

W-G-F-KKM mapping, intersection theorems and minimax inequalities in *FC*-space

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Abstract

By using basic *KKM* theorem, a new matching theorem and some minimax inequalities for set-valued mappings defined on the *FC*-spaces are proved under very weak assumptions. These results generalized many known results from the recent literature.

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1. Introduction

The famous *KKM* theorem [1] and its generalizations are of fundamental importance in modern nonlinear analysis. Later many authors have studied lots of *KKM* theorems and their equivalent forms and discussed the properties of corresponding *KKM* mapping. In 1983, Hovath [2], replacing convex hulls by contract subsets, gave a purely topological version of the *KKM* theorem. Motivated by the work of Hovath, in 1996, Ding [3] introduced *H*-space and studied generalized *H-KKM* mapping. In 1997, Tan [4] studied a class generalized *G-KKM* mapping from a nonempty set *X* to a *G*-convex space and gave some new generalized *G-KKM* theorems and their applications to minimax inequalities and saddle point problems. After this, Chang and Yen [5] made a systematic study of the class *KKM*(*X*, *Y*). In [6], Lin, Ko, and Park further extended the result of Chang and Yen by introducing the concepts of generalized *G-KKM* mapping with

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respect to T . Recently, in [7], Balaj studied weakly KKM mapping with respect to T in G -convex space. In terms of the various versions of KKM theorem, in 1989, Park [8] introduced the concept of generalized KKM mapping obtaining thus generalized KKM theorems and generalized matching theorems. In [9], he further studied the equivalence among KKM theorem, matching theorem, coincidence theorem and minimax inequality involving the upper semicontinuous mappings. In 1999, Verma [10] was first to obtain an intersection theorem involving R - KKM mapping in G - H -space.

From contractible subsets to H -space, H -space to G -convex space, it is, indeed, a processing that the linear structure and the convex structure of the space have being weakened. Recently, Ding [11] introduced FC -space which extended G -convex space further and proved the corresponding KKM theorem. From this, many new KKM type theorems and applications were founded in FC -spaces (see [11–17]).

In this paper, by introducing W - G - F - KKM mapping with respect to T which extends the corresponding notion of Ding in [13], we will prove a new matching theorem and some intersection theorems in noncompact FC -spaces and study some properties of KKM mappings. As applications, some new minimax inequalities are established. The results presented in this paper extend some corresponding known results in the literature.

2. Preliminaries

For a nonempty set X , 2^X denotes the class of all nonempty subsets of X and $\langle X \rangle$ denotes the class of all nonempty finite subsets of X . For simplicity, in the paper, $\{e_0, \dots, e_n\}$ denotes canonical orthogonal base of Euclid space R^{n+1} , for each $A \in \langle X \rangle$, let $A = \{x_0, \dots, x_n\} \in \langle X \rangle$, $B = \{x_{i_0}, \dots, x_{i_k}\} \in \langle A \rangle$, Δ_n denotes the standard n -dimensional simplex $\text{co}(\{e_0, \dots, e_n\})$ with respect to A , $\text{card } A = n + 1$, Δ_B denotes the convex hull of $\{e_{i_0}, \dots, e_{i_k}\}$ with respect to $B = \{x_{i_0}, \dots, x_{i_k}\}$.

If X and Y are topological spaces, a mapping $T : X \rightarrow 2^Y$ is said to be:

- (i) upper semicontinuous (u.s.c.) if the set $\{x \in X : T(x) \cap F \neq \emptyset\}$ is closed in X , for each closed subset F of Y ;
- (ii) lower semicontinuous (l.s.c.) if the set $\{x \in X : T(x) \cap V \neq \emptyset\}$ is open in X , for each open subset V of Y .

The above statements are obviously equivalent to the following statements:

- (i') upper semicontinuous (u.s.c.) if the set $\{x \in X : T(x) \subset V\}$ is open in X , for each open subset V of Y ;
- (ii') lower semicontinuous (l.s.c.) if the set $\{x \in X : T(x) \subset F\}$ is closed in X , for each closed subset V of Y .

The following definition was introduced by Ding in [11].

Definition 2.1. (X, φ_A) is said to be a finitely continuous space (for short, FC -space) if X is a topological space and for each $A \in \langle X \rangle$ where some elements may be the same, there exists a continuous mapping $\varphi_A : \Delta_n \rightarrow X$. A subset D of (X, φ_A) is said to be FC -subspace of X , this means that for each $A \in \langle X \rangle$, each $B \in A \cap D$, then $\varphi_A(\Delta_B) \subset D$.

Remark 2.1. Suppose D is an FC -subspace of X , then (D, φ_P) is an FC -space, where $P \in \langle D \rangle$, φ_P is one of the given mappings collective $\{\varphi_A\}_{A \in \langle X \rangle}$. In fact, for any $P \in \langle D \rangle$, then $P \in \langle X \rangle$, thus, there exists a continuous mapping $\varphi_P: \Delta_P \rightarrow X$, where $p = \text{card } P$. Since D is an FC -subspace of X , $P \subset P \cap D$, so $\varphi_P(\Delta_P) \subset D$, i.e., φ_P is a mapping from Δ_P to D for any $P \in \langle D \rangle$.

For the convenience to contrast, we list the following notion introduced in [13] by Ding which was a generalization of the corresponding notion of Chang and Yen in [5].

Definition 2.2. Let (X, φ_A) be an FC -space, Y a nonempty set and $T, S: X \rightarrow 2^Y$ two mappings. We say that S is a generalized KKM mapping with respect to T if for each $A \in \langle X \rangle$, each $B \in \langle A \rangle$, $T(\varphi_A(\Delta_B)) \subset S(B)$. If Y is a topological space, $T: X \rightarrow 2^Y$ is said to have the F - KKM property if for any mapping $S: X \rightarrow 2^Y$ generalized KKM with respect to T , the family $\{\bar{S}(z): z \in X\}$ has finite intersection property.

Definition 2.3. Let (X, φ_A) be an FC -space, Y a nonempty set and $T, S: X \rightarrow 2^Y$ two mappings. We say that S is weakly generalized F - KKM mapping with respect to T (for short, W - G - F - KKM mapping with respect to T) if for each $A \in \langle X \rangle$, each $B \in A$, for any $x \in \varphi_A(\Delta_B)$, $T(x) \cap S(B) \neq \emptyset$.

Remark 2.2. Obviously, each generalized KKM mapping with respect to T is W - G - F - KKM mapping with respect to T , and therefore, this notion extends Definition 2.2 of Ding.

Definition 2.4. Let (X, φ_A) be an FC -space, Y a nonempty subset and $S: Y \rightarrow 2^X$. Then S is a generalized F - KKM mapping if for each finite subset $\tilde{A} = \{y_0, \dots, y_n\}$ of Y , there exists a finite subset $A = \{x_0, \dots, x_n\}$ of X such that for any subset $B = \{x_{i_0}, \dots, x_{i_k}\}$ of A ,

$$\varphi_A(\text{co}\{e_{i_0}, \dots, e_{i_k}\}) \subset \bigcup_{j=0}^k S(y_{i_j}).$$

Remark 2.3. The difference of Definitions 2.2 and 2.4 lies in: (1) in Definition 2.2, S is respect to T , (2) in Definition 2.2, S is a mapping from FC -space to a nonempty set, but Definition 2.4 is inverse.

Definition 2.5. Let (X, φ_A) be an FC -space, Y a nonempty set and $\beta \in R$. We say that a function $f: X \times Y \rightarrow R \cup \{-\infty, +\infty\}$ is F - β -quasi-convex on X if for each $\lambda < \beta$ and any $y \in Y$, $A \in \langle \{x \in X: f(x, y) < \lambda\} \rangle$, $B \in \langle A \rangle$, then $\varphi_A(\Delta_B) \subset \{x \in X: f(x, y) < \lambda\}$.

Definition 2.6. Let (X, φ_A) be an FC -space, Y a nonempty subset $f: X \times Y \rightarrow R \cup \{-\infty, +\infty\}$ and $g: Y \times X \rightarrow R \cup \{-\infty, +\infty\}$. Then for $\gamma \in R$:

- (1) f is γ -generalized F -quasi-convex (respectively γ -generalized F -quasi-concave) in the second variable y if for each finite subset $\tilde{A} = \{y_0, \dots, y_n\}$ of Y , there exists a subset $A = \{x_0, \dots, x_n\}$ of X such that for any subset $B = \{x_{i_0}, \dots, x_{i_k}\}$ of $\{x_0, \dots, x_n\}$ and any $x_0 \in \varphi_A(\text{co}\{e_{i_0}, \dots, e_{i_k}\})$, $\gamma \leq \max_{0 \leq j \leq k} f(x_0, y_{i_j})$ (respectively $\gamma \geq \max_{0 \leq j \leq k} f(x_0, y_{i_j})$).
- (2) g is γ -generalized F -quasi-convex (respectively γ -generalized F -quasi-concave) in the first variable y if the function $h: X \times Y \rightarrow R \cup \{-\infty, +\infty\}$, defined by $h(w, z) = g(z, w)$ for

all $(w, z) \in X \times Y$ is γ -generalized F -quasi-convex (respectively γ -generalized F -quasi-concave) in the second variable z .

Remark 2.4. The above F -KKM mapping, F - β -quasi-convex, γ -generalized F -quasi-convex (respectively γ -generalized F -quasi-concave) function generalizes the corresponding notion on G -convex space in [18] to FC -spaces.

3. KKM type theorems and intersection theorems

The following is the classical KKM theorem due to Knaster, Kuratowski and Mazurkiewicz in [1].

Lemma 3.1. Let F_0, \dots, F_n be closed (or open) subsets of Δ_n such that $\text{co}\{e_{i_0}, \dots, e_{i_k}\} \subset \bigcup_{j=0}^k F_{i_j}$ for any choice of $0 \leq i_0, \dots, i_k \leq n$. Then $\bigcap_{i=0}^n F_i \neq \emptyset$.

Now we prove the following KKM type theorem and matching theorem in FC -spaces by using the classical KKM theorem.

Theorem 3.1. Let (X, φ_A) be an FC -space, Y a nonempty set. If $S: Y \rightarrow 2^X$ is a generalized F -KKM mapping with closed (respectively open) values, then $\{S(x)\}_{x \in X}$ has finite intersection property.

Proof. For any $\tilde{A} = \{y_0, \dots, y_n\} \in \langle Y \rangle$, by the definition of generalized F -KKM mappings, there exists $A = \{x_0, \dots, x_n\} \in \langle X \rangle$ such that for any $B = \{x_{i_0}, \dots, x_{i_k}\}$,

$$\varphi_A \left(\text{co} \left\{ e_{i_0}, \dots, e_{i_k} \subset \bigcup_{j=0}^k S(y_{i_j}) \right\} \right)$$

holds.

Therefore

$$\text{co}\{e_{i_0}, \dots, e_{i_k}\} \subset \bigcup_{j=0}^k \varphi_A^{-1} S(y_{i_j}).$$

Since $\varphi_A^{-1} S(y_{i_j})$ is closed (respectively open) in the compact set Δ_n . Applying the KKM principle, we have

$$\bigcap_{i=0}^n \varphi_A^{-1} S(y_i) \neq \emptyset$$

which implies $\bigcap_{i=0}^n S(y_i) \neq \emptyset$. This completes the proof. \square

Remark 3.1. If X is G -convex space, the conclusion goes back to Theorem 2.2 of Tan in [18].

The following matching theorem plays a fundamental role in our paper.

Theorem 3.2. Let (X, φ_A) be an FC -space, $A = \{x_0, \dots, x_n\} \in \langle X \rangle$ and $\{M_x: x \in A\}$ an open or closed cover of X . Then there exists a nonempty B of A such that $\varphi_A(\Delta_B) \cap \bigcap \{M_x: x \in B\} \neq \emptyset$.

Proof. First, we prove the case that $\{M_x: x \in A\}$ is a closed cover of X . Consider an n -simplex $\Delta_n = \text{co}(\{e_0, \dots, e_n\})$. For each $z \in \Delta_A$, let $I(z) = \{i: \varphi_A(z) \in M_{x_i}\}$, it is easy to know $I(z) \neq \emptyset$. Define $T: \Delta_A \rightarrow \Delta_A$ as: $\forall z \in \Delta_A$, Tz is the convex hull of $\{e_i: i \in I(z)\}$. For each $z \in \Delta_A$, $\bigcup\{M_{x_i}: i \notin I(z)\}$ is closed, so $U_z = \Delta_A \setminus \varphi_A^{-1}(\bigcup\{M_{x_i}: i \notin I(z)\})$ is the open neighborhood of z . If $z' \in U_z$, then $z' \notin \varphi_A^{-1}(\bigcup\{M_{x_i}: i \notin I(z)\})$, i.e., $\varphi_A(z') \notin \bigcup\{M_{x_i}: i \notin I(z)\}$. Hence, if $i \notin I(z)$, then $\varphi_A(z') \notin M_{x_i}$, thus, $i \notin I(z')$, therefore $I(z') \subset I(z)$. By the definition of $T(z)$, we have $T(z') \subset T(z)$. It follows that T is u.s.c. since for any $z \in \Delta_A$, there exists a neighborhood U_z of z such that for any $z' \in U_z$, $T(z') \subset T(z)$. Now, for all $z \in \Delta_A$, still by the definition of T , $T(z)$ is the nonempty compact convex set of Δ_A , applying Kakutani fixed point theorem, there exists $z_0 \in \Delta_A$ such that $z_0 \in T(z_0)$. Denote $B = \{x_i: i \in I(z_0)\}$. Since $T(z_0) = \text{co}\{e_i: i \in I(z_0)\}$, we have $z_0 \in \text{co}\{e_i: i \in I(z_0)\}$, i.e., $z_0 \in \Delta_B$. By the definition of $I(z_0)$, for each $i \in I(z_0)$, $\varphi_A(z_0) \in M_{x_i}$, hence $\varphi_A(z_0) \in \bigcap_{i \in I(z_0)} M_{x_i} = \bigcap\{M_x: x \in B\}$. Therefore, when $\{M_z: z \in A\}$ is a closed cover of X , we have $\varphi_A(\Delta_B) \cap \bigcap\{M_z: z \in B\} \neq \emptyset$ for some $B \in \langle A \rangle$.

Next we prove the case that $\{M_x: x \in A\}$ is an open cover. For each $x \in A$, let $F(x) = X \setminus M_x$. Suppose that the conclusion is false. Then for every subset $B = \{x_{i_0}, \dots, x_{i_k}\} \in \langle A \rangle$, we have

$$\varphi_A(\text{co}\{e_{i_0}, \dots, e_{i_k}\}) \subset X \setminus \bigcap\{M_x: x \in B\} = \bigcup_{j=0}^k F(x_{i_j}).$$

That is $\text{co}\{e_{i_0}, \dots, e_{i_k}\} \subset \bigcup_{j=0}^k \varphi_A^{-1}F(x_{i_j})$. Therefore, by the KKM principle, we have

$$\bigcap_{i=0}^n \varphi_A^{-1}F(x_{i_i}) \neq \emptyset$$

which implies $\bigcap_{i=0}^n F(x_{i_i}) \neq \emptyset$, so $\bigcup\{M_x: x \in A\} \neq X$, this is a contradiction and the proof is complete. \square

Applying the above matching theorem, we give the following intersection theorems.

Theorem 3.3. Let (X, φ_A) be a compact FC-space, Y a nonempty set, $T, S: X \rightarrow 2^Y$ two mappings satisfying the following conditions:

- (i) S is W - G - F - KKM mapping with respect to T ;
- (ii) for each $z \in X$ the set $\{x \in X: T(x) \cap S(z) \neq \emptyset\}$ is closed.

Then there exists $x_0 \in X$ such that $T(x_0) \cap S(z) \neq \emptyset$ for each $z \in X$.

Proof. For each $z \in X$, denote

$$M_z = \{x \in X: T(x) \cap S(z) = \emptyset\}.$$

Suppose the conclusion does not hold, by (ii), the family $\{M_z: z \in X\}$ is an open cover of X ; since X is compact, there is a set $A \in \langle X \rangle$ such that $\bigcup\{M_z: z \in A\} = X$. By Theorem 3.2, there exists a nonempty subset B of A and a point

$$x_0 \in \varphi_A(\Delta_B) \cap \bigcap\{M_z: z \in B\}.$$

Since S is W - G - F - KKM mapping with respect to T , $x_0 \in \varphi_A(\Delta_B)$, we get $T(x_0) \cap S(B) \neq \emptyset$. On the other hand, $x_0 \in \bigcap\{M_z: z \in B\}$, we have $T(x_0) \cap S(z) = \emptyset$ for each $z \in B$, thus, $T(x_0) \cap S(B) = \emptyset$. The obtained contradiction completes the proof. \square

Theorem 3.4. Let (X, φ_A) be a compact FC-space, Y be a nonempty topological space, $T, S: X \rightarrow 2^Y$ two mappings satisfying the following conditions:

- (i) S is W - G - F - KKM mapping with respect to T ;
- (ii) T is upper semicontinuous and S has closed values.

Then there exists $x_0 \in X$ such that $T(x_0) \cap S(z) \neq \emptyset$ for each $z \in X$.

Proof. For any $z \in X$, $S(z)$ is closed, it follows that the set $\{x \in X: T(x) \cap S(z) \neq \emptyset\}$ is closed since T is upper semicontinuous. Conclusion holds by Theorem 3.3. \square

Theorem 3.5. Let (X, φ_A) be an FC-space, and $\varphi_A(\Delta_n)$ an FC-subspace for each $A \in \langle X \rangle$, Y be a nonempty set. $T, S: X \rightarrow 2^Y$ are two mappings satisfying the following conditions:

- (i) S is W - G - F - KKM mapping with respect to T ;
- (ii) the sets $\{x \in X: T(x) \cap S(z) \neq \emptyset\}$ are either all closed or all open, for all $z \in X$.

Then for each $A \in \langle X \rangle$ there exists a point $x_0 \in \varphi_A(\Delta_n)$ such that $T(x_0) \cap S(z) \neq \emptyset$ for each $z \in \varphi_A(\Delta_n)$.

Proof. For $P \in \langle \varphi_A(\Delta_n) \rangle$, $(\varphi_A(\Delta_n), \varphi_P)$ is an FC-space and compact since Δ_n is compact and φ_A continuous. Therefore, we need only to prove that $S|_{\varphi_A(\Delta_n)}$ is W - G - F - KKM mapping with respect to $T|_{\varphi_A(\Delta_n)}$ then conclusion holds by repeating the proof of Theorem 3.3. In fact, $\forall P \in \varphi_A(\Delta_n)$, $\forall Q \in \langle P \rangle$, $\forall x \in \varphi_P(\Delta_Q)$, obviously, $x \in \varphi_A(\Delta_n)$; again, $Q \in \langle P \rangle$, then $Q \in \varphi_A(\Delta_n) \subset X$. Therefore, for any $Q \in \langle P \rangle$ and any $x \in \varphi_P(\Delta_Q)$, we have

$$T|_{\varphi_A(\Delta_n)}(x) \cap S|_{\varphi_A(\Delta_n)}(Q) = T(x) \cap S(Q) \neq \emptyset,$$

i.e., $S|_{\varphi_A(\Delta_n)}$ is W - G - F - KKM mapping with respect to $T|_{\varphi_A(\Delta_n)}$. \square

Theorem 3.6. Let (X, φ_A) be an FC-space, and $\varphi_A(\Delta_n)$ be an FC-subspace for each $A \in \langle X \rangle$, Y is a nonempty topological space, $T, S: X \rightarrow 2^Y$ are two mappings satisfying the following conditions:

- (i) S is W - G - F - KKM mapping with respect to T ;
- (ii) either T is upper semicontinuous and S is closed-valued or T is lower semicontinuous and S is open-valued.

Then for each $A \in \langle X \rangle$ there exists a point $x_0 \in \varphi_A(\Delta_n)$ such that $T(x_0) \cap S(z) \neq \emptyset$ for each $z \in \varphi_A(\Delta_n)$.

Proof. Applying Theorem 3.5, the proof is similar to Theorem 3.4. \square

The following proposition shows the relation of generalized F - KKM mapping and γ -generalized F -quasi-convex (respectively γ -generalized F -quasi-concave).

Proposition 3.1. Let (X, φ_A) be an FC-space, Y a nonempty set, $\gamma \in R$ and $f: X \times Y \rightarrow R \cup \{-\infty, +\infty\}$. Then the following are equivalent:

- (1) the mapping $S: Y \rightarrow 2^X$, defined by $S(y) = \{x \in X: f(x, y) \leq \gamma\}$ (respectively $S(y) = \{x \in X: f(x, y) \geq \gamma\}$) for all $y \in Y$, is a generalized F -KKM mapping;
 (2) f is γ -generalized F -quasi-concave (respectively γ -generalized F -quasi-convex) in the second variable y .

Proof. (1) \Rightarrow (2). For any finite subset $\tilde{A} = \{y_0, \dots, y_n\}$ of Y , there exists $A = \{x_0, \dots, x_n\} \subset X$, $\varphi_A(\text{co}\{e_{i_0}, \dots, e_{i_k}\}) \subset \bigcup_{j=0}^k S(y_{i_j})$ such that for any $z_0 \in \varphi_A(\text{co}\{e_{i_0}, \dots, e_{i_k}\})$, we have $z_0 \in \bigcup_{j=0}^k S(y_{i_j})$. It follows that $z_0 \in S(y_{i_m})$ for some $0 \leq m \leq k$ and hence $f(z_0, y_{i_m}) \leq \gamma$ (respectively $f(z_0, y_{i_m}) \geq \gamma$). Hence $\min_{0 \leq j \leq k} f(z_0, y_{i_j}) \leq \gamma$ (respectively $\max_{0 \leq j \leq k} f(z_0, y_{i_j}) \geq \gamma$) so that f is γ -generalized F -quasi-concave (respectively γ -generalized F -quasi-convex) in the second variable y .

(2) \Rightarrow (1). Since f is γ -generalized F -quasi-concave (respectively γ -generalized F -quasi-convex) in the second variable y , for any finite subset $\tilde{A} = \{y_0, \dots, y_n\}$ of Y , there exists $A = \{x_0, \dots, x_n\} \subset X$, for any subset $B = \{x_{i_0}, \dots, x_{i_k}\}$ of A such that for any $z_0 \in \varphi_A(\text{co}\{e_{i_0}, \dots, e_{i_k}\})$, $\min_{0 \leq j \leq k} f(z_0, y_{i_j}) \leq \gamma$ (respectively $\max_{0 \leq j \leq k} f(z_0, y_{i_j}) \geq \gamma$). It follows that for some $m \in \{0, \dots, k\}$, $z_0 \in S(y_{i_m}) \subset \bigcup_{j=0}^k S(y_{i_j})$. Hence

$$\varphi_A(\text{co}\{e_{i_0}, \dots, e_{i_k}\}) \subset \bigcup_{j=0}^k S(y_{i_j})$$

so that S is a generalized F -KKM mapping. \square

4. Applications to minimax inequalities

In the next theorem, as in the other minimax theorems established, we shall suppose $\inf_x \sup_y f(x, y) > -\infty$. As to the case $\inf_x \sup_y f(x, y) = -\infty$, all these results remain true evidently.

Theorem 4.1. Let (X, φ_A) be a compact FC -space, Y a topological space. Let $T: X \rightarrow 2^Y$ be a u.s.c. mapping, $f, g: X \times Y \rightarrow \mathbb{R}$ be two functions and $\beta = \inf_{x \in X} \sup_{y \in T(x)} f(x, y)$. Suppose that:

- (i) for each $z \in X$, $g(z, \cdot)$ is u.s.c. on Y ;
 (ii) for any $\lambda < \beta$, $y \in T(x)$, if for each $A \in \langle X \rangle$ and $B \in \langle A \cap \{x \in X: g(x, y) < \lambda\} \rangle$ one has $\varphi_A(\Delta_B) \subset \{x \in X: f(x, y) < \lambda\}$.

Then the following statements hold:

- (1) $\inf_{x \in X} \sup_{y \in T(x)} f(x, y) \leq \sup_{x \in X} \inf_{z \in X} \sup_{y \in T(x)} g(z, y)$.
 (2) Moreover, if T is compact-valued, then there exists $x_0 \in X$ such that

$$\inf_{x \in X} \sup_{y \in T(x)} f(x, y) \leq \inf_{z \in X} \sup_{y \in T(x_0)} g(z, y).$$

Proof. Let $\lambda < \beta$ be fixed and define $S: X \rightarrow 2^Y$ by

$$S(z) = \{y \in Y: g(z, y) \geq \lambda\}, \quad z \in X.$$

By (i), $S(z)$ is closed for each $z \in X$. We show that S is W - G - F - KKM mapping with respect to T . Suppose, on the contrary, that there exist $A \in \langle X \rangle$, $B \in \langle A \rangle$ and $\bar{x} \in \varphi_A(\Delta_B)$ such that $T(\bar{x}) \cap S(B) = \emptyset$. Then for each $y \in T(\bar{x})$, $B \subset \{z \in X: g(z, y) < \lambda\}$, hence,

$$B \in \left\langle A \cap \{x \in X: g(x, y) < \lambda\} \right\rangle.$$

Consequently, by (2),

$$\bar{x} \in \varphi_A(\Delta_B) \subset \{x \in X: f(x, y) < \lambda\}$$

for all $y \in T(\bar{x})$.

Hence $\sup_{y \in T(\bar{x})} f(\bar{x}, y) \leq \lambda$, which contradicts $\lambda < \beta$.

By Theorem 3.4, there exists a point $x_0 \in X$ such that $T(x_0) \cap S(z) \neq \emptyset$ for all $z \in X$. Hence, we have that $\lambda < \inf_{z \in X} \sup_{y \in T(x_0)} g(z, y)$, and thereby

$$\lambda \leq \sup_{x \in X} \inf_{z \in X} \sup_{y \in T(x)} g(z, y),$$

we proved part (1).

Further, if $T(x)$ is compact for all $x \in X$, then $x \rightarrow \inf_{z \in X} \sup_{y \in T(x)} g(z, y)$ is u.s.c. on X because T is u.s.c. on X and $g(\cdot, y)$ is u.s.c. on Y (see [23, Proposition 3.1.21]). Since X is compact there exists $x_0 \in X$ such that

$$\inf_{z \in X} \sup_{y \in T(x_0)} g(z, y) = \sup_{x \in X} \inf_{z \in X} \sup_{y \in T(x)} g(z, y).$$

Therefore, part (2) follows part (1). \square

Corollary 4.1. Let (X, φ_A) be a compact FC-space, Y a topological space. Let $T: X \rightarrow 2^Y$ be a u.s.c. mapping, $f, g: X \times Y \rightarrow \mathbb{R}$ be two functions and $\beta = \inf_{x \in X} \sup_{y \in T(x)} f(x, y)$. Suppose that:

- (i) for each $z \in X$, $g(z, \cdot)$ is u.s.c. on Y ;
- (ii) $f(z, y) \leq g(z, y)$ for all $(z, y) \in X \times T(X)$;
- (iii) $f|_{X \times T(X)}$ is F - β -quasi-convex on X .

Then the following statements hold:

- (1) $\inf_{x \in X} \sup_{y \in T(x)} f(x, y) \leq \sup_{x \in X} \inf_{z \in X} \sup_{y \in T(x)} g(z, y)$.
- (2) Moreover, if T is compact-valued, then there exists $x_0 \in X$ such that

$$\inf_{x \in X} \sup_{y \in T(x)} f(x, y) \leq \inf_{z \in X} \sup_{y \in T(x_0)} g(z, y).$$

Proof. We need only to prove that (ii) and (iii) imply (i) in Theorem 4.1. If $\lambda < \beta$, $y \in T(x)$ and $A \in \langle X \rangle$ and $B \in \langle A \cap \{x \in X: g(x, y) < \lambda\} \rangle$, then by (ii), $B \in \langle A \cap \{x \in X: f(x, y) < \lambda\} \rangle$, by (iii), $\varphi_A(\Delta_B) \subset \{x \in X: f(x, y) < \lambda\}$.

When $X = Y$ and T is an identity mapping, Theorem 4.1 and Corollary 4.1 become Fan's minimax inequality [19]. If X, Y are convex subsets of topological vector space, our results reduce to corresponding minimax inequalities of Ha [20], Liu [21]. If X is a G -convex space, our results go back to minimax inequalities of Kim [22]. \square

Theorem 4.2. Let (X, φ_A) be an FC-space and $\varphi_A(\Delta_n)$ is an FC-subspace, Y is a topological space. Let $T: X \rightarrow 2^Y$ be an l.s.c. mapping, $f, g: X \times Y \rightarrow R$ be two functions and $\beta = \inf_{x \in X} \sup_{y \in T(x)} f(x, y)$. Suppose that:

- (i) for each $x \in X$, $g(x, \cdot)$ is l.s.c. on Y ;
- (ii) for any $\lambda < \beta$, $y \in T(x)$ and if for each $A \in \langle X \rangle$ and $B \in \langle A \cap \{x \in X: g(x, y) < \lambda\} \rangle$ one has $\varphi_A(\Delta_B) \subset \{x \in X: f(x, y) < \lambda\}$.

Then the following statements hold:

$$\inf_{x \in X} \sup_{y \in T(x)} f(x, y) \leq \inf_{A \in \langle X \rangle} \sup_{x \in \varphi_A(\Delta_n)} \min_{z \in \varphi_A(\Delta_n)} \sup_{y \in T(x)} g(z, y).$$

Proof. Let $\lambda < \beta$ be fixed and define $S: X \rightarrow 2^Y$ by

$$S(z) = \{y \in Y: g(z, y) > \lambda\}, \quad z \in X.$$

By (i), $S(x)$ is open for each $x \in X$. We show that S is W - G - F - KKM with respect to T . Suppose, on the contrary, that there exist $A \in \langle X \rangle$ and $\bar{x} \in \varphi_A(\Delta_B)$ such that $T(\bar{x}) \cap S(B) = \emptyset$. Then for each $y \in T(\bar{x})$, $B \subset \{x \in X: g(x, y) \leq \lambda\}$. Hence, $B \in \langle A \cap \{x \in X: g(x, y) < \lambda\} \rangle$. By (ii), for all $y \in T(\bar{x})$

$$\bar{x} \in \varphi_A(\Delta_B) \subset \{x \in X: f(x, y) \leq \lambda\}.$$

Therefore, $\sup_{y \in T(\bar{x})} f(\bar{x}, y) \leq \lambda$, which contradicts $\lambda < \beta$.

By Theorem 3.6, for each $A \in \langle X \rangle$ there exists a point $x_A \in \varphi_A(\Delta_n)$ such that $T(x_A) \cap S(z) \neq \emptyset$ for all $z \in \varphi_A(\Delta_n)$. Consequently, $\min_{z \in A} \sup_{y \in T(x_A)} g(z, y) > \lambda$, whence

$$\sup_{x \in \varphi_A(\Delta_A)} \min_{z \in \varphi_A(\Delta_n)} \sup_{y \in T(x)} g(z, y) > \lambda$$

for all $A \in \langle X \rangle$ and the proof of the theorem is complete. \square

Corollary 4.2. Let (X, φ_N) be an FC-space and $\varphi_A(\Delta_n)$ is an FC-subspace, Y is a topological space. Let $T: X \rightarrow 2^Y$ be an l.s.c. mapping, $f, g: X \times Y \rightarrow R$ be two functions and $\beta = \inf_{x \in X} \sup_{y \in T(x)} f(x, y)$. Suppose that:

- (i) for each $x \in X$, $g(x, \cdot)$ is l.s.c. on Y ;
- (ii) $f(z, y) \leq g(z, y)$ for all $(z, y) \in X \times T(X)$;
- (iii) $f|_{X \times T(X)}$ is F - β -quasi-convex on X .

Then the following statements hold:

$$\inf_{x \in X} \sup_{y \in T(x)} f(x, y) \leq \inf_{A \in \langle X \rangle} \sup_{x \in \varphi_A(\Delta_n)} \min_{z \in \varphi_A(\Delta_n)} \sup_{y \in T(x)} g(z, y).$$

Proof. Similar to Corollary 4.1. \square

Theorem 4.3. Let (X, φ_A) be an FC-space and $\varphi_A(\Delta_n)$ is an FC-subspace, Y is a topological space. Let $T: X \rightarrow 2^Y$ be a u.s.c. mapping, $f, g: X \times Y \rightarrow R$ be two functions and $\beta = \inf_{x \in X} \sup_{y \in T(x)} f(x, y)$. Suppose that:

- (i) for each $x \in X$, $g(x, \cdot)$ is u.s.c. on Y ;
- (ii) for any $\lambda < \beta$, $y \in T(x)$ and if for each $A \in \langle X \rangle$ and $B \in \langle A \cap \{x \in X: g(x, y) < \lambda\} \rangle$ one has $\varphi_A(\Delta_B) \subset \{x \in X: f(x, y) < \lambda\}$.

Then

$$\inf_{x \in X} \sup_{y \in T(x)} f(x, y) \leq \inf_{A \in \langle X \rangle} \sup_{x \in \varphi_A(\Delta_n)} \min_{z \in \varphi_A(\Delta_n)} \sup_{y \in T(x)} g(z, y).$$

Proof. Repeat the proof of Theorem 4.2. \square

Corollary 4.3. Let (X, φ_A) be an FC-space and $\varphi_A(\Delta_n)$ a FC-subspace, Y is a topological space. Let $T: X \rightarrow 2^Y$ be a u.s.c. mapping, $f, g: X \times Y \rightarrow \mathbb{R}$ be two functions and $\beta = \inf_{x \in X} \sup_{y \in T(x)} f(x, y)$. Suppose that:

- (i) for each $x \in X$, $g(x, \cdot)$ is u.s.c. on Y ;
- (ii) $f(z, y) \leq g(z, y)$ for all $(z, y) \in X \times T(X)$;
- (iii) $f|_{X \times T(X)}$ is F - β -quasi-convex on X .

Then

$$\inf_{x \in X} \sup_{y \in T(x)} f(x, y) \leq \inf_{A \in \langle X \rangle} \sup_{x \in \varphi_A(\Delta_n)} \min_{z \in \varphi_A(\Delta_n)} \sup_{y \in T(x)} g(z, y).$$

References

- [1] B. Knaster, K. Kuratowski, S. Mazurkiewicz, Ein Beweis des Fixpunktsatzes Für n -dimensionale Simplexe, *Fund. Math.* 14 (1929) 132–137.
- [2] C. Hovath, Some results on multivalued mappings and inequalities without convexity, in: B.L. Lin, S. Simons (Eds.), *Nonlinear and Convex Analysis*, in: *Lecture Notes in Pure and Appl. Math.*, vol. 106, Dekker, New York, 1987, pp. 99–106.
- [3] X.P. Ding, Coincidence theorems and equilibria of generalized games, *Indian J. Pure Appl. Math.* 27 (1996) 1057–1071.
- [4] K.K. Tan, G - KKM theorem, minimax inequalities and saddle points, *Nonlinear Anal.* 30 (1997) 4151–4160.
- [5] T.H. Chang, C.L. Yen, KKM property and fixed point theorem, *J. Math. Anal. Appl.* 203 (1996) 224–235.
- [6] L.J. Lin, C.J. Ko, S. Park, Coincidence theorems for set-valued maps with G - KKM property on generalized convex space, *Discuss. Math. Differ. Incl.* 18 (1998) 69–85.
- [7] M. Balaj, Weakly G - KKM mappings, G - KKM properties and minimax inequalities, *J. Math. Anal. Appl.* 294 (2004) 237–245.
- [8] S. Park, Generalizations of Ky Fan's matching theorems and their applications, *J. Math. Anal. Appl.* 141 (1989) 164–176.
- [9] S. Park, Foundations of the KKM theory via coincidence of composites of upper semicontinuous maps, *J. Korean Math. Soc.* 31 (1994) 493–520.
- [10] R.U. Verma, Some results on R - KKM mappings and R - KKM selections and their applications, *J. Math. Anal. Appl.* 232 (1999) 428–433.
- [11] X.P. Ding, Maximal elements theorems in product FC-space and generalized games, *J. Math. Anal. Appl.* 305 (2005) 29–42.
- [12] X.P. Ding, Maximal elements of G_{KKM} -majorized mappings in product FC-spaces and applications (I), *Nonlinear Anal.* (2007), doi: 10.1016/j.na.2006.06.037, in press.
- [13] X.P. Ding, T.M. Ding, KKM type theorems and generalized vector equilibrium problems in noncompact FC-spaces, *J. Math. Anal. Appl.* (2007), doi: 10.1016/j.jmaa.2006.09.059, in press.
- [14] M. Fang, N.J. Huang, KKM type theorems with applications to generalized vector equilibrium problems in FC-spaces, *Nonlinear Anal.* (2007), doi: 10.1016/j.na.2006.06.040, in press.

- [15] Q.B. Zhang, C.Z. Cheng, Some fixed-point theorems and minimax inequalities in FC -space, J. Math. Anal. Appl. 328 (2) (2007) 1369–1377.
- [16] X.P. Ding, Weak Pareto equilibria for generalized constrained multiobjective games in locally FC -spaces, Nonlinear Anal. 65 (2006) 538–545.
- [17] X.P. Ding, Maximal elements of G_{KKM} -majorized mappings in product FC -spaces and applications (II), Nonlinear Anal. (2007), doi: 10.1016/j.na.2006.10.025, in press.
- [18] K.K. Tan, G - KKM theorem, minimax inequalities and saddle points, Nonlinear Anal. 30 (1997) 4151–4160.
- [19] Ky Fan, A minimax inequality and applications, in: O. Shisha (Ed.), Inequalities, vol. III, Academic Press, San Diego, 1972, pp. 103–113.
- [20] C.W. Ha, On a minimax inequality of Ky Fan, Proc. Amer. Math. Soc. 99 (1987) 680–682.
- [21] F.C. Liu, On a form of KKM principle and SupInfSup inequalities of Von Neumann and of Ky Fan type, J. Math. Anal. Appl. 155 (1991) 420–436.
- [22] J. Kim, KKM theorems and minimax inequalities in G -convex spaces, Nonlinear Anal. Forum 6 (2001) 135–142.
- [23] J.P. Aubin, J. Ekeland, Applied Nonlinear Analysis, Wiley, New York, 1984.