Available online at www.sciencedirect.com
SCIENCE DIRECT®

J. Math. Anal. Appl. 311 (2005) 683–702

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Well posedness of balance laws with boundary

Rinaldo M. Colombo *, Massimiliano D. Rosini

Department of Mathematics, Brescia University, Via Branze 38, 25133 Brescia, Italy

Received 30 November 2004

Available online 2 April 2005

Submitted by G. Chen

Abstract

Consider the initial-boundary value problem for a Temple system of balance laws. Aim of this paper is to prove the well posedness of this problem for large times and without requiring the total variation of the initial data be small.

Keywords: Balance laws; Initial boundary value problem for conservation laws

1. Introduction

This paper is devoted to the well posedness of the following initial-boundary value problem for a nonlinear system of balance laws:

$$\begin{cases} \partial_t u + \partial_x f(u) = g(t,x,u), & (t,x) \in \Omega, \\ u(t_0,x) = \bar{u}(x), & x \geqslant \Psi(t_0), \\ u(t,\Psi(t)) = \tilde{u}(t), & t \geqslant t_0, \end{cases} \tag{1.1}$$

where $t_0 \in \mathbb{R}$, $\Omega = \{(t,x) \in \mathbb{R}^2\colon\ t \geqslant t_0 \text{ and } x \geqslant \Psi(t)\}$ and u denotes the unknown vector function. The present result extends and unifies those obtained in [12,13].

* Corresponding author.
E-mail address: rinaldo@ing.unibs.it (R.M. Colombo).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2005.03.008

We consider the conservation law with boundary

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & (t,x) \in \Omega, \\ u(t_0, x) = \bar{u}(x), & x \geqslant \Psi(t_0), \\ u(t, \Psi(t)) = \tilde{u}(t), & t \geqslant t_0, \end{cases} \tag{1.2}$$

and the source part

$$\begin{cases} \partial_t u = g(t,x,u), & (t,x) \in \Omega, \\ u(t_0, x) = \bar{u}(x), & x \geqslant \Psi(t_0), \\ u(t, \Psi(t)) = \tilde{u}(t), & t \geqslant t_0, \end{cases} \tag{1.3}$$

separately. Indeed, the well posedness of (1.1) is proved below under those assumptions on f, respectively on g, that make (1.2), respectively (1.3), well posed. Besides, we require a sort of compatibility between the conservation law (1.2) and the ordinary differential equation (1.3). Namely we ask that there exists a domain which is invariant for both (1.2) and (1.3).

This assumption replaces other compatibility conditions (dissipativity, diagonal dominance) found in the literature, see [16, §13.8] for a survey of related results. On the other hand, in the present setting, the total variation and the $\mathbf{L}^\infty$ norm of the solution may well grow exponentially with time, see (6) in Theorem 2.3. Aiming at the well posedness on the whole time interval $[t_0, +\infty[$ we necessarily require on (1.2) hypotheses that ensure the well posedness for large data. Therefore, in view of [5,6,12], we assume that (1.2) is a Temple system.

A further motivation for the present result is given by several traffic flow models, see [3,4,9,10]. Indeed, macroscopic continuum models are often stated through conservation laws. The role of source terms is then justified by the presence of entries/exits or by inhomogeneities in the road, see [4].

We follow here the definition of solution to the boundary value problem (1.2) proposed in [17]. This approach is completely independent from the choice of any viscosity operator and is questionable in the case of gas dynamics, where the role of the boundary layer can hardly be neglected. On the contrary, in traffic models the boundary is usually the first entry to a highway and no boundary layer seem to play any role.

2. Preliminaries and main result

We introduce the following assumptions on the convective part (1.2):

(F) Let $\mathcal{U}$ be the closure of an open subset of $\mathbb{R}^n$, $f : \mathcal{U} \mapsto \mathbb{R}^n$ be smooth and such that $\partial_t u + \partial_x f(u) = 0$ is a Temple system, i.e.
- (F_1) The system is strictly hyperbolic in $\mathcal{U}$, i.e. the Jacobian Df has n real eigenvalues $\lambda_1, \ldots, \lambda_n$ and $\sup_{u \in \mathcal{U}} \lambda_i(u) < \inf_{u \in \mathcal{U}} \lambda_{i+1}(u)$, for all $i = 1, \ldots, n-1$.
- (F_2) For $i = 1, \ldots, n$, the i-shock curve and the i-rarefaction curve coincide.
- (F_3) In $\mathcal{U}$, there exists a system of Riemann coordinates $\{w_1, \ldots, w_n\}$, such that $\frac{\partial u}{\partial w_i}$ is parallel to r_i, r_i being the right eigenvector corresponding to λ_i, for $i = 1, \ldots, n$.

Recall the definition of solution to the convective problem (1.2), see [1,17]. Below, we let $u(x\pm) = \lim_{\xi \to x^\pm} u(\xi)$ for any function $u \in \mathbf{BV}(\mathbb{R}, \mathbb{R}^n)$.

Definition 2.1. Let $u : \Omega \mapsto \mathcal{U}$ be such that for a.e. $t \in [t_0, +\infty[$, $x \mapsto u(t,x)$ is in $\mathbf{BV}([\Psi(t), +\infty[, \mathbb{R}^n)$. u solves the convective problem (1.2) if

(1) it is a weak entropic solution to (1.2) in Ω,
(2) it coincides with $\bar{u}$ at time $t = t_0$,
(3) it satisfies the boundary condition: for a.e. $\tau \geqslant t_0$, $u(\tau, \Psi(\tau)+) = w(t,x)$ for all $(t,x) \in \Omega$ such that

$$\begin{cases} x - \Psi(\tau) > D_-\Psi(\tau) \cdot (t - \tau), \\ t > \tau, \end{cases}$$

where w is the self-similar Lax solution to the Riemann problem

$$\begin{cases} \partial_t w + \partial_x f(w) = 0, & t \geqslant \tau,\ x \in \mathbb{R}, \\ w(\tau, x) = \begin{cases} \tilde{u}(\tau) & \text{if } x < \Psi(\tau), \\ u(\tau, \Psi(\tau)+) & \text{if } x > \Psi(\tau). \end{cases} \end{cases}$$

$D_-\Psi(t) = \liminf_{h \to 0^-} \frac{\Psi(t+h) - \Psi(t)}{h}$ is the lower left Dini derivative. At (1), for the definition of weak entropic solution see [7,11] or Definition 2.4 below.

On (1.3) we assume (here, $|\cdot|$ denotes the norm (2.2) in $\mathbb{R}^n$):

(G) The source term $g : [t_0, +\infty[\times \mathbb{R} \times \mathcal{U} \mapsto \mathbb{R}^n$ is such that
- (G_1) For a.e. $t \in [t_0, +\infty[$ and all $x \in \mathbb{R}$, $g(t,x,0) = 0$.
- (G_2) For all $(x,u) \in \mathbb{R} \times \mathcal{U}$ the map $t \mapsto g(t,x,u)$ is measurable.
- (G_3) For a.e. $t \in [t_0, +\infty[$ and all $u \in \mathcal{U}$, the map $x \mapsto g(t,x,u)$ is uniformly $\mathbf{BV}(\mathbb{R}, \mathbb{R}^n)$, i.e. there exists a finite positive measure μ such that for a.e. $t \in [t_0, +\infty[$, for all $x_1, x_2 \in \mathbb{R}$ with $x_1 \leqslant x_2$ and for all $u \in \mathcal{U}$,

$$\bigl|g(t, x_2+, u) - g(t, x_1-, u)\bigr| \leqslant \mu\bigl([x_1, x_2]\bigr).$$

- (G_4) For a.e. $t \in [t_0, +\infty[$ and $x \in \mathbb{R}$, the map $u \mapsto g(t,x,u)$ is locally Lipschitz and sublinear in $\mathcal{U}$, i.e. for every compact subset K of $\mathcal{U}$, there exists a function $l_K \in \mathbf{L}^\infty_{\mathrm{loc}}([t_0, +\infty[, \mathbb{R})$ such that for a.e. $t \in [t_0, +\infty[$, all $x \in \mathbb{R}$ and all $u_1, u_2 \in K$,

$$\bigl|g(t,x,u_2) - g(t,x,u_1)\bigr| \leqslant l_K(t) \cdot |u_2 - u_1|$$

and there exists a function $l \in \mathbf{L}^1_{\mathrm{loc}}([t_0, +\infty[, \mathbb{R})$ such that for a.e. $t \in [t_0, +\infty[$, all $x \in \mathbb{R}$ and all $u \in \mathcal{U}$,

$$\bigl|g(t,x,u)\bigr| \leqslant l(t) \cdot |u|.$$

Assumption (G_1) ensures that $\mathbf{L}^1$ is invariant with respect to (1.3); in [4] this assumption is relaxed. Given $(t,x) \in \Omega$ we define

$$\alpha(t,x) = \inf\bigl\{s > t_0\colon \bigl(\theta s + (1-\theta)t, x\bigr) \in \Omega,\ \forall \theta \in [0,1]\bigr\}. \tag{2.1}$$

Introduce the following definition of solution to (1.3).

Definition 2.2. By solution to (1.3) we mean a map $u : \Omega \mapsto \mathcal{U}$ such that for all $(\tau, x) \in \Omega$ the map $t \mapsto u(t, x)$ is an absolutely continuous Carathéodory solution [18, §1] of

$$\begin{cases} \partial_t u = g(t, x, u), & t \in \left]\alpha(\tau, x), \tau\right[, \\ u(\alpha(\tau, x), x) = \begin{cases} \tilde{u}(\alpha(\tau, x)) & \text{if } \alpha(\tau, x) > t_0, \\ \bar{u}(x) & \text{if } \alpha(\tau, x) = t_0. \end{cases} \end{cases}$$

Here, the role of the boundary condition is analogous to that in Definition 2.1. Indeed, as it is usual, we consider the source as generating waves with 0 speed. Therefore, the trace $u(t, \psi(t)+)$ of the solution on the boundary of Ω may differ from the boundary data $\tilde{u}(t)$ only at those points $(t, \psi(t))$ where the boundary has positive speed. In the following, we let

$$\begin{aligned} &|v| = \max_{i=1,\dots,n} |v_i| && \text{for } v \in \mathbb{R}^n, \\ &\|u\| = \left|w(u)\right| && \text{for } u \in \mathcal{U}, \\ &\mathrm{TV}(u) = \textstyle\sum_{i=1}^n \text{ total variation of } w_i\big(u(\cdot)\big) && \text{for } u : \mathbb{R} \mapsto \mathcal{U}. \end{aligned} \tag{2.2}$$

On any compact subset of $\mathcal{U}$, $\|\cdot\|$ (and, respectively $\mathrm{TV}(\cdot)$) is equivalent to the usual Euclidean norm (respectively total variation) because of (F_3).

It is useful to consider the set $\mathcal{D}_t$ of triples $(\bar{u}, \tilde{u}, \Psi)$, where $t \geqslant t_0$ and

$$\begin{aligned} &\bar{u} \in \mathbf{L}^1\big(\big[\Psi(t), +\infty\big[, \mathcal{U}\big) \cap \mathbf{BV}\big(\big[\Psi(t), +\infty\big[, \mathcal{U}\big), \\ &\tilde{u} \in \mathbf{L}^1\big([t, +\infty[, \mathcal{U}\big) \cap \mathbf{BV}\big([t, +\infty[, \mathcal{U}\big), \\ &\Psi \in \mathbf{C}^0\big([t, +\infty[, \mathbb{R}\big). \end{aligned}$$

For $M > 0$, introduce for later use the set

$$\mathcal{D}_{t,M} = \left\{(\bar{u}, \tilde{u}, \Psi) \in \mathcal{D}_t \colon \mathrm{TV}(\bar{u}) + \left\|\bar{u}\big(\Psi(t)\big) - \tilde{u}(t)\right\| + \mathrm{TV}(\tilde{u}) \leqslant M\right\}.$$

As in [12], we further require a sort of *compatibility* between the convective part (1.2) and the source term (1.3).

(U) (U_1) The set $\mathcal{U}$ is invariant with respect to (1.2).
    (U_2) The set $\mathcal{U}$ is invariant with respect to (1.3).

In (U_1), invariance means that any data $(\bar{u}, \tilde{u}, \Psi)$ with values in $\mathcal{U}$ leads to a solution u to (1.2) valued in $\mathcal{U}$. Equivalently, if $\bar{u}([\Psi(t_0), +\infty[) \subseteq \mathcal{U}$ and $\tilde{u}([t_0, +\infty[) \subseteq \mathcal{U}$, then $u(\Omega) \subseteq \mathcal{U}$. Recall that a closed set $\mathcal{U}$ is invariant with respect to (1.2) if and only if any Riemann problem with data in $\mathcal{U}$ yields a solution attaining values in $\mathcal{U}$. For a treatment of invariant domains for conservation laws, we refer to [19], where a necessary and sufficient condition for the invariance of $\mathcal{U}$ is proved. Due to this condition, in the present case (U_1) could be replaced by the assumption that the boundary of $\mathcal{U}$ be the juxtaposition of Lax curves.

Similarly, in (U_2), invariance means that any data $(\bar{u}, \tilde{u}, \Psi)$ valued in $\mathcal{U}$ yields a solution u to (1.3) with values in $\mathcal{U}$. Therefore, (U_2) could be replaced by the classical Nagumo condition, see [20], stating that g needs to point towards $\mathcal{U}$ all along the boundary $\partial\mathcal{U}$ of $\mathcal{U}$.

Remark that, in both cases, $\mathcal{U}$ needs neither be convex nor compact in the u coordinates.

Below we show that (1.1) generates a process F,

$$F:\{(\mathbf{p},t_1,t_2):\ \mathbf{p}\in\mathcal{D}_{t_1},\ t_2\geqslant t_1\geqslant t_0\}\mapsto\bigcup_{t\geqslant t_0}\mathcal{D}_t,$$
$$\big((\bar u,\tilde u,\Psi),t_1,t_2\big)\mapsto\big(u(t_2),\mathcal{T}_{t_2-t_1}\tilde u,\mathcal{T}_{t_2-t_1}\Psi\big),$$

u being a solution to (1.1) with data $(\bar u,\tilde u,\Psi)$ at time t_1, and $\mathcal{T}_t$ being the translation operator, i.e. $(\mathcal{T}_t\tilde u)(s)=\tilde u(t+s)$ and $(\mathcal{T}_t\Psi)(s)=\Psi(t+s)$. Moreover, if $\mathbf{p}\in\mathcal{D}_{t_1}$, then $F(\mathbf{p},t_1,t_2)\in\mathcal{D}_{t_2}$.

We are now ready to state the main result of this paper.

Theorem 2.3. *Let* (1.1) *satisfy assumptions* (F), (G) *and* (U). *Then, there exists a unique evolution operator F with the properties*:

(1) *For all $t\in[t_0,+\infty[$ and $(\bar u,\tilde u,\Psi)\in\mathcal{D}_{t_0}$ the function $u:\Omega\mapsto\mathcal{U}$ defined by $(u(t,\cdot),\mathcal{T}_{t-t_0}\tilde u,\mathcal{T}_{t-t_0}\Psi)=F((\bar u,\tilde u,\Psi),t_0,t)$ is a weak entropic solution to* (1.1).
(2) *For all t_1,t_2,t_3 with $t_3\geqslant t_2\geqslant t_1\geqslant t_0$, $F(F(\mathbf{p},t_1,t_2),t_2,t_3)=F(\mathbf{p},t_1,t_3)$ for all $\mathbf{p}\in\mathcal{D}_{t_1}$, while for all $t\geqslant t_0$, $F(\mathbf{p},t,t)=\mathbf{p}$ for all $\mathbf{p}\in\mathcal{D}_t$.*
(3) *If $\bar u$ and $\tilde u$ are piecewise constant and if Ψ is piecewise linear and continuous, then for small times the corresponding solution u coincides with the function obtained by piecing together the solutions to the Riemann problems on the points of jump of $\bar u$ and at $(t_0,\Psi(t_0))$.*

Moreover, for every $T,M>0$, there exist constants L,C such that

(5) *Fix two triples $(\bar u,\tilde u,\Psi)$ and $(\bar u',\tilde u',\Psi')$ in $\mathcal{D}_{t_0,M}$ and call u,u' the solutions to* (1.1) *yielded by F.*
   (a) *If $\tilde u=\tilde u'$ then, for any $t\in[t_0,T]$,*
   $$\big\|u(t)-u'(t)\big\|_{\mathbf{L}^1}\leqslant L\cdot\big(\|\bar u-\bar u'\|_{\mathbf{L}^1}+\|\Psi-\Psi'\|_{\mathbf{C}^0}\big).$$
   (b) *If Ψ,Ψ' are Lipschitz with constants $\mathcal{L},\mathcal{L}'$ and $t,t'\in[t_0,T]$, then*
   $$\begin{aligned}\big\|u(t)-u'(t')\big\|_{\mathbf{L}^1}\leqslant{}& L\cdot\big(\|\bar u-\bar u'\|_{\mathbf{L}^1}+\|\Psi-\Psi'\|_{\mathbf{C}^0}\big)\\&+L\cdot(1+\mathcal{L}+\mathcal{L}')\big(\|\tilde u-\tilde u'\|_{\mathbf{L}^1}+|t-t'|\big).\end{aligned}$$
(6) *For any data $(\bar u,\tilde u,\Psi)\in\mathcal{D}_{t_0,M}$, the solution yielded by F satisfies*
$$\begin{aligned}\big\|u(t)\big\|_{\mathbf{L}^\infty}&\leqslant e^{C\int_{t_0}^t l(\tau)\,d\tau}\big(\|\bar u\|_{\mathbf{L}^\infty}+\|\tilde u\|_{\mathbf{L}^\infty}\big),\\ \mathrm{TV}\big(u(t)\big)&\leqslant e^{C(t-t_0)}\big(\mathrm{TV}(\bar u)+\big\|\bar u\big(\Psi(t_0)\big)-\tilde u(t_0)\big\|+\mathrm{TV}(\tilde u)\big)\\&\quad+Ce^{C(t-t_0)}\mu(\mathbb{R})(t-t_0).\end{aligned}$$
(7) *If $\mathcal{U}$ is compact, then C does not depend on T.*

We recall the definition of weak solution to (1.1) and to the corresponding Riemann problem.

Definition 2.4. $u : \Omega \mapsto \mathcal{U}$ is a solution of the problem (1.1) if

(1) for any function $\varphi \in \mathbf{C}_c^\infty(\Omega \cup\,]-\infty, t_0[\, \times \mathbb{R})$,

$$\int_{t_0}^{+\infty} \int_{\Psi(t)}^{+\infty} \left[\partial_t \varphi(t,x) u(t,x) + \partial_x \varphi(t,x) f\big(u(t,x)\big)\right] dx\, dt + \int_{t_0}^{+\infty} \int_{\Psi(t)}^{+\infty} \varphi(t,x) g\big(t,x,u(t,x)\big)\, dx\, dt + \int_{\Psi(t_0)}^{+\infty} \varphi(t_0,x) \bar{u}(x)\, dx = 0,$$

(2) for a.e. $\tau \in [t_0, +\infty[$, the Riemann problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & t \geqslant \tau,\ x \in \mathbb{R}, \\ u(\tau, x) = \begin{cases} \tilde{u}(\tau) & \text{if } x < \Psi(\tau), \\ u(\tau, \Psi(\tau)+) & \text{if } x > \Psi(\tau), \end{cases} \end{cases}$$

admits a solution with waves all slower than the boundary at τ, in the sense of (iii) in Definition 2.1.

Given an entropy–entropy flux pair (η, q) (see [7,12,16]), the weak solution u is *entropic* if for any $\varphi \in \mathbf{C}_c^\infty(\Omega \cup\,]-\infty, t_0[\, \times \mathbb{R})$, with $\varphi \geqslant 0$,

$$\int_{t_0}^{+\infty} \int_{\Psi(t)}^{+\infty} \left[\partial_t \varphi(t,x) \eta\big(u(t,x)\big) + \partial_x \varphi(t,x) q\big(u(t,x)\big)\right] dx\, dt + \int_{t_0}^{+\infty} \int_{\Psi(t)}^{+\infty} \varphi(t,x) D\eta\big(u(t,x)\big) g\big(t,x,u(t,x)\big)\, dx\, dt + \int_{\Psi(t_0)}^{+\infty} \eta\big(\varphi(t_0,x)\big) u_0(x)\, dx \geqslant 0.$$

Definition 2.5. Fix m in $\mathbb{R}$ and let $\Omega = \{(t,x) \in \mathbb{R}^2\colon\ t \geqslant 0,\ x \geqslant mt\}$. Let $\bar{u}, \tilde{u}$ in $\mathcal{U}$ be fixed. The solution to the Riemann problem with boundary

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & (t,x) \in \Omega, \\ u(0,x) = \bar{u}, & x \geqslant 0, \\ u(t, mt) = \tilde{u}, & t \geqslant 0, \end{cases} \tag{2.3}$$

is the restriction to Ω of the Lax solution to the standard Riemann problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & (t,x) \in [0, +\infty[\, \times \mathbb{R}, \\ u(0,x) = \begin{cases} \tilde{u} & \text{if } x < 0, \\ \bar{u} & \text{if } x > 0. \end{cases} \end{cases} \tag{2.4}$$

Remark that if the boundary is not a straight line, then the restriction to Ω of the solution to (2.4) not necessarily solves (2.3) in the sense of Definition 2.1.

3. Technical proofs

In this section, ε is sufficiently small and fixed, all estimates being uniform in ε. $\bar{\mathcal{L}}$ denotes an upper bound for the Lipschitz constants of all the boundaries Ψ. The limit $\varepsilon \to 0$ and the case of a continuous boundary will be considered only in the final part of the section. Below, we write $\mathcal{D}$ instead of $\mathcal{D}_t$ for notational simplicity.

3.1. *The convective part*

We let $t_0 = 0$ throughout this paragraph.

Following [12], we introduce an ε-grid in $w(\mathcal{U})$. More precisely, by (U) we know that $w(\mathcal{U})$ is the Cartesian product of closed possibly unbounded intervals: $w(\mathcal{U}) = \prod_{i=1}^n \mathcal{I}_i$. For all i, introduce in each $\mathcal{I}_i$ a finite set $\mathcal{I}_i^\varepsilon$ with the properties

(i) $]w_i - \varepsilon, w_i + \varepsilon[\cap \mathcal{I}_i^\varepsilon \neq \emptyset$ for any $w_i \in \mathcal{I}_i$;

(ii) there exists a positive δ^ε such that $\min_{w_i', w_i'' \in \mathcal{I}_i^\varepsilon,\, w_i' \neq w_i''} |w_i' - w_i''| > \delta^\varepsilon$;

(iii) $$\min \mathcal{I}_i^\varepsilon = \begin{cases} -1/\varepsilon & \text{if } \inf \mathcal{I}_i = -\infty, \\ \inf \mathcal{I}_i & \text{if } \inf \mathcal{I}_i \in \mathcal{I}_i, \\ \inf \mathcal{I}_i + \varepsilon & \text{if } \inf \mathcal{I}_i \notin \mathcal{I}_i, \end{cases}$$

$$\max \mathcal{I}_i^\varepsilon = \begin{cases} 1/\varepsilon & \text{if } \sup \mathcal{I}_i = \infty, \\ \sup \mathcal{I}_i & \text{if } \sup \mathcal{I}_i \in \mathcal{I}_i, \\ \sup \mathcal{I}_i - \varepsilon & \text{if } \sup \mathcal{I}_i \notin \mathcal{I}_i. \end{cases}$$

Finally, we call the set $\mathcal{G}^\varepsilon = \prod_{i=1}^n \mathcal{I}_i^\varepsilon$ an ε*-grid* in $w(\mathcal{U})$.

Fix an ε-grid $\mathcal{G}^\varepsilon$. As in [6,12], we consider Riemann problems

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(0,x) = \begin{cases} u^l & \text{if } x < 0, \\ u^r & \text{if } x > 0, \end{cases} \end{cases} \tag{3.1}$$

with data $w^l = w(u^l)$ and $w^r = w(u^r)$ in $\mathcal{G}^\varepsilon$. Introduce the states $u^0, \ldots, u^n$ through their Riemann coordinates $w^0, \ldots, w^n$ as follows:

$$\begin{array}{lll} u^0 = u^l & : & w^0 = \left(w_1^l, w_2^l, \ldots, w_{n-1}^l, w_n^l\right), \quad \text{i.e. } w^0 = w(u^l), \\ u^1 & : & w^1 = \left(w_1^r, w_2^l, \ldots, w_{n-1}^l, w_n^l\right), \\ \vdots & & \\ u^{n-1} & : & w^{n-1} = \left(w_1^r, w_2^r, \ldots, w_{n-1}^r, w_n^l\right), \\ u^n = u^r & : & w^n = \left(w_1^r, w_2^r, \ldots, w_{n-1}^r, w_n^r\right), \quad \text{i.e. } w^n = w(u^r). \end{array} \tag{3.2}$$

Note that $w^i \in \mathcal{G}^\varepsilon$ for $i = 0, \ldots, n$.

The exact weak entropic solution to (3.1) is the juxtaposition of the n solutions to the n scalar Riemann problems

$$\begin{cases} \partial_t s_i + \partial_x f_i(u^{i-1}; s_i) = 0, \\ s_i(0,x) = \begin{cases} 0 & \text{if } x < 0, \\ \sigma_i & \text{if } x > 0, \end{cases} \end{cases}$$

where $f_i(u^{i-1}; s) = \int_0^s \lambda_i(\mathcal{L}_i(u^{i-1}, s))\,ds$ and σ_i satisfies

$$\mathcal{L}_i(u^{i-1}, \sigma_i) = u^i \quad \text{for } i = 1, \ldots, n, \tag{3.3}$$

where we denoted by $\sigma \mapsto \mathcal{L}_i(u_0, \sigma)$ the ith generalized Lax curve exiting u_0, parameterized through the signed arc length σ.

We now aim at the definition of a piecewise constant weak solution to (3.1), whose entropy defect is $\mathcal{O}(\varepsilon)$. Let $s \mapsto f_i^\varepsilon(u^{i-1}; s)$ be the piecewise linear function that coincides with $s \mapsto f_i(u^{i-1}; s)$ on $\mathcal{G}^\varepsilon$.

A weak, possibly nonentropic, solution to (3.1) is obtained gluing the n (weak entropic) exact solutions to the n (approximate) Riemann problems

$$\begin{cases} \partial_t s_i + \partial_x f_i^\varepsilon(u^{i-1}; s_i) = 0, \\ s_i(0, x) = \begin{cases} 0 & \text{if } x < 0, \\ \sigma_i & \text{if } x > 0, \end{cases} \end{cases} \tag{3.4}$$

where σ_i is defined in (3.3). Let

$$\mathcal{D}_M(\mathcal{G}^\varepsilon) = \left\{ (\bar{u}, \tilde{u}, \Psi) \in \mathcal{D}\colon \begin{cases} (\bar{u}, \tilde{u}, \Psi) \in \mathbf{PC} \times \mathbf{PC} \times \mathbf{PLC}, \\ w(\bar{u})(\mathbb{R}) \subseteq \mathcal{G}^\varepsilon,\ w(\tilde{u})(\mathbb{R}) \subseteq \mathcal{G}^\varepsilon, \\ \mathrm{TV}(\bar{u}, \tilde{u}, \Psi) \leqslant M, \\ |\dot{\Psi}| = \max\{\hat{\lambda} + 1, \bar{\mathcal{L}}\} \end{cases} \right\}. \tag{3.5}$$

Above, $\mathbf{PC}$ is the set of piecewise constant functions $[0, +\infty[\mapsto \mathbb{R}^n$ with finitely many jumps. $\hat{\lambda}$ is an upper bound for all characteristic speeds on a compact set to be precisely chosen below, see the proof of Lemma 3.10. Note that if $u \in \mathbf{PC} \cap \mathbf{L}^1([0, +\infty[)$ then u has compact support. $\mathbf{PLC}$ is the set of piecewise linear and continuous functions $[0, +\infty[\mapsto \mathbb{R}$ with finitely many corners on any compact interval.

To construct an approximate solution to (1.2), the standard wave front tracking procedure [6,12,13], see also [1,2,5,7,8,11,14,15], can now be started. First, fix an ε-grid $\mathcal{G}^\varepsilon$ and approximate the given triple $(\bar{u}, \tilde{u}, \Psi)$ in (1.1) through a triple $(\bar{u}^\varepsilon, \tilde{u}^\varepsilon, \Psi^\varepsilon)$ in $\mathcal{D}_M(\mathcal{G}^\varepsilon)$.

At time $t = 0$, at every point $x > \Psi^\varepsilon(0)$ where $\bar{u}^\varepsilon$ has a jump, we approximately solve the Riemann problem (3.1) with $u^l = \bar{u}^\varepsilon(x-)$ and $u^r = \bar{u}^\varepsilon(x+)$ by means of the exact solutions to the n Riemann problems (3.4). Similarly, at $(0, \Psi^\varepsilon(0))$ we approximately solve the Riemann problem with boundary restricting to $\Omega^\varepsilon = \{(t, x) \in \mathbb{R}^2\colon t \geqslant 0 \text{ and } x \geqslant \Psi^\varepsilon(t)\}$ the juxtaposition of the solutions to (3.4) with $u^l = \tilde{u}^\varepsilon(0+)$ and $u^r = \bar{u}^\varepsilon(\Psi^\varepsilon(0)+)$.

Patching together these solutions, we obtain a piecewise constant approximate solution of (1.2) on Ω^ε up to the first time t_1 at which one of the following interactions takes place:

(I) two or more waves collide in the interior of Ω^ε;
(II) one or more waves hits the boundary;
(III) the value of the boundary condition changes
  (III.1) where the slope of the boundary is positive,
  (III.2) where the slope of the boundary is negative;
(IV) the slope of the boundary changes.

In case (I), the approximate solution is extended beyond t_1 by solving again the corresponding Riemann problem. In cases (II), (III) and (IV) the extension beyond t_1 is achieved

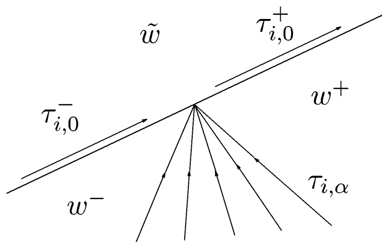

Fig. 1. Notation for case (II).

applying the Riemann solver above to the Riemann problem with boundary arising at $(t_1, \Psi^\varepsilon(t_1))$. We prove below that this procedure can be iterated leading to an approximate solution $u^\varepsilon(t,x)$ defined on all Ω^ε. To this aim, we need to provide the usual bounds on the total variation and on the number of interaction points.

First we prove that the total variation of the approximate solution is bounded for all t uniformly in ε. Fix some positive time $\bar{t}$. The approximate solution u^ε at time $\bar{t}$ and the approximate boundary condition have the form

$$u^\varepsilon = \sum_{\alpha=1}^{n} u_\alpha \chi_{[x_{\alpha-1}, x_\alpha[} \quad \text{and} \quad \tilde{u}^\varepsilon = \sum_{\alpha=1}^{n} \tilde{u}_\alpha \chi_{[t_{\alpha-1}, t_\alpha[},$$

where $t_0 = \bar{t}$ and $x_0 = \psi(\bar{t})$. For $\alpha = 1, \ldots,$ call $\sigma_{i,\alpha}$ (respectively $\tilde{\sigma}_{i,\alpha}$) the total size of the i-waves $\sigma^h_{i,\alpha}$, $h = 1, \ldots,$ in the Riemann problem between u_α and $u_{\alpha+1}$ at x_α (respectively t_α) as defined by (3.3). According to Definition 2.1, there may well be a jump between the trace $u^\varepsilon(t, \Psi^\varepsilon(t)+)$ of u^ε at the boundary and the boundary data $\tilde{u}^\varepsilon(t)$. Call $\sigma_{i,0}$ the total size of the i-waves in the solution of the Riemann problem (3.4) with $u^l = u^\varepsilon(t, \Psi^\varepsilon(t)+)$ and $u^r = \tilde{u}^\varepsilon(t)$.

For notational simplicity, in the sequel we omit ε. Following [6], we introduce for later use the quantity $\tau_{i,\alpha}$ (respectively $\tilde{\tau}_{i,\alpha}$ and $\tau_{i,0}$) as the signed length of the wave $\sigma_{i,\alpha}$ (respectively $\tilde{\sigma}_{i,\alpha}$ and $\sigma_{i,0}$) measured in the space of the Riemann coordinates. More precisely, set $u^l = u_{\alpha-1}$ and $u^r = u_\alpha$, then $\tau_{i,\alpha}$ is the signed length of the segment between w^{i-1} and w^i as defined in (3.2).

Define now the following functionals:

$$V = \sum_{i=1}^{n} \sum_{\alpha \geqslant 0} |\tau_{i,\alpha}| \quad \text{and} \quad \tilde{V} = \sum_{i=1}^{n} \sum_{\beta > 0} |\tilde{\tau}_{i,\beta}|, \tag{3.6}$$

where we omitted the various dependencies on $\mathbf{p}$, $\bar{t}$ and ε. Note that the waves with index $\alpha = 0$ are considered as located along the boundary.

Proposition 3.1. *Along any approximate solution u, the map $t \mapsto V(t) + \tilde{V}(t)$ is nonincreasing.*

Proof. For any fixed $\bar{t} > 0$, let $\Delta V = V(\bar{t}+) - V(\bar{t}-)$ and $\Delta \tilde{V} = \tilde{V}(\bar{t}+) - \tilde{V}(\bar{t}-)$. Consider the cases (I)–(IV) separately:

(I) Clearly, $\Delta \tilde{V} = 0$. By [6, Paragraph 2], we have that $\Delta V \leqslant 0$.

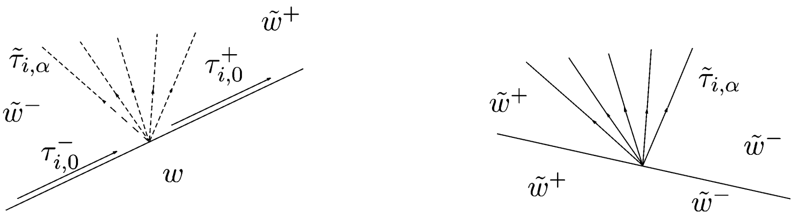

Fig. 2. On the left, case (III.1) where $\dot{\Psi} > 0$ and, on the right, case (III.2) where $\dot{\Psi} < 0$.

(II) Again, $\Delta\tilde{V} = 0$. By [13, Proposition 3.1], we obtain that $\Delta V \leqslant 0$.
(III) Here, $\Delta V \leqslant \sum_i |\tilde{\tau}_{i,\tilde{\alpha}}| = -\Delta\tilde{V}$ as in [13, Proposition 3.1], see Fig. 2.
(IV) In this case, $\Delta V = 0$ and $\Delta\tilde{V} = 0$.

The proof is complete. □

By the proposition above, $\mathcal{D}_M(\mathcal{G}^\varepsilon)$ is positively invariant as long as the approximate solution is defined.

To bound the number of interaction points, introduce

$$Q = \sum_{i>j,\, x_\alpha < x_\beta} |\tau_{i,\alpha}\tau_{j,\beta}| \quad \text{and} \quad \Upsilon = Q + \hat{M}\tilde{V}.$$

Q is the Glimm interaction potential, $\tau_{i,\alpha}$ is the jump in the ith Riemann coordinate of the discontinuity located at x_α and the functional $\tilde{V}$ is defined in (3.6). Above, $\hat{M}$ is chosen so that $\hat{M} > 2\mathrm{TV}(u(t))$ for all $t \in [t_0, T]$, which is available by Lemma 3.9.

The map $t \mapsto \Upsilon(t)$ is nonincreasing along any approximate solution belonging to $\mathcal{D}_M(\mathcal{G}^\varepsilon)$ in any strip $](h-1)\varepsilon, h\varepsilon[$, $h = 1, \ldots, N$. Moreover, at interactions

$$\Delta\Upsilon \leqslant -\sum_{\substack{\tau_\alpha^-, \tau_\beta^- \text{ interact and are} \\ \text{of different families}}} \left|\tau_\alpha^- \tau_\beta^-\right| \quad \text{in cases (I), (II), (IV),}$$

$$\Delta\Upsilon \leqslant -\sum_{\tilde{\tau}_\alpha \text{ entering } \Omega} \sum_{\alpha>0} \left|\tilde{\tau}_\alpha \tau_\alpha^-\right| \quad \text{in case (III).}$$

Therefore, at each interaction $\Delta\Upsilon < -(\delta_\varepsilon)^2$ and the total number of interactions is bounded.

By [6,12], for every $\varepsilon > 0$, the above algorithm yields a semigroup $S^\varepsilon : [0, +\infty[\, \times \mathcal{D}_M(\mathcal{G}^\varepsilon) \mapsto \mathcal{D}_M(\mathcal{G}^\varepsilon)$ whose orbits approximately solve (1.2).

We now prove the Lipschitz continuous dependence of the approximate solutions uniform in ε by means of the now classical technique based on *pseudopolygonals*, see [1,5–8, 11–14].

Definition 3.2. Let $a < b$. An *elementary path in* **PC** is a map

$$\begin{aligned} \gamma :]a, b[&\mapsto \mathbf{PC} \\ \theta &\mapsto \textstyle\sum_\alpha u^\alpha \chi_{[x_{\alpha-1}(\theta), x_\alpha(\theta)[} \end{aligned} \quad \text{with} \quad \begin{aligned} &x_\alpha(\theta) = \bar{x}_\alpha + \theta\xi_\alpha, \\ &x_{\alpha-1}(\theta) < x_\alpha(\theta) \quad \forall\theta. \end{aligned}$$

Fix $T > 0$ and assume that $\Psi', \Psi'' \in \mathbf{PLC}$ do not coincide on $[0, T]$. The *elementary path in* **PLC** joining Ψ' and Ψ'' on $[0, T]$ is the curve

$$\gamma(\theta)(t) = \begin{cases} \Psi'(t) + [\![[\![\Psi''(t) - \Psi'(t)]\!]_+ + \theta]\!]_+ & \text{if } \theta < 0, \\ \Psi''(t) + [\![[\![\Psi'(t) - \Psi''(t)]\!]_+ - \theta]\!]_+ & \text{if } \theta > 0, \end{cases}$$

defined for $|\theta| \leqslant \|\Psi' - \Psi''\|_{\mathbf{C}^0([0,T])}$, where $[\![x]\!]_+ = \max(x, 0)$. If $\Psi' = \Psi''$, the map γ defined by $\gamma(\theta) = \Psi'$ for all θ is also an elementary path in **PLC**.

Call π_i, $i = 1, 2, 3$, the three canonical projections defined in $\mathcal{D}_M(\mathcal{G}^\varepsilon)$. An *elementary path in* $\mathcal{D}_M(\mathcal{G}^\varepsilon)$ is a map $\gamma :]a, b[\mapsto \mathcal{D}_M(\mathcal{G}^\varepsilon)$ such that $\pi_i \circ \gamma$ is a **PC**-elementary path for $i = 1, 2$, and a **PLC**-elementary path for $i = 3$.

A continuous map $\gamma : [a, b] \mapsto \mathcal{D}_M(\mathcal{G}^\varepsilon)$ is a *pseudopolygonal in* $\mathcal{D}_M(\mathcal{G}^\varepsilon)$ if there exist countably many disjoint open intervals $J_h \subseteq]a, b[$ such that $]a, b[\setminus \bigcup_h J_h$ is countable and the restriction of γ to each J_h is an elementary path in $\mathcal{D}_M(\mathcal{G}^\varepsilon)$.

By [1, Proposition 3], any two triples in $\mathcal{D}_M(\mathcal{G}^\varepsilon)$ can be joined by a pseudopolygonal contained in $\mathcal{D}_M(\mathcal{G}^\varepsilon)$. Furthermore, S^ε preserves pseudopolygonals: if γ is a pseudopolygonal, then so is $S_t^\varepsilon \circ \gamma$, for all $t \geqslant 0$.

Consider a pseudopolygonal γ joining two triples in $\mathcal{D}_M(\mathcal{G}^\varepsilon)$. Introduce the shift speed of the boundary

$$\kappa(\gamma) = \begin{cases} 0 & \text{if } \theta \mapsto \pi_3 \circ \gamma(\theta) \text{ is constant,} \\ 1 & \text{otherwise.} \end{cases} \tag{3.7}$$

Define the *generalized shift speeds*

$$\eta_{i,\alpha} = \max\{\kappa, |\xi_{i,\alpha}|\}, \quad \eta_{i,0} = \kappa, \quad \tilde{\eta}_\alpha = \kappa + 2\mathcal{L}|\tilde{\xi}_\alpha|, \tag{3.8}$$

where $\xi_{i,\alpha}$ is the horizontal shift speed of the ith wave $\sigma_{i,\alpha}$ at x_α, $\tilde{\xi}_\alpha$ is the vertical shift speed of the jump at t_α in the boundary condition and $\mathcal{L} = \max\{L_{\Psi'}, L_{\Psi''}, \hat{\lambda} + 1\}$, similarly to (3.5).

Along a pseudopolygonal, through

$$\bar{\Upsilon}_\eta(\gamma) = \sum_{i,\alpha} |\sigma_{i,\alpha}\eta_{i,\alpha}| W_{i,\alpha} \quad \text{and} \quad \tilde{\Upsilon}_\eta(\gamma) = \sum_{i,\tilde{\alpha}} |\tilde{\sigma}_{i,\tilde{\alpha}}\tilde{\eta}_{\tilde{\alpha}}| \tilde{W}_{i,\tilde{\alpha}}$$

define the functionals

$$\Upsilon_\eta(\gamma) = \bar{\Upsilon}_\eta(\gamma) + \tilde{\Upsilon}_\eta(\gamma), \tag{3.9}$$

$$\Xi_\varepsilon(\gamma) = \int_a^b \Upsilon_\eta(\gamma(\theta))\, d\theta, \tag{3.10}$$

$$\|\gamma\|_\varepsilon = \int_a^b \big(\Upsilon_\eta(\gamma(\theta)) + \kappa(\gamma(\theta))\big)\, d\theta, \tag{3.11}$$

$W_{i,\alpha}, \tilde{W}_{i,\tilde{\alpha}}$ being weights bounded uniformly in ε, see (3.17).

Call below $\ell_X(\gamma)$ the length of the curve γ with respect to the distance in the metric space X. For instance, in $\mathcal{D}$, we consider the metric

$$d(\mathbf{p}', \mathbf{p}'') = \|\bar{u}'' - \bar{u}'\|_{\mathbf{L}^1} + \|\tilde{u}'' - \tilde{u}'\|_{\mathbf{L}^1} + \|\Psi'' - \Psi'\|_{\mathbf{C}^0}. \tag{3.12}$$

Lemma 3.3. *Fix a positive M. Then, there exists a positive constant C such that for all $\mathbf{p}_1, \mathbf{p}_2 \in \mathcal{D}_M$ with $\pi_3 \mathbf{p}_i$ having Lipschitz constant L_i, for all pseudopolygonal $\gamma : [a,b] \mapsto \mathcal{D}_M$ joining $\mathbf{p}_1$ to $\mathbf{p}_2$ and for all small ε, the following estimates hold*:

$$\begin{aligned}
&\|\gamma\|_\varepsilon \geqslant \frac{1}{C} \cdot \ell_{\mathcal{D}}(\gamma),\\
&\|\gamma\|_\varepsilon \leqslant C \cdot \bigl(\ell_{\mathbf{L}^1}(\gamma_1) + \bigl(1 + \max\{L_1, L_2\}\bigr)\ell_{\mathbf{L}^1}(\gamma_2) + \ell_{\mathbf{C}^0}(\gamma_3)\bigr),\\
&\Xi_\varepsilon(\gamma) \geqslant \frac{1}{C} \cdot \ell_{\mathbf{L}^1}(\gamma_1),\\
&\Xi_\varepsilon(\gamma) \leqslant C \cdot \bigl(\ell_{\mathbf{L}^1}(\gamma_1) + \ell_{\mathbf{L}^1}(\gamma_2) + \bigl(\mathrm{TV}(\mathbf{p}_{1|[0,T]}) + \mathrm{TV}(\mathbf{p}_{2|[0,T]})\bigr)\ell_{\mathbf{C}^0}(\gamma_3)\bigr),
\end{aligned}$$

where $\pi_i \circ \gamma = \gamma_i$.

Above, referring to the choice (2.2) of the norms, we denoted

$$\mathrm{TV}(\mathbf{p}_{|[t_0,T]}) = \mathrm{TV}(\bar{u}) + \bigl\|\bar{u}\bigl(\Psi(t_0+)\bigr) - \tilde{u}(t_0+)\bigr\| + \mathrm{TV}(\tilde{u}_{|[t_0,T]}).$$

It immediately follows that the metric on $\mathcal{D}_M(\mathcal{G}^\varepsilon)$ defined by

$$d_\eta^\varepsilon(\mathbf{p}_1, \mathbf{p}_2) = \inf\bigl\{\|\gamma\|_\varepsilon \colon \gamma \text{ pseudopolygonal joining } \mathbf{p}_1 \text{ to } \mathbf{p}_2\bigr\}$$

is equivalent to the distance (3.12), see also [1,5–7,13].

Due to the possible “*movement*” of the boundary, below it is necessary to consider one more type of interaction, namely the points where

(V) the boundary stops shifting, i.e. where κ passes from 1 to 0.

The following interaction estimates, see Figs. 1–3 for the notation:

$$\begin{aligned}
&\text{(I):} \quad \Bigl|\sum_{\alpha>0} \sigma^+_{i,\alpha}\Bigr| \leqslant \Bigl(1 + K \sum_{k\neq i}\Bigl|\sum_{\alpha>0} \tau^-_{k,\alpha}\Bigr|\Bigr)\Bigl|\sum_{\alpha>0} \sigma^-_{i,\alpha}\Bigr|,\\
&\text{(II):} \quad \bigl|\sigma^+_{i,0}\bigr| \leqslant \Bigl(1 + K \sum_{k\neq i}\Bigl|\sum_{\alpha\geqslant 0} \tau^-_{k,\alpha}\Bigr|\Bigr)\Bigl|\sum_{\alpha\geqslant 0} \sigma^-_{i,\alpha}\Bigr|,\\
&\text{(III.1):} \quad \bigl|\sigma^+_{i,0}\bigr| \leqslant \Bigl(1 + K \sum_{k\neq i}\bigl(\bigl|\tau^-_{k,0} + \tilde{\tau}_{k,\tilde{\alpha}}\bigr|\bigr)\Bigr)\bigl(\bigl|\sigma^-_{i,0} + \tilde{\sigma}_{i,\tilde{\alpha}}\bigr|\bigr),
\end{aligned}$$

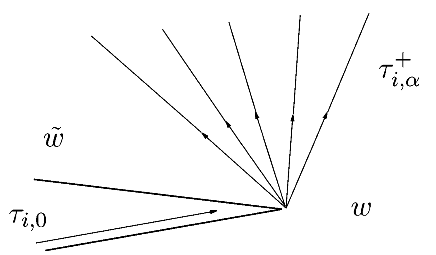

Fig. 3. Notation for case (IV).

$$\text{(III.2):}\quad \left|\sum_{\alpha>0}\sigma^+_{i,\alpha}\right| = |\tilde\sigma_{i,\tilde\alpha}|,$$

$$\text{(IV):}\quad \left|\sum_{\alpha>0}\sigma^+_{i,\alpha}\right| = \left|\sigma^-_{i,0}\right| \quad\text{and}\quad \sigma^+_{i,0}=0 \tag{3.13}$$

hold for a suitable positive constant K. The former estimate comes from [6, formula (5.7)], while the others are refinements of analogous results in [13].

Proposition 3.4. *Consider a point $P_* = (t_*, x_*)$ of interaction. Let $u(t,x)$ be the approximate solution to* (1.2) *defined for $t < t_*$ by extending backward the shocks and for $t \geqslant t_*$ by solving the approximate Riemann problem. Then*

$$\begin{aligned}\text{(I):}\quad \sum_{\alpha>0}\left|\sigma^+_{i,\alpha}\eta^+_{i,\alpha}\right| \leqslant{}& \left(1+K\sum_{k\neq i}\left|\sum_{\alpha>0}\tau^-_{k,\alpha}\right|\right)^2\sum_{\alpha>0}\left|\sigma^-_{i,\alpha}\eta^-_{i,\alpha}\right| \\ &+ K\left|\sum_{\alpha}\tau^-_{i,\alpha}\right|\sum_{k\neq i}\sum_{\alpha>0}\left|\sigma^-_{k,\alpha}\eta^-_{k,\alpha}\right|,\end{aligned} \tag{3.14}$$

$$\text{(II):}\quad \left|\sigma^+_{i,0}\eta^+_{i,0}\right| \leqslant \left(1+K\sum_{k\neq i}\left|\sum_{\alpha\geqslant 0}\tau^-_{k,\alpha}\right|\right)^2\sum_{\alpha\geqslant 0}\left|\sigma^-_{i,\alpha}\eta^-_{i,\alpha}\right|, \tag{3.15}$$

$$\text{(III.1):}\quad \left|\sigma^+_{i,0}\eta^+_{i,0}\right| \leqslant \left(1+K\sum_{k\neq i}\left|\tau^-_{k,0}+\tilde\tau^-_{k,\tilde\alpha}\right|\right)^2\left(\left|\sigma^-_{i,0}\eta^-_{i,0}\right| + |\tilde\sigma_{i,\tilde\alpha}\tilde\eta_{\tilde\alpha}|\right). \tag{3.16}$$

Proof. We consider the various cases separately.

(I) If $\eta^+_{i,\alpha} = |\xi^+_{i,\alpha}|$, (3.14) follows from [6, formula (5.8)]. If $\eta^+_{i,\alpha} = \kappa$, then we assume $\eta^-_{i,\alpha} = \kappa$, since in the case $\eta^+_{i,\alpha} > \kappa$ the right-hand side in (3.14) becomes greater. Now, (3.14) follows from [6, formula (5.8)] setting for all i,α, $\xi^-_{i,\alpha} = 1$, which implies $\xi^+_{i,\alpha} = 1$.

(II) If $\kappa = 0$, then $\eta^+_{i,0} = 0$ and (3.15) holds. If $\kappa = 1$, then, by (3.8), we have $\eta^\pm_{i,0} = 1$ and $\eta^-_{i,\alpha} \geqslant 1$ for $\alpha > 0$. Thus (II) in (3.13) implies (3.15), indeed

$$\begin{aligned}&\left(1+K\sum_{k\neq i}\left|\sum_{\alpha\geqslant 0}\tau^-_{k,\alpha}\right|\right)^2\sum_{\alpha\geqslant 0}\left|\sigma^-_{i,\alpha}\eta^-_{i,\alpha}\right| \\ &\quad\geqslant \left(1+K\sum_{k\neq i}\left|\sum_{\alpha\geqslant 0}\tau^-_{k,\alpha}\right|\right)\left|\sigma^+_{i,0}\right| \geqslant \left|\sigma^+_{i,0}\eta^+_{i,0}\right|.\end{aligned}$$

(III.1) In this case, $\sigma^+_{i,\alpha} = 0$ for any $\alpha > 0$. If $\kappa = 0$, then $\eta^+_{i,0} = 0$ and hence (3.16) follows. If on the other hand $\kappa = 1$, by (3.8) we have $\tilde\eta_\alpha \geqslant 1$, $\alpha > 0$, and $\eta^\pm_{i,0} = 1$. Thus

$$\left(1+K\sum_{k\neq i}\left|\tau^-_{k,0}+\tilde\tau_{k,\tilde\alpha}\right|\right)^2\left(\left|\sigma^-_{i,0}\eta^-_{i,0}\right| + \left|\tilde\sigma^-_{i,\tilde\alpha}\tilde\eta_{\tilde\alpha}\right|\right)$$

$$\geqslant \left(1 + K\sum_{k\neq i}\left|\tau_{k,0}^- + \tilde{\tau}_{k,\tilde{\alpha}}\right|\right)\left|\sigma_{i,0}^+\right| \geqslant \left|\sigma_{i,0}^+\eta_{i,0}^+\right|. \qquad \Box$$

We now specialize the choice of the approximating boundary Ψ. Indeed, let T^ε be such that $\lim_{\varepsilon\to 0^+} T^\varepsilon = +\infty$ and particularize (3.5) as $\dot{\Psi}(t) = -\max\{\hat{\lambda}+1, L_\Psi\}$ for $t \geqslant T^\varepsilon$. $\tilde{u}$ and $\bar{u}$ have bounded support, hence there exists a time $\hat{T}^\varepsilon$ (with $\hat{T}^\varepsilon > T^\varepsilon$) such that no interaction takes place for $t > \hat{T}^\varepsilon$, see [21].

Following [6], assign weight 1 at all waves in $u(\hat{T}^\varepsilon, \cdot)$. Next consider a point P_* of interaction and suppose that the weights $W_{i,\alpha}^+$ of the waves exiting the interaction are already assigned. The incoming waves are weighted as follows. If no i-wave exits the interaction, each i-wave that enters the interaction is assigned weight $W_{i,\alpha}^- = 1$. In the other cases let

$$\begin{aligned}
&\text{(I)} && W_{i,\alpha}^- = \left(1 + K\sum_{k\neq i}\left|\sum_{\alpha>0}\tau_{k,\alpha}^-\right|\right)^2 \max_{\alpha>0} W_{i,\alpha}^+ \\
& && \qquad + K\sum_{k\neq i}\left(\left|\sum_{\alpha>0}\tau_{k,\alpha}^-\right| \max_{\alpha>0} W_{k,\alpha}^+\right), \\
&\text{(II)} && W_{i,\alpha}^- = \left(1 + K\sum_{k\neq i}\left|\sum_{\alpha\geqslant 0}\tau_{k,\alpha}^-\right|\right)^2 W_{i,0}^+, \\
&\text{(III.1)} && W_{i,0}^- = \left(1 + K\sum_{k\neq i}\left|\tau_{k,0}^- + \tilde{\tau}_{k,\tilde{\alpha}}^-\right|\right)^2 W_{i,0}^+, \\
& && \tilde{W}_{i,\tilde{\alpha}} = \left(1 + K\sum_{k\neq i}\left|\tau_{k,0}^- + \tilde{\tau}_{k,\tilde{\alpha}}^-\right|\right)^2 W_{i,0}^+, \\
&\text{(III.2)} && \tilde{W}_{i,\tilde{\alpha}} = \max\left\{W_{i,\alpha}^+ \colon \sigma_{i,\alpha} \text{ exits the interaction}\right\}, \\
&\text{(IV)} && W_{i,0}^- = \max\left\{W_{i,\alpha}^+ \colon \sigma_{i,\alpha} \text{ exits the interaction}\right\}.
\end{aligned} \tag{3.17}$$

In case (V), since there is no interaction, it is not necessary to define weights.

Proposition 3.5. *Fix an elementary path γ. Let an interaction take place at P_*. Let $\Upsilon_\eta(t) = \Upsilon_\eta(S_t^\varepsilon \circ \gamma)$, where Υ_η is defined in* (3.9)*, and $\kappa(t) = \kappa(S_t^\varepsilon \circ \gamma)$, κ being defined in* (3.7)*. Then in any of the cases* (I)–(V)*,*

$$\Upsilon_\eta(t_*+) \leqslant \Upsilon_\eta(t_*-) \quad \textit{and} \quad \Upsilon_\eta(t_*+) + \kappa(t_*+) \leqslant \Upsilon_\eta(t_*-) + \kappa(t_*-).$$

Proof. Since κ can only decreases passing from 1 to 0, it is sufficient to show that $\Delta\Upsilon_\eta \leqslant 0$ in all cases.

(I) In this case $\Delta\tilde{\Upsilon}_\eta^\varepsilon = 0$ and κ remains constant. Moreover $\Delta\bar{\Upsilon}_\eta \leqslant 0$. Indeed, as proved in [6, Paragraph 6] and [13, Proposition 3.6], by (I) in (3.17), it holds that, with obvious notation,

$$\sum_\alpha \left|\sigma_{i,\alpha}^+\eta_{i,\alpha}^+\right| W_{i,\alpha}^+ \leqslant \sum_\alpha \left|\sigma_{i,\alpha}^-\eta_{i,\alpha}^-\right| W_{i,\alpha}^-.$$

(II) We refer to Fig. 1. As before, $\Delta\tilde{\Upsilon}_\eta = 0$ and κ remains constant. Furthermore $\Delta\bar{\Upsilon}_\eta \leqslant 0$. In fact, using (3.15) and (II) in (3.17) we have

$$\begin{aligned}
\left|\sigma^+_{i,0}\eta^+_{i,0}\right| W^+_{i,0} &\leqslant \left(1 + K\sum_{k\neq i}\left|\sum_{\alpha\geqslant 0}\tau^-_{k,\alpha}\right|\right)^2 \sum_{\alpha\geqslant 0}\left|\sigma^-_{i,\alpha}\eta^-_{i,\alpha}\right| W^+_{i,0}\\
&= \sum_{\alpha\geqslant 0}\left|\sigma^-_{i,\alpha}\eta^-_{i,\alpha}\right| W^-_{i,\alpha}.
\end{aligned}$$

(III) First consider case (III.1), see Fig. 2, left. In this case $\Delta\Upsilon_\eta \leqslant 0$ because for (3.16) and (III.1) in (3.17) we have

$$\begin{aligned}
\left|\sigma^+_{i,0}\eta^+_{i,0}\right| W^+_{i,0} &\leqslant \left(1 + K\sum_{k\neq i}\left|\tau^-_{k,0} + \tilde{\tau}^-_{k,\tilde{\alpha}}\right|\right)^2 \left(\left|\sigma^-_{i,0}\eta^-_{i,0}\right| + \left|\tilde{\sigma}_{i,\tilde{\alpha}}\tilde{\eta}_{\tilde{\alpha}}\right|\right) W^+_{i,0}\\
&= \left|\sigma^-_{i,0}\eta^-_{i,0}\right| W^-_{i,0} + \left|\tilde{\sigma}^-_{i,\tilde{\alpha}}\tilde{\eta}^-_{\tilde{\alpha}}\right| \tilde{W}_{i,\tilde{\alpha}}.
\end{aligned}$$

Consider case (III.2), see Fig. 2, right. By (III.2) in (3.17) and in (3.13), we immediately obtain $\Delta\bar{\Upsilon}_\eta \leqslant -\Delta\tilde{\Upsilon}_\eta$.

(IV) We refer to Fig. 3. In this case, by (IV) in (3.13) and since $\xi^+_{i\alpha} = \kappa$, $\Delta\bar{\Upsilon}_\eta \leqslant 0$ and $\Delta\tilde{\Upsilon}_\eta = 0$. Hence $\Delta\Upsilon_\eta \leqslant 0$.

(V) In this case $\Delta\bar{\Upsilon}_\eta = \Delta\tilde{\Upsilon}_\eta = 0$ because the waves do not change sizes. □

As a consequence of Proposition 3.5, the length of $S^\varepsilon \circ \gamma$ computed as in (3.11) is nonincreasing as a function of time.

3.2. The source term

We approximate g as

$$g^\varepsilon(t,x,u) = \sum_{k\in\mathbb{Z}} \frac{1}{\varepsilon}\left(\int_{(k-1)\varepsilon}^{k\varepsilon} g(t,\xi,u)\,d\xi\right)\cdot \chi_{](k-1)\varepsilon,k\varepsilon]}(x) \tag{3.18}$$

and consider the approximate problem

$$\begin{cases} \partial_t u = g^\varepsilon(t,x,u), & (t,x)\in\Omega,\\ u(0,x) = \bar{u}^\varepsilon(x), & x \geqslant \Psi^\varepsilon(t_0),\\ u(t,\Psi^\varepsilon(t)) = \tilde{u}^\varepsilon(t), & t\geqslant t_0, \end{cases} \tag{3.19}$$

where $(\bar{u}^\varepsilon, \tilde{u}^\varepsilon, \Psi^\varepsilon)$ are as in the previous paragraph. In [12, Lemma 4.3] the following lemma is proved.

Lemma 3.6. *Let g be as in* (G). *Then g^ε satisfies* (G) *with* (G_3) *modified as follows*: *if $h,k\in\mathbb{Z}$ and $h\leqslant k$, for all $x_1 \in\,]h\varepsilon,(h+1)\varepsilon]$ and $x_2\in\,]k\varepsilon,(k+1)\varepsilon]$ we have*

$$\left|g^\varepsilon(t,x_2,u) - g^\varepsilon(t,x_1,u)\right| \leqslant 3\mu\big(\big[h\varepsilon,(k+1)\varepsilon\big]\big). \tag{3.20}$$

Below, $\operatorname{spt}(u)$ denotes the support of the function u.

Lemma 3.7. *The differential equation* (3.19) *generates the map*

$$\begin{aligned}\Sigma^\varepsilon : \mathcal{I}\times\mathcal{D} &\mapsto \mathbf{L}^1\cap\mathbf{BV}\big(\big[\Psi(t_0),+\infty\big[,\mathcal{U}\big),\\ (t_0,t),\mathbf{p} &\mapsto \Sigma^\varepsilon_{t_0,t}\mathbf{p}\end{aligned}\tag{3.21}$$

in the sense that for all $(\bar u^\varepsilon,\tilde u^\varepsilon,\Psi^\varepsilon)\in\mathcal{D}$*, the map* $t\mapsto\Sigma^\varepsilon_{t_0,t}(\bar u^\varepsilon,\tilde u^\varepsilon,\Psi^\varepsilon)$ *is the solution to* (3.19)*. For all* $R>0$ *and* $T>t_0$*, there exist a positive* $\tilde l\in\mathbf{L}^1_{\mathrm{loc}}([t_0,+\infty[)$ *and constants* $C,\hat M>0$*, both independent from* ε*, such that for all* $t\in[t_0,T]$ *and* $\mathbf{p}=(\bar u,\tilde u,\Psi)\in\mathcal{D}$ *with* $\mathrm{TV}(\mathbf{p}_{|[t_0,T]})\leqslant R$*,*

$$\big\|\Sigma^\varepsilon_{t_0,t}\mathbf{p}\big\|_{\mathbf{L}^\infty}\leqslant e^{\int_{t_0}^t\tilde l(\tau)\,d\tau}\cdot\|\bar u\|_{\mathbf{L}^\infty}+\sup_{\tau\in[t_0,t]}e^{\int_\tau^t\tilde l(s)\,ds}\cdot\big\|\tilde u(\tau)\big\|,\tag{3.22}$$

$$\mathrm{spt}\big(\Sigma^\varepsilon_{t_0,t}\mathbf{p}\big)\subseteq\mathrm{spt}(\bar u)\cup\Psi\big(\mathrm{spt}(\tilde u)\cap[t_0,t]\big),\tag{3.23}$$

$$\begin{aligned}\mathrm{TV}\big(\Sigma^\varepsilon_{t_0,t}\mathbf{p}\big)&\leqslant e^{C(t-t_0)}\cdot\big(1+C(t-t_0)\big)\cdot\mathrm{TV}(\mathbf{p}_{|[t_0,t]})\\ &\quad+e^{C(t-t_0)}\cdot 9L_w\cdot n\mu(\mathbb{R})\cdot(t-t_0).\end{aligned}\tag{3.24}$$

Finally, there exists an ε*-grid* $\bar{\mathcal{G}}^\varepsilon$ *such that*

$$(\bar u,\tilde u,\Psi)\in\mathcal{D}_M(\mathcal{G}^\varepsilon)\quad\Rightarrow\quad\big(\Sigma^\varepsilon_{t_0,t}(\bar u,\tilde u,\Psi),\mathcal{T}_{t-t_0}\tilde u,\mathcal{T}_{t-t_0}\Psi\big)\in\mathcal{D}_{\hat M}(\bar{\mathcal{G}}^\varepsilon).\tag{3.25}$$

Proof. By the standard theory of ordinary differential equations, there exists a compact K in the space of the conserved quantities such that the solutions to (3.19) with data $\mathbf{p}$ with $\mathrm{TV}(\mathbf{p}_{|[t_0,T]})\leqslant R$ attain values in K for all $t\in[0,T]$. Let $\tilde K=w(K)$ and denote by L_w, L_u the Lipschitz constants of the maps $w\to u$ and $u\to w$ restricted on K and $\tilde K$, respectively.

Now, we use the formulation of (1.3) in the Riemann coordinates, i.e.

$$\partial_t w=\tilde g^\varepsilon(t,x,w),\tag{3.26}$$

where $\tilde g^\varepsilon(t,x,w)=D_u w(t,x)g^\varepsilon(t,x,u(w))$ satisfies conditions analogous to (G) on g. In fact, (G_1) and (G_2) are immediate. Condition (G_3) holds modified as in (3.20), with $L_w\mu$ in place of μ. Concerning (G_4), for any $w,w_1,w_2\in\tilde K$,

$$\big|\tilde g^\varepsilon(t,x,w_2)-\tilde g^\varepsilon(t,x,w_1)\big|\leqslant\tilde l_K(t)\cdot|w_2-w_1|,\tag{3.27}$$

$$\big|\tilde g^\varepsilon(t,x,w)\big|\leqslant\tilde l(t)\cdot|w|,\tag{3.28}$$

where $\tilde l_{\tilde K}(t)=(\sup_K\|D_u^2w\|\cdot\sup_K|u|+L_w)\cdot L_u\cdot l_K(t)$ and $\tilde l(t)=c\cdot L_w\cdot L_u\cdot l(t)$, for a suitable constant $c>0$.

We consider now (3.22). Let $u(t)=\Sigma^\varepsilon_{t_0,t}\mathbf{p}$. (2.1), (2.2) and (3.28) imply

$$\text{if }\alpha(t,x)=t_0,\quad\big\|u(t,x)\big\|\leqslant\big\|\bar u(x)\big\|+\int_{t_0}^t\tilde l(\tau)\cdot\big\|u(\tau,x)\big\|\,d\tau,$$

$$\text{if }\alpha(t,x)>t_0,\quad\big\|u(t,x)\big\|\leqslant\big\|\tilde u\big(\alpha(t,x)\big)\big\|+\int_{\alpha(t,x)}^t\tilde l(\tau)\cdot\big\|u(\tau,x)\big\|\,d\tau.$$

By Grönwall lemma and passing to $\mathbf{L}^\infty$ norm, the inequality (3.22) follows.

(3.23) follows from (F_3), (G_1) and Definition 2.2.

Consider (3.24). Fix R and T. If $u \in \mathcal{D}$ with $\|\bar u\|_{\mathbf{L}^\infty} + \|\tilde u\|_{\mathbf{L}^\infty} \leqslant R$, then (3.22) implies that for $t \in [t_0, T]$, the solution $w(t) = w(u(t))$ to (3.26) with data $\bar w(x) = w(\bar u(x))$ and $\tilde w(t) = w(\tilde u(t))$, attains values in the compact set

$$\tilde K = w(\mathcal{U}) \cap \bigl[-Re^{\int_{t_0}^t \tilde l(\tau)\,d\tau}, Re^{\int_{t_0}^t \tilde l(\tau)\,d\tau}\bigr]^n.$$

Define $K = u(\tilde K)$ and note that $K \subseteq \mathcal{U}$. We seek an upper bound for $\sum_i |w(t, x_{i-1}) - w(t, x_i)|$, where $x_0 \geqslant \Psi(t)$ and $x_i > x_{i-1}$ for all i. Let $h_i \in \mathbb{Z}$ be such that $x_i \in [h_i\varepsilon, (h_i+1)\varepsilon[$. Note that $h_{i-1} \leqslant h_i$ and, by (2.1), $\alpha(t, x_{i-1}) \geqslant \alpha(t, x_i)$ for all i. Let i_0 be the smallest index such that $\alpha(t, x_{i_0}) = t_0$. Then, following the same lines of [12, Lemma 4.4], we obtain for any fixed $i > i_0$,

$$\begin{aligned}
h_{i-1} = h_i, &\quad \bigl|w(t, x_{i-1}) - w(t, x_i)\bigr| \leqslant e^{\int_{t_0}^t \tilde l_K(\tau)\,d\tau} \bigl|w(t_0, x_{i-1}) - w(t_0, x_i)\bigr|,\\
h_{i-1} < h_i, &\quad \bigl|w(t, x_{i-1}) - w(t, x_i)\bigr| \leqslant e^{\int_{t_0}^t \tilde l_K(\tau)\,d\tau} \bigl|w(t_0, x_{i-1}) - w(t_0, x_i)\bigr|\\
&\qquad + 3L_w e^{\int_{t_0}^t \tilde l_K(\tau)\,d\tau} n\mu\bigl([h_{i-1}, h_i]\varepsilon\bigr)(t - t_0).
\end{aligned}$$

Choose now $i \leqslant i_0$. By the same procedure we get, if $h_{i-1} = h_i$,

$$\begin{aligned}
&\bigl|w(t, x_{i-1}) - w(t, x_i)\bigr|\\
&\quad\leqslant e^{\int_{\alpha(t,x_{i-1})}^t \tilde l_K(\tau)\,d\tau} \bigl|w\bigl(\alpha(t, x_{i-1}), x_{i-1}\bigr) - w\bigl(\alpha(t, x_{i-1}), x_i\bigr)\bigr|\\
&\quad\leqslant e^{\int_{\alpha(t,x_{i-1})}^t \tilde l_K(\tau)\,d\tau} \bigl\|\tilde u\bigl(\alpha(t, x_{i-1})\bigr) - \tilde u\bigl(\alpha(t, x_i)\bigr)\bigr\|\\
&\qquad + e^{\int_{\alpha(t,x_{i-1})}^t \tilde l_K(\tau)\,d\tau} \bigl|w\bigl(\alpha(t, x_{i-1}), x_i\bigr) - w\bigl(\alpha(t, x_i), x_i\bigr)\bigr|\\
&\quad\leqslant e^{\int_{\alpha(t,x_{i-1})}^t \tilde l_K(\tau)\,d\tau} \cdot \bigl\|\tilde u\bigl(\alpha(t, x_{i-1})\bigr) - \tilde u\bigl(\alpha(t, x_i)\bigr)\bigr\|\\
&\qquad + e^{\int_{\alpha(t,x_i)}^t \tilde l_K(\tau)\,d\tau} \cdot \int\limits_{\alpha(t,x_i)}^{\alpha(t,x_{i-1})} \tilde l(\tau)\,d\tau \cdot \|\tilde u\|_{L^\infty(t_0,t)},
\end{aligned}$$

while in the case $h_{i-1} < h_i$, by [12, (4.20)] and Lemma 3.6,

$$\begin{aligned}
&\bigl|w(t, x_{i-1}) - w(t, x_i)\bigr|\\
&\quad\leqslant e^{\int_{\alpha(t,x_{i-1})}^t \tilde l_K(\tau)\,d\tau} \bigl|w\bigl(\alpha(t, x_{i-1}), x_{i-1}\bigr) - w\bigl(\alpha(t, x_i), x_i\bigr)\bigr|\\
&\qquad + e^{\int_{\alpha(t,x_i)}^t \tilde l_K(\tau)\,d\tau} \cdot 3L_w \cdot \mu\bigl([h_{i-1}, h_i]\varepsilon\bigr) \cdot \bigl(t - \alpha(t, x_{i-1})\bigr)\\
&\quad\leqslant e^{\int_{t_0}^t \tilde l_K(\tau)\,d\tau} \cdot \bigl\|\tilde u\bigl(\alpha(t, x_{i-1})\bigr) - \tilde u\bigl(\alpha(t, x_i)\bigr)\bigr\|\\
&\qquad + e^{\int_{t_0}^t \tilde l_K(\tau)\,d\tau} \cdot \int\limits_{\alpha(t,x_i)}^{t_0} \tilde l(\tau)\,d\tau \cdot \|\tilde u\|_{L^\infty(t_0,t)}\\
&\qquad + e^{\int_{t_0}^t \tilde l_K(\tau)\,d\tau} \cdot 3L_w \cdot n\mu\bigl([h_{i-1}, h_i]\varepsilon\bigr) \cdot (t - t_0).
\end{aligned}$$

Summing up over i we get

$$\begin{aligned}
&\sum_i \bigl|w(t,x_{i-1}) - w(t,x_i)\bigr| \\
&\quad\leqslant e^{\int_{t_0}^t \tilde{l}_K(\tau)\,d\tau} \cdot \bigl(\mathrm{TV}(\bar{u}) + \bigl\|\bar{u}(t_0+) - \tilde{u}(t_0+)\bigr\| + \mathrm{TV}(\tilde{u}_{|[t_0,t]})\bigr) \\
&\qquad + C e^{\int_{t_0}^t \tilde{l}_K(\tau)\,d\tau} \cdot \int_{t_0}^{t} \tilde{l}(\tau)\,d\tau \cdot \mathrm{TV}(\tilde{u}_{|[t_0,t]}) \\
&\qquad + e^{\int_{t_0}^t \tilde{l}_K(\tau)\,d\tau} \cdot 9L_w \cdot n\mu\bigl(\bigl[\Psi(t), +\infty\bigr[\bigr) \cdot (t-t_0) \\
&\quad\leqslant e^{\int_{t_0}^t \tilde{l}_K(\tau)\,d\tau} \cdot \left(1 + C\int_{t_0}^{t} \tilde{l}(\tau)\,d\tau\right) \cdot \bigl(\mathrm{TV}(\bar{u}) + \bigl\|\bar{u}(t_0+) - \tilde{u}(t_0+)\bigr\| + \mathrm{TV}(\tilde{u}_{|[t_0,t]})\bigr) \\
&\qquad + e^{\int_{t_0}^t \tilde{l}_K(\tau)\,d\tau} \cdot 9L_w \cdot n\mu(\mathbb{R}) \cdot (t-t_0).
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\mathrm{TV}\bigl(u(t)\bigr) \leqslant{}& e^{C(t-t_0)} \cdot \bigl(1 + C(t-t_0)\bigr) \\
&\times \bigl(\mathrm{TV}(\bar{u}) + \bigl\|\bar{u}\bigl(\Psi(t_0+)\bigr) - \tilde{u}(t_0+)\bigr\| + \mathrm{TV}(\tilde{u}_{|[t_0,t]})\bigr) \\
&+ e^{C(t-t_0)} \cdot 9L_w n \cdot \mu(\mathbb{R}) \cdot (t-t_0).
\end{aligned}$$

Using (2.2), we obtain (3.24) for a suitable $C \geqslant 9nL_w$.

Concerning (3.25), with a slight abuse of notation, let $\Sigma^{\varepsilon}_{t_0,t}$ act on $\mathcal{U}$ instead of on functions valued in $\mathcal{U}$. Then, $\Sigma^{\varepsilon}_{t_0,t}(\mathcal{G}^{\varepsilon})$ is a finite set and is contained in a suitable ε-grid $\bar{\mathcal{G}}^{\varepsilon}$. □

3.3. *Operator splitting*

An approximate solution to (1.1) is constructed through the following operator splitting scheme. Fix positive ε, M and an ε-grid $\mathcal{G}^{\varepsilon}$. Let $\mathbf{p} = (\bar{u}, \tilde{u}, \Psi) \in \mathcal{D}_M(\mathcal{G}^{\varepsilon})$. Let $h > k$ be in $\mathbb{N}$ and for $t_0 \in [k\varepsilon, (k+1)\varepsilon[$ define recursively

$$F^{\varepsilon}_{t_0,t}\mathbf{p} = \begin{cases} S^{\varepsilon}_{t-t_0}\mathbf{p} & \text{if } t \in [t_0, (k+1)\varepsilon[, \\ (\Sigma^{\varepsilon}_{t_0,t}(S^{\varepsilon}_{t-t_0}\mathbf{p}), \mathcal{T}_{t-t_0}\tilde{u}, \mathcal{T}_{t-t_0}\Psi) & \text{if } t = (k+1)\varepsilon, \\ S^{\varepsilon}_{t-h\varepsilon}(\bigcirc_{i=k+1}^{h-1} F^{\varepsilon}_{i\varepsilon,(i+1)\varepsilon}) F^{\varepsilon}_{t_0,(k+1)\varepsilon} u & \text{if } t \in [h\varepsilon, (h+1)\varepsilon[. \end{cases} \tag{3.29}$$

Concerning the grid, refine it recursively. Indeed start with an initial datum $\mathbf{p} \in \mathcal{D}(\mathcal{G}^{\varepsilon})$ assigned at time t_0. For $t \in [t_0, (k+1)\varepsilon[$, $F^{\varepsilon}_{t_0,t}\mathbf{p}$ attains values in the same grid $\mathcal{G}^{\varepsilon}$. At time $(k+1)\varepsilon$ we apply the o.d.e. solver $\Sigma^{\varepsilon}_{t_0,(k+1)\varepsilon}$ and at the same time pass to another ε-grid $\mathcal{G}^{\varepsilon}_1 = \bar{\mathcal{G}}^{\varepsilon}$, according to (3.25).

Recursively, if $F^{\varepsilon}_{t_0,h\varepsilon}\mathbf{p}$ attains values in $\mathcal{G}^{\varepsilon}_m$, then $F^{\varepsilon}_{t_0,t}\mathbf{p}$ is valued in the same grid for all $t \in [h\varepsilon, (h+1)\varepsilon[$. Applying $\Sigma^{\varepsilon}_{h\varepsilon,(h+1)\varepsilon}$ we pass to another ε-grid $\mathcal{G}^{\varepsilon}_{m+1} = \bar{\mathcal{G}}^{\varepsilon}_m$.

Lemma 3.8. *Let $T > t_0$. The operator $F^\varepsilon : \mathcal{I} \times \mathcal{D} \mapsto \mathcal{D}$ is well defined and can be written as $F^\varepsilon_{t_0,t}(\bar u, \tilde u, \Psi) = (u^\varepsilon(t), \mathcal{T}_{t-t_0}\tilde u, \mathcal{T}_{t-t_0}\Psi)$. Moreover, the total number of discontinuities is finite on any strip $[t_0, T] \times \mathbb{R}$.*

Proof. For (3.29) to be well defined, it is necessary to check that all compositions are possible: indeed, for all $\mathbf{p} \in \mathcal{D}$, $S^\varepsilon_t \mathbf{p}$ is in $\mathcal{D}$ as well as $\Sigma^\varepsilon_{k\varepsilon,(k+1)\varepsilon}\mathbf{p}$, thanks to Lemma 3.7.

The use of a discrete grid at each convective step ensures that the number of interactions remains finite over all the time interval $[0, T]$. □

Lemma 3.9. *For all $R > 0$ and $T > t_0$, there exist positive $\tilde l \in \mathbf{L}^1([t_0, T])$ and a constant C, both independent from ε, such that for $t \in [t_0, T]$ and for $\mathbf{p} = (\bar u, \tilde u, \Psi) \in \mathcal{D}$ with $\|\tilde u\|_{\mathbf{L}^\infty} + \|\bar u\|_{\mathbf{L}^\infty} \leqslant R$, the function u defined by $(u(t), \mathcal{T}_{t-t_0}\tilde u, \mathcal{T}_{t-t_0}\Psi) = F^\varepsilon_{t_0,t}\mathbf{p}$ satisfies*

$$\left\|u(t)\right\|_{\mathbf{L}^\infty} \leqslant e^{\int_{t_0}^t \tilde l(\tau)\,d\tau} \cdot \left(\|\tilde u\|_{\mathbf{L}^\infty} + \|\bar u\|_{\mathbf{L}^\infty}\right), \tag{3.30}$$

$$\begin{aligned}\mathrm{TV}\big(u(t)\big) \leqslant{}& e^{C(t-t_0)} \cdot \big(1 + C(t-t_0)\big) \cdot \mathrm{TV}(\mathbf{p}_{|[t_0,t]}) \\ &+ e^{C(t-t_0)} \cdot 9 L_w n \cdot \mu(\mathbb{R}) \cdot (t - t_0).\end{aligned} \tag{3.31}$$

Proof. The first estimate follows from Proposition 3.1 and (3.22). Similarly, to prove (3.31) we use Proposition 3.1 and (3.24). □

In particular, the previous lemma provides an upper bound of the total variation of the approximate solution uniform in ε. By Helly compactness theorem, the above lemmas yield an existence result to (1.1). We now proceed towards an estimate of the Lipschitz constant for F^ε uniform in ε.

Lemma 3.10. *Fix $M > 0$, $N \in \mathbb{N}$ and let $T = t_0 + N\varepsilon$. Consider $\mathbf{p}_1, \mathbf{p}_2$ in $\mathcal{D}_M(\mathcal{G}^\varepsilon)$ with $\max\{\mathrm{TV}(\mathbf{p}_{1|[0,T]}), \mathrm{TV}(\mathbf{p}_{2|[0,T]})\} \leqslant R$ and a pseudopolygonal γ joining $\mathbf{p}_1$ to $\mathbf{p}_2$. Then, for all $t \in [t_0, T]$, there exist weights uniformly bounded from above by a quantity dependent from M and T but not from ε, such that for all $t \in [t_0, T]$,*

$$\left\|F^\varepsilon_{t_0,t} \circ \gamma\right\|_\varepsilon \leqslant e^{\int_{t_0}^t \tilde l(\tau)\,d\tau} \cdot \|\gamma\|_\varepsilon, \qquad \Xi_\varepsilon\big(F^\varepsilon_{t_0,t} \circ \gamma\big) \leqslant e^{\int_{t_0}^t \tilde l(\tau)\,d\tau} \cdot \Xi_\varepsilon(\gamma).$$

Thanks to the construction above, this proof is entirely similar to that of [12, Lemma 4.7].

Proof of Theorem 2.3. Let $\varepsilon_\nu = 2^{-\nu}$ for $\nu \in \mathbb{N}$. For any data construct a sequence of approximate solutions by means of (3.29). A standard argument, see [2,7,8,14,15], shows that this is a Cauchy sequence in $\mathbf{L}^1$ and that it converges to a weak entropic solution of (1.1), proving points (1)–(3).

Consider now point (5)(b), with $\mathbf{p} = (\bar u, \tilde u, \Psi)$, $\mathbf{p}' = (\bar u', \tilde u', \Psi')$ and Ψ, Ψ' having Lipschitz constants $\mathcal{L}, \mathcal{L}'$. Then

$$\begin{aligned}\left\|u(t) - u'(t)\right\|_{\mathbf{L}^1} &\leqslant d(F_{t_0,t}\mathbf{p}, F_{t_0,t}\mathbf{p}') \leqslant C \lim_{\nu\to+\infty} d_{\varepsilon_\nu}\big(F_{t_0,t}\mathbf{p}_{\varepsilon_\nu}, F_{t_0,t}\mathbf{p}'_{\varepsilon_\nu}\big) \\ &\leqslant C e^{\int_{t_0}^t \tilde l(\tau)\,d\tau} \lim_{\nu\to+\infty} d_{\varepsilon_\nu}\big(\mathbf{p}_{\varepsilon_\nu}, \mathbf{p}'_{\varepsilon_\nu}\big) \leqslant C e^{\int_{t_0}^t \tilde l(\tau)\,d\tau} d(\mathbf{p}, \mathbf{p}')\end{aligned}$$

$$\begin{aligned}&\leqslant Ce^{\int_{t_0}^{t}\tilde{l}(\tau)\,d\tau}\left(\|\bar{u}-\bar{u}'\|_{\mathbf{L}^1}+\|\Psi-\Psi'\|_{\mathbf{C}^0}\right)\\&\quad+Ce^{\int_{t_0}^{t}\tilde{l}(\tau)\,d\tau}\left(1+\max\{\mathcal{L},\mathcal{L}'\}\right)\|\tilde{u}-\tilde{u}'\|_{\mathbf{L}^1}.\end{aligned}$$

Point (5)(a) follows, in the case $\tilde{u}=\tilde{u}'$, approximating Ψ and Ψ' through suitable sequence of Lipschitz functions converging uniformly on $[t_0, T]$.

Finally, point (6) follows from Lemma 3.9. □

References

[1] D. Amadori, R.M. Colombo, Continuous dependence for 2×2 conservation laws with boundary, J. Differential Equations 138 (1997) 229–266.

[2] D. Amadori, L. Gosse, G. Guerra, Global BV entropy solutions and uniqueness for hyperbolic systems of balance laws, Arch. Ration. Mech. Anal. 162 (2002) 327–366.

[3] A. Aw, M. Rascle, Resurrection of "second order" models of traffic flow, SIAM J. Appl. Math. 60 (2000) 916–938 (electronic).

[4] P. Bagnerini, R.M. Colombo, A. Corli, On the role of source terms in continuum traffic flow models, preprint, 2005.

[5] P. Baiti, A. Bressan, The semigroup generated by a Temple class system with large data, Differential Integral Equations 10 (1997) 401–418.

[6] S. Bianchini, The semigroup generated by a Temple class system with non-convex flux function, Differential Integral Equations 13 (2000) 1529–1550.

[7] A. Bressan, Hyperbolic Systems of Conservation Laws: The One-Dimensional Cauchy Problem, Oxford Lecture Series in Mathematics and Its Applications, vol. 20, Oxford Univ. Press, Oxford, 2000.

[8] A. Bressan, R.M. Colombo, The semigroup generated by 2×2 conservation laws, Arch. Ration. Mech. Anal. 133 (1995) 1–75.

[9] R.M. Colombo, A 2×2 hyperbolic traffic flow model, Math. Comput. Modelling 35 (2002) 683–688, Traffic flow—modelling and simulation.

[10] R.M. Colombo, Hyperbolic phase transitions in traffic flow, SIAM J. Appl. Math. 63 (2002) 708–721.

[11] R.M. Colombo, Wave front tracking in systems of conservation laws, Appl. Math. 49 (2004) 501–537, course delivered at the school Mathematical Theory in Fluid Mechanics.

[12] R.M. Colombo, A. Corli, On a class of hyperbolic balance laws, J. Hyperbolic Differential Equations 1 (2004) 725–745.

[13] R.M. Colombo, A. Groli, On the initial boundary value problem for Temple systems, Nonlinear Anal. 56 (2004) 569–589.

[14] R.M. Colombo, N.H. Risebro, Continuous dependence in the large for some equations of gas dynamics, Comm. Partial Differential Equations 23 (1998) 1693–1718.

[15] G. Crasta, B. Piccoli, Viscosity solutions and uniqueness for systems of inhomogeneous balance laws, Discrete Contin. Dyn. Syst. 3 (1997) 477–502.

[16] C.M. Dafermos, Hyperbolic Conservation Laws in Continuum Physics, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Sciences), vol. 325, Springer-Verlag, Berlin, 2000.

[17] F. Dubois, P. Lefloch, Boundary conditions for nonlinear hyperbolic systems of conservation laws, J. Differential Equations 71 (1988) 93–122.

[18] A.F. Filippov, Differential Equations with Discontinuous Righthand Sides, Kluwer Academic, Dordrecht, 1988.

[19] D. Hoff, Invariant regions for systems of conservation laws, Trans. Amer. Math. Soc. 289 (1985) 591–610.

[20] M. Nagumo, Über die Lage der Integralkurven gewöhnlicher Differentialgleichungen, Proc. Phys.-Math. Soc. Japan (3) 24 (1942) 551–559.

[21] D. Serre, Solutions à variations bornées pour certains systèmes hyperboliques de lois de conservation, J. Differential Equations 68 (1987) 137–168.