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# Inclusion and argument properties for certain subclasses of meromorphic functions associated with a family of multiplier transformations

Nak Eun Cho [a], Oh Sang Kwon [b], H.M. Srivastava [c,*]

[a] *Department of Applied Mathematics, Pukyong National University, Pusan 608-737, South Korea*
[b] *Department of Mathematics, Kyungsung University, Pusan 608-736, South Korea*
[c] *Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3P4, Canada*

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**Abstract**

Making use of a multiplier transformation, which is defined here by means of the Hadamard product (or convolution), the authors introduce some new subclasses of meromorphic functions and investigate their inclusion relationships and argument properties. Some integral-preserving properties in a given sector are also considered.


*Keywords:* Analytic functions; Meromorphic functions; Subordination between analytic functions; Starlike functions; Convex functions; Close-to-convex functions; Multiplier transformation; Choi–Saigo–Srivastava operator; Meromorphic strongly starlike functions

* Corresponding author.
*E-mail addresses:* necho@pknu.ac.kr (N.E. Cho), oskwon@ks.ac.kr (O.S. Kwon), harimsri@math.uvic.ca (H.M. Srivastava).

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## 1. Introduction

Let $\Sigma$ denote the class of functions of the following form:

$$f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_k z^k,$$

which are *analytic* in the *punctured* open unit disk

$$\mathbb{D}=\{z\colon\ z\in\mathbb{C}\text{ and }0<|z|<1\}=\mathbb{U}\setminus\{0\}.$$

A function $f\in\Sigma$ is said to be *meromorphic strongly starlike of order* $\alpha$ in $\mathbb{D}$ if it satisfies the following condition:

$$\left|\arg\left(-\frac{zf'(z)}{f(z)}\right)\right|<\frac{\pi}{2}\alpha\quad(0<\alpha\leqq 1;\ z\in\mathbb{U}).$$

We denote by $\Sigma^*(\alpha)$ the subclass of $\Sigma$ consisting of all meromorphic strongly starlike functions of order $\alpha$ in $\mathbb{D}$. Also we note that

$$\Sigma^*(1)=:\Sigma^*$$

is the well-known class of meromorphic starlike functions in $\mathbb{D}$ (see, for details, [6]).

For

$$n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}\quad\big(\mathbb{N}:=\{1,2,3,\ldots\}\big),$$

we define the *multiplier transformation* $D_\lambda^n$ of functions $f\in\Sigma$ by

$$D_\lambda^n f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}\left(\frac{k+1+\lambda}{\lambda}\right)^n a_k z^k\quad(\lambda>0;\ z\in\mathbb{D}).$$

Obviously, we have

$$D_\lambda^m\big(D_\lambda^n f(z)\big)=D_\lambda^{m+n}f(z)\quad(m,n\in\mathbb{N}_0;\ \lambda>0).$$

The operators $D_\lambda^n$ and $D_1^n$ are the multiplier transformations introduced and studied earlier by Sarangi and Uralegaddi [16] and Uralegaddi and Somanatha ([20] and [21]), respectively. Analogous to $D_\lambda^n$, we here define a new multiplier transformation $\mathcal{I}_{\lambda,\mu}^n$ as follows.

Put

$$f_n(z)=\frac{1}{z}+\sum_{k=0}^{\infty}\left(\frac{k+1+\lambda}{\lambda}\right)^n z^k\quad(n\in\mathbb{N}_0;\ \lambda>0)$$

and let the associated function $f_{n,\mu}^\dagger$ be so defined that the Hadamard product (or convolution):

$$f_n(z)*f_{n,\mu}^\dagger(z)=\frac{1}{z(1-z)^\mu}\quad(\mu>0;\ z\in\mathbb{D}).$$

Then, analogous to $D_\lambda^n$, we have

$$\mathcal{I}_{\lambda,\mu}^n f(z):=f_{n,\mu}^\dagger(z)*f(z).\tag{1.1}$$

We note that

$$\mathcal{I}_{1,2}^{0} f(z) = zf'(z) + 2f(z) \quad \text{and} \quad \mathcal{I}_{1,2}^{1} f(z) = f(z).$$

It is easily verified from the definition (1.1) that

$$z\big(\mathcal{I}_{\lambda,\mu}^{n+1} f(z)\big)' = \lambda \mathcal{I}_{\lambda,\mu}^{n} f(z) - (\lambda+1)\mathcal{I}_{\lambda,\mu}^{n+1} f(z) \tag{1.2}$$

and

$$z\big(\mathcal{I}_{\lambda,\mu}^{n} f(z)\big)' = \mu \mathcal{I}_{\lambda,\mu+1}^{n} f(z) - (\mu+1)\mathcal{I}_{\lambda,\mu}^{n} f(z). \tag{1.3}$$

The definition (1.1) of the multiplier transformation $\mathcal{I}_{\lambda,\mu}^{n}$ is motivated essentially by the Choi–Saigo–Srivastava operator [3] for analytic functions, which includes a simpler integral operator studied earlier by Noor [12] and others (cf. [8,9,13]).

Let $\mathcal{N}$ be the class of analytic functions $h$ with $h(0) = 1$, which are convex and univalent in $\mathbb{U}$ and for which

$$\Re\big\{h(z)\big\} > 0 \quad (z \in \mathbb{U}).$$

For functions $f$ and $g$ analytic in

$$\mathbb{U} := \mathbb{D} \cup \{0\},$$

we say that $f$ is *subordinate* to $g$, and write

$$f \prec g \quad \text{in } \mathbb{U} \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a *Schwarz function* $w(z)$, which (by definition) is analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad \big|w(z)\big| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(w(z)\big) \quad (z \in \mathbb{U}).$$

It is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \quad \Longrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function $g$ is *univalent* in $\mathbb{U}$, then (see, e.g., [11, p. 4])

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \quad \Longleftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Making use of the principle of subordination between analytic functions, we introduce the following new subclasses:

$$\Sigma_{\lambda,\mu}^{l}(l; h) \quad \text{and} \quad \Sigma_{\lambda,\mu}^{l}(l; A, B; \alpha)$$

of the class $\Sigma$.

Let the functions $g_1, \ldots, g_l$ be in the class $\Sigma$. Then we say that the functions $g_1, \ldots, g_l$ are in the subclass $\Sigma_{\lambda,\mu}^{l}(l; h)$ if they satisfy the following subordination condition:

$$-\frac{z\big(\mathcal{I}_{\lambda,\mu}^{n} g_k(z)\big)'}{\frac{1}{l}\sum_{j=1}^{l} \mathcal{I}_{\lambda,\mu}^{n} g_j(z)} \prec h(z) \quad (z \in \mathbb{U};\ k = 1, \ldots, l;\ h \in \mathcal{N}), \tag{1.4}$$

where

$$z\sum_{j=1}^{l}\mathcal{I}_{\lambda,\mu}^{n}g_j(z)\neq 0 \quad (z\in\mathbb{U}).$$

In particular, we set

$$\Sigma_{\lambda,\mu}^{n}\left(l;\frac{1+Az}{1+Bz}\right)=:\Sigma_{\lambda,\mu}^{n}(l;A,B)\quad(-1<B<A\leqq 1;\ z\in\mathbb{U}). \tag{1.5}$$

We note that

$$\Sigma_{1,2}^{1}\left(1;\left(\frac{1+z}{1-z}\right)^{\alpha}\right)=\Sigma^{*}(\alpha)\quad(0<\alpha\leqq 1;\ z\in\mathbb{U})$$

and

$$\Sigma_{1,2}^{1}\left(1;\frac{1+z}{1-z}\right)=\Sigma^{*}\quad(z\in\mathbb{U})$$

for the familiar subclasses $\Sigma^{*}(\alpha)$ $(0<\alpha\leqq 1)$ and $\Sigma^{*}$ of the class $\Sigma$.

Next, we denote by $\Sigma_{\lambda,\mu}^{n}(l;A,B;\alpha)$ the class of functions $f\in\Sigma$ satisfying the following inequality:

$$\left|\arg\left(-\frac{z\big(\mathcal{I}_{\lambda,\mu}^{n}f(z)\big)'}{\frac{1}{l}\sum_{j=1}^{l}\mathcal{I}_{\lambda,\mu}^{n}g_j(z)}\right)\right|<\frac{\pi}{2}\alpha$$
$$\big(z\in\mathbb{U};\ 0<\alpha\leqq 1;\ g_j\in\Sigma_{\lambda,\mu}^{n}(l;A,B);\ j=1,\ldots,l\big). \tag{1.6}$$

We note that, for appropriate choices of the parameters involved in (1.6), the class $\Sigma_{\lambda,\mu}^{n}(l;A,B;\alpha)$ can be reduced to that of meromorphic close-to-convex functions introduced and studied by Libera and Robertson [7] and Singh [18]. Furthermore, for some interesting developments related to the classes

$$\Sigma_{\lambda,\mu}^{n}(l;h)\quad\text{and}\quad\Sigma_{\lambda,\mu}^{n}(l;A,B;\alpha),$$

the reader can be referred to the works of (for example) Bharati and Rajagopal [2] and Padmanabhan and Parvatham [14].

In the present paper, we give some argument properties of meromorphic functions belonging to the class $\Sigma$ which contain the basic inclusion relationships among the classes

$$\Sigma_{\lambda,\mu}^{n}(l;h)\quad\text{and}\quad\Sigma_{\lambda,\mu}^{n}(l;A,B;\alpha).$$

The integral-preserving properties of the operator $\mathcal{I}_{\lambda,\mu}^{n}$ defined by (1.1) are also considered. Furthermore, we obtain the previous results of Bajpai [1] and Goel and Sohi [5] as special cases.

## 2. The main inclusion properties and their consequences

The following results will be required in our investigation.

**Lemma 1** (Eenigenberg et al. [4]). *Let $h$ be convex univalent in $\mathbb{U}$ with*

$$h(0)=1 \quad \textit{and} \quad \Re\{\lambda h(z)+\nu\}>0 \quad (z\in\mathbb{U};\ \lambda,\nu\in\mathbb{C}).$$

*If $q$ is analytic in $\mathbb{U}$ with $q(0)=1$, then the following subordination*:

$$q(z)+\frac{zq'(z)}{\lambda q(z)+\nu}\prec h(z) \quad (z\in\mathbb{U})$$

*implies that*

$$q(z)\prec h(z) \quad (z\in\mathbb{U}).$$

**Lemma 2** (Miller and Mocanu [10]). *Let $h$ be convex univalent in $\mathbb{U}$ and $\omega$ be analytic in $\mathbb{U}$ with*

$$\Re\{\omega(z)\}\geqq 0.$$

*If $q$ is analytic in $\mathbb{U}$ and*

$$q(0)=h(0),$$

*then the following subordination*:

$$q(z)+\omega(z)zq'(z)\prec h(z) \quad (z\in\mathbb{U})$$

*implies that*

$$q(z)\prec h(z) \quad (z\in\mathbb{U}).$$

**Lemma 3** (Cf., e.g., Takahashi and Nunokawa [19, p. 653]). *Let $q$ be analytic in $\mathbb{U}$ with*

$$q(0)=1 \quad \textit{and} \quad q(z)\neq 0 \quad (z\in\mathbb{U}).$$

*If there exist two points $z_1, z_2\in\mathbb{U}$ such that*

$$-\frac{\pi}{2}\alpha_1=\arg\bigl(q(z_1)\bigr)<\arg\bigl(q(z)\bigr)<\arg\bigl(q(z_2)\bigr)=\frac{\pi}{2}\alpha_2 \tag{2.1}$$

*for some $\alpha_1$ and $\alpha_2$ $(\alpha_1,\alpha_2>0)$ and for all $z$ $(|z|<|z_1|=|z_2|)$, then*

$$\frac{z_1q'(z_1)}{q(z_1)}=-i\left(\frac{\alpha_1+\alpha_2}{2}\right)m \quad \textit{and} \quad \frac{z_2q'(z_2)}{q(z_2)}=i\left(\frac{\alpha_1+\alpha_2}{2}\right)m, \tag{2.2}$$

*where*

$$m\geqq\frac{1-|b|}{1+|b|} \quad \textit{and} \quad b=i\tan\frac{\pi}{4}\left(\frac{\alpha_2-\alpha_1}{\alpha_1+\alpha_2}\right). \tag{2.3}$$

First of all, with the help of Lemmas 1 and 2, we obtain the following inclusion relationships.

**Proposition 1.** *Let* $h \in \mathcal{N}$ *with*

$$\max_{z \in \mathbb{U}} \big(\Re\{h(z)\}\big) < \min(\lambda + 1, \mu + 1) \quad (\lambda, \mu > 0).$$

*Then the following inclusion relationships hold true*:

$$\Sigma^n_{\lambda,\mu+1}(l; h) \subset \Sigma^n_{\lambda,\mu}(l; h) \subset \Sigma^{n+1}_{\lambda,\mu}(l; h).$$

**Proof.** We begin by showing that

$$\Sigma^n_{\lambda,\mu+1}(l; h) \subset \Sigma^n_{\lambda,\mu}(l; h).$$

Let

$$g_j \in \Sigma^n_{\lambda,\mu}(l; h) \quad (j = 1, \ldots, l)$$

and set

$$p_k(z) = -\frac{z\big(\mathcal{I}^n_{\lambda,\mu} g_k(z)\big)'}{\frac{1}{l}\sum_{j=1}^{l} \mathcal{I}^n_{\lambda,\mu} g_j(z)} \quad (k = 1, \ldots, l),$$

where $p_k$ $(k = 1, \ldots, l)$ is analytic in $\mathbb{U}$ with

$$p_k(0) = 1 \quad (k = 1, \ldots, l).$$

By using the identity (1.3), we get

$$\frac{1}{l}\sum_{j=1}^{l} \big(\mathcal{I}^n_{\lambda,\mu} g_j(z)\big) p_k(z) - (\mu + 1)\mathcal{I}^n_{\lambda,\mu} g_k(z) = -\mu \mathcal{I}^n_{\lambda,\mu+1} g_k(z) \quad (k = 1, \ldots, l). \tag{2.4}$$

Upon differentiating both sides of (2.4) with respect to $z$, and then simplifying, we have

$$p_k(z) + \frac{z p_k'(z)}{-\frac{1}{l}\sum_{j=1}^{l} p_j(z) + \mu + 1} = -\frac{z\big(\mathcal{I}^n_{\lambda,\mu+1} g_k(z)\big)'}{\frac{1}{l}\sum_{j=1}^{l} \mathcal{I}^n_{\lambda,\mu+1} g_j(z)} \prec h(z) \quad (z \in \mathbb{U};\ k = 1, \ldots, l), \tag{2.5}$$

since

$$g_k \in \Sigma^n_{\lambda,\mu+1}(l; h) \quad (k = 1, \ldots, l).$$

Since $h$ is convex in $\mathbb{U}$, for any $z_0 \in \mathbb{U}$, there exists a point $\zeta_0 \in \mathbb{U}$ such that

$$q(z_0) + \frac{z_0 q'(z_0)}{-q(z_0) + \mu + 1} = h(\zeta_0),$$

where

$$q(z) = \frac{1}{l}\sum_{j=1}^{l} p_j(z).$$

Thus we find from Lemma 1 that $q \prec h$ in $\mathbb{U}$. Applying Lemma 2 with

$$\omega(z) = \frac{1}{-q(z) + \mu + 1}$$

to (2.5) again, it follows that

$$p_k \prec h \quad \text{in } \mathbb{U} \quad \text{for all } k \ (k = 1, \dots, l),$$

which implies that

$$g_k \in \Sigma^n_{\lambda,\mu}(l; h) \quad (k = 1, \dots, l)$$

whenever

$$\max_{z \in \mathbb{U}} \bigl(\Re\{h(z)\}\bigr) < \mu + 1 \quad (\mu > 0).$$

Next, we prove that

$$z \sum_{j=1}^{l} \mathcal{I}^n_{\lambda,\mu} g_j(z) \neq 0 \quad (z \in \mathbb{U}).$$

Since

$$g_k \in \Sigma^n_{\lambda,\mu+1}(l; h) \quad (k = 1, \dots, l)$$

and $h$ is convex in $\mathbb{U}$, there exists a point $\zeta_0 \in \mathbb{U}$ such that, for any $z_0 \in \mathbb{U}$,

$$r(z_0) := -\frac{z_0 \bigl(\sum_{j=1}^{l} \mathcal{I}^n_{\lambda,\mu+1} g_j(z_0)\bigr)'}{\sum_{j=1}^{l} \mathcal{I}^n_{\lambda,\mu+1} g_j(z_0)} = h(\zeta_0),$$

and hence $r \prec h$ in $\mathbb{U}$. We note also that

$$\sum_{j=1}^{l} \mathcal{I}^n_{\lambda,\mu} g_j(z) = \frac{\mu}{z^{\mu+1}} \int_0^z t^{\mu} \sum_{j=1}^{l} \mathcal{I}^n_{\lambda,\mu+1} g_j(t)\, dt.$$

Thus, by applying a known result [5, Theorem 1] (see also [15]), we conclude that

$$z \sum_{j=1}^{l} \mathcal{I}^n_{\lambda,\mu} g_j(z) \neq 0 \quad (z \in \mathbb{U}).$$

To prove the second inclusion relationship asserted by Proposition 1, let

$$g_k \in \Sigma^n_{\lambda,\mu}(l; h) \quad (k = 1, \dots, l)$$

and put

$$s_k(z) = -\frac{z\bigl(\mathcal{I}^{n+1}_{\lambda,\mu} g_k(z)\bigr)'}{\frac{1}{l}\sum_{j=1}^{l} \mathcal{I}^{n+1}_{\lambda,\mu} g_j(z)} \quad (k = 1, \dots, l),$$

where $s_k$ $(k = 1, \dots, l)$ is analytic in $\mathbb{U}$ with

$$s_k(0) = 1 \quad (k = 1, \dots, l).$$

Then, by using the arguments similar to those detailed above with (1.2), it follows that

$$s_k \prec h \quad \text{in } \mathbb{U} \quad \text{for all } k \ (k = 1, \dots, l),$$

that is,

$$g_k \in \Sigma_{\lambda,\mu}^{n+1}(l;h) \quad (k=1,\dots,l)$$

whenever

$$\max_{z\in\mathbb{U}}\bigl(\Re\{h(z)\}\bigr) < \lambda+1 \quad (\lambda>0).$$

Thus we have completed the proof of Proposition 1. □

If we take

$$h(z)=\frac{1+Az}{1+Bz} \quad (-1<B<A\leqq 1)$$

in Proposition 1, we obtain the following result involving the function class defined by (1.5).

**Corollary 1.** *Let*

$$\frac{1+A}{1+B} < \min(\lambda+1,\mu+1) \quad (\lambda,\mu>0;\ -1<B<A\leqq 1).$$

*Then*

$$\Sigma_{\lambda,\mu+1}^{n}(l;A,B)\subset \Sigma_{\lambda,\mu}^{n}(l;A,B)\subset \Sigma_{\lambda,\mu}^{n+1}(l;A,B).$$

**Proposition 2.** *Let $h\in\mathcal{N}$ with*

$$\max_{z\in\mathbb{U}}\bigl(\Re\{h(z)\}\bigr) < c+1 \quad (c>0).$$

*Then*

$$g_k \in \Sigma_{\lambda,\mu}^{n}(l;h) \quad (k=1,\dots,l) \quad \Longrightarrow \quad F_c(g_k)\in \Sigma_{\lambda,\mu}^{n}(l;h),$$

*where $F_c$ is the integral operator defined by*

$$F_c(g_k)=F_c(g_k)(z) := \frac{c}{z^{c+1}}\int_0^z t^c g_k(t)\,dt \quad (k=1,\dots,l;\ c>0). \tag{2.6}$$

**Proof.** Suppose that

$$g_k \in \Sigma_{\lambda,\mu}^{n}(l;h) \quad (k=1,\dots,l)$$

and set

$$p_k(z) = -\frac{z\bigl(\mathcal{I}_{\lambda,\mu}^{n}F_c(g_k)(z)\bigr)'}{\frac{1}{l}\sum_{j=1}^{l}\mathcal{I}_{\lambda,\mu}^{n}F_c(g_j)(z)} \quad (k=1,\dots,l). \tag{2.7}$$

From (2.6), we have

$$z\bigl(\mathcal{I}_{\lambda,\mu}^{n}F_c(g_k)(z)\bigr)' = c\,\mathcal{I}_{\lambda,\mu}^{n}g_k(z) - (c+1)\mathcal{I}_{\lambda,\mu}^{n}F_c(g_k)(z), \tag{2.8}$$

where $p_k$ $(k = 1, \dots, l)$ is analytic in $\mathbb{U}$ with

$$p_k(0) = 1 \quad (k = 1, \dots, l).$$

Then, by applying (2.8) to (2.7), we get

$$\frac{1}{l}\sum_{j=1}^{l}\big(\mathcal{I}_{\lambda,\mu}^{n}F_c(g_j)(z)\big)p_k(z) - (c+1)\mathcal{I}_{\lambda,\mu}^{n}F_c(g_k)(z) = -c\mathcal{I}_{\lambda,\mu}^{n}g_k(z) \\ (k = 1, \dots, l). \tag{2.9}$$

By differentiating both sides of (2.9) with respect to $z$, and then simplifying, we obtain

$$p_k(z) + \frac{zp_k'(z)}{-\frac{1}{l}\sum_{j=1}^{l}p_j(z) + c + 1} = -\frac{z\big(\mathcal{I}_{\lambda,\mu}^{n}g_k(z)\big)'}{-\frac{1}{l}\sum_{j=1}^{l}\mathcal{I}_{\lambda,\mu}^{n}g_j(z)} \quad (k = 1, \dots, l).$$

We note also that

$$\sum_{j=1}^{l}\mathcal{I}_{\lambda,\mu}^{n}F_c(g_j)(z) = \frac{c}{z^{c+1}}\int_0^z t^c\left(\sum_{j=1}^{l}\mathcal{I}_{\lambda,\mu}^{n}g_j(t)\right)dt.$$

Therefore, by the same arguments as in the proof of Proposition 1, we conclude that Proposition 2 holds true as stated above. □

From Proposition 2, we immediately have the following consequence.

**Corollary 2.** *Let*

$$1 + A < (c+1)(1+B) \quad (c > 0;\ -1 < B < A \leqq 1).$$

*Then, for the function class* $\Sigma_{\lambda,\mu}^{n}(l; A, B)$ *defined by* (1.5),

$$g_k \in \Sigma_{\lambda,\mu}^{n}(l; A, B) \quad (k = 1, \dots, l) \\ \Longrightarrow \quad F_c(g_k) \in \Sigma_{\lambda,\mu}^{n}(l; A, B) \quad (k = 1, \dots, l),$$

*where* $F_c$ *is the integral operator defined by* (2.6).

**Remark 1.** By setting

$$n = \lambda = l = 1, \qquad \mu = 2, \quad \text{and} \quad B \mapsto A$$

in Corollary 2, we arrive at a result of Goel and Sohi [5], which includes the result given earlier by Bajpai [1] as a *further* special case.

## 3. Argument properties and their consequences

**Theorem 1.** *Let* $0 < \delta_1, \delta_2 \leqq 1$ *and*

$$1 + A < (\mu+1)(1+B) \quad (\mu > 0;\ -1 < B < A \leqq 1).$$

*If a function $f \in \Sigma$ satisfies the following two-sided inequality*:

$$-\frac{\pi}{2}\delta_1 < \arg\left(-\frac{z\left(\mathcal{I}_{\lambda,\mu+1}^n f(z)\right)'}{\frac{1}{l}\sum_{j=1}^{l}\mathcal{I}_{\lambda,\mu+1}^n g_j(z)}\right) < \frac{\pi}{2}\delta_2,$$

*where*

$$g_k \in \Sigma_{\lambda,\mu+1}^n(l; A, B) \quad (k = 1, \ldots, l),$$

*then*

$$-\frac{\pi}{2}\alpha_1 < \arg\left(-\frac{z\left(\mathcal{I}_{\lambda,\mu}^n f(z)\right)'}{\frac{1}{l}\sum_{j=1}^{l}\mathcal{I}_{\lambda,\mu}^n g_j(z)}\right) < \frac{\pi}{2}\alpha_2,$$

*where*

$$\alpha_1 \quad \textit{and} \quad \alpha_2 \quad (0 < \alpha_1, \alpha_2 \leqq 1)$$

*are the solutions of the following equations*:

$$\delta_1 = \alpha_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|b|)\cos(\frac{\pi}{2}t_1)}{2\left(\frac{A-1}{1-B}+\mu+1\right)(1+|b|)+(\alpha_1+\alpha_2)(1-|b|)\sin(\frac{\pi}{2}t_1)}\right) \tag{3.1}$$

*and*

$$\delta_2 = \alpha_2 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|b|)\cos(\frac{\pi}{2}t_1)}{2\left(\frac{A-1}{1-B}+\mu+1\right)(1+|b|)+(\alpha_1+\alpha_2)(1-|b|)\sin(\frac{\pi}{2}t_1)}\right) \tag{3.2}$$

*when $b$ is given by* (2.3) *and*

$$t_1 = \frac{2}{\pi}\sin^{-1}\left(\frac{A-B}{(\mu+1)(1-B^2)-(1-AB)}\right). \tag{3.3}$$

**Proof.** Let

$$p(z) = -\frac{z\left(\mathcal{I}_{\lambda,\mu}^n f(z)\right)'}{\frac{1}{l}\sum_{j=1}^{l}\mathcal{I}_{\lambda,\mu}^n g_j(z)} \quad \text{and} \quad q(z) = \frac{1}{l}\sum_{j=1}^{l} q_j(z),$$

where

$$q_k(z) = -\frac{z\left(\mathcal{I}_{\lambda,\mu}^n g_k(z)\right)'}{\frac{1}{l}\sum_{j=1}^{l}\mathcal{I}_{\lambda,\mu}^n g_j(z)} \quad (k = 1, \ldots, l).$$

Making use of (1.3), we readily have

$$\frac{1}{l}\sum_{j=1}^{l}\left(\mathcal{I}_{\lambda,\mu}^n g_j(z)\right)p(z) - (\mu+1)\mathcal{I}_{\lambda,\mu}^n f(z) = -\mu\mathcal{I}_{\lambda,\mu+1}^n f(z). \tag{3.4}$$

By differentiating both sides of (3.4) with respect to $z$, and then simplifying, we obtain

$$-\frac{z\left(\mathcal{I}_{\lambda,\mu+1}^n f(z)\right)'}{\frac{1}{l}\sum_{j=1}^{l}\mathcal{I}_{\lambda,\mu+1}^n g_j(z)} = p(z) + \frac{zp'(z)}{-q(z)+\mu+1}.$$

Since

$$g_k \in \Sigma^n_{\lambda,\mu+1}(l; A, B) \quad (k = 1, \ldots, l),$$

by Corollary 1, we see that

$$g_k \in \Sigma^n_{\lambda,\mu}(l; A, B) \quad (k = 1, \ldots, l).$$

Therefore, we get

$$q(z) \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U};\ -1 < B < A \leqq 1).$$

Hence we observe from [17] that

$$\left| q(z) - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2} \quad (z \in \mathbb{U};\ -1 < B < A \leqq 1). \tag{3.5}$$

Thus, by using (3.5), we have

$$-q(z) + \mu + 1 = \rho e^{i\frac{\pi\phi}{2}},$$

where

$$\mu + 1 - \frac{1 + A}{1 + B} < \rho < \mu + 1 + \frac{A - 1}{1 - B} \quad \text{and} \quad -t_1 < \phi < t_1,$$

$t_1$ being given by (3.3).

We note that $p$ is analytic in $\mathbb{U}$ with $p(0) = 1$. Let $w = h(z)$ be the function which maps $\mathbb{U}$ onto the angular domain

$$\left\{ w\colon -\frac{\pi}{2}\delta_1 < \arg(w) < \frac{\pi}{2}\delta_2 \right\} \quad \text{with } h(0) = 1.$$

Applying Lemma 2 for this function $h$ with

$$\omega(z) = \frac{1}{-q(z) + \mu + 1},$$

we see that

$$\Re\{p(z)\} > 0 \quad (z \in \mathbb{U}),$$

and hence

$$p(z) \neq 0 \quad (z \in \mathbb{U}).$$

If there exist two points $z_1, z_2 \in \mathbb{U}$ such that the condition (2.1) is satisfied, then (by Lemma 3) we obtain (2.2) under the restriction (2.3). Hence we have

$$\begin{aligned}
&\arg\left( p(z_1) + \frac{z_1 p'(z_1)}{-q(z_1) + \mu + 1} \right) \\
&\quad = -\frac{\pi}{2}\alpha_1 + \arg\left( 1 - i\frac{\alpha_1 + \alpha_2}{2} m \left(\rho e^{i\frac{\pi\phi}{2}}\right)^{-1} \right) \\
&\quad \leqq -\frac{\pi}{2}\alpha_1 - \tan^{-1}\left( \frac{(\alpha_1 + \alpha_2) m \sin(\frac{\pi}{2}(1 - \phi))}{2\rho + (\alpha_1 + \alpha_2) m \cos(\frac{\pi}{2}(1 - \phi))} \right)
\end{aligned}$$

$$\begin{aligned}
&\leqq -\frac{\pi}{2}\alpha_1 - \tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|b|)\cos(\frac{\pi}{2}t_1)}{2\left(\frac{A-1}{1-B}+\mu+1\right)(1+|b|)+(\alpha_1+\alpha_2)(1-|b|)\sin(\frac{\pi}{2}t_1)}\right)\\
&= -\frac{\pi}{2}\delta_1
\end{aligned}$$

and

$$\begin{aligned}
&\arg\left(p(z_2)+\frac{z_2p'(z_2)}{-q(z_2)+\mu+1}\right)\\
&\quad \geqq \frac{\pi}{2}\alpha_2 + \tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|b|)\cos(\frac{\pi}{2}t_1)}{2\left(\frac{A-1}{1-B}+\mu+1\right)(1+|b|)+(\alpha_1+\alpha_2)(1-|b|)\sin(\frac{\pi}{2}t_1)}\right)\\
&\quad = \frac{\pi}{2}\delta_2,
\end{aligned}$$

where we have used the inequality (2.3), $\delta_1$, $\delta_2$, and $t_1$ being given by (3.1), (3.2), and (3.3), respectively. These obviously contradict the assumption of Theorem 1. The proof of Theorem 1 is thus completed. □

If we let $\delta_1 = \delta_2$ in Theorem 1, we easily obtain the following consequence.

**Corollary 3.** *Let* $0 < \delta \leqq 1$ *and*

$$1 + A < (\mu+1)(1+B) \quad (\mu > 0;\ -1 < B < A \leqq 1).$$

*If a function* $f \in \Sigma$ *satisfies the following inequality*:

$$\left|\arg\left(-\frac{z\left(\mathcal{I}^n_{\lambda,\mu+1}f(z)\right)'}{\frac{1}{l}\sum_{j=1}^{l}\mathcal{I}^n_{\lambda,\mu+1}g_j(z)}\right)\right| < \frac{\pi}{2}\delta,$$

*where*

$$g_k \in \Sigma^n_{\lambda,\mu+1}(l; A, B) \quad (k = 1, \ldots, l),$$

*then*

$$\left|\arg\left(-\frac{z\left(\mathcal{I}^n_{\lambda,\mu}f(z)\right)'}{\frac{1}{l}\sum_{j=1}^{l}\mathcal{I}^n_{\lambda,\mu}g_j(z)}\right)\right| < \frac{\pi}{2}\alpha,$$

*where* $\alpha$ $(0 < \alpha \leqq 1)$ *is the solution of the following equation*:

$$\delta = \alpha + \frac{2}{\pi}\tan^{-1}\left(\frac{\alpha\cos(\frac{\pi}{2}t_1)}{\frac{A-1}{1-B}+\mu+1+\alpha\sin(\frac{\pi}{2}t_1)}\right) \tag{3.6}$$

*when* $t_1$ *is given by* (3.3).

The proof of Theorem 2 below is similar to that of Theorem 1, and so the details may be omitted.

**Theorem 2.** *Let* $0 < \delta_1, \delta_2 \leqq 1$ *and*

$$1 + A < (\lambda+1)(1+B) \quad (\lambda > 0;\ -1 < B < A \leqq 1).$$

*If a function* $f \in \Sigma$ *satisfies the following two-sided inequality*:

$$-\frac{\pi}{2}\delta_1 < \arg\left(-\frac{z\big(\mathcal{I}^n_{\lambda,\mu} f(z)\big)'}{\frac{1}{l}\sum_{j=1}^{l} \mathcal{I}^n_{\lambda,\mu} g_j(z)}\right) < \frac{\pi}{2}\delta_2,$$

*where*

$$g_k \in \Sigma^n_{\lambda,\mu}(l; A, B) \quad (k = 1, \ldots, l),$$

*then*

$$-\frac{\pi}{2}\alpha_1 < \arg\left(-\frac{z\big(\mathcal{I}^{n+1}_{\lambda,\mu} f(z)\big)'}{\frac{1}{l}\sum_{j=1}^{l} \mathcal{I}^{n+1}_{\lambda,\mu} g_j(z)}\right) < \frac{\pi}{2}\alpha_2,$$

*where*

$$\alpha_1 \quad \textit{and} \quad \alpha_2 \quad (0 < \alpha_1, \alpha_2 \leqq 1)$$

*are the solutions of Eqs.* (3.1) *and* (3.2) *with* $\mu = \lambda$.

From Theorem 1 (or Corollary 3) and Theorem 2, we immediately obtain the following inclusion relationships.

**Corollary 4.** *Let*

$$\frac{1+A}{1+B} < \min(\lambda + 1, \mu + 1) \quad (\lambda, \mu > 0;\ -1 < B < A \leqq 1).$$

*Then the following inclusion relationships hold true*:

$$\Sigma^n_{\lambda,\mu+1}(l; A, B; \alpha) \subset \Sigma^n_{\lambda,\mu}(l; A, B; \alpha) \subset \Sigma^{n+1}_{\lambda,\mu}(l; A, B; \alpha).$$

Next, we prove the following argument property.

**Theorem 3.** *Let* $0 < \delta_1, \delta_2 \leqq 1$ *and*

$$1 + A < (c + 1)(1 + B) \quad (c > 0;\ -1 < B < A \leqq 1).$$

*If a function* $f \in \Sigma$ *satisfies the following two-sided inequality*:

$$-\frac{\pi}{2}\delta_1 < \arg\left(-\frac{z\big(\mathcal{I}^n_{\lambda,\mu} f(z)\big)'}{\frac{1}{l}\sum_{j=1}^{l} \mathcal{I}^n_{\lambda,\mu} g_j(z)}\right) < \frac{\pi}{2}\delta_2,$$

*where*

$$g_k \in \Sigma^n_{\lambda,\mu}(l; A, B) \quad (k = 1, \ldots, l),$$

*then*

$$-\frac{\pi}{2}\alpha_1 < \arg\left(-\frac{z\big(\mathcal{I}^n_{\lambda,\mu} F_c(f)(z)\big)'}{\frac{1}{l}\sum_{j=1}^{l} \mathcal{I}^n_{\lambda,\mu} F_c(g_j)(z)}\right) < \frac{\pi}{2}\alpha_2,$$

*where* $F_c$ *is the integral operator defined by* (2.6), *and*

$$\alpha_1 \quad \textit{and} \quad \alpha_2 \quad (0 < \alpha_1, \alpha_2 \leqq 1)$$

*are the solutions of Eqs.* (3.1) *and* (3.2) *with* $\mu = c$.

**Proof.** Let

$$p(z) = -\frac{z\big(\mathcal{I}^n_{\lambda,\mu} F_c(f)(z)\big)'(z)}{\frac{1}{l}\sum_{j=1}^{l} \mathcal{I}^n_{\lambda,\mu} F_c(g_j)(z)} \quad \text{and} \quad q(z) = \frac{1}{n}\sum_{k=1}^{n} q_k(z),$$

where

$$q_k(z) = -\frac{z\big(\mathcal{I}^n_{\lambda,\mu} F_c(g_k)\big)'(z)}{\frac{1}{l}\sum_{j=1}^{l} \mathcal{I}^n_{\lambda,\mu} F_c(g_j)(z)} \quad (k = 1, \ldots, l).$$

Using the identity (2.8), we obtain

$$\frac{1}{l}\sum_{j=1}^{l}\big(\mathcal{I}^n_{\lambda,\mu} F_c(g_j)(z)\big)p(z) - (c+1)\mathcal{I}^n_{\lambda,\mu} F_c(f)(z) = -c\mathcal{I}^n_{\lambda,\mu} f(z). \tag{3.7}$$

By differentiating both sides of (3.7) with respect to $z$, and then simplifying, we get

$$-\frac{z\big(\mathcal{I}^n_{\lambda,\mu} f(z)\big)'}{\frac{1}{l}\sum_{j=1}^{l} \mathcal{I}^n_{\lambda,\mu} g_j(z)} = p(z) + \frac{zp'(z)}{-q(z)+c+1}.$$

Since

$$g_k \in \Sigma^n_{\lambda,\mu}(l; A, B) \quad (k = 1, \ldots, l),$$

by Proposition 2, we know that

$$F_c(g_k) \in \Sigma^n_{\lambda,\mu}(l; A, B) \quad (k = 1, \ldots, l).$$

Hence we find that

$$q(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{U};\ -1 < B < A \leqq 1).$$

The remaining part of the proof of Theorem 3 is similar to that in the proof of Theorem 1, and so we omit the details involved. □

For the special case when $\delta_1 = \delta_2$, Theorem 3 reduces to the following form.

**Corollary 5.** *Let* $0 < \delta \leqq 1$ *and*

$$1 + A < (c+1)(1+B) \quad (c > 0;\ -1 < B < A \leqq 1).$$

*If a function* $f \in \Sigma$ *satisfies the following inequality*:

$$\left|\arg\left(-\frac{z\big(\mathcal{I}^n_{\lambda,\mu} f(z)\big)'}{\frac{1}{l}\sum_{j=1}^{l} \mathcal{I}^n_{\lambda,\mu} g_j(z)}\right)\right| < \frac{\pi}{2}\delta,$$

*where*

$$g_k \in \Sigma^n_{\lambda,\mu}(l; A, B) \quad (k = 1, \ldots, l),$$

*then*

$$\left|\arg\left(-\frac{z\big(\mathcal{I}^n_{\lambda,\mu} F_c(f)(z)\big)'}{\frac{1}{l}\sum_{j=1}^{l} \mathcal{I}^n_{\lambda,\mu} F_c(g_j)(z)}\right)\right| < \frac{\pi}{2}\alpha,$$

*where* $F_c$ *is the integral operator defined by* (2.6)*, and* $\alpha$ $(0 < \alpha \leqq 1)$ *is the solution of Eq.* (3.6) *with* $\mu = c$.

**Remark 2.** Corollary 6 below is an obvious consequence of Corollary 5.

**Corollary 6.** *Let*

$$1 + A < (c+1)(1+B) \quad (c > 0;\ -1 < B < A \leqq 1).$$

*Then*

$$f \in \Sigma^n_{\lambda,\mu}(l; A, B; \alpha) \quad \Longrightarrow \quad F_c(f) \in \Sigma^n_{\lambda,\mu}(l; A, B; \alpha),$$

*where* $F_c$ *is the integral operator defined by* (2.6).

**Remark 3.** From Theorem 3 or Corollary 6, we observe that every function $f$ in $\Sigma^n_{\lambda,\mu}(l; A, B; \alpha)$ preserves the angles under the integral operator defined by (2.6). If we put

$$n = \lambda = l = \alpha = 1, \qquad \mu = 2, \quad \text{and} \quad B \mapsto A$$

in Corollary 6, we obtain the result given earlier by Goel and Sohi [5].

Finally, we state Theorem 4 below, the proof of which is much akin to that of Theorem 1.

**Theorem 4.** *Let* $0 < \delta_1, \delta_2 \leqq 1$, $\gamma \geqq 0$ *and*

$$1 + A < (\mu+1)(1+B) \quad (\mu > 0;\ -1 < B < A \leqq 1).$$

*If a function* $f \in \Sigma$ *satisfies the following two-sided inequality*:

$$-\frac{\pi}{2}\delta_1 < \arg\left(-\left[\gamma \frac{z(\mathcal{I}^n_{\lambda,\mu+1} f(z))'}{\frac{1}{l}\sum_{j=1}^{l} \mathcal{I}^n_{\lambda,\mu+1} g_j(z)} + (1-\gamma)\frac{z(\mathcal{I}^n_{\lambda,\mu} f(z))'}{\frac{1}{l}\sum_{j=1}^{l} \mathcal{I}^n_{\lambda,\mu} g_j(z)}\right]\right) < \frac{\pi}{2}\delta_2,$$

*where*

$$g_k \in \Sigma^n_{\lambda,\mu+1}(l; A, B) \quad (k = 1, \ldots, l),$$

*then*

$$-\frac{\pi}{2}\alpha_1 < \arg\left(-\frac{z(\mathcal{I}^n_{\lambda,\mu} f(z))'}{\frac{1}{l}\sum_{j=1}^{l} \mathcal{I}^n_{\lambda,\mu} g_j(z)}\right) < \frac{\pi}{2}\alpha_2,$$

*where* $\alpha_1$ *and* $\alpha_2$ $(0 < \alpha_1, \alpha_2 \leqq 1)$ *are the solutions of the following equations*:

$$\delta_1 = \alpha_1 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|b|)\gamma\cos(\frac{\pi}{2}t_1)}{2(\frac{A-1}{1-B} + \mu + 1)(1+|b|) + (\alpha_1+\alpha_2)(1-|b|)\sin(\frac{\pi}{2}t_1)}\right) \tag{3.8}$$

*and*

$$\delta_2 = \alpha_2 + \frac{2}{\pi}\tan^{-1}\left(\frac{(\alpha_1+\alpha_2)(1-|b|)\gamma\cos(\frac{\pi}{2}t_1)}{2(\frac{A-1}{1-B} + \mu + 1)(1+|b|) + (\alpha_1+\alpha_2)(1-|b|)\sin(\frac{\pi}{2}t_1)}\right) \tag{3.9}$$

*when* $b$ *and* $t_1$ *are given by* (2.3) *and* (2.12)*, respectively.*

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## References

[1] S.K. Bajpai, A note on a class of meromorphic univalent functions, Rev. Roumaine Math. Pures Appl. 22 (1977) 295–297.
[2] R. Bharati, R. Rajagopal, Meromorphic functions and differential subordination, in: S. Ponnusamy (Ed.), New Trends in Geometric Function Theory and Applications, World Scientific, Singapore, 1990, pp. 10–17.
[3] J.H. Choi, M. Saigo, H.M. Srivastava, Some inclusion properties of a certain family of integral operators, J. Math. Anal. Appl. 276 (2002) 432–445.
[4] P. Eenigenberg, S.S. Miller, P.T. Mocanu, M.O. Reade, On a Briot–Bouquet differential subordination, in: General Inequalities, vol. 3, Oberwolfach, 1981, in: Internat. Ser. Numer. Math., vol. 64, Birkhäuser, Basel, 1983, pp. 339–348; see also Rev. Roumaine Math. Pures Appl. 29 (1984) 567–573.
[5] R.M. Goel, N.S. Sohi, On a class of meromorphic functions, Glasnik Mat. Ser. III 17 (1981) 19–28.
[6] A.W. Goodman, Univalent Functions, vols. I and II, Mariner, Tampa, 1983.
[7] R.J. Libera, M.S. Robertson, Meromorphic close-to-convex functions, Michigan Math. J. 8 (1961) 167–176.
[8] J.-L. Liu, The Noor integral and strongly starlike functions, J. Math. Anal. Appl. 261 (2001) 441–447.
[9] J.-L. Liu, K.I. Noor, Some properties of Noor integral operator, J. Natur. Geom. 21 (2002) 81–90.
[10] S.S. Miller, P.T. Mocanu, Differential subordinations and univalent functions, Michigan Math. J. 28 (1981) 157–171.
[11] S.S. Miller, P.T. Mocanu, Differential Subordination: Theory and Applications, Monogr. Textbooks Pure Appl. Math., vol. 225, Dekker, New York, 2000.
[12] K.I. Noor, On new classes of integral operators, J. Natur. Geom. 16 (1999) 71–80.
[13] K.I. Noor, M.A. Noor, On integral operators, J. Math. Anal. Appl. 238 (1999) 341–352.
[14] K.S. Padmanabhan, R. Parvatham, On analytic functions and differential subordination, Bull. Math. Soc. Sci. Math. R. S. Roumanie 31 (1987) 237–248.
[15] G.S. Sălăgean, Integral operators and meromorphic functions, Rev. Roumaine Math. Pures Appl. 33 (1999) 135–140.
[16] S.M. Sarangi, S.B. Uralegaddi, Certain differential operators for meromorphic functions, Bull. Calcutta Math. Soc. 88 (1996) 333–336.
[17] H. Silverman, E.M. Silvia, Subclasses of starlike functions subordinate to convex functions, Canad. J. Math. 37 (1985) 48–61.
[18] R. Singh, Meromorphic close-to-convex functions, J. Indian Math. Soc. (N.S.) 33 (1969) 13–20.
[19] N. Takahashi, M. Nunokawa, A certain connection between starlike and convex functions, Appl. Math. Lett. 16 (2003) 653–655.
[20] B.A. Uralegaddi, C. Somanatha, New criteria for meromorphic starlike functions, Bull. Austral. Math. Soc. 43 (1991) 137–140.
[21] B.A. Uralegaddi, C. Somanatha, Certain differential operators for meromorphic functions, Houston J. Math. 17 (1991) 279–284.