in G \times H^\infty(G)$, we set $\varphi(x)(g) = g(x)$. It is clear that $\varphi \in \operatorname{Hol}(G, H^\infty(G)^*)$, and that

$$\varphi'(x)(\upsilon)(g) = g'(x) \quad \text{for } (x, \upsilon, g) \in G \times E \times H^\infty(G).$$

We first choose $\varepsilon > 0$ such that for all $u \in D$, where $B(\varepsilon) = \{z \in E\colon\ \|z\| \leqslant \varepsilon\}$. Now, for $u \in D$, $g \in H^\infty(G)$ such that $\|g\| \leqslant 1$, and $h \in E \setminus \{0\}$, we have $\varepsilon h/\|h\| \in B(\varepsilon)$ and, by the Cauchy integral formula,

$$\bigl|g'(u)\bigl(h/\|h\|\bigr)\bigr| = \left|(2\pi i)^{-1} \int\limits_{|\mu|=\varepsilon} g\bigl(u + \mu h/\|h\|\bigr)\mu^{-2}\, d\mu\right| \leqslant (1/\varepsilon)\|g\| \leqslant 1/\varepsilon.$$

This shows that $|g'(u)(h)| \leqslant (1/\varepsilon)\|h\|$ for $(u,h) \in D \times E$. Finally, if $(x,y) \in D \times D$, then

$$\begin{aligned}\rho_G(x,y) &\leqslant \sup\bigl\{\bigl\|\varphi'\bigl[tx+(1-t)y\bigr](y-x)\bigr\|\colon t \in \langle 0,1\rangle\bigr\}\\ &\leqslant \sup\bigl\{\bigl|g'\bigl[tx+(1-t)\bigr](y-x)\bigr|\colon t \in \langle 0,1\rangle,\ g \in H^\infty(G),\ \|g\| \leqslant 1\bigr\}\\ &\leqslant (1/\varepsilon)\|x-y\|. \qquad \square\end{aligned}$$

Next, we shall prove

Theorem 4.1. *Let E be a separable complex Banach space, let $G \subset E$ be a nonempty (not necessarily bounded) domain and let $D \subset G$ be a bounded domain such that $\partial G \cap \partial D \neq \emptyset$. Let E_0 be a dense countable subset of E and let $G_0 = G \cap E_0$. Assume that $w \in \partial G \cap \partial D$, $w_n \in G$ for $n \in \mathbb{N}$ and $\lim_{n\to\infty} w_n = w$. Then there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that the limits*

$$\lim_{k\to\infty}\bigl[\rho_G(y,w_{n_k}) - \rho_G(x,w_{n_k})\bigr]$$

and

$$\lim_{m\to\infty} \lim{}_{k\to\infty}\bigl[\rho_G(y^{(m)},w_{n_k}) - \rho_G(x^{(m)},w_{n_k})\bigr]$$

exist and are equal for each points $x, y \in G$ and for each sequences $\{x^{(m)}, y^{(m)}\} \subset G_0 \times G_0$ such that $\lim_{m\to\infty} x^{(m)} = x$ and $\lim_{m\to\infty} y^{(m)} = y$. The sets

$$H\bigl(x,\{w_{n_k}\},w;\rho_G,R\bigr) = \Bigl\{y \in G\colon \lim_{k\to\infty}\bigl[\rho_G(y,w_{n_k}) - \rho_G(x,w_{n_k})\bigr] < (1/2)\log R\Bigr\},$$

$x \in G$, $R > 0$,we call CRF-horospheres in G.

Proof. If $x, y, z \in G$, then $|\rho_G(y,z) - \rho_G(x,z)| \leqslant \rho_G(x,y)$. Hence we obtain that

$$\begin{aligned}-\infty < -\rho_G(x,y) &\leqslant \varliminf_{z\to w}\bigl[\rho_G(y,z) - \rho_G(x,z)\bigr]\\ &\leqslant \varlimsup_{z\to w}\bigl[\rho_G(y,z) - \rho_G(x,z)\bigr] \leqslant \rho_G(x,y) < +\infty. \qquad (4.1)\end{aligned}$$

First we prove that there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that the limits $\lim_{k\to\infty}[\rho_G(y,w_{n_k}) - \rho_G(x,w_{n_k})]$ exists for each $(x,y) \in G_0 \times G_0$.

Indeed, let $G_0 \times G_0 = \{x^{(n)}, y^{(n)}\}$. By (4.1), there exists a subsequence $\{w_{n,1}\}$ of $\{w_n\}$ such that the limit $\lim_{n\to\infty}[\rho_G(y^{(1)},w_{n,1}) - \rho_G(x^{(1)},w_{n,1})]$ exists. By induction, using analogous consideration, we conclude that there exist a subsequence $\{w_{n,k}\}$ of $\{w_{n,k-1}\}$ such that the limit $\lim_{n\to\infty}[\rho_G(y^{(k)},w_{n,k}) - \rho_G(x^{(k)},w_{n,k})]$ exists. Thus, for each $k \in \mathbb{N}$, there exists a subsequence $\{w_{n,k}\}$ of $\{w_n\}$ such that, for each $i = 1,\dots,k$, the limits $\lim_{n\to\infty}[\rho_G(y^{(i)},w_{n,k}) - \rho_G(x^{(i)},w_{n,k})]$ exist. Consequently, the diagonal sequence $\{w_{n,n}\}$ is a required subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that the limit $\lim_{k\to\infty}[\rho_G(y,w_{n_k}) - \rho_G(x,w_{n_k})]$ exists for each $(x,y) \in G_0 \times G_0$.

Let now $x, y \in G$ be arbitrary and fixed, let $\{x^{(m)}, y^{(m)}\} \subset G_0 \times G_0$ be an arbitrary and fixed sequence such that $\lim_{m\to\infty} x^{(m)} = x$ and $\lim_{m\to\infty} y^{(m)} = y$ and let $\{w_{n_k}\}$ be a sequence defined above. Since, for each $m, k \in \mathbb{N}$,

$$\begin{aligned}
&\rho_G(y^{(m)}, w_{n_k}) - \rho_G(x^{(m)}, w_{n_k}) - A_m - B_m \\
&\quad \leqslant \rho_G(y, w_{n_k}) - \rho_G(x, w_{n_k}) \\
&\quad \leqslant \rho_G(y^{(m)}, w_{n_k}) - \rho_G(x^{(m)}, w_{n_k}) + A_m + B_m,
\end{aligned}$$

thus we have

$$Q_m - A_m - B_m \leqslant \underline{C} \leqslant \bar{C} \leqslant Q_m + A_m + B_m, \tag{4.2}$$

where

$$\begin{aligned}
&Q_m = \lim_{k\to\infty} \bigl[\rho_G(y^{(m)}, w_{n_k}) - \rho_G(x^{(m)}, w_{n_k})\bigr], \\
&A_m = \rho_G(x^{(m)}, x),\ B_m = \rho_G(y^{(m)}, y), \quad m \in \mathbb{N}, \\
&\underline{C} = \lim_{k\to\infty} \bigl[\rho_G(y, w_{n_k}) - \rho_G(x, w_{n_k})\bigr], \\
&\bar{C} = \lim_{k\to\infty} \bigl[\rho_G(y, w_{n_k}) - \rho_G(x, w_{n_k})\bigr].
\end{aligned}$$

Let us observe that the conditions $\lim_{m\to\infty} x^{(m)} = x$ and $\lim_{m\to\infty} y^{(m)} = y$ imply that $\lim_{m\to\infty} A_m = \lim_{m\to\infty} B_m = 0$. Indeed, we first assume that $\lim_{m\to\infty} x^{(m)} = x$, so that there exist $B(x, r) \subset G$, $r > 0$, and $m_0 \in \mathbb{N}$, such that $D = B(x, r/2)$ lies strictly inside G, and, for each $m > m_0$, $x^{(m)} \in D$. Now, we are able to apply Lemma 4.1 to have that there exists a positive number ε such that $\rho_G(x^{(m)}, x) \leqslant (1/\varepsilon)\|x^{(m)} - x\|$ for each $m > m_0$. Hence $\lim_{m\to\infty} A_m = 0$. Identically, we show that $\lim_{m\to\infty} B_m = 0$.

Therefore inequalities (4.2) implies

$$\liminf_{m\to\infty} Q_m \leqslant \limsup_{m\to\infty} Q_m \leqslant \underline{C} \leqslant \bar{C} \leqslant \liminf_{m\to\infty} Q_m \leqslant \limsup_{m\to\infty} Q_m$$

and, consequently, $\lim_{m\to\infty} Q_m = \underline{C} = \bar{C}$, as required. □

Our version of Julia's lemma is

Theorem 4.2. *Let E be a separable complex Banach space, let $G \subset E$ be a nonempty* (*not necessarily bounded*) *domain, let $f \in \mathrm{Hol}(G, E)$ and let $f(G)$ be contained in some open bounded subset D of G. Assume that $\partial G \cap \partial D \neq \emptyset$, $w, \upsilon \in \partial G \cap \partial D$, $w_n \in G$ for $n \in \mathbb{N}$, $\lim_{n\to\infty} w_n = w$, $\lim_{n\to\infty} f(w_n) = \upsilon$ and, for some $x \in G$ and $\alpha > 0$,*

$$\lim_{n\to\infty} \bigl[\rho_G(x, w_n) - \rho_G\bigl(x, f(w_n)\bigr)\bigr] \leqslant (1/2)\log\alpha < +\infty.$$

Then there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that, for each $R > 0$,

$$f\bigl[H\bigl(x, \{w_{n_k}\}, w; \rho_G, R\bigr)\bigr] \subset H\bigl(x, \bigl\{f(w_{n_k})\bigr\}, \upsilon; \rho_G, \alpha R\bigr).$$

Proof. Indeed, since CRF-pseudometrics are a Schwarz–Pick system [10,23,40], in virtue of Theorem 4.1, for some subsequence $\{w_{n_k}\}$ of the sequence $\{w_n\}$ and for each $y \in H(x, \{w_{n_k}\}, w; \rho_G, R)$, we have

$$\begin{aligned}
&\lim_{k\to\infty}\bigl[\rho_G\bigl(f(y), f(w_{n_k})\bigr) - \rho_G\bigl(x, f(w_{n_k})\bigr)\bigr]\\
&\quad\leqslant \lim_{k\to\infty}\bigl[\rho_G(y, w_{n_k}) - \rho_G\bigl(x, f(w_{n_k})\bigr)\bigr]\\
&\quad\leqslant \lim_{k\to\infty}\bigl[\rho_G(y, w_{n_k}) - \rho_G(x, w_{n_k})\bigr] + \lim_{k\to\infty}\bigl[\rho_G(x, w_{n_k}) - \rho_G\bigl(x, f(w_{n_k})\bigr)\bigr]\\
&\quad\leqslant \lim_{k\to\infty}\bigl[\rho_G(y, w_{n_k}) - \rho_G(x, w_{n_k})\bigr] + (1/2)\log\alpha\\
&\quad\leqslant (1/2)\log(\alpha R).
\end{aligned}$$

This concludes the proof. □

Now, as an application of Section 3, Theorem 4.1 and Lemma 4.1, we prove the following new version of Wolff's theorem.

Theorem 4.3. *Let E be a separable complex Banach space, $G \subset E$ a nonempty (not necessarily bounded) domain and $D \subset G$ a open bounded convex set. Assume that $f \in \mathrm{Hol}(G, D)$ is a compact map having no fixed points in G. Then there exist $w \in \partial G \cap \partial D$ and sequence $\{w(t_n, a_n)\} \subset \mathrm{Appr}(f)$ such that $\lim_{n\to\infty} t_n = 1$,*

$$\lim_{n\to\infty} w(t_n, a_n) = \lim_{n\to\infty} f\bigl[w(t_n, a_n)\bigr] = w$$

and, for each $x \in G$, $R > 0$ and $k \in \mathbb{N}$,

$$f^{[k]}\bigl[H\bigl(x, \bigl\{w(t_n, a_n)\bigr\}, w; \rho_G, R\bigr)\bigr] \subset H\bigl(x, \bigl\{w(t_n, a_n)\bigr\}, w, \rho_G, R\bigr).$$

Proof. By Theorems 3.3 and 4.1 (up to a subsequence of the sequence $\{w(t_n, a_n)\}$ if it is necessary), without loss of generality, we may assume that, for each $x \in G$ and $R > 0$, the CRF-horospheres $H(x, \{w(t_n, a_n)\}, w; \rho_G, R)$ exist. Suppose that $y \in H(x, \{w(t_n, a_n)\}, w; \rho_G, R)$. Then since the limit $\lim_{n\to\infty} h_{t_n,a_n}(y) = f(y)$ holds, there exist $B(f(y), r) \subset G$, $r > 0$, and $n_0 \in \mathbb{N}$, such that $D = B(f(y), r/2)$ lies strictly inside G, and, for each $n > n_0$, $h_{t_n,a_n}(y) \in D$. By Lemma 4.1, there exists $\varepsilon > 0$ such that

$$\rho_G\bigl(f(y), h_{t_n,a_n}(y)\bigr) \leqslant (1/\varepsilon)\bigl\| f(y) - h_{t_n,a_n}(y)\bigr\| \quad \text{for } n > n_0.$$

Hence $\lim_{n\to\infty} \rho_G(f(y), h_{t_n,a_n}(y)) = 0$ and since CRF-pseudometrics are a Schwarz–Pick system, we conclude that

$$\begin{aligned}
&\lim_{n\to\infty}\bigl[\rho_G\bigl(f(y), w(t_n, a_n)\bigr) - \rho_G\bigl(x, w(t_n, a_n)\bigr)\bigr]\\
&\quad\leqslant \lim_{n\to\infty}\bigl[\rho_G\bigl(f(y), (h_{t_n,a_n})(y)\bigr)\\
&\qquad + \rho_G\bigl((h_{t_n,a_n})(y), (h_{t_n,a_n})\bigl(w(t_n, a_n)\bigr)\bigr) - \rho_G\bigl(x, w(t_n, a_n)\bigr)\bigr]\\
&\quad\leqslant \lim_{n\to\infty}\bigl[\rho_G\bigl(f(y), h_{t_n,a_n}(y)\bigr) + \rho_G\bigl(y, w(t_n, a_n)\bigr) - \rho_G\bigl(x, w(t_n, a_n)\bigr)\bigr]\\
&\quad= \lim_{n\to\infty}\bigl[\rho_G\bigl(y, w(t_n, a_n)\bigr) - \rho_G\bigl(x, w(t_n, a_n)\bigr)\bigr].
\end{aligned}$$

Therefore

$$f\bigl[H\bigl(x, \bigl\{w(t_n, a_n)\bigr\}, w; \rho_G, R\bigr)\bigr] \subset H\bigl(x, \bigl\{w(t_n, a_n)\bigr\}, w, \rho_G, R\bigr),$$

and so, by induction, for each $i \in \mathbb{N} \cup \{0\}$,

$$f^{[i+1]}\big[H\big(x, \{w(t_n, a_n)\}, w; \rho_G, R\big)\big] \subset f^{[i]}\big[H\big(x, \{w(t_n, a_n)\}, w, \rho_G, R\big)\big],$$

where $f^{[0]} = I_E$. This implies the assertion. $\square$

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