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An existence result for elliptic equations with *VMO*-coefficients

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Abstract

In this paper we prove an existence and uniqueness theorem for the Dirichlet problem for linear elliptic equations (in unbounded domains) whose leading coefficients are locally *VMO* and satisfy a suitable condition at infinity.
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1. Introduction

Consider the Dirichlet problem

$$\begin{cases} u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega), \\ Lu = f, \quad f \in L^p(\Omega), \end{cases} \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^n$, $n \geqslant 3$, with a suitable regularity property, $p \in]1, +\infty[$ and L is the uniformly elliptic differential operator defined by

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$$L = -\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + a. \tag{1.2}$$

If Ω is bounded, problem (1.1) has been widely investigated by several authors under various hypotheses on the leading coefficients a_{ij}. In particular, when the a_{ij}'s are of class $C^0(\bar{\Omega})$, a complete reference for classical results on this subject is [8, Chapter 9]. If the coefficients a_{ij} are required to be discontinuous, a relevant existence and uniqueness theorem can be found in a paper by Miranda [9], where the author assumed that the a_{ij}'s belong to $W^{1,n}(\Omega)$ and that $p = 2$. This result was later generalized in different ways, supposing that the derivatives of the leading coefficients belong to certain wider spaces.

More recently, a relevant contribution to the theory has been given in [6,7], under the assumption that the a_{ij}'s are of class VMO, $a_i = a = 0$ and $p \in]1, +\infty[$ (recall here that VMO contains both spaces $C^0(\bar{\Omega})$ and $W^{1,n}(\Omega)$).

If the set Ω is unbounded, problem (1.1) has for instance been studied in [10], where the leading coefficients satisfy restrictions similar to those in [9] and $p = 2$, and in [3], where the investigation has been extended to the case $p \in]1, +\infty[$. The aim of this paper is to generalize the above quoted results in [6,7] to the case of unbounded open sets. In fact, under the hypotheses that the coefficients a_{ij} are in $VMO_{\mathrm{loc}}(\bar{\Omega})$ and satisfy a suitable condition at infinity, a $W^{2,p}$-estimate for the solutions of (1.1) will be obtained, and an existence and uniqueness result will be deduced.

2. Notation and function spaces

Let Ω be an open subset of $\mathbb{R}^n$, and let $\Sigma(\Omega)$ be the collection of all Lebesgue measurable subsets of Ω. For each $A \in \Sigma(\Omega)$, we denote by $|A|$ and χ_A the Lebesgue measure and the characteristic function of A, respectively. Moreover, put

$$A(x, r) = A \cap B(x, r) \quad \forall x \in \mathbb{R}^n, \ \forall r \in \mathbb{R}_+,$$

where $B(x, r)$ is the open ball of $\mathbb{R}^n$ of radius r centered at x, and

$$B_r = B(0, r) \quad \forall r \in \mathbb{R}_+.$$

If $X(\Omega)$ is a space of functions defined on Ω, we denote by $X_{\mathrm{loc}}(\bar{\Omega})$ the space of all functions $g : \Omega \to \mathbb{R}$ such that $\zeta g \in X\ (\Omega)$ for every $\zeta \in \mathfrak{D}(\bar{\Omega})$, where $\mathfrak{D}(\bar{\Omega}) = \{\xi_{|\Omega} \mid \xi \in C_0^\infty(\mathbb{R}^n)\}$.

For $p \in [1, +\infty[$ and $\lambda \in [0, n[$, denote by $M^{p,\lambda}(\Omega)$ the set of all functions $g \in L^p_{\mathrm{loc}}(\bar{\Omega})$ such that

$$\|g\|_{M^{p,\lambda}(\Omega)} = \sup_{\substack{\rho \in]0,1] \\ x \in \Omega}} \rho^{-\lambda/p} \|g\|_{L^p(\Omega(x,\rho))} < +\infty, \tag{2.1}$$

endowed with the norm defined by (2.1). Moreover, $M_0^{p,\lambda}(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $M^{p,\lambda}(\Omega)$. It is known (see, e.g., [11, Lemma 2.1]) that a function g (in $M^{p,\lambda}(\Omega)$) belongs to $M_0^{p,\lambda}(\Omega)$ if and only if

$$\lim_{t \to 0^+} \left(p_g(t) + \left\| (1 - \zeta_{1/t}) g \right\|_{M^{p,\lambda}(\Omega)} \right) = 0, \tag{2.2}$$

where

$$p_g(t) = \sup_{\substack{A \in \Sigma(\Omega) \\ \sup_{x \in \Omega} |A(x,1)| \leqslant t}} \|\chi_A g\|_{M^{p,\lambda}(\Omega)}, \quad t \in \mathbb{R}_+,$$

and ζ_r, $r \in \mathbb{R}_+$, is a function in $C_0^\infty(\mathbb{R}^n)$ such that

$$0 \leqslant \zeta_r \leqslant 1, \quad \zeta_{r|B_r} = 1, \quad \operatorname{supp} \zeta_r \subset B_{2r}.$$

Thus a *modulus of continuity* of $g \in M_0^{p,\lambda}(\Omega)$ is a function

$$\sigma_0[g] : \mathbb{R}_+ \to \mathbb{R}_+$$

such that

$$\sigma_0[g](t) \geqslant p_g(t) + \left\| (1 - \zeta_{1/t}) g \right\|_{M^{p,\lambda}(\Omega)} \quad \forall t \in \mathbb{R}_+, \qquad \lim_{t \to 0^+} \sigma_0[g](t) = 0.$$

In the following we will put $M^p(\Omega) = M^{p,0}(\Omega)$ and $M_0^p(\Omega) = M_0^{p,0}(\Omega)$.

If Ω has the property

$$\left|\Omega(x,\rho)\right| \geqslant C\rho^n \quad \forall x \in \Omega, \ \forall \rho \in]0,1], \tag{2.3}$$

where C is some positive constant independent of x and ρ, one can consider the space $BMO(\Omega, t)$, $t \in \mathbb{R}_+$, consisting of all functions g in $L^1_{\mathrm{loc}}(\bar{\Omega})$ such that

$$[g]_{BMO(\Omega,t)} = \sup_{\substack{x \in \Omega \\ r \in]0,t]}} \fint_{\Omega(x,r)} \left| g - \fint_{\Omega(x,r)} g \right| < +\infty, \tag{2.4}$$

where

$$\fint_{\Omega(x,r)} g = \left|\Omega(x,r)\right|^{-1} \int_{\Omega(x,r)} g.$$

If $g \in BMO(\Omega) = BMO(\Omega, t_C)$, with

$$t_C = \sup_{t \in \mathbb{R}_+} \left(\sup_{\substack{x \in \Omega \\ r \in]0,t]}} \frac{r^n}{|\Omega(x,r)|} \leqslant \frac{1}{C} \right),$$

we say that g is in $VMO(\Omega)$ when

$$\lim_{t \to 0^+} [g]_{BMO(\Omega,t)} = 0.$$

Moreover, a function $\eta[g] : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *modulus of continuity* of g in $VMO(\Omega)$ if

$$\eta[g](t) \geqslant [g]_{BMO(\Omega,t)} \quad \forall t \in \mathbb{R}_+, \qquad \lim_{t \to 0^+} \eta[g](t) = 0.$$

A more detailed account of properties of the above defined function spaces can be found in [11,12].

3. Some a priori bounds

Let Ω be an unbounded open subset of $\mathbb{R}^n$, $n \geqslant 3$, with the uniform $C^{1,1}$-regularity property, and let $p \in]1, +\infty[$. Consider in Ω the operator

$$L = -\sum_{i,j=1}^n a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i \frac{\partial}{\partial x_i} + a, \tag{3.1}$$

and suppose that the coefficients of L satisfy the following conditions:

$$(\mathrm{H}_1)\quad \begin{cases} a_{ij} = a_{ji} \in L^\infty(\Omega) \cap VMO_{\mathrm{loc}}(\bar{\Omega}), \quad i,j = 1,\dots,n, \\ \exists \nu \in \mathbb{R}_+ \colon \displaystyle\sum_{i,j=1}^n a_{ij}\xi_i\xi_j \geqslant \nu|\xi|^2 \quad \text{a.e. in } \Omega,\ \forall \xi \in \mathbb{R}^n, \end{cases}$$

there exist functions e_{ij}, $i,j = 1,\dots,n$, g and $\mu \in \mathbb{R}_+$ such that

$$(\mathrm{H}_2)\quad \begin{cases} e_{ij} = e_{ji} \in L^\infty(\Omega), \quad (e_{ij})_{x_h} \in M_0^{s,n-s}(\Omega), \quad \text{with } s \in]2,n],\ i,j,h = 1,\dots,n, \\ \displaystyle\sum_{i,j=1}^n e_{ij}\xi_i\xi_j \geqslant \mu|\xi|^2 \quad \text{a.e. in } \Omega,\ \forall \xi \in \mathbb{R}^n, \\ g \in L^\infty(\Omega), \quad g_0 = \operatorname*{ess\,inf}_{\Omega} g > 0, \\ \displaystyle\lim_{r\to+\infty} \sum_{i,j=1}^n \|e_{ij} - g a_{ij}\|_{L^\infty(\Omega\setminus B_r)} = 0, \end{cases}$$

$$(\mathrm{H}_3)\quad \begin{cases} a_i \in M_0^{t_1}(\Omega), \quad \text{with } t_1 > n \text{ if } p \leqslant n,\ t_1 = p \text{ if } p > n, \\ a = a' + b, \quad \text{where} \\ a' \in M_0^{t_2}(\Omega), \quad \text{with } t_2 > n/2 \text{ if } p \leqslant n/2 \text{ and } t_2 = p \text{ if } p > n/2, \\ b \in L^\infty(\Omega), \quad b_0 = \operatorname*{ess\,inf}_{\Omega} b > 0. \end{cases}$$

Observe that under assumptions (H_1) and (H_3) the operator $L : W^{2,p}(\Omega) \to L^p(\Omega)$ is bounded (see for instance [5, Theorem 3.2]). Put

$$L_0 = -\sum_{i,j=1}^n a_{ij}\frac{\partial^2}{\partial x_i \partial x_j}, \qquad E = -\sum_{i,j=1}^n e_{ij}\frac{\partial^2}{\partial x_i \partial x_j} \tag{3.2}$$

and

$$\begin{cases} h_{ij}^r = \zeta_{2r} a_{ij} + (1-\zeta_{2r}) e_{ij}, \quad i,j = 1,\dots,n, \\ H^r = -\displaystyle\sum_{i,j=1}^n h_{ij}^r \frac{\partial^2}{\partial x_i \partial x_j}. \end{cases} \tag{3.3}$$

For each $r \in \mathbb{R}_+$ the coefficients h_{ij}^r are symmetric, elliptic (with ellipticity constant equal to $\min\{\nu,\mu\}$) and belong to $L^\infty(\Omega) \cap VMO(\Omega)$ (see [2, Lemma 4.2]).

Lemma 3.1. *Suppose that conditions* (H_1)–(H_3) *hold. Then for each positive real number* r *there exists* $c(r) \in \mathbb{R}_+$ *such that*

$$\begin{aligned} &\|\zeta_r u\|_{W^{2,p}(\Omega)} \leqslant c(r)\big(\big\|(L_0+b)(\zeta_r u)\big\|_{L^p(\Omega)} + \|\zeta_r u\|_{L^p(\Omega)}\big) \\ &\quad \forall u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega), \end{aligned} \tag{3.4}$$

where $c(r)$ *depends on* n, p, Ω, ν, $\|a_{ij}\|_{L^\infty(\Omega)}$, $\|b\|_{L^\infty(\Omega)}$, $\eta[\zeta_{2r}a_{ij}]$.

Proof. Let $r \in \mathbb{R}_+$ and $u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega)$. By Theorem 5.1 of [5] and by (3.3), there exists $c_1(r) \in \mathbb{R}_+$, depending on n, p, Ω, ν, $\|a_{ij}\|_{L^\infty(\Omega)}$, $\eta[\zeta_{2r}a_{ij}]$, such that

$$\begin{aligned}\|\zeta_r u\|_{W^{2,p}(\Omega)} &\leqslant c_1(r)\big(\big\|H^r(\zeta_r u)\big\|_{L^p(\Omega)} + \|\zeta_r u\|_{L^p(\Omega)}\big)\\ &= c_1(r)\big(\big\|\zeta_{2r}L_0(\zeta_r u) + (1-\zeta_{2r})E(\zeta_r u)\big\|_{L^p(\Omega\cap B_{2r})} + \|\zeta_r u\|_{L^p(\Omega)}\big)\\ &= c_1(r)\big(\big\|L_0(\zeta_r u)\big\|_{L^p(\Omega)} + \|\zeta_r u\|_{L^p(\Omega)}\big). \qquad (3.5)\end{aligned}$$

The statement follows now from (3.5). □

In the following, for each function f of n variables $x_1, \dots, x_n$, we put

$$f_x = \left(\sum_{i=1}^n f_{x_i}^2\right)^{1/2} \quad \text{and} \quad f_{xx} = \left(\sum_{i,j=1}^n f_{x_i x_j}^2\right)^{1/2}.$$

Lemma 3.2. *Suppose that conditions* (H_1)–(H_3) *hold. Then there exist positive real numbers* r_0 *and* c_0 *such that*

$$\begin{aligned}&\big\|(1-\zeta_{r_0})u\big\|_{W^{2,p}(\Omega)} \leqslant c_0\big\|(L_0+b)\big((1-\zeta_{r_0})u\big)\big\|_{L^p(\Omega)}\\ &\quad \forall u \in W^{2,p}(\Omega)\cap \mathring{W}^{1,p}(\Omega), \qquad (3.6)\end{aligned}$$

where r_0 *and* c_0 *depend on* $n, p, \Omega, \mu, g_0, b_0, s, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|b\|_{L^\infty(\Omega)}, \sigma_0[(e_{ij})_x]$.

Proof. Let u be any element of $W^{2,p}(\Omega)\cap \mathring{W}^{1,p}(\Omega)$. It follows from Theorem 5.1 of [3] that there exist $c_1 \in \mathbb{R}_+$ and a bounded open subset $\Omega_1 \Subset \Omega$ such that

$$\begin{aligned}\big\|(1-\zeta_r)u\big\|_{W^{2,p}(\Omega)} &\leqslant c_1\big(\big\|(E+gb)\big((1-\zeta_r)u\big)\big\|_{L^p(\Omega)}\\ &\quad + \big\|(1-\zeta_r)u\big\|_{L^p(\Omega_1)}\big) \quad \forall r \in \mathbb{R}_+, \qquad (3.7)\end{aligned}$$

where c_1 and Ω_1 depend on $n, p, \Omega, \mu, g_0, b_0, s, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|b\|_{L^\infty(\Omega)}, \sigma_0[(e_{ij})_x]$. If r_1 is a positive real number such that Ω_1 is contained in $\Omega\cap B_{r_1}$, we have that

$$\big\|(1-\zeta_r)u\big\|_{L^p(\Omega_1)} = 0 \quad \forall r \geqslant r_1. \qquad (3.8)$$

Applying (3.7) and (3.8), it follows that

$$\begin{aligned}\big\|(1-\zeta_r)u\big\|_{W^{2,p}(\Omega)} &\leqslant c_1\big(\big\|(E-gL_0)\big((1-\zeta_r)u\big)\big\|_{L^p(\Omega)}\\ &\quad + \|g\|_{L^\infty(\Omega)}\big\|(L_0+b)\big((1-\zeta_r)u\big)\big\|_{L^p(\Omega)}\big). \qquad (3.9)\end{aligned}$$

The statement is now a consequence of assumption (H_2). □

We can now prove the main result of this section.

Theorem 3.3. *Suppose that conditions* (H_1)–(H_3) *hold. Then there exist a positive real number* c *and a bounded open subset* $\Omega_0 \subset \Omega$ *such that*

$$\|u\|_{W^{2,p}(\Omega)} \leqslant c\big(\|Lu\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega_0)}\big) \quad \forall u \in W^{2,p}(\Omega)\cap \mathring{W}^{1,p}(\Omega), \qquad (3.10)$$

where c *and* Ω_0 *depend on* $n, p, \Omega, \nu, \mu, g_0, b_0, s, t_1, t_2, \|a_{ij}\|_{L^\infty(\Omega)}, \|e_{ij}\|_{L^\infty(\Omega)}, \|g\|_{L^\infty(\Omega)}, \|b\|_{L^\infty(\Omega)}, \eta[\zeta_{2r_0}a_{ij}], \sigma_0[(e_{ij})_x], \sigma_0[a_i], \sigma_0[a']$, *with* r_0 *as in the statement of Lemma* 3.2.

Proof. Let u be any element of $W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega)$ and choose $r_0 \in \mathbb{R}_+$ as in the statement of Lemma 3.2. It follows from (3.4), (3.6) and the usual product derivatives that

$$\begin{aligned}
\|u\|_{W^{2,p}(\Omega)} &\leqslant \|\zeta_{r_0} u\|_{W^{2,p}(\Omega)} + \big\|(1-\zeta_{r_0})u\big\|_{W^{2,p}(\Omega)} \\
&\leqslant c(r_0)\big(\big\|(L_0+b)(\zeta_{r_0}u)\big\|_{L^p(\Omega)} + \|\zeta_{r_0}u\|_{L^p(\Omega)}\big) \\
&\quad + c_0\big(\big\|(L_0+b)\big((1-\zeta_{r_0})u\big)\big\|_{L^p(\Omega)}\big) \\
&\leqslant c_1\big(\big\|(L_0+b)u\big\|_{L^p(\Omega)} + \|\zeta_{r_0}u\|_{L^p(\Omega)} \\
&\quad + \big\|(\zeta_{r_0})_x u_x\big\|_{L^p(\Omega)} + \big\|(\zeta_{r_0})_{xx}u\big\|_{L^p(\Omega)}\big),
\end{aligned} \tag{3.11}$$

with $c_1 \in \mathbb{R}_+$ depending on n, p, Ω, ν, μ, g_0, b_0, s, $\|a_{ij}\|_{L^\infty(\Omega)}$, $\|e_{ij}\|_{L^\infty(\Omega)}$, $\|g\|_{L^\infty(\Omega)}$, $\|b\|_{L^\infty(\Omega)}$, $\eta[\zeta_{2r_0}a_{ij}]$, $\sigma_0[(e_{ij})_x]$.

If Ω_1 is a bounded open subset of Ω with the cone property for which $\operatorname{supp}\zeta_{r_0} \cap \Omega \subset \Omega_1$, by (3.11) we obtain that there exists $c_2 \in \mathbb{R}_+$, dependent on the same parameters as c_1, such that

$$\|u\|_{W^{2,p}(\Omega)} \leqslant c_2\big(\big\|(L_0+b)u\big\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega_1)} + \|u_x\|_{L^p(\Omega_1)}\big). \tag{3.12}$$

On the other hand, from Theorem 3.2 in [5] and Corollary 3.5 in [11], it can be deduced that for each $\varepsilon \in \mathbb{R}_+$ there exist $c(\varepsilon) \in \mathbb{R}_+$ and a bounded open subset $\Omega_\varepsilon \Subset \Omega$, with the cone property, such that

$$\left\|\sum_{i=1}^n a_i u_{x_i} + a'u\right\|_{L^p(\Omega)} \leqslant \varepsilon\|u\|_{W^{2,p}(\Omega)} + c(\varepsilon)\big(\|u_x\|_{L^p(\Omega_\varepsilon)} + \|u\|_{L^p(\Omega_\varepsilon)}\big), \tag{3.13}$$

where $c(\varepsilon)$ and Ω_ε depend on $\varepsilon, n, p, \Omega, t_1, t_2, \sigma_0[a_i], \sigma_0[a']$. Therefore (3.12) and (3.13) yield that there exist $c_3 \in \mathbb{R}_+$ and a bounded open subset $\Omega_2 \subset \Omega$, with the cone property, such that

$$\|u\|_{W^{2,p}(\Omega)} \leqslant c_3\big(\|Lu\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega_2)} + \|u_x\|_{L^p(\Omega_2)}\big), \tag{3.14}$$

with c_3 and Ω_2 depending on n, p, Ω, ν, μ, g_0, b_0, s, t_1, t_2, $\|a_{ij}\|_{L^\infty(\Omega)}$, $\|e_{ij}\|_{L^\infty(\Omega)}$, $\|g\|_{L^\infty(\Omega)}$, $\|b\|_{L^\infty(\Omega)}$, $\eta[\zeta_{2r_0}a_{ij}]$, $\sigma_0[(e_{ij})_x]$, $\sigma_0[a_i]$, $\sigma_0[a']$. Moreover, there exists a positive real number $K = K(p, \Omega_2)$ such that

$$\|u_x\|_{L^p(\Omega_2)} \leqslant K\varepsilon\|u\|_{W^{2,p}(\Omega_2)} + \frac{K}{\varepsilon}\|u\|_{L^p(\Omega_2)} \tag{3.15}$$

for all $\varepsilon \in]0, 1]$ (see for instance [1, Theorem 4.14]). The statement follows now from (3.14) and (3.15). □

4. A uniqueness and existence theorem

In the proof of our main theorem we will need the following result of operator theory.

Lemma 4.1. *Let X and Y be Banach spaces, and let*

$$\mathcal{L}: \tau \in [0,1] \mapsto L_\tau \in B(X, Y)$$

be a continuous function such that

$$N(L_\tau) = \{0\} \quad \text{and} \quad R(L_\tau) \text{ is closed}, \quad \text{for each } \tau \in [0,1], \tag{4.1}$$

where $N(L_\tau)$ and $R(L_\tau)$ are the kernel and the range of the operator L_τ, respectively. Then there exists a positive real number C_0 such that

$$\|x\|_X \leqslant C_0\|L_\tau x\|_Y \quad \forall x \in X,\ \forall \tau \in [0,1]. \tag{4.2}$$

Proof. It is clearly enough to prove that there exists a positive real number C_0 such that

$$\|L_\tau x\|_Y \geqslant C_0^{-1} \quad \forall x \in S,\ \forall \tau \in [0,1], \tag{4.3}$$

where $S = \{x \in X\colon \|x\|_X = 1\}$. Assume by contradiction that for each $n \in \mathbb{N}$ there are $\tau_n \in [0,1]$ and $x_n \in S$ such that

$$\|L_{\tau_n} x_n\|_Y < \frac{1}{n}. \tag{4.4}$$

Consider a subsequence $(\tau_{n_k})_{k\in\mathbb{N}}$ of $(\tau_n)_{n\in\mathbb{N}}$ which converges to a certain $\tau_0 \in [0,1]$. Since

$$\begin{aligned}\|L_{\tau_0} x_{n_k}\|_Y &\leqslant \big\|(L_{\tau_0} - L_{\tau_{n_k}})x_{n_k}\big\|_Y + \|L_{\tau_{n_k}} x_{n_k}\|_Y \\ &\leqslant \|L_{\tau_0} - L_{\tau_{n_k}}\|_{B(X,Y)} + \|L_{\tau_{n_k}} x_{n_k}\|_Y,\end{aligned}$$

it follows from (4.4) that

$$\lim_{k\to+\infty} L_{\tau_0} x_{n_k} = 0. \tag{4.5}$$

On the other hand, hypothesis (4.1) yields that L_τ^{-1} belongs to $B(R(L_\tau), X)$ for each $\tau \in [0,1]$; thus by (4.5) we obtain that

$$\lim_{k\to+\infty} x_{n_k} = \lim_{k\to+\infty} L_{\tau_0}^{-1}(L_{\tau_0} x_{n_k}) = 0,$$

a contradiction since $x_{n_k} \in S$ for each k. The lemma is proved. □

For our purposes, we state here as a lemma the following uniqueness result, that was proved in [4].

Lemma 4.2. *Suppose that conditions* (H_1)–(H_3) *hold, and also that the assumption*

$$\exists a_0 \in \mathbb{R}_+ \quad \textit{such that} \quad a \geqslant a_0 \quad \textit{a.e. in } \Omega \tag{4.6}$$

is satisfied. Then the solution of the problem

$$\begin{cases} u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega), \\ Lu = 0 \end{cases}$$

is zero in Ω.

Theorem 4.3. *Suppose that conditions* (H_1)–(H_3), (4.6) *hold, and also that* $g \in \mathrm{Lip}(\bar{\Omega})$. *Then there exists a unique solution for the problem*

$$(\mathcal{D}) \quad \begin{cases} u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega), \\ Lu = f, \quad f \in L^p(\Omega). \end{cases}$$

Proof. For each $\tau \in [0,1]$ put

$$L_\tau = \tau(gL) + (1-\tau)(E + gb).$$

The function

$$\tau \in [0,1] \mapsto L_\tau \in B\big(W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega), L^p(\Omega)\big)$$

is clearly continuous; moreover, it is easy to show that the coefficients of each operator L_τ satisfies the hypotheses of Lemma 4.2, and hence $N(L_\tau) = \{0\}$. On the other hand, it follows

from Theorem 3.3 that $R(L_\tau)$ is closed for any $\tau \in [0,1]$, so that Lemma 4.1 can be used to obtain the existence of $C_0 \in \mathbb{R}_+$ such that

$$\|u\|_{W^{2,p}(\Omega)} \leqslant C_0 \|L_\tau u\|_{L^p(\Omega)} \quad \forall u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega),\ \forall \tau \in [0,1]. \tag{4.7}$$

An application of Theorem 5.2 of [3] to the operator $E + gb$ yields now that the problem

$$(\mathcal{D}_0) \quad \begin{cases} w \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega), \\ (E + gb)w = h, \quad h \in L^p(\Omega), \end{cases}$$

is uniquely solvable. This latter result and estimate (4.7) allow to use the method of continuity along a parameter in order to prove that the problem

$$(\mathcal{D}_1) \quad \begin{cases} u \in W^{2,p}(\Omega) \cap \mathring{W}^{1,p}(\Omega), \\ (gL)u = gf, \quad f \in L^p(\Omega), \end{cases}$$

is likewise uniquely solvable. The proof is complete. □

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