

A limit theorem for Szegő polynomials with respect to convolution of point masses with the Fejér kernel

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Abstract

Two recently-proposed methods for estimating the m frequencies of a trigonometric signal using Szegő polynomials of fixed degree $k > m$ consist of multiplying the moments of the n -truncated periodogram by the moments of the Poisson kernel and the wrapped Gaussian, respectively, in an effort to address the non-convergence of the polynomials as $n \rightarrow \infty$. These methods are seen to be equivalent to convolution of point masses with approximate identities, suggesting a general method. We characterize the limit polynomial for the case when the approximate identity is the Fejér kernel, extending recent results of the author for the case of the Poisson kernel. Moreover, the limit is seen to be the same as in the former case.

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1. Introduction

A general approach for estimating the m normalized frequencies θ_j of a complex sinusoidal signal using Szegő polynomials is based on work of Wiener [10] and Levinson [6] and has been developed over the years. The idea is to generate a sequence of monic, orthogonal polynomials, the Szegő polynomials, from the signal, or from measures related to the signal, and to use the arguments of the roots as frequency estimates. We denote, by $P_k(z, \mu)$, the Szegő polynomial

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of degree k with respect to the measure μ . If measures, μ_n are constructed from the signal such that

$$\lim_{n \rightarrow \infty} \mu_n = \sum_{j=1}^m \alpha_j \delta_{\theta_j} =: \mu_\delta, \quad (1)$$

where the limit is in the weak-star sense, and where δ_θ is the point mass at $\theta \in [-\pi, \pi)$ and $\alpha_j > 0$ for $j = 1, 2, 3, \dots, m$, then any limit polynomial of the sequence $\{P_k(z, \mu_n)\}_{n=1}^\infty$ will have m zeros at the $e^{i\theta_j}$ for $j = 1, 2, 3, \dots, m$ when $k \geq m$. When $k > m$ (m is generally assumed unknown) it may thus be possible to distinguish the m “signal” zeros of $P_k(z, \mu_n)$ from the $k - m$ “extraneous” zeros for large n . The behavior of the extraneous zeros and the related problem of non-convergence of $\{P_k(z, \mu_n)\}_{n=1}^\infty$ are two areas that have received attention in the recent literature.

Real trigonometric signals of the form $x(\ell) = \sum_{j=-M}^M \beta_j e^{i\ell\theta_j}$, where $\beta_{-j} = \bar{\beta}_j \neq 0$ and $\theta_j = \theta_{-j}$, are considered in [4,5,7–9], where measures are defined via the *periodogram* $\psi'_n(\theta) = \frac{1}{2\pi n} |\sum_{k=0}^{n-1} x(k) e^{-ik\theta}|^2$. It was shown in [4] that the weak-star convergence (1) holds for this measure. However, as was shown by example in [7], $P_k(z, \mu_n)$ do not, in general, converge for $k > 2M + 1$. Indeed it is shown by construction in [1] that even strong convergence of measures does not guarantee a unique polynomial limit. This non-convergence has been addressed with several modifications [5,8,9].

In the *R-process* [5], and the *V-process* [9], the ℓ th moment $\frac{1}{n} \hat{\psi}_n(\ell)$ is multiplied by the moments of the Poisson kernel and the *wrapped* Gaussian, respectively, and limits are taken first as $n \rightarrow \infty$. By weak-star convergence of the periodogram, it is readily seen that the situation $n = \infty$ gives rise to Szegő polynomials with respect to convolution of these kernels with the measure $\sum_{j=-M}^M |\beta_j|^2 \delta_{\theta_j}$. Furthermore, it is easy to show (see [1] for a simple proof) that $\lim_{h \rightarrow 0} P_k(z, \mu_h)$ exists for any measure $\mu_h \rightarrow \mu_\delta$, where μ_δ is given in (1) and where h is a continuous or discrete parameter with $h \rightarrow 0$, if the moments $\hat{\mu}_h$ have power series representations about $h = 0$. It follows that $\lim_{h \rightarrow 0} P_k(z, \psi_h * \mu_\delta)$ exists for any approximate identity ψ_h whose moments have this property. It can be shown that this property does not hold for the periodogram, with $h = 1/n$. On the other hand, the moments of the Poisson kernel, $\hat{\psi}_r(\ell) = r^{|\ell|}$ with $r \rightarrow 1$, are polynomial functions of $h = 1 - r$. In [1] it is shown that if the θ_j are distinct, then convolution of the point masses μ_δ with the Poisson kernel yields the limit

$$P_{k-m}(z, \nu) \prod_{j=1}^m (z - e^{i\theta_j}), \quad (2)$$

where ν is the absolutely continuous measure with density

$$\frac{d\nu}{d\theta} = \sum_{j=1}^m \alpha_j \prod_{p \neq j}^m |e^{i\theta} - e^{i\theta_p}|^2. \quad (3)$$

In light of the above remarks, (2) and (3) now give the limit for the *R-process* applied to m complex sinusoids. Note that the “extra” limit factor is itself a Szegő polynomial with respect to the measure ν . In this article we extend this result and show that convolution with the Fejér kernel also yields the limit defined by (2) and (3).

1.1. Background

Given a measure, μ , on the unit circle, the k th Szegő polynomial $P_k(z, \mu)$, with respect μ , is the polynomial in the complex variable z which attains the minimum

$$\min_{p \in \Lambda_k} \int_{-\pi}^{\pi} |p(e^{i\theta})|^2 d\mu(\theta) = \int_{-\pi}^{\pi} |P_k(e^{i\theta}, \mu)|^2 d\mu(\theta), \quad (4)$$

where Λ_k is the set of monic polynomials of degree k . The $P_k(z, \mu)$ are uniquely defined if μ is supported on $m > k$ points. Note that for $m < k$, any polynomial with m zeros at the point mass locations will attain the minimum of zero in (4).

The *prediction error power*, which we denote $\rho_k(\mu)$ is a multiple of the minimum attained in (4):

$$\rho_k(\mu) := \frac{1}{2\pi} \int_{-\pi}^{\pi} |P_k(e^{i\theta}, \mu)|^2 d\mu. \quad (5)$$

The $P_k(z, \mu)$ are also characterized by the orthogonality property

$$\int_{-\pi}^{\pi} P_k(e^{i\theta}, \mu) \overline{p}(e^{i\theta}) d\mu(\theta) = 0 \quad (6)$$

for any polynomial p of degree less than k .

The *reflection coefficients* of the $P_k(z, \mu)$ are the constant terms

$$R_k(\mu) := P_k(0, \mu). \quad (7)$$

We denote by $P_k^*(z, \mu)$ the *reverse polynomial*: $P_k^*(z, \mu) = z^k \overline{P_k}(1/z, \mu)$. The zeros of P^* are obtained from those of P by reflection in the unit circle. Equivalently, the coefficients are reversed and conjugated. Note that we have $|P_k^*(e^{i\theta}, \mu)| = |P_k(e^{i\theta}, \mu)|$, and that $P_k^*(0, \mu) = 1$.

2. Convolution with Fejér kernel

Let μ_δ be defined as in (1) and let $\phi_n(\theta)$ denote the Fejér kernel:

$$\phi_n(\theta) := \begin{cases} \frac{1}{n} \left[\frac{\sin(n\theta/2)}{\sin(\theta/2)} \right]^2, & \theta \neq 0, \\ n, & \theta = 0, \end{cases} \quad (8)$$

for $n = 1, 2, 3, \dots$, and define

$$\mu_n := \phi_n * \mu_\delta. \quad (9)$$

Note that the weak-star convergence (1) necessarily holds for μ_n . We will use the notation

$$\zeta := e^{i\theta} \quad (10)$$

throughout this paper. We will show that the limit of the Szegő polynomials, $P_k(z, \mu_n)$ of fixed degree $k > m$ with respect μ_n , as $n \rightarrow \infty$, is also given by (2) and (3). The situation mirrors that of the Poisson kernel, but with some significant differences.

We make use of the following representations of ϕ_n and spectral factorization of $d\mu_n$. From (8) it is easy to see that

$$\phi_n(\theta) = \frac{2}{n} \frac{(1 - \cos n\theta)}{|\zeta - 1|^2}, \quad (11)$$

hence

$$d\mu_n = \frac{2 \sum_{j=1}^m \alpha_j (1 - \cos n(\theta - \theta_j)) \prod_{p \neq j}^m |\zeta - e^{i\theta_p}|^2}{\prod_{j=1}^m |\zeta - e^{i\theta_j}|^2} d\theta. \quad (12)$$

Using (11) we can write

$$\begin{aligned} \phi_n &= \frac{1}{n} \left[\frac{e^{in\theta/2} - e^{-in\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}} \right]^2 \\ &= \frac{1}{n} \left| \frac{e^{in\theta} - 1}{e^{i\theta} - 1} \right|^2 \\ &= \frac{1}{n} \left| \frac{\zeta^n - 1}{\zeta - 1} \right|^2 \end{aligned} \quad (13)$$

$$= \frac{1}{n} |\zeta^{n-1} + \zeta^{n-2} + \dots + 1|^2. \quad (14)$$

Using (14) we can express the convolution (9) as

$$\begin{aligned} d\mu_n &= \frac{1}{n} \sum_{j=1}^m \alpha_j |e^{i(n-1)(\theta-\theta_j)} + e^{i(n-2)(\theta-\theta_j)} + \dots + 1|^2 d\theta \\ &= \frac{1}{n} \sum_{j=1}^m \alpha_j |e^{i(n-1)\theta_j} \zeta^{(n-1)} + e^{i(n-2)\theta_j} \zeta^{(n-2)} + \dots + 1|^2 d\theta \\ &= \frac{1}{n} \sum_{j=1}^m \alpha_j |\zeta^{(n-1)} + e^{i\theta_j} \zeta^{(n-2)} + \dots + e^{i(n-1)\theta_j}|^2 d\theta. \end{aligned} \quad (15)$$

Since the density in (15) is a non-negative trigonometric polynomial, it has a spectral factorization

$$\mu'_n = |g_n(\zeta)|^2 = \frac{1}{n} \sum_{j=1}^m \alpha_j |\zeta^{(n-1)} + e^{i\theta_j} \zeta^{(n-2)} + \dots + e^{i(n-1)\theta_j}|^2, \quad (16)$$

where $g_n(z)$ is a polynomial of degree $n-1$ with all its zeros outside the unit circle [2,3].

The moments of ϕ_n are

$$\hat{\phi}_n(\ell) = \begin{cases} 1 - \frac{|\ell|}{n}, & |\ell| \leq n, \\ 0, & |\ell| > n, \end{cases} \quad (17)$$

so by [1, Proposition 2.1], $\lim_{n \rightarrow \infty} P_k(z, \mu_n)$ exists, and we can write

$$\lim_{n \rightarrow \infty} P_k(z, \mu_n) = Q(z) \prod_{j=1}^m (z - e^{i\theta_j}), \quad (18)$$

and, without loss of generality,

$$P_k(z, \mu_n) = Q_n(z) \prod_{j=1}^m (z - w_j^{(n)}), \quad (19)$$

where Q_n and Q are polynomials of degree $k - m$ with $Q_n \rightarrow Q$ and $w_j^{(n)} \rightarrow e^{i\theta_j}$ for $j = 1, 2, 3, \dots, m$.

We will need to show that the $w_j^{(n)}$ converge at the rate $1/n$. As in the case of the Poisson kernel of [1], this will follow from the uniform boundedness of the reflection coefficients away from the unit circle, which we show using arguments adapted from [7].

Lemma 2.1. *Let μ_n be given in (9), where μ_δ is the sum of point masses in (1). Then*

$$|R_k(\mu_n)| \leq \left(1 - \frac{1}{4^m}\right)^{1/2}. \quad (20)$$

Proof. Using (5) and (16), we have

$$\begin{aligned} \rho_k(\mu) &= \frac{1}{2\pi} \int |P_k(\zeta, \mu_n)|^2 d\mu_n \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |P_k^*(\zeta, \mu_n)|^2 |g_n(\zeta)|^2 d\theta \end{aligned} \quad (21)$$

$$\geq \frac{1}{2\pi} |P_k^*(0, \mu_n)|^2 |g_n(0)|^2 = \frac{1}{2\pi n} \sum_{j=1}^m \alpha_j, \quad (22)$$

where we have used the subharmonicity of the integrand in (21). On the other hand, (5) and (12) with the minimum characterization of $\rho_k(\mu_n)$ in (4) give

$$\begin{aligned} \rho_k(\mu_n) &\leq \frac{1}{n\pi} \int_{-\pi}^{\pi} \prod_{j=1}^m |\zeta - e^{i\theta_j}|^2 \frac{\sum_{j=1}^m \alpha_j (1 - \cos n(\theta - \theta_j)) \prod_{p \neq j}^m |\zeta - e^{i\theta_p}|^2}{\prod_{j=1}^m |\zeta - e^{i\theta_j}|^2} d\theta \\ &= \frac{4^m \sum_{j=1}^m \alpha_j}{n}. \end{aligned} \quad (23)$$

The estimates (22) and (23), and the relation [2] $|R_{k+1}(\mu_r)|^2 = 1 - \rho_k(\mu_r)/\rho_{k+1}(\mu_r)$, now give (20). $\square$

The following corollary now follows from [1, Corollary 2.2].

Corollary 2.1. *Let $w_j^{(r)}$ be given in (19) and μ_n defined in (9) where the θ_j are distinct for $j = 1, 2, 3, \dots, m$. Then there exist constants N_j , for $j = 1, 2, \dots, m$, such that*

$$|w_j^{(n)} - e^{i\theta_j}| \leq \frac{N_j}{n}. \quad (24)$$

The following is the main result of this paper.

Theorem 2.1. Let μ_n be given in (9), where μ_δ is the sum of point masses in (1) with the θ_j distinct for $j = 1, 2, 3, \dots, m$. Then

$$\lim_{n \rightarrow \infty} P_k(z, \mu_n) = P_{k-m}(z, \nu) \prod_{j=1}^m (z - e^{i\theta_j}), \quad (25)$$

where ν is defined in (3).

In characterizing the limit of the $P_k(z, \mu_n)$, we will exploit the orthogonality property and show that as $n \rightarrow \infty$, the “extraneous” factors Q_n defined in (19), converge to a polynomial which satisfies (6) for the measure defined in (3). The orthogonality property with representation (12) gives

$$\int Q_n(\zeta) \overline{T(\zeta)} \prod_{j=1}^m (\zeta - w_j^{(n)}) \frac{\sum_{j=1}^m \alpha_j (1 - \cos n(\theta - \theta_j)) \prod_{p \neq j}^m |\zeta - e^{i\theta_p}|^2}{\prod_{j=1}^m |\zeta - e^{i\theta_j}|^2} d\theta = 0$$

for any polynomial $T(z)$ of degree $k-1$ or less. In particular, if $T(z) = t(z) \prod_{j=1}^m (z - e^{i\theta_j})$ with $t(z)$ an arbitrary polynomial of degree $k-m-1$ or less, we have

$$\int \gamma(n, \zeta) d\theta = 0, \quad (26)$$

where we define

$$\gamma(n, \zeta) := Q_n(\zeta) \overline{t(\zeta)} \frac{\prod_{j=1}^m (\zeta - w_j^{(n)})}{\prod_{j=1}^m (\zeta - e^{i\theta_j})} \sum_{j=1}^m \alpha_j (1 - \cos n(\theta - \theta_j)) \prod_{p \neq j}^m |\zeta - e^{i\theta_p}|^2. \quad (27)$$

Before proving Theorem 2.1 we state and prove some supporting results. We first write simple bounds involving the signal zeros and points on the unit circle.

As a result of the relationship between chord length and arc length between points on the unit circle we have

$$\frac{|\theta - \theta_j|}{2} < |\zeta - e^{i\theta_j}| < |\theta - \theta_j| \quad \text{for } 0 < |\theta - \theta_j| < \pi. \quad (28)$$

With Corollary 2.1 and the fact that $|w_j^{(n)}| < 1$ we have

$$|\zeta - w_j^{(n)}| \leq |\zeta - e^{i\theta_j}| + |w_j^{(n)} - e^{i\theta_j}| \leq |\theta - \theta_j| + \frac{N_j}{n}$$

and $|\zeta - w_j^{(n)}| < 2$, so that

$$|\zeta - w_j^{(n)}| \leq \min \left\{ 2, |\theta - \theta_j| + \frac{N_j}{n} \right\}. \quad (29)$$

We investigate the convergence behavior of $\gamma(n, \zeta)$ by considering the factors of the form

$$\frac{(\zeta - w_j^{(n)})(1 - \cos n(\theta - \theta_j))}{\zeta - e^{i\theta_j}} \quad \text{and} \quad \frac{\prod_{j=1}^m (\zeta - w_j^{(r)})}{\prod_{j=1}^m |\zeta - e^{i\theta_j}|^2},$$

neither of which converge in $\mathcal{L}_1$, nor is the latter bounded.

Lemma 2.2. Let the convergence $w_j^{(n)} \rightarrow e^{i\theta_j}$ be as in (24) for $j = 1, 2, \dots, m$. There exist constants, $M_1, M_2, \dots$, such that for all $\theta \neq \theta_j$, and for all $n = 1, 2, \dots$,

$$\left| (\zeta - w_j^{(n)}) \frac{1 - \cos n(\theta - \theta_j)}{\zeta - e^{i\theta_j}} \right| \leq M_j. \quad (30)$$

Proof. Assume without loss of generality that $\theta_j = 0$. Suppose first that $0 < |\theta| \leq 1/n$. Using (29) we obtain

$$\left| (\zeta - w_j^{(n)}) \frac{1 - \cos n\theta}{\zeta - 1} \right| \leq \frac{2}{n} (N_j + 1) \left| \frac{1 - \cos n\theta}{\theta} \right|. \quad (31)$$

Since $|\frac{1 - \cos \theta}{\theta}| < 1$, we have $\max_{|\theta| < 1/n} |\frac{1 - \cos n\theta}{\theta}| < n$. This, with (31) gives

$$\left| (\zeta - w_j^{(n)}) \frac{1 - \cos n\theta}{\zeta - 1} \right| < 2(N_j + 1) \quad \text{for } |\theta| \leq 1/n. \quad (32)$$

Now suppose that $|\theta| > 1/n$. With (24) and (28) we obtain

$$\left| \frac{\zeta - w_j^{(n)}}{\zeta - 1} \right| \leq 1 + \frac{|w_j^{(n)} - 1|}{|\zeta - 1|} < 1 + \frac{2N_j}{n\theta} < 1 + 2N_j.$$

Thus

$$\left| (\zeta - w_j^{(n)}) \frac{1 - \cos n\theta}{\zeta - 1} \right| \leq 2(1 + 2N_j). \quad (33)$$

Equations (32) and (33) now give (30) with $M_j = 2 + 4N_j$. $\square$

Lemma 2.3. Let $I_j(n) = (\theta_j - 1/n, \theta_j + 1/n)$ and let $\mathcal{X}_n$ be the indicator function for the set $[-\pi, \pi) \setminus \bigcup_{j=1}^m I_j(n)$ and let the convergence $w_j^{(n)} \rightarrow e^{i\theta_j}$ be as in (24) for $j = 1, 2, \dots, m$. Then

$$\lim_{n \rightarrow \infty} \frac{\prod_{j=1}^m (\zeta - w_j^{(n)})}{\prod_{j=1}^m (\zeta - e^{i\theta_j})} \mathcal{X}_n(\theta) = 1 \quad \text{in } \mathcal{L}_1(d\theta).$$

Proof. Since

$$\frac{\prod_{j=1}^m (\zeta - w_j^{(n)})}{\prod_{j=1}^m (\zeta - e^{i\theta_j})} \mathcal{X}_n(\theta) \rightarrow 1$$

pointwise except at the θ_j , we need only show that the function is bounded in n . The lemma will then follow from Lebesgue's Bounded Convergence Theorem. With (28), the convergence in (24) gives

$$\left| \frac{\zeta - w_j^{(n)}}{\zeta - e^{i\theta_j}} \right| \mathcal{X}_n(\theta) \leq \left(1 + \left| \frac{w_j^{(n)} - e^{i\theta_j}}{\zeta - e^{i\theta_j}} \right| \right) \mathcal{X}_n(\theta) < 1 + 2N_j.$$

Thus, $\frac{\zeta - w_j^{(n)}}{\zeta - e^{i\theta_j}} \mathcal{X}_n(\theta)$ is bounded for each $j = 1, 2, \dots, m$. $\square$

Lemma 2.4. Let $f_n \rightarrow f$ in $\mathcal{L}_1(d\theta)$, and let g_n be a bounded sequence of functions with $|g_n(\theta)| \leq M$ for all n and θ . Then

$$\lim_{n \rightarrow \infty} \int f_n g_n d\theta = \lim_{n \rightarrow \infty} \int f g_n d\theta,$$

provided the limits exist.

Proof. We have

$$\left| \int f_n g_n d\theta - \int f g_n d\theta \right| \leq \int |f_n g_n - f g_n| d\theta \leq M \int |f_n - f| d\theta.$$

Now let $n \rightarrow \infty$. $\square$

Proof of Theorem 2.1. The result will be proved if we can show that

$$\int Q(\zeta) \overline{t(\zeta)} \sum_{j=1}^m \alpha_j \prod_{p \neq j}^m |\zeta - e^{i\theta_p}|^2 d\theta = 0, \quad (34)$$

where $Q_n(\zeta) \rightarrow Q(\zeta)$ with $Q_n(\zeta)$ defined in (19), and $t(z)$ is an arbitrary polynomial of degree $k - m - 1$.

Let $I_j(n)$ and $\mathcal{X}_n$ be defined as in Lemma 2.3, and suppose that n is large enough so that the $I_j(n)$ are disjoint. Then (26) gives

$$\int \gamma(n, \zeta) \mathcal{X}_n(\theta) d\theta + \int_{\bigcup_{j=1}^m I_j(n)} \gamma(n, \zeta) d\theta = 0. \quad (35)$$

We will use Lemma 2.2 to show that $\gamma(n, \zeta)$ is bounded on each $I_j(n)$ uniformly in n , from which it will follow that both integrals in (35) converge to 0. Given $s \in \{1, 2, \dots, m\}$, we can write

$$\frac{\prod_{j=1}^m (\zeta - w_j^{(n)})}{\prod_{j=1}^m (\zeta - e^{i\theta_j})} = \left(\frac{\zeta - w_s^{(n)}}{\zeta - e^{i\theta_s}} \right) \frac{\prod_{j \neq s}^m (\zeta - w_j^{(n)})}{\prod_{j \neq s}^m (\zeta - e^{i\theta_j})}$$

and

$$\begin{aligned} & \sum_{j=1}^m \alpha_j (1 - \cos n(\theta - \theta_j)) \prod_{p \neq j}^m |\zeta - e^{i\theta_p}|^2 \\ &= \alpha_s (1 - \cos n(\theta - \theta_s)) \prod_{p \neq s} |\zeta - e^{i\theta_p}|^2 + \sum_{j \neq s} \alpha_j (1 - \cos n(\theta - \theta_j)) \prod_{p \neq j} |\zeta - e^{i\theta_p}|^2 \\ &= \alpha_s (1 - \cos n(\theta - \theta_s)) \prod_{p \neq s} |\zeta - e^{i\theta_p}|^2 \\ & \quad + |\zeta - e^{i\theta_s}|^2 \sum_{j \neq s} \alpha_j (1 - \cos n(\theta - \theta_j)) \prod_{p \neq j, s} |\zeta - e^{i\theta_p}|^2. \end{aligned}$$

Hence, with γ defined in (27),

$$\begin{aligned} \gamma(n, \zeta) = Q_n(\zeta) \overline{t(\zeta)} \frac{\prod_{j \neq s}^m (\zeta - w_j^{(n)})}{\prod_{j \neq s}^m (\zeta - e^{i\theta_j})} \left\{ \left(\frac{\zeta - w_s^{(n)}}{\zeta - e^{i\theta_s}} \right) \alpha_s (1 - \cos n(\theta - \theta_s)) \prod_{p \neq s} |\zeta - e^{i\theta_p}|^2 \right. \\ \left. + (\zeta - w_s^{(n)}) (\overline{\zeta - e^{i\theta_s}}) \sum_{j \neq s} \alpha_j (1 - \cos n(\theta - \theta_j)) \prod_{p \neq j, s} |\zeta - e^{i\theta_p}|^2 \right\}. \end{aligned}$$

Now suppose $\theta \in I_s(n)$. Then $|\theta - \theta_j| > d/2$ for $s \neq j$, where $d = \min_{i \neq j} |\theta_i - \theta_j|$. So Lemma 2.2 with the course estimates $\frac{|\theta - \theta_j|}{2} < |\zeta - e^{i\theta_j}| \leq 2$ (given in (28) and (29)), now give

$$|\gamma(n, \zeta)| \leq |Q_n(\zeta) \overline{t(\zeta)}| \frac{(2^{m-1})}{(\frac{d}{4})^{m-1}} \left\{ (M_s) 2^{2(m-1)} + (2)(2) \sum_{j \neq s} \alpha_j (2) (2^{2(m-2)}) \right\}. \quad (36)$$

Since $Q_n(\zeta) \overline{t(\zeta)} \rightarrow Q(\zeta) \overline{t(\zeta)}$ uniformly, we see that γ is bounded uniformly in n for $\theta \in I_s(n)$ for $s = 1, 2, \dots, m$. It follows that $\lim_{n \rightarrow \infty} \int_{\bigcup I_j(n)} \gamma(n, \theta) d\theta = 0$, which, with (35), gives

$$\lim_{n \rightarrow \infty} \int \gamma(n, \theta) \mathcal{X}_n d\theta = 0. \quad (37)$$

We now express $\gamma(n, \theta) \mathcal{X}_n$ as the sum of two terms, one of which converges in $\mathcal{L}_1$. From the definition of γ in (27) we write

$$\begin{aligned} \gamma(n, \theta) = Q_n(\zeta) \overline{t(\zeta)} \prod_{j=1}^m \frac{(\zeta - w_j^{(n)})}{(\zeta - e^{i\theta_j})} \sum_{j=1}^m \alpha_j \prod_{p \neq j} |\zeta - e^{i\theta_p}|^2 \\ - Q_n(\zeta) \overline{t(\zeta)} \prod_{j=1}^m \frac{(\zeta - w_j^{(n)})}{(\zeta - e^{i\theta_j})} \sum_{j=1}^m \alpha_j \cos n(\theta - \theta_j) \prod_{p \neq j} |\zeta - e^{i\theta_p}|^2. \end{aligned} \quad (38)$$

By Lemma 2.3,

$$Q_n(\zeta) \overline{t(\zeta)} \prod_{j=1}^m \frac{(\zeta - w_j^{(n)})}{(\zeta - e^{i\theta_j})} \mathcal{X}_n \rightarrow Q(\zeta) \overline{t(\zeta)} \quad \text{in } \mathcal{L}_1.$$

Thus, regarding the first term on the right-hand side of (38), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int Q_n(\zeta) \overline{t(\zeta)} \prod_{j=1}^m \frac{(\zeta - w_j^{(n)})}{(\zeta - e^{i\theta_j})} \sum_{j=1}^m \alpha_j \prod_{p \neq j} |\zeta - e^{i\theta_p}|^2 \mathcal{X}_n d\theta \\ = \int Q(\zeta) \overline{t(\zeta)} \sum_{j=1}^m \alpha_j \prod_{p \neq j} |\zeta - e^{i\theta_p}|^2 d\theta, \end{aligned} \quad (39)$$

and regarding the second term on the right-hand side of (38), Lemma 2.4 gives

$$\begin{aligned} \lim_{n \rightarrow \infty} \int Q_n(\zeta) \overline{t(\zeta)} \prod_{j=1}^m \frac{(\zeta - w_j^{(n)})}{(\zeta - e^{i\theta_j})} \sum_{j=1}^m \alpha_j \cos n(\theta - \theta_j) \prod_{p \neq j} |\zeta - e^{i\theta_p}|^2 \mathcal{X}_n d\theta \\ = \lim_{n \rightarrow \infty} \int Q(\zeta) \overline{t(\zeta)} \sum_{j=1}^m \alpha_j \cos n(\theta - \theta_j) \prod_{p \neq j} |\zeta - e^{i\theta_p}|^2 d\theta. \end{aligned} \quad (40)$$

Now

$$\lim_{n \rightarrow \infty} \int Q(\zeta) \overline{t(\zeta)} \sum_{j=1}^m \alpha_j \cos n(\theta - \theta_j) \prod_{p \neq j}^m |\zeta - e^{i\theta_p}|^2 d\theta = 0. \quad (41)$$

To see this, consider the $\mathcal{L}^2$ function $f(\theta) = Q(\zeta) \overline{t(\zeta)} \prod_{p \neq j}^m |\zeta - e^{i\theta_p}|^2$. We have

$$\int \sum_{j=1}^m \alpha_j f(\theta) e^{in(\theta - \theta_j)} d\theta = \sum_{j=1}^m \alpha_j e^{in\theta_j} \hat{f}(n) \rightarrow 0$$

by the Riemann–Lebesgue Lemma. Equation (41) is then apparent by expanding $e^{in(\theta - \theta_j)}$ using Euler’s formula and considering real and imaginary parts of f .

Equations (37)–(41) then give

$$\int Q(\zeta) \overline{t(\zeta)} \sum_{j=1}^m \alpha_j \prod_{p \neq j}^m |\zeta - e^{i\theta_p}|^2 d\theta = 0. \quad \square$$

3. Remarks

In [9] it is shown that for the V -process in the case of real sinusoidal signals, a multiple of m of the zeros actually approach the signal zeros. For example, if $k/m = p$, the limit polynomial is $\prod_{j=1}^m (z - e^{i\theta_j})^p$. Thus, in light of the remarks in the introduction, if ψ_h with $h \rightarrow 0$ is the wrapped Gaussian, $P_k(z, \psi_h) \rightarrow \prod_{j=1}^m (z - e^{i\theta_j})^p$, which is not of the form (25), since the zeros of $P_{k-m}(z, \nu)$ lie strictly inside the unit circle for the absolutely continuous measure ν in (3).

The following simple examples illustrate different zero behavior for polynomials with respect to different kernels $\psi_h (= \psi_h * \delta_0)$, so corresponds to a single point mass at $\theta = 0$.

- ψ_h is either the Poisson or Fejér kernel: As noted, the limits are the same. Theorem 2.1 gives $d\nu = d\theta$, and

$$\lim_{h \rightarrow 0} P_k(z, \psi_h) = z^{k-1}(z - 1).$$

Thus all extraneous zeros approach the origin.

- ψ_h is a kernel such that, for any $\varepsilon > 0$, there exists h_0 such that ψ_h is supported entirely on $(-\varepsilon, \varepsilon)$ for $h < h_0$. Then

$$\lim_{h \rightarrow 0} P_k(z, \psi_h) = (z - 1)^k.$$

This follows from Fejér’s Convex Hull Theorem. So in this case, all zeros approach the single signal zero. Examples of this type of kernel are the triangular and rectangular “windows” of engineering applications.

- ψ_h is the wrapped Gaussian. Conjecture:

$$\lim_{h \rightarrow 0} P_k(z, \psi_h) = (z - 1)^k.$$

This is based on simulations and the fact that the moments expanded about $h = 0$ agree with those of the triangular window to first order in h .

The moments of the Poisson and Fejér kernels also agree to first order in h , and we conjecture that

Szegő polynomial limits for convolutions of point masses with approximate identities whose moments are analytic about $h = 0$ are determined completely by the first-order expansion of the moments.

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