

Fourier multipliers and periodic solutions of delay equations in Banach spaces

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Abstract

In this paper we characterize the existence and uniqueness of periodic solutions of inhomogeneous abstract delay equations and establish maximal regularity results for strong solutions. The conditions are obtained in terms of R -boundedness of linear operators determined by the equations and L^p -Fourier multipliers. Periodic mild solutions are also studied and characterized.

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1. Introduction

The aim of this paper is the study of the equation

$$u'(t) = Au(t) + Fu_t + f(t), \quad t \in \mathbb{R}, \quad (1.1)$$

where $(A, D(A))$ is a (unbounded) linear operator on a Banach space X , $u_t(\cdot) = u(t + \cdot)$ on $[-r, 0]$, $r > 0$, and the delay operator F is supposed to belong to $\mathcal{B}(L^p([-r, 0], X), X)$ for some $1 \leq p < \infty$. The state space $L^p([-r, 0], X)$ is a typical choice with regards to certain applications (e.g., to control theory, or to numerical methods, see [12]).

First studies on Eq. (1.1) goes back to J. Hale [10] and G. Webb [20]. Recent references on partial differential equations with delay can be found in [22]. The problem to find conditions

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for all solutions of (1.1) to be periodic arises naturally from recent studies on the existence of (almost) periodic solutions of evolution equations, see, e.g., [17,18].

Recently, a significant progress has been made in finding sufficient conditions for operator valued functions to be L^p -Fourier multipliers, see [3,21], the monograph [8] and literature therein. In particular, in [4] the theory of operator-valued Fourier multipliers is applied to obtain results on the hyperbolicity of delay equations and in [16] to obtain stability of linear control systems in Banach spaces.

On the other hand, various connections of periodicity for differential equations and Fourier multipliers were recently noticed in the work by Arendt and Bu [3,13,14].

In this paper we are able to give necessary and sufficient conditions in order to obtain existence and uniqueness of periodic solutions for Eq. (1.1) in the space $L^p(\mathbb{T}, X)$. In contrast with above papers on the subject, we do not assume that A generates a C_0 -semigroup. Instead, our results involves *UMD*-spaces and *R*-boundedness, which are not too restrictive conditions for applications concerning nonlinear problems (cf. [8,15]). We remark that the Fourier multiplier approach used here allows to give a direct treatment of the equation, in contrast with the approach using the correspondence between (1.1) and the solutions of the abstract Cauchy problem

$$\mathcal{U}'(t) = \mathcal{AU}(t) + \mathcal{F}(t), \quad t \geq 0,$$

where $\mathcal{A} = \begin{pmatrix} A & F \\ 0 & d/d\sigma \end{pmatrix}$; see [5,19] and references therein.

In the second part, we study mild solutions for Eq. (1.1). Our main result shows a characterization of periodic mild solutions in terms of L^p -Fourier multipliers when the operator A involved is the generator of a strongly continuous semigroup.

2. Preliminaries

We denote by \mathbb{T} the group defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. There is an obvious identification between functions on \mathbb{T} and 2π -periodic functions on \mathbb{R} . We consider the interval $[0, 2\pi)$ as a model for \mathbb{T} .

Given $1 \leq p < \infty$, we denote by $L^p(\mathbb{T}, X)$ the space of all Bochner measurable vector-valued, p -integrable functions on \mathbb{T} .

For a function $f \in L^1(\mathbb{T}; X)$, we denote by $\hat{f}(k)$, $k \in \mathbb{Z}$, the k th Fourier coefficient of f :

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t) dt,$$

for all $k \in \mathbb{Z}$ and $t \in \mathbb{R}$.

Denote $f_\tau(t) := f(t + \tau)$, $\tau \in \mathbb{T}$; then it follows from the definition that $\hat{f}_\tau(k) = e^{ik\tau} \hat{f}(k)$.

Let $f \in L^p(\mathbb{T}, X)$. Then by Fejér's theorem, one has

$$f = \lim_{n \rightarrow \infty} \sigma_n(f) \tag{2.1}$$

in $L^p(\mathbb{T}, X)$ where

$$\sigma_n(f) := \frac{1}{n+1} \sum_{m=0}^n \sum_{k=-m}^m e_k \hat{f}(k)$$

with $e_k(t) := e^{ikt}$.

A Banach space X is said to be *UMD*, if the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for some (and then all) $p \in (1, \infty)$. Here the Hilbert transform H of a function $f \in \mathcal{S}(\mathbb{R}, X)$, the Schwartz space of rapidly decreasing X -valued functions, is defined by

$$Hf := \frac{1}{\pi} P V \left(\frac{1}{t} \right) * f.$$

These spaces are also called \mathcal{HT} spaces. It is a well-known theorem that the set of Banach spaces of class \mathcal{HT} coincides with the class of *UMD* spaces. This has been shown by Bourgain [6] and Burkholder [7].

Let X, Y be Banach spaces. We denote by $\mathcal{B}(X, Y)$ the space of all bounded linear operators from X to Y . When $X = Y$, we write simply $\mathcal{B}(X)$. For a linear operator A on X , we denote the domain by $D(A)$ and its resolvent set by $\rho(A)$, and for $\lambda \in \rho(A)$, we write $R(\lambda, A) = (\lambda I - A)^{-1}$. By $[D(A)]$ we denote the domain of A equipped with the graph norm.

Definition 2.1. Let X and Y be Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{B}(X, Y)$ is called *R*-bounded, if there is a constant $C > 0$ and $p \in [1, \infty)$ such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and for all independent, symmetric, $\{-1, 1\}$ -valued random variables r_j on a probability space $(\Omega, \mathcal{M}, \mu)$ the inequality

$$\left\| \sum_{j=1}^N r_j T_j x_j \right\|_{L^p(\Omega, Y)} \leq C \left\| \sum_{j=1}^N r_j x_j \right\|_{L^p(\Omega, X)} \quad (2.2)$$

is valid. The smallest such C is called *R*-bound of \mathcal{T} , we denote it by $R_p(\mathcal{T})$.

We remark that large classes of classical operators are *R*-bounded (cf. [9] and references therein). Hence, this assumption is not too restrictive for the applications that we consider in this article.

Remark 2.2. Several properties of *R*-bounded families can be found in the recent monograph of Denk et al. [8]. For the reader's convenience, we summarize here from [8, Section 3] some results.

(a) If $\mathcal{T} \subset \mathcal{B}(X, Y)$ is *R*-bounded then it is uniformly bounded, with

$$\sup\{\|T\|: T \in \mathcal{T}\} \leq R_p(\mathcal{T}).$$

(b) The definition of *R*-boundedness is independent of $p \in [1, \infty)$.

(c) When X and Y are Hilbert spaces, $\mathcal{T} \subset \mathcal{B}(X, Y)$ is *R*-bounded if and only if \mathcal{T} is uniformly bounded.

(d) Let X, Y be Banach spaces and $\mathcal{T}, \mathcal{S} \subset \mathcal{B}(X, Y)$ be *R*-bounded. Then

$$\mathcal{T} + \mathcal{S} = \{T + S: T \in \mathcal{T}, S \in \mathcal{S}\}$$

is *R*-bounded as well, and $R_p(\mathcal{T} + \mathcal{S}) \leq R_p(\mathcal{T}) + R_p(\mathcal{S})$.

(e) Let X, Y, Z be Banach spaces, and $\mathcal{T} \subset \mathcal{B}(X, Y)$ and $\mathcal{S} \subset \mathcal{B}(Y, Z)$ be *R*-bounded. Then

$$\mathcal{ST} = \{ST: T \in \mathcal{T}, S \in \mathcal{S}\}$$

is *R*-bounded, and $R_p(\mathcal{ST}) \leq R_p(\mathcal{S})R_p(\mathcal{T})$.

A very useful tool in connection with R -boundedness is the contraction principle of Kahane, which we state as a lemma. A proof can be found in [8, Lemma 3.5].

Lemma 2.3. *Let X be a Banach space, $n \in \mathbb{N}$, $x_j \in X$, r_j independent, symmetric, $\{-1, 1\}$ -valued random variables on a probability space $(\Omega, \mathcal{M}, \mu)$, and $\alpha_j, \beta_j \in \mathbb{C}$ such that $|\alpha_j| \leq |\beta_j|$, for each $j = 1, \dots, N$. Then*

$$\left\| \sum_{j=1}^N \alpha_j r_j x_j \right\|_{L^p(\Omega, X)} \leq 2 \left\| \sum_{j=1}^N \beta_j r_j x_j \right\|_{L^p(\Omega, X)}.$$

The constant 2 can be omitted in case where α_j and β_j are real.

Definition 2.4. For $1 \leq p < \infty$, we say that a sequence $\{M_k\}_{k \in \mathbb{Z}} \subset \mathcal{B}(X, Y)$ is an L^p -multiplier if, for each $f \in L^p(\mathbb{T}, X)$, there exists $u \in L^p(\mathbb{T}, Y)$ such that

$$\hat{u}(k) = M_k \hat{f}(k) \quad \text{for all } k \in \mathbb{Z}.$$

The following theorem, due to Arendt and Bu [3, Theorem 1.3], is the discrete analogue of the operator-valued version of Mikhlin's theorem due to Weis [21] and plays an important role in our investigations.

Theorem 2.5. *Let X, Y be UMD spaces and let $\{M_k\}_{k \in \mathbb{Z}} \subseteq \mathcal{B}(X, Y)$. If the sets $\{M_k\}_{k \in \mathbb{Z}}$ and $\{k(M_{k+1} - M_k)\}_{k \in \mathbb{Z}}$ are R -bounded, then $\{M_k\}_{k \in \mathbb{Z}}$ is an L^p -multiplier for $1 < p < \infty$.*

3. A criterion for periodic solutions

We consider in this section the equation

$$u'(t) = Au(t) + Fu_t + f(t), \quad t \in \mathbb{T}, \quad (3.1)$$

where $A : D(A) \subseteq X \rightarrow X$ is a linear, closed operator; $f \in L^p(\mathbb{T}, X)$, $p \geq 1$; for $r_{2\pi} := 2\pi N$ (some $N \in \mathbb{N}$) $F : L^p([-r_{2\pi}, 0], X) \rightarrow X$ is a linear, bounded operator and u_t is an element of $L^p([-r_{2\pi}, 0], X)$ which is defined as $u_t(\theta) = u(t + \theta)$ for $-r_{2\pi} \leq \theta \leq 0$. Note that we identify the space $L^p(\mathbb{T}, X)$ of vector-valued functions defined on $[0, 2\pi)$ to their periodic extension to \mathbb{R} .

We denote

$$H^{1,p}(\mathbb{T}; X) = \{u \in L^p(\mathbb{T}; X) : \exists v \in L^p(\mathbb{T}; X), \hat{v}(k) = ik\hat{u}(k) \text{ for all } k \in \mathbb{Z}\}.$$

As shown in [3, pp. 226–227], for every $u \in H^{1,p}(\mathbb{T}; X)$ there exists a unique $v \in H^{1,p}(\mathbb{T}; X)$ such that

$$u(t) = u(0) + \int_0^t v(\xi) d\xi, \quad \text{a.a. } t \in [0, 2\pi],$$

and $u(0) = u(2\pi)$. We will identify u with this continuous representative.

Definition 3.1. We say that a function $u \in H^{1,p}(\mathbb{T}; X)$ is a strong L^p -solution of (3.1) if $u(t) \in D(A)$ and Eq. (3.1) holds for a.a. $t \in [0, 2\pi)$.

Denote by $e_\lambda(t) := e^{i\lambda t}$ for all $\lambda \in \mathbb{R}$, and define the operators $\{B_\lambda\}_{\lambda \in \mathbb{R}} \subseteq \mathcal{B}(X)$ by

$$B_\lambda x = F(e_\lambda x), \quad \text{for all } \lambda \in \mathbb{R} \text{ and } x \in X. \quad (3.2)$$

Defining the *real spectrum* of (3.1) by

$$\sigma(\Delta) = \{s \in \mathbb{R}: isI - A - B_s \in \mathcal{B}(D(A), X) \text{ is not invertible}\},$$

we prove the following result.

Proposition 3.2. *Let A be a closed linear operator defined on a UMD space X . Suppose that $\mathbb{Z} \cap \sigma(\Delta) = \emptyset$. Then the following assertions are equivalent:*

- (i) $\{ik(ikI - A - B_k)^{-1}\}_{k \in \mathbb{Z}}$ is an L^p -multiplier for $1 < p < \infty$.
- (ii) $\{ik(ikI - A - B_k)^{-1}\}_{k \in \mathbb{Z}}$ is R -bounded.

Proof. By [3, Proposition 1.11] it follows that (i) implies (ii). Conversely, define $M_k = ik(C_k - A)^{-1}$, where $C_k := ikI - B_k$. By Theorem 2.5 is sufficient to prove that the set $\{k(M_{k+1} - M_k)\}_{k \in \mathbb{Z}}$ is R -bounded. In fact, we claim first that the set $\{B_k\}_{k \in \mathbb{Z}}$ is R -bounded since given $x_j \in D(A)$ we have

$$\begin{aligned} \left\| \sum_{j=1}^N r_j B_j x_j \right\|_{L^p(0,1,X)}^p &= \int_0^1 \left\| \sum_{j=1}^N r_j(t) F(e_j x_j) \right\|_X^p dt \\ &= \int_0^1 \left\| F \left(\sum_{j=1}^N r_j(t) e_j x_j \right) \right\|_X^p dt \\ &\leq \|F\|^p \int_0^1 \left\| \sum_{j=1}^N r_j(t) e_j x_j \right\|_{L^p(-r_{2\pi}, 0, X)}^p dt \\ &= \|F\|^p \int_0^1 \int_{-r_{2\pi}}^0 \left\| \sum_{j=1}^N r_j(t) e_j(s) x_j \right\|_X^p ds dt \\ &= \|F\|^p \int_{-r_{2\pi}}^0 \int_0^1 \left\| \sum_{j=1}^N r_j(t) e_j(s) x_j \right\|_X^p dt ds \\ &= \|F\|^p \int_{-r_{2\pi}}^0 \left\| \sum_{j=1}^N r_j e_j(s) x_j \right\|_{L^p(0,1,X)}^p ds. \end{aligned}$$

By Kahane's contraction principle (Lemma 2.3) we obtain

$$\begin{aligned} \left\| \sum_{j=1}^N r_j B_j x_j \right\|_{L^p(0,1,X)}^p &\leq 2 \|F\|^p \int_{-r_{2\pi}}^0 \left\| \sum_{j=1}^N r_j x_j \right\|_{L^p(0,1,X)}^p ds \\ &= 2r_{2\pi} \|F\|^p \left\| \sum_{j=1}^N r_j x_j \right\|_{L^p(0,1,X)}^p. \end{aligned}$$

We conclude that

$$R_p(\{B_k\}_{k \in \mathbb{Z}}) \leq (2r_{2\pi})^{1/p} \|F\| \quad (3.3)$$

and the claim is proved. Next we note the following identities:

$$\begin{aligned} k[M_{k+1} - M_k] &= k[i(k+1)(C_{k+1} - A)^{-1} - ik(C_k - A)^{-1}] \\ &= k(C_{k+1} - A)^{-1}[i(k+1)(C_k - A) - ik(C_{k+1} - A)](C_k - A)^{-1} \\ &= k(C_{k+1} - A)^{-1}[ik(C_k - C_{k+1}) + i(C_k - A)](C_k - A)^{-1} \\ &= k(C_{k+1} - A)^{-1}(C_k - C_{k+1})ik(C_k - A)^{-1} + ik(C_{k+1} - A)^{-1} \\ &= k(C_{k+1} - A)^{-1}(B_{k+1} - B_k - iI)ik(C_k - A)^{-1} + ik(C_{k+1} - A)^{-1}. \end{aligned}$$

Since products and sums of R -bounded sequences are R -bounded (cf. Remark 2.2), the proof is finished. \square

Proposition 3.3. *Let X be a Banach space and let $A: D(A) \subset X \rightarrow X$ be a closed linear operator. Suppose that for every $f \in L^p(\mathbb{T}, X)$, there exists a unique strong L^p -solution of (3.1) for $1 < p < \infty$. Then*

- (i) $\mathbb{Z} \cap \sigma(\Delta) = \emptyset$,
- (ii) $\{ik(ikI - A - B_k)^{-1}\}_{k \in \mathbb{Z}}$ is R -bounded.

Proof. (i) Follows the same lines of [3, Theorem 2.3]. We give the proof for the sake of completeness. Let $k \in \mathbb{Z}$ and $y \in X$. Define $f(t) = e_k(t)y$. By hypothesis, there exists $u \in H^{1,p}(\mathbb{T}, X)$ such that $u(t) \in D(A)$ and $u'(t) = Au(t) + Fu_t + f(t)$. Taking Fourier transform on both sides, we have $\hat{u}(k) \in D(A)$ and, since F is linear and bounded, we obtain (cf. (3.2))

$$ik\hat{u}(k) = A\hat{u}(k) + F(e_k\hat{u}(k)) + \hat{f}(k) = A\hat{u}(k) + B_k\hat{u}(k) + \hat{f}(k) = A\hat{u}(k) + B_k\hat{u}(k) + y.$$

Thus $ikI - A - B_k$ is surjective for all $k \in \mathbb{Z}$. Let $x \in D(A)$. If $(ik - A - B_k)x = 0$, that is $Ax + B_kx = ikx$, then $u(t) = e_k(t)y$ defines a periodic solution of (3.1). In fact, since $u_t(\theta) = e^{ik\theta}u(t)$ we obtain $u_t = e_ku(t)$ and then $Fu_t = F(e_ku(t)) = B_ku(t)$. Hence

$$Au(t) + Fu_t = e^{ikt}Ax + e^{ikt}B_kx = ik e^{ikt}x = u'(t).$$

Hence $u \equiv 0$ by the assumption of uniqueness, and thus $x = 0$.

(ii) By Proposition 3.2 is sufficient to show that the set $\{ik(ikI - A - B_k)^{-1}\}_{k \in \mathbb{Z}}$ is an L^p -multiplier. Let $f \in L^p(\mathbb{T}, X)$. By hypothesis, there exists a unique $u \in H^{1,p}(\mathbb{T}, X)$ such that $u(t) \in D(A)$ and Eq. (3.1) is valid. Taking Fourier transforms, we deduce that $\hat{u}(k) \in D(A)$ and $(ik - A - B_k)\hat{u}(k) = \hat{f}(k)$ for all $k \in \mathbb{Z}$. Hence

$$ik\hat{u}(k) = ik(ik - A - B_k)^{-1}\hat{f}(k)$$

for all $k \in \mathbb{Z}$. On the other hand, since $u \in H^{1,p}(\mathbb{T}, X)$, there exists $v \in L^p(\mathbb{T}, X)$ such that $\hat{v}(k) = ik\hat{u}(k)$. This proves the claim. \square

Our main result in this section establishes that the converse of Proposition 3.3 is true, provided X is an UMD space.

Theorem 3.4. *Let X be a UMD space and let $A: D(A) \subset X \rightarrow X$ be a closed linear operator. Then the following assertions are equivalent for $1 < p < \infty$:*

- (i) For every $f \in L^p(\mathbb{T}, X)$, there exists a unique strong L^p -solution of (3.1).
 (ii) $\mathbb{Z} \cap \sigma(\Delta) = \emptyset$ and $\{ik(ikI - A - B_k)^{-1}\}_{k \in \mathbb{Z}}$ is R -bounded.

Proof. Let $f \in L^p(\mathbb{T}, X)$. Define $N_k = (ikI - A - B_k)^{-1}$. By Proposition 3.2, the family $\{M_k := ikN_k\}_{k \in \mathbb{Z}}$ is an L^p -multiplier. By [3, Lemma 2.2], this is equivalent to the fact that the family $\{N_k\}_{k \in \mathbb{Z}}$ is an L^p -multiplier that maps $L^p(\mathbb{T}, X)$ into $H^{1,p}(\mathbb{T}, X)$, i.e., there exists $u \in H^{1,p}(\mathbb{T}, X)$ such that

$$\hat{u}(k) = N_k \hat{f}(k) = (ikI - A - B_k)^{-1} \hat{f}(k). \quad (3.4)$$

In particular, $u \in L^p(\mathbb{T}, X)$ and there exists $v \in L^p(\mathbb{T}, X)$ such that

$$\hat{u}'(k) := \hat{v}(k) = ik\hat{u}(k). \quad (3.5)$$

We claim that the family $\{B_k N_k\}_{k \in \mathbb{Z}}$ is an L^p -multiplier. In fact, it is clear that $\{B_k N_k\}_{k \in \mathbb{Z}}$ is R -bounded. On the other hand, since $\{B_k\}_{k \in \mathbb{Z}}$ is R -bounded (cf. the proof of Proposition 3.2) the identity

$$k(B_{k+1}N_{k+1} - B_k N_k) = B_{k+1}(kN_{k+1}) - B_k(kN_k)$$

shows that $\{k(B_{k+1}N_{k+1} - B_k N_k)\}_{k \in \mathbb{Z}}$ is also R -bounded. Then the claim follows from Theorem 2.5.

By Fejer's theorem (cf. (2.1)) one has in $L^p([-r_{2\pi}, 0], X)$,

$$u_t(\theta) = u(t + \theta) = \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{m=0}^n \sum_{k=-m}^m e^{ikt} e^{ik\theta} \hat{u}(k).$$

Hence in $L^p(\mathbb{T}, X)$ we obtain

$$u_t = \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{m=0}^n \sum_{k=-m}^m e^{ikt} e_k \hat{u}(k).$$

Then, since F is linear and bounded

$$Fu_t = \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{m=0}^n \sum_{k=-m}^m e^{ikt} F(e_k \hat{u}(k)) = \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{m=0}^n \sum_{k=-m}^m e^{ikt} B_k \hat{u}(k).$$

By (3.4) and (3.5) we have

$$\hat{u}'(k) = ik\hat{u}(k) = A\hat{u}(k) + B_k \hat{u}(k) + \hat{f}(k)$$

for all $k \in \mathbb{Z}$. Then using that A is closed we conclude that $u(t) \in D(A)$ (cf. [3, Lemma 3.1]) and, from the uniqueness theorem of Fourier coefficients, that (3.1) is valid for a.a. $t \in \mathbb{T}$.

To show uniqueness, let $u \in L^p(\mathbb{T}, D(A)) \cap H^{1,p}(\mathbb{T}, X)$ be such that $u'(t) = Au(t) + Fu_t$, $t \in \mathbb{T}$, then $\hat{u}(k) \in D(A)$ and $ik\hat{u}(k) = A\hat{u}(k) + B_k \hat{u}(k)$. Since $\mathbb{Z} \cap \sigma(\Delta) = \emptyset$ this implies that $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$ and thus $u = 0$. \square

The solution $u(\cdot)$ given in Theorem 3.4 actually satisfies the following maximal regularity property.

Corollary 3.5. *In the context of Theorem 3.4, if condition (ii) is fulfilled, we have u' , Au , $Fu_{(\cdot)} \in L^p(\mathbb{T}, X)$. Moreover, there exists a constant $C > 0$ independent of $f \in L^p(\mathbb{T}, X)$ such that*

$$\|u'\|_{L^p(\mathbb{T}, X)} + \|Au\|_{L^p(\mathbb{T}, X)} + \|Fu_{(\cdot)}\|_{L^p(\mathbb{T}, X)} \leq C \|f\|_{L^p(\mathbb{T}, X)}. \quad (3.6)$$

Proof. The first statement follows from the proof of Theorem 3.4. We verify this for Au . From Eq. (3.4) we have $\hat{u}(k) = (ikI - A - B_k)^{-1} \hat{f}(k)$; hence

$$A\hat{u}(k) = ikN_k \hat{f}(k) - B_k N_k \hat{f}(k) - \hat{f}(k).$$

Since $g \in L^p(\mathbb{T}, X)$ and $\{ikN_k\}$, $\{B_k N_k\}$ are L^p -multipliers, the claim follows. On the other hand, $Fu_t = u'(t) - Au(t) - f(t)$, which implies that $Fu_{(\cdot)} \in L^p(\mathbb{T}, X)$. The second statement is a consequence of the closed graph theorem. \square

Remark 3.6. From the inequality (3.6) we deduce that the operator L defined by

$$D(L) = H^{1,p}(\mathbb{T}; X) \cap L^p(\mathbb{T}, [D(A)]), \quad (3.7)$$

$$(Lu)(t) = u'(t) - Au(t) - Fu_t \quad (3.8)$$

is an isomorphism onto. Indeed, since A is closed, the space $H^{1,p}(\mathbb{T}; X) \cap L^p(\mathbb{T}, [D(A)])$ becomes a Banach space under the norm

$$\|u\| := \|u\|_p + \|u'\|_p + \|Au\|_p.$$

We remark that such isomorphisms are crucial for the handling of nonlinear evolution equations (see [1]).

In the case of a Hilbert space, Theorem 3.4 takes a particularly simple form. It is remarkable that it corresponds essentially to the case where $X = \mathbb{C}$.

Corollary 3.7. *Let H be Hilbert space and let $A : D(A) \subset X \rightarrow X$ be a closed linear operator. Then the following assertions are equivalent for $1 < p < \infty$:*

- (i) *For every $f \in L^p(\mathbb{T}, X)$, there exists a unique strong L^p -solution of (3.1).*
- (ii) $\mathbb{Z} \cap \sigma(\Delta) = \emptyset$ and

$$\sup_{k \in \mathbb{Z}} \|k(ikI - A - B_k)^{-1}\| < \infty. \quad (3.9)$$

Proof. This is a consequence of Plancherel's theorem. \square

Example 3.8. Let A be a closed linear operator defined on a Hilbert space H and suppose that $i\mathbb{Z} \subset \rho(A)$ and $\sup_k \|A(ik - A)^{-1}\| =: M < \infty$. From the identity

$$ikI - A - B_k = (ik - A)(I - B_k(ik - A)^{-1})$$

it follows that $ikI - A - B_k$ is invertible whenever $\|B_k(ik - A)^{-1}\| < 1$.

Next observe that

$$\|B_k\| \leq r_{2\pi}^{1/p} \|F\|.$$

Hence

$$\|B_k(ik - A)^{-1}\| = \|B_k A^{-1} A(ik - A)^{-1}\| \leq r_{2\pi}^{1/p} \|F\| \|A^{-1}\| M =: \alpha.$$

Therefore, under the condition

$$\|F\| < \frac{1}{\|A^{-1}\| M r_{2\pi}^{1/p}} \quad (3.10)$$

we obtain that $\mathbb{Z} \cap \sigma(\Delta) = \emptyset$, and the identity

$$\begin{aligned} (ikI - A - B_k)^{-1} &= (ik - A)^{-1} (I - B_k(ik - A)^{-1}) \\ &= (ik - A)^{-1} \sum_{n=0}^{\infty} [B_k(ik - A)^{-1}]^n. \end{aligned} \quad (3.11)$$

It follows that

$$\|k(ikI - A - B_k)^{-1}\| \leq \|ik(ik - A)^{-1}\| \sum_{n=0}^{\infty} \|B_k(ik - A)^{-1}\|^n \leq \frac{1+M}{1-\alpha},$$

and hence condition (ii) in Corollary 3.7 is satisfied.

The above example can be adapted to obtain the following criterion in case of *UMD* spaces.

Theorem 3.9. *Let X be a *UMD* space and let $A : D(A) \subset X \rightarrow X$ be a closed linear operator such that $i\mathbb{Z} \subset \rho(A)$ and $R_p(\{A(ik - A)^{-1}\}_{k \in \mathbb{Z}}) =: M < \infty$. Suppose that*

$$\|F\| < \frac{1}{(2r_{2\pi})^{1/p} \|A^{-1}\| M}. \quad (3.12)$$

Then for every $f \in L^p(\mathbb{T}, X)$, there exists a unique strong L^p -solution of (3.1).

Proof. Since R -boundedness implies uniform boundedness by Remark 2.2(a), we obtain $\sup_k \|A(ik - A)^{-1}\| \leq R_p(A(ik - A)^{-1}) = M$. Also note that (3.12) implies (3.10). Then we obtain that $ikI - A - B_k$ is invertible for all $k \in \mathbb{Z}$ and the identity (3.11) is valid.

Using Remark 2.2, and induction we have

$$\begin{aligned} R_p(ik(ik - A)^{-1} [B_k(ik - A)^{-1}]^n) \\ \leq R_p(ik(ik - A)^{-1}) [R_p(B_k A^{-1} A(ik - A)^{-1})]^n \\ \leq R_p(ik(ik - A)^{-1}) [R_p(B_k A^{-1})]^n [R_p(A(ik - A)^{-1})]^n \\ \leq R_p(ik(ik - A)^{-1}) \|A^{-1}\|^n [R_p(B_k)]^n [R_p(A(ik - A)^{-1})]^n. \end{aligned}$$

By (3.3) we obtain

$$\begin{aligned} R_p(ik(ik - A)^{-1} [B_k(ik - A)^{-1}]^n) &\leq R_p(ik(ik - A)^{-1}) \|A^{-1}\|^n ((2r_{2\pi})^{1/p} \|F\|)^n M^n \\ &= R_p(ik(ik - A)^{-1}) \alpha^n, \end{aligned}$$

where $\alpha := \|A^{-1}\| (2r_{2\pi})^{1/p} \|F\| M$. Finally by (3.12), Remark 2.2, and taking into account that R -boundedness is preserved by convergence in the strong operator topology, one has

$$R_p((ikI - A - B_k)^{-1}) \leq R_p(ik(ik - A)^{-1}) \frac{1}{1-\alpha} \leq \frac{M+1}{1-\alpha}. \quad (3.13)$$

This proves that $\{(ikI - A - B_k)^{-1}\}$ is R -bounded and the conclusion follows from Theorem 3.4. \square

To close this section, and as an application, we want to compare the periodic problem

$$u'(t) = Au(t) + f(t), \quad t \in \mathbb{T}, \quad (3.14)$$

with the delay equation (3.1). As a direct consequence of Theorem 3.9 and [3, Theorem 2.3] we have the following result.

Corollary 3.10. Assume that X is a UMD space. Let $1 < p < \infty$. If for each $f \in L^p(\mathbb{T}, X)$ there is a unique strong L^p -solution of Eq. (3.14) and condition (3.12) is satisfied, then for all $f \in L^p(\mathbb{T}, X)$ there is a unique strong L^p -solution of Eq. (3.1).

4. Periodic mild solutions

In this section we consider mild solutions of Eq. (3.1) in the following sense.

Definition 4.1. Let A be a closed linear operator on X . A function $u \in C(\mathbb{T}, X)$ is called a 2π -periodic mild solution of the problem (3.1) if

$$\begin{cases} \int_0^t u(s) ds \in D(A), \\ u(t) = A \int_0^t u(s) ds + \int_0^t F u_s ds + \int_0^t f(s) ds + u(0), \end{cases} \quad (4.1)$$

for all $0 \leq t \leq 2\pi$.

Lemma 4.2. Let $u \in C(\mathbb{T}, X)$. Then

$$\widehat{F(u_s)}(k) = B_k \hat{u}(k).$$

Proof. Follows easily taking into account that for all $\theta \in [-r_{2\pi}, 0]$ we have

$$(e_k \hat{u}(k))(\theta) = \frac{1}{2\pi} \int_0^{2\pi} e^{ik(\theta-s)} u(s) ds = \frac{1}{2\pi} \int_0^{2\pi} e^{-iks} u_s(\theta) ds,$$

and hence, since F is bounded,

$$B_k \hat{u}(k) = F(e_k \hat{u}(k)) = F\left(\frac{1}{2\pi} \int_0^{2\pi} e^{-iks} u_s ds\right) = \frac{1}{2\pi} \int_0^{2\pi} e^{-iks} F u_s ds. \quad \square$$

From the above lemma and following the same lines as in the proof of [3, Proposition 3.2] we can prove the following result.

Proposition 4.3. Let $u \in C(\mathbb{T}, X)$ be given. Assume that $\overline{D(A)} = X$. Then u is a 2π -periodic mild solution of (3.1) if and only if

$$\hat{u}(k) \in D(A) \quad \text{and} \quad (ik - A - B_k) \hat{u}(k) = \hat{f}(k) \quad \text{for all } k \in \mathbb{Z}. \quad (4.2)$$

Proof. Suppose u is a 2π -periodic mild solution. Letting $t = 2\pi$, we obtain $\hat{u}(0) \in D(A)$ and by Lemma 4.2:

$$A \hat{u}(0) + B_0 \hat{u}(0) + \hat{f}(0) = 0. \quad (4.3)$$

In particular, it shows that (4.2) is satisfied in case $k = 0$. Define

$$v(t) = \int_0^t u(s) ds \quad \text{and} \quad g(t) = u(t) - u(0) - \int_0^t f(s) ds - \int_0^t Fu_s ds.$$

Then by [3, Lemma 3.1], $\hat{v}(k) \in D(A)$ and $A\hat{v}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}, k \neq 0$. But, for $k \neq 0$ we have

$$\hat{v}(k) = \frac{-1}{ik} \hat{u}(0) + \frac{1}{ik} \hat{u}(k)$$

and

$$\begin{aligned} \hat{g}(k) &= \hat{u}(k) - \left[\frac{1}{ik} \hat{f}(k) - \frac{1}{ik} \hat{f}(0) \right] - \left[\frac{1}{ik} B_k \hat{u}(k) - \frac{1}{ik} B_0 \hat{u}(0) \right] \\ &= \hat{u}(k) - \frac{1}{ik} \hat{f}(k) - \frac{1}{ik} B_k \hat{u}(k) + \frac{1}{ik} \hat{f}(0) + \frac{1}{ik} B_0 \hat{u}(0). \end{aligned}$$

Then by (4.3), $\frac{1}{ik} A\hat{u}(k) = \hat{u}(k) - \frac{1}{ik} \hat{f}(k) - \frac{1}{ik} B_k \hat{u}(k)$ and hence (4.2) is satisfied for all $k \neq 0$.

Conversely, suppose (4.2) is valid and let $x^* \in D(A^*)$. Define

$$w(s) = \langle u(s), A^* x^* \rangle + \langle f(s), x^* \rangle + \langle Fu_s, x^* \rangle \quad \text{and} \quad g(t) = \int_0^t w(s) - \langle u(t), x^* \rangle.$$

Then, by Lemma 4.2 and hypothesis we obtain

$$\hat{w}(k) = \langle \hat{u}(k), A^* x^* \rangle + \langle \hat{f}(k), x^* \rangle + \langle B_k \hat{u}(k), x^* \rangle = ik \langle \hat{u}(k), x^* \rangle,$$

for all $k \in \mathbb{Z}$. In particular, $\hat{w}(0) = 0$ and therefore

$$\hat{g}(k) = \frac{-1}{ik} \hat{w}(0) + \frac{1}{ik} \hat{w}(k) - \langle \hat{u}(k), x^* \rangle = 0$$

for all $k \neq 0$. We conclude that g is constant. Then $g(t) = g(0) = -\langle u(0), x^* \rangle$. We have proved that u is a weak* mild solution. It follows from [2, Proposition B.10] that u is a mild solution of (3.1) and the proof is finished. \square

The following result is an immediate consequence.

Proposition 4.4. Suppose $\overline{D(A)} = X$. Let $1 \leq p < \infty$ and assume that for all $f \in L^p(\mathbb{T}, X)$ there exists a unique mild solution of (3.1). Then $\mathbb{Z} \cap \sigma(\Delta) = \emptyset$ and $\{(ik - A - B_k)^{-1}\}_{k \in \mathbb{Z}}$ is an L^p -multiplier.

Proof. As in the proof of Proposition 3.3 we obtain that $\mathbb{Z} \cap \sigma(\Delta) = \emptyset$. Let $f \in L^p(\mathbb{T}, X)$ and u be the mild solution of (3.1). It follows from (4.2) that

$$\hat{u}(k) = (ik - A - B_k)^{-1} \hat{f}(k)$$

for all $k \in \mathbb{Z}$. The claim follows. \square

The main result in this section shows that if A generates a C_0 -semigroup then the converse of the above proposition is true.

Theorem 4.5. *Let A be the generator of a C_0 -semigroup $\{T(t)\}_{t \geq 0}$ and let $1 \leq p < \infty$. Then the following assertions are equivalent:*

- (i) *For all $f \in L^p(\mathbb{T}, X)$ there exists a unique 2π -periodic mild solution u of (3.1).*
- (ii) *$\mathbb{Z} \cap \sigma(\Delta) = \emptyset$ and $\{(ik - A - B_k)^{-1}\}_{k \in \mathbb{Z}}$ is an L^p -multiplier.*

Proof. Let $f \in L^p(\mathbb{T}, X)$ and define

$$f_N(t) = \frac{1}{N+1} \sum_{m=0}^N \sum_{k=-m}^m e^{ikt} \hat{f}(k).$$

Then $f = \lim_{N \rightarrow \infty} f_N$ in $L^p(\mathbb{T}, X)$. Define

$$u_N(t) = \frac{1}{N+1} \sum_{m=0}^N \sum_{k=-m}^m e^{ikt} (ik - A - B_k)^{-1} \hat{f}(k), \quad (4.4)$$

then $u_N \in C^1(\mathbb{T}, X)$ and satisfies

$$u_N(t) = T(t)u_N(0) + \int_0^t T(t-s)f_N(s)ds + \int_0^t T(t-s)(u_N)_s ds, \quad (4.5)$$

and u_N is even a strong solution of (3.1). Now by hypothesis, $u = \lim_{N \rightarrow \infty} u_N$ exists in $L^p(\mathbb{T}, X)$.

Since $u_N(0) = u_N(2\pi)$ we obtain taking $t = 2\pi$ in (4.5),

$$(I - T(2\pi))u_N(0) = \int_0^{2\pi} T(2\pi - s)f_N(s)ds + \int_0^{2\pi} T(2\pi - s)(u_N)_s ds,$$

where the right-hand side converges as $N \rightarrow \infty$. On the other hand, multiplication of (4.5) by $T(2\pi - t)$ and integration over \mathbb{T} yields

$$\begin{aligned} T(2\pi)u_N(0) &= \int_0^{2\pi} T(2\pi - t)u_N(t)dt - \int_0^{2\pi} T(2\pi - s) \int_0^t T(t-s)f_N(s)ds dt \\ &\quad - \int_0^{2\pi} T(2\pi - s) \int_0^t T(t-s)(u_N)_s ds dt \end{aligned}$$

and the right-hand side of this equality also converges as $N \rightarrow \infty$. This shows that $u_N(0) = (I - T(2\pi))u_N(0) + T(2\pi)u_N(0)$ tends to some $x \in X$. Hence (4.5) implies that u satisfies

$$u(t) = T(t)x + \int_0^t T(t-s)f(s)ds + \int_0^t T(t-s)u_s ds, \quad (4.6)$$

and $u(0) = u(2\pi)$. Finally, by [11, Lemma 2.11] this condition is equivalent to saying that u is a mild solution. Uniqueness of the solution follows from (4.2). The proof is complete. \square

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