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Monotonization of flux, entropy and numerical schemes for conservation laws

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Abstract

Using the concept of monotonization, families of two step and k-step finite volume schemes for scalar hyperbolic conservation laws are constructed and analyzed. These families contain the FORCE scheme and give an alternative to the MUSTA scheme. These schemes can be extended to systems of conservation law.

Keywords: Finite volumes; Finite differences; Riemann solvers; Conservation laws; Monotonization

1. Introduction

Let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function and consider the scalar conservation law

$$u_t + f(u)_x = 0, \quad t > 0,\ x \in \mathbb{R}, \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where $u_0 \in BV(\mathbb{R}) \cap L^1_{\mathrm{loc}}(\mathbb{R})$. This problem has been extensively studied in the last decade. In general problem (1.1) and (1.2) does not admit regular solutions because of two reasons:

(i) The characteristics do not fill the entire space, giving raise to empty regions where the solutions cannot be determined by the initial data.
(ii) The characteristics intersect and lead to multivalued solutions.

In order to overcome these difficulties, the concept of weak solutions is introduced. To construct such solutions with simple data, one faces difficulties (i) and (ii). In order to overcome (i), one make use of the invariance property of Eq. (1.1), that is it is invariant under the transformation $x \to \alpha x$, $t \to \alpha t$ for any $\alpha > 0$. This leads to a self similar

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solution in the variable $\xi = x/t$ is called the rarefaction wave. This solution is not determined by the initial data and is used to determine the solution in the empty space.

In order to overcome (ii), discontinuities are introduced in the regions where the characteristics intersect. Now using the definition of weak solutions, it is possible to determine the lines of discontinuity called shocks. Shocks must satisfy the Rankine Hugoniot condition connecting the tangent to the line of discontinuity with the jumps of the solution across it.

The next problem is the nonuniqueness of weak solutions. This nonuniqueness basically comes from the way one fills the empty region by a solution. Now by looking at the underlying physical phenomena a concept of an "Entropy" criterion is introduced which uniquely defines the solution in the empty region for simple data. Using the above concepts, existence of a unique entropy solution is proved for arbitrary data.

There are three fundamental methods used to obtain the existence and uniqueness of an entropy solution.

(1) *The Hamilton–Jacobi method.* Here one assumes that the flux f is C^2 and strictly convex and considers the Hamilton–Jacobi equation:

$$v_t + f(v_x) = 0, \quad x \in \mathbb{R},\ t > 0, \tag{1.3}$$

$$v(x,0) = v_0(x), \quad x \in \mathbb{R}, \tag{1.4}$$

where $v_0(x) = \int_0^x u_0(\theta)\,d\theta$. This equation has a unique viscosity solution and an explicit formula was given by Hopf [2]. Then letting $u = \frac{\partial v}{\partial x}$, Lax and Oleinik proved that u is the unique entropy solution of Eqs. (1.1), (1.2) [2].

(2) *The vanishing viscosity method.* Here one looks at the nonlinear parabolic equation

$$u_t + f(u)_x = \varepsilon u_{xx}, \quad x \in \mathbb{R},\ t > 0, \tag{1.5}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}. \tag{1.6}$$

Kruzkov [4] proves that there exists a unique solution u^ε of Eqs. (1.5), (1.6) converging in L^1_{loc} to a unique entropy solution of Eqs. (1.1), (1.2) as $\varepsilon \to 0$.

(3) *Numerical schemes.* In this method numerical schemes are derived using space and time discretizations of Eqs. (1.1), (1.2). These scheme calculate an approximation solution which converges to the unique entropy solution. Lax and Friedrichs, Godunov, Enquist and Osher, Roe and others contributed to this approach.

In this paper we concentrate on numerical schemes to solve Eqs. (1.1), (1.2) and we will construct schemes whose numerical fluxes can be evaluated by point evaluations of the flux function f contrarily to many numerical schemes in which numerical flux evaluations involve calculations of integrals, maxima or minima of f. This property of using only point evaluations of the numerical flux is crucial for extending without too much complexity a numerical scheme to systems.

In Section 2 we introduce the concept of monotonization which leads us to a new definition of entropy solution. This approach can lead to the concept of entropy for systems. In Section 3 we construct a first two step monotonization scheme which is actually the Force scheme [5,6]. In Section 4 this scheme is generalized to a family of two step monotonization schemes and analyzed. In Section 5 these numerical schemes are extended to a family of k-step monotonization schemes which gives an alternative to the MUSTA scheme [7]. These schemes are easy to extend to systems (Section 6).

2. Monotonization and entropy

A basic ingredient in studying a numerical scheme is the study of the Riemann problem. Let $a, b, x_0 \in \mathbb{R}$ and let

$$u_0(x) = \begin{cases} a & \text{if } x < x_0, \\ b & \text{if } x \geqslant x_0, \end{cases} \tag{2.1}$$

and the solution set $R(a,b,x_0)$ associated to (2.1) is given by

$$R(a,b,x_0) = \big\{u;\ u \text{ is a weak solution of Eqs. (1.1), (2.1)}\big\}.$$

In general $R(a,b,x_0)$ can have more than one solution. This set $R(a,b,x_0)$ turns out to be an important tool in deriving finite volume schemes. This is a well studied method (for details see [3]).

Let $h>0$, $\Delta t>0$, $\lambda=\Delta t/h$ and discretize the domain into disjoint rectangles R_{in} with length h and breadth Δt. That is for $i\in\mathbb{Z}$, $x\geqslant 0$ let $R_{in}=I_i\times J_n$ where

$$x_{i+1/2}=ih,\qquad t_n=n\Delta t,\qquad I_i=[x_{i+1/2},x_{i+3/2}),\qquad J_n=\big[n\Delta t,(n+1)\Delta t\big).$$

λ must satisfy the CFL condition: let

$$\lambda\sup_{\theta\in[-M,M]}\big|f'(\theta)\big|\leqslant 1, \tag{2.2}$$

where $M=\|u_0\|_\infty$.

We discretize u_0 by $\{u_i^0\}_{i\in\mathbb{Z}}$ defined as

$$u_i^0=\frac{1}{h}\int\limits_{I_i}u_0(x)\,dx$$

and we assume that for $0\leqslant j\leqslant n$, $\{u_i^j\}_{i\in\mathbb{Z}}$ are known. In order to define $\{u_i^{n+1}\}_{i\in\mathbb{Z}}$, we choose a solution $w_i^n\in R(u_i^n,u_{i+1}^n,x_{i+3/2})$ for each $i\in\mathbb{Z}$ and define

$$w(x,t)=w_i^n(x,t)\quad\text{for }(x,t)\in R_{in}.$$

Then from the CFL condition, w is well defined and is a solution of Eq. (1.1) for $x\in\mathbb{R}$, $n\Delta t<t<(n+1)\Delta t$ with initial condition $w(x,t_n)=u_i^n$ for $x\in I_i$. Now we define $\{u_i^{n+1}\}_{i\in\mathbb{Z}}$ as

$$u_i^{n+1}=\frac{1}{h}\int\limits_{I_i}w(x,t_{n+1})\,dx.$$

Since w satisfies (1.1) in $\mathbb{R}\times(n\Delta t,(n+1)\Delta t)$ and hence integrating (1.1) over R_{in} to obtain the following formula

$$u_i^{n+1}=u_i^n-\lambda\left(\frac{1}{\Delta t}\int\limits_{n\Delta t}^{(n+1)\Delta t}f\big(w(x_{i+3/2},t)\big)\,dt-\frac{1}{\Delta t}\int\limits_{n\Delta t}^{(n+1)\Delta t}f\big(w(x_{i+1/2},t)\big)\,dt\right). \tag{2.3}$$

The evaluation of u_i^{n+1} from (2.3) depends heavily on the choice of the Riemann problem solution $\{w_i^n\}$. Hence in general different choices of $\{w_i^n\}$ give raise to different sets of $\{u_i^n\}$.

In fact one can generate infinitely many L^1-contractive convergent schemes provided f has both increasing as well as decreasing parts [1].

If the flux f is strictly monotone then the characteristics of Eq. (1.1) do not intersect the lines $x=x_{i+1/2}$ for $n\Delta t\leqslant t<(n+1)\Delta t$. Hence for any $w_i^n\in R(u_i^n,u_{i+1}^n,x_{i+3/2})$

$$w_i^n(x_{i+3/2},t)=\begin{cases}u_i^n & \text{if } f'>0,\\ u_{i+1}^n & \text{if } f'<0,\end{cases}$$

and scheme (2.3) reduces to the standard upstream scheme

$$u_i^{n+1}=\begin{cases}u_i^n-\lambda(f(u_{i+1}^n)-f(u_i^n)) & \text{if } f'<0,\\ u_i^n-\lambda(f(u_i^n)-f(u_{i+1}^n)) & \text{if } f'>0.\end{cases} \tag{2.4}$$

As a consequence of this, the scheme is independent of the choice of the Riemann data solution $\{w_i^n\}$ and converges in L^1_{loc} to the unique entropy solution.

If f is strictly convex and the $\{w_i^n\}$ are chosen so that they satisfy the Lax–Oleinik, Kruzkov entropy condition, then scheme (2.3) reduces to the finite volume scheme

$$u_i^{n+1}=u_i^n-\lambda\big(F^G\big(u_i^n,u_{i+1}^n\big)-F^G\big(u_{i-1}^n,u_i^n\big)\big) \tag{2.5}$$

where $F^G(a,b)$ is the Godunov flux defined by

$$F^G(a,b)=\begin{cases}\min_{\theta\in(a,b)}f(\theta) & \text{if } a\leqslant b,\\ \max_{\theta\in(a,b)}f(\theta) & \text{if } a\geqslant b.\end{cases} \tag{2.6}$$

In fact from a theorem of Oleinik (see [3]), even if f is not convex, the Godunov scheme (2.5), (2.6) converges in L^1_{loc} to the unique entropy solution of problem (1.1), (1.2).

Besides its optimal properties in terms of numerical diffusion, a drawback of the Godunov scheme is that its numerical flux (2.6) cannot be written in terms of point values of the function f, a point which becomes critical when extending the method to systems.

On the other hand we notice from scheme (2.4) that if the flux is monotone, then the choice of the Riemann problem solution is irrelevant and the flux can be calculated in terms of point values of f. Therefore we convert problem (1.1) to a problem having a monotone flux function. This procedure is called *monotonization*.

Given $M > 0$, $\alpha \in \mathbb{R}$, denote $f_\alpha(u)$ by $f_\alpha(u) = f(u) - u/\alpha$ and choose α such that

$$\sup_{u\in[-M,M]} \left|f'(u)\right| < |\alpha|. \tag{2.7}$$

Let u be a solution of Eqs. (1.1), (1.2) with $\|u_0\|_\infty \leqslant M$, and consider the change of variables

$$x = X + \alpha\tau, \quad t = \tau, \quad u(x,t) = v(X,\tau).$$

Then v satisfies

$$v_\tau + f_{1/\alpha}(v)_X = 0 \quad \text{for } X \in \mathbb{R},\ \tau > 0, \tag{2.8}$$

$$v(X,0) = u_0(X) \quad \text{for } X \in \mathbb{R}. \tag{2.9}$$

Furthermore $|v(x,0)| = |u_0(x)| \leqslant M$ and from (2.7), $f_{\frac{1}{\alpha}}(v) = f(v) - \alpha v$ is a strictly monotone function for $v \in [-M, M]$. Hence the finite volume scheme for Eqs. (2.8), (2.9) does not depend on the choice of the Riemann data solution. Furthermore it produces a solution v_h which converges in L^1_{loc} to the unique entropy solution v of problem (2.8), (2.9). Therefore $u(x,t) = v(x - \alpha t, t)$ is the unique entropy solution for problem (1.1), (1.2).

Consequently we can state the following alternative definition of the entropy solution.

Definition *(Entropy solution).* Let $u \in L^1_{\mathrm{loc}} \cap L^\infty$ be a weak solution of problem (1.1), (1.2). Then u is said to be an entropy solution to this problem, if $u(x,t) = v(x - \alpha t, t)$ for all α such that $f_{1/\alpha}$ is strictly monotone in $[-\|u_0\|_\infty, \|u_0\|_\infty]$ and v is the unique solution of problem (2.8), (2.9) obtained after convergence of the solution of the upstream finite volume scheme (2.4) applied to problem (2.8), (2.9).

In the above analysis we change the variables so that in the new variables, the flux function becomes strictly monotone. This allows us to reduce the finite volume scheme on rectangular space-time meshes to a simple upstream numerical scheme. Now in the next section we reverse the order, namely we keep the equation as it is but change the rectangular mesh to parallelogram meshes to obtain a numerical scheme which lies between Godunov and Lax–Friedrichs schemes in terms of numerical viscosity.

3. A first two step monotonization scheme

3.1. Formulation of the two step monotonization scheme

We assume again that the initial data satisfies $u_i^0 \in [-M, M]$ for some $M > 0$ and we introduce some more notation

$$x_i = (i + 1/2)h, \qquad t_{n+1/2} = (n + 1/2)\Delta t,$$
$$p_i^n = (x_{i+1/2}, n\Delta t), \qquad p_i^{n+1/2} = \big(x_i, (n + 1/2)\Delta t\big),$$
$$P_i^{n+1/2} = \text{parallelogram with vertices } p_i^n,\ p_{i+1}^n,\ p_{i+1}^{n+1/2},\ p_i^{n+1/2},$$
$$P_i^n = \text{ parallelogram with vertices } p_i^{n+1/2},\ p_{i+1}^{n+1},\ p_i^{n+1},\ p_{i-1}^{n+1/2},$$

as shown in Fig. 3.1.

We assume that all the characteristics emanating from p_i^n (respectively $p_i^{n+1/2}$) do not intersect the line segments $[p_i^n, p_i^{n+1/2}]$, $[p_{i-1}^n, p_{i-1}^{n+1/2}]$ (respectively $[p_i^{n+1/2}, p_{i+1}^{n+1}]$, $[p_{i-1}^{n+1/2}, p_i^{n+1}]$). This implies the following condition

$$\lambda \sup_{u\in[-M,M]} \left|f'(u)\right| \leqslant 1. \tag{3.1}$$

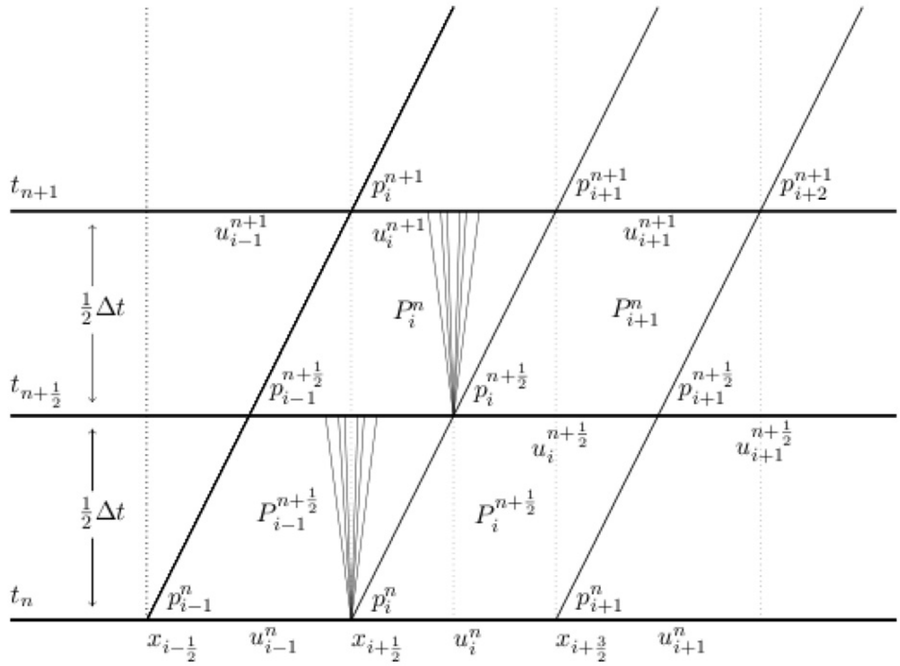

Fig. 3.1. Notation for the two step monotonization scheme.

With the above notation and assumption, we can now derive the two step monotonization scheme.

Assume that $\{u_i^k\}_{i\in\mathbb{Z}}$ for $0 \leqslant k \leqslant n$, are given with $u_i^k \in [-M, M]$. As in Section 2 let $w_i^n \in R(u_i^n, u_{i+1}^n, x_{i+3/2})$ (which need not be an entropy solution) be any solution and define

$$w(x,t) = w_i^n(x,t) \quad \text{for } (x,t) \in P_i^{n+1/2}.$$

Then from condition (3.1), w is a well-defined weak solution and let

$$u_i^{n+1/2} = \frac{1}{h}\int_{x_i}^{x_{i+1}} w\big(x, (n+1/2)\Delta t\big)\,dx.$$

Mass conservation in the cells gives

$$0 = \int_{P_i^{n+1/2}} \big(w_t + f(w)_x\big)\,dx\,dt = \int_{\partial P_i^{n+1/2}} \big(w\nu_t + f(w)\nu_x\big)\,ds.$$

Evaluating the integral on the boundary of $P_i^{n+1/2}$ and using the CFL condition (3.1) (characteristics do not intersect the line segments, see Fig. 3.1) we obtain

$$\begin{aligned} u_i^{n+1/2} &= u_i^n - \frac{\lambda}{2}\left[\frac{1}{\Delta t}\int_{p_{i+1}^n}^{p_{i+1}^{n+1/2}} \left(f(w) - \frac{1}{\lambda}w\right)dt - \frac{1}{\Delta t}\int_{p_i^n}^{p_i^{n+1/2}} \left(f(w) - \frac{1}{\lambda}w\right)dt\right] \\ &= u_i^n - \frac{\lambda}{2}\left[\left(f\big(u_{i+1}^n\big) - \frac{1}{\lambda}u_{i+1}^n\right) - \left(f\big(u_i^n\big) - \frac{1}{\lambda}u_i^n\right)\right] \\ &= \frac{u_i^n + u_{i+1}^n}{2} - \frac{\lambda}{2}\big[f\big(u_{i+1}^n\big) - f\big(u_i^n\big)\big]. \end{aligned}$$

Now repeating the argument with data $\{u_i^{n+1/2}\}$ at $t_{n+1/2}$ and $u_i^{n+1} = \frac{1}{h}\int_{x_{i+1/2}}^{x_{i+3/2}} w(x,(n+1)\Delta t)\,dx$ we have

$$u_i^{n+1} = u_{i-1}^{n+1/2} - \frac{\lambda}{2}\left[\left(f\left(u_i^{n+1/2}\right) - \frac{u_i^{n+1/2}}{\lambda}\right) - \left(f\left(u_{i-1}^{n+1/2}\right) - \frac{u_{i-1}^{n+1/2}}{\lambda}\right)\right]$$
$$= \frac{u_{i-1}^{n+1/2} + u_i^{n+1/2}}{2} - \frac{\lambda}{2}\left[f\left(u_i^{n+1/2}\right) - f\left(u_{i-1}^{n+1/2}\right)\right].$$

Thus the two step monotonization scheme reads

$$u_i^{n+1/2} = \frac{u_i^n + u_{i+1}^n}{2} - \frac{\lambda}{2}\left[f\left(u_{i+1}^n\right) - f\left(u_i^n\right)\right],$$
$$u_i^{n+1} = \frac{u_{i-1}^{n+1/2} + u_i^{n+1/2}}{2} - \frac{\lambda}{2}\left[f\left(u_i^{n+1/2}\right) - f\left(u_{i-1}^{n+1/2}\right)\right]. \tag{3.2}$$

Observe that there is a backward shift in evaluating u_i^{n+1} and that it needs only point evaluations of the flux function f.

The two step scheme (3.2) was already introduced and analyzed in [5,6] and it can be written in a compact form as follows.

For a, b in $\mathbb{R}$ let

$$H(a,b) = a - \frac{\lambda}{2}\left(f_\lambda(b) - f_\lambda(a)\right) = \frac{a+b}{2} - \frac{\lambda}{2}\left(f(b) - f(a)\right). \tag{3.3}$$

Then scheme (3.2) can be rewritten as

$$u_i^{n+1/2} = u_i^n - \frac{\lambda}{2}\left[f_\lambda\left(u_{i+1}^n\right) - f_\lambda\left(u_i^n\right)\right],$$
$$u_i^{n+1} = u_{i-1}^{n+1/2} - \frac{\lambda}{2}\left[f_\lambda\left(u_i^{n+1/2}\right) - f_\lambda\left(u_{i-1}^{n+1/2}\right)\right], \tag{3.4}$$

or

$$u_i^{n+1/2} = H\left(u_i^n, u_{i+1}^n\right), \qquad u_i^{n+1} = H\left(u_{i-1}^{n+1/2}, u_i^{n+1/2}\right),$$

so we formulate the two step monotonization scheme in the compact form

$$u_i^{n+1} = H\left(H\left(u_{i-1}^n, u_i^n\right), H\left(u_i^n, u_{i+1}^n\right)\right). \tag{3.5}$$

3.2. Convergence of the two step monotonization scheme

Concerning convergence we have the main result.

Theorem 3.1. *Let $u_0 \in BV(\mathbb{R})$ and $\|u\|_\infty \leqslant M$. Then under the CFL condition* (3.1) *the two step finite volume scheme $\{u_i^n\}$ given in* (3.5) *converges to the unique entropy solution.*

Proof. Let $a, b \in [-M, M]$, then from (3.1) we have

$$\frac{\partial H}{\partial a}(a,b) = \frac{1}{2}\left(1 + \lambda f'(a)\right) \geqslant 0, \qquad \frac{\partial H}{\partial b}(a,b) = \frac{1}{2}\left(1 - \lambda f'(b)\right) \geqslant 0 \tag{3.6}$$

and hence H is a nondecreasing function in each of its argument. Therefore from (3.5) the scheme is a three point monotone scheme. Let $g(X,Y,Z) = H(H(X,Y), H(Y,Z))$, then

$$g(X,X,X) = H\left(H(X,X), H(X,X)\right) = H(X,X) = X. \tag{3.7}$$

The scheme is L^∞-stable, since, when assuming $u_i^n \in [-M, M]$ for all $i \in \mathbb{Z}$, from (3.6) we can write

$$-M = g(-M,-M,-M) \leqslant g\left(u_{i-1}^n, u_i^n, u_{i+1}^n\right) = u_i^{n+1} \leqslant g(M,M,M) = M.$$

Finally let us write the scheme in conservative form. Expanding (3.5) we obtain

$$\begin{aligned} u_i^{n+1} &= H\big(u_{i-1}^n, u_i^n\big) - \frac{\lambda}{2}\big(f_\lambda\big(H\big(u_i^n, u_{i+1}^n\big)\big) - f_\lambda\big(H\big(u_{i-1}^n, u_i^n\big)\big)\big) \\ &= u_{i-1}^n - \frac{\lambda}{2}\big[f_\lambda\big(u_i^n\big) + f_\lambda\big(H\big(u_i^n, u_{i+1}^n\big)\big) - f_\lambda\big(u_{i-1}^n\big) - f_\lambda\big(H\big(u_{i-1}^n, u_i^n\big)\big)\big] \\ &= u_i^n - \frac{\lambda}{2}\left[f_\lambda\big(u_i^n\big) + f_\lambda\big(H\big(u_i^n, u_{i+1}^n\big)\big) + \frac{2u_i^n}{\lambda} - f_\lambda\big(u_{i-1}^n\big) - f_\lambda\big(H\big(u_{i-1}^n, u_i^n\big)\big) - \frac{2u_{i-1}^n}{\lambda}\right]. \end{aligned}$$

Therefore

$$u_i^{n+1} = u_i^n - \lambda\big[F_\lambda\big(u_i^n, u_{i+1}^n\big) - F_\lambda\big(u_{i-1}^n, u_i^n\big)\big] \tag{3.8}$$

where the numerical flux $F_\lambda(a,b)$ is given by

$$\begin{aligned} F_\lambda(a,b) &= \frac{1}{2}\left[f_\lambda(a) + f_\lambda\big(H(a,b)\big) + \frac{2a}{\lambda}\right] \\ &= \frac{1}{2}\left[f(a) + f\big(H(a,b)\big) - \frac{H(a,b)}{\lambda} + \frac{a}{\lambda}\right] \\ &= \frac{1}{2}\left[f(a) + f\big(H(a,b)\big) - \frac{b-a}{2\lambda} + \frac{1}{2}\big(f(b) - f(a)\big)\right], \end{aligned}$$

or

$$F_\lambda(a,b) = \frac{1}{4}\left[f(a) + f(b) + 2f\big(H(a,b)\big) + \frac{a-b}{\lambda}\right]. \tag{3.9}$$

From (3.7) $F_\lambda(a,a) = f(a)$ so the flux is consistent and consequently the solution of the two step monotonization scheme (3.2) (which can be written alternatively as (3.3), (3.5) or (3.8), (3.9)) converges to the unique entropy solution of problem (1.1), (1.2). This proves the theorem. □

3.3. Comparison with the Lax–Friedrichs (LF) and the two step Lax–Wendroff–Richtnzer (LWR) scheme

On one hand the two step monotonization scheme (3.8), (3.9) gives

$$u_i^{n+1} = \frac{u_{i-1}^n + 2u_i^n + u_{i+1}^n}{4} - \frac{\lambda}{4}\big[f\big(u_{i+1}^n\big) + 2f\big(H\big(u_i^n, u_{i+1}^n\big)\big) - f\big(u_{i-1}^n\big) - 2f\big(H\big(u_{i-1}^n, u_i^n\big)\big)\big]. \tag{3.10}$$

On the other hand the two step LWR scheme (put $\alpha = \beta = 1/2$ in Eq. (2.19) of [3]) is given by

$$w_i^{n+1} = u_i^n - \lambda\big(f\big(H\big(u_i^n, u_{i+1}^n\big)\big) - f\big(H\big(u_{i-1}^n, u_i^n\big)\big)\big),$$

while the LF scheme reads

$$v_i^{n+1} = \frac{u_{i-1}^n + u_{i+1}^n}{2} - \frac{\lambda}{2}\big(f\big(u_{i+1}^n\big) - f\big(u_{i-1}^n\big)\big).$$

It follows that

$$u_i^{n+1} = \frac{w_i^{n+1} + v_i^{n+1}}{2}.$$

Hence the solution given by scheme (3.8), (3.9) is the average of that given by the LF and LWR schemes. This remark was already made by Toro [5]. Therefore even though the LWR scheme is not L^∞-stable, by taking its average with the L^∞-stable LF scheme we obtain a L^∞-stable convergent scheme.

In terms of numerical viscosity, the numerical viscosity coefficient $Q_{i+1/2}$ of the two step monotonization scheme (3.8), (3.9) is determined by

$$F_\lambda(a,b) = \frac{1}{2}\left(f(a) + f(b) - \frac{Q_{i+1/2}}{\lambda}(b-a)\right).$$

Hence

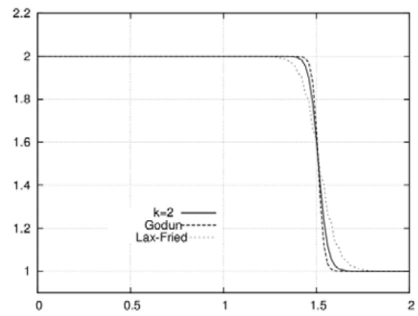

Fig. 3.2. Comparison of the Godunov, Lax–Friedrich and the two step monotonization scheme for $f(u) = u^2/2$ with initial condition $u(x,0) = 2$ if $x < 0$ and 1 if $x > 0$. The solution is shown at $t = 1$ and was calculated with $h = 2/100$, $\Delta t = 2/300$.

$$\begin{aligned}
\frac{(b-a)}{\lambda} Q_{i+1/2} &= f(a) + f(b) - 2F_\lambda(a,b) \\
&= f(a) + f(b) - \frac{1}{2}\left(f(a) + f(b) + 2f\big(H(a,b)\big) + \frac{a-b}{\lambda}\right) \\
&= \frac{1}{2}\big(f(a) + f(b)\big) - f\big(H(a,b)\big) - \frac{a-b}{2\lambda} \\
&= \frac{1}{2}\big(f(a) + f(b)\big) - f\left(a - \frac{\lambda}{2}\big(f_\lambda(b) - f_\lambda(a)\big)\right) - \frac{a-b}{2\lambda} \\
&= \frac{1}{2}\big(f(a) + f(b)\big) - f(a) + \frac{\lambda}{2} f'(\xi)\big(f_\lambda(b) - f_\lambda(a)\big) - \frac{a-b}{2\lambda} \\
&= \frac{1}{2}\big(f(b) - f(a)\big) + \frac{\lambda}{2} f'(\xi)\big(f(b) - f(a)\big) - \frac{b-a}{2} f'(\xi) + \frac{b-a}{2\lambda} \\
&= \frac{1}{2}\big(1 + \lambda f'(\xi)\big)\big(f(b) - f(a)\big) + \frac{1}{2\lambda}\big(1 - \lambda f'(\xi)\big)(b-a).
\end{aligned}$$

Therefore

$$Q_{i+1/2} = \frac{\lambda}{2}\big(1 + \lambda f'(\xi)\big)\frac{f(b) - f(a)}{b-a} + \frac{1}{2}\big(1 - \lambda f'(\xi)\big).$$

Let $f(u) = u$ and denote by $Q^G_{i+1/2}$ and $Q^{LF}_{i+1/2}$ the numerical viscosity coefficient of the Godunov and the Lax–Friedrichs schemes respectively. Then we have

$$Q^G_{i+1/2} = \lambda \leqslant \frac{1+\lambda^2}{2} = Q_{i+1/2} \leqslant 1 = Q^{LF}_{i+1/2}.$$

This shows that the performance of scheme (3.8), (3.9) is better than the Lax–Friedrichs scheme in terms of numerical viscosity as can be observed on Fig. 3.2.

4. Generalized two step monotonization schemes

There are many ways to generalize the two step monotonization presented in the previous section. In this section we generalize it to a family of two step schemes.

Let $\gamma_1, \gamma_2 \in [0,1]$ satisfying $\gamma_1 + \gamma_2 = 1$ and $\beta_1, \beta_2 \in [-1,1]$. Given the discretization steps h, Δt of space and time, we further discretize time by dividing the time step into two substeps $\gamma_1 \Delta t, \gamma_2 \Delta t$, and we move the space discretization point p_i^n to $p_i^{n+1/2}$ by length $\beta_1 h$ at time $t_n + \gamma_1 \Delta t$ and further move $p_i^{n+1/2}$ by length $\beta_2 h$ back to one of the discretization points at time $t_n + (\gamma_1 + \gamma_2)\Delta t = t_{n+1}$. See Fig. 4.1 for $\gamma_1 = 1/3$, $\gamma_2 = 2/3$, $\beta_1 = 1/2$, $\beta_2 = -1/2$ and Fig. 3.1 for $\gamma_1 = \gamma_2 = 1/2$, $\beta_1 = \beta_2 = 1/2$. In this way we build line segments which, in addition of the lines $t = t_n$, $t = \gamma_1 \Delta t$, $t = t_{n+1}$ will form the boundaries of the control volumes for the two step finite volume sheme.

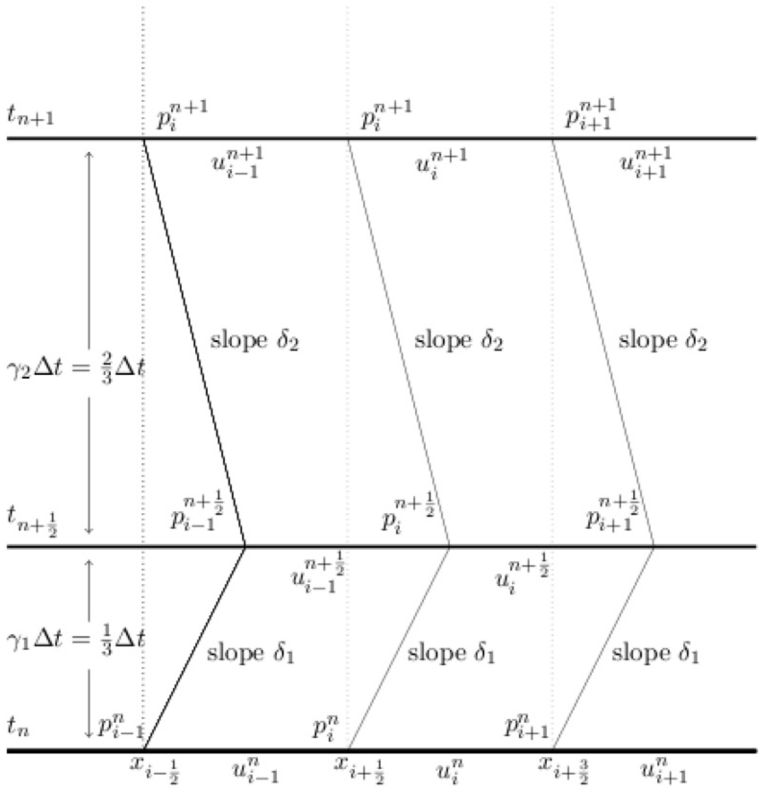

Fig. 4.1. Control volumes for a generalized two step monotonization scheme with $\gamma_1 = 1/3$, $\gamma_2 = 2/3$, $\beta_1 = 1/2$, $\beta_2 = -1/2$.

$\gamma_l, \beta_l, l = 1, 2$, are chosen also in order to satisfy the CFL condition

$$\gamma_l \lambda \sup_{u\in[-M,M]} \left|f'(u)\right| \leqslant \min(|\beta_l|, (1 - |\beta_l|)), \quad l = 1, 2, \tag{4.1}$$

in order to ensure that, for $l = 1$, the characteristics leaving p_i^n, and for $l = 2$, that leaving $p_i^{n+1/2}$ do not intersect the line segments.

For $l = 1, 2$, let $\delta_l = \frac{\gamma_l \lambda}{\beta_l}$. δ_1 is the slope of the segment $[p_i^n, p_i^{n+1/2}]$ and δ_2 is the slope of the segment connecting $p_i^{n+1/2}$ to one of the discretization points at time t_{n+1}.

For $a, b \in \mathbb{R}$ define

$$H_{\delta_l}(a, b) = \begin{cases} a - \gamma_l \lambda (f_{\delta_l}(b) - f_{\delta_l}(a)) & \text{if } \delta_l > 0, \\ b - \gamma_l \lambda (f_{\delta_l}(b) - f_{\delta_l}(a)) & \text{if } \delta_l < 0, \end{cases} \quad l = 1, 2. \tag{4.2}$$

We can generate four different families of convergent schemes, depending on the signs of the β's.

Scheme 1. $\beta_1 \geqslant 0$, $\beta_2 \geqslant 0$, $\beta_1 + \beta_2 = 1$.

If $|u_i^n| \leqslant M$, from the CFL condition (4.1), Scheme 1 reads

$$u_i^{n+1/2} = H_{\delta_1}(u_i^n, u_{i+1}^n) = u_i^n - \gamma_1 \lambda (f_{\delta_1}(u_{i+1}^n) - f_{\delta_1}(u_i^n)),$$
$$u_i^{n+1} = H_{\delta_2}(u_{i-1}^{n+1/2}, u_i^{n+1/2}) = u_{i-1}^{n+1/2} - \gamma_2 \lambda (f_{\delta_2}(u_i^{n+1/2}) - f_{\delta_2}(u_{i-1}^{n+1/2})).$$

Note that the case $\beta_l = \gamma_l = 1/2$, $l = 1, 2$, corresponds to the two step scheme presented in the previous section.

Scheme 2. $\beta_1 > 0$, $\beta_2 < 0$ with $\beta_1 = |\beta_2|$.

If $|u_i^n| \leqslant M$, under the CFL condition (4.1), Scheme 2 reads

$$u_i^{n+1/2} = H_{\delta_1}\big(u_i^n, u_{i+1}^n\big) = u_i^n - \gamma_1\lambda\big(f_{\delta_1}\big(u_{i+1}^n\big) - f_{\delta_1}\big(u_i^n\big)\big),$$
$$u_i^{n+1} = H_{\delta_2}\big(u_{i-1}^{n+1/2}, u_i^{n+1/2}\big) = u_i^{n+1/2} - \gamma_2\lambda\big(f_{\delta_2}\big(u_i^{n+1/2}\big) - f_{\delta_2}\big(u_{i-1}^{n+1/2}\big)\big).$$

This case corresponds to the situation shown in Fig. 4.1.

The other two cases are $\beta_1 \leqslant 0$, $\beta_2 \leqslant 0$, $\beta_1+\beta_2 = -1$ and $\beta_1 < 0$, $\beta_2 > 0$, $\beta_2 = |\beta_1|$ and they can be dealt exactly as above.

5. Generalized k-step monotonization schemes

We now generalize the method to k steps.

Let $k \geqslant 1$ be an integer and for $l = 1, \dots, k$, let $0 \leqslant \gamma_l \leqslant 1$ satisfying $\sum_{l=1}^k \gamma_l = 1$. We introduce subintervals of t_n, t_{n+1} denoted by $[t_{n+\frac{l}{k}}, t_{n+\frac{l+1}{k}}]$, $l = 0, \dots, k-1$, with $t_{n+\frac{l}{k}} = t_n + \sum_{\ell=1}^l \gamma_\ell \Delta t$, $l = 1, \dots, k-1$.

Let $X_1 < X_2 < X_3$ be any three consecutive space discretization points. Thus they satisfy $X_3 - X_2 = X_2 - X_1 = h$.

We now introduce *admissible curves*. $\rho : [t_n, t_{n+1}] \to \mathbb{R}$ is said to be an admissible curve if

(1) ρ is continuous and $\rho(t_n) = X_2$,
(2) $\rho|_{[t_n+\frac{l}{k}, t_n+\frac{l+1}{k}]}$ is a line segment for $0 \leqslant l \leqslant k-1$,
(3) $\rho(t_{n+1}) \in \{X_1, X_2, X_3\}$.

Examples of admissible curves are shown in Figs. 3.1, 4.1, 5.1, 5.2.

Denote

$$\Gamma(X_1, X_2, X_3, \gamma_1, \dots, \gamma_k, \lambda) = \big\{\rho : [t_n, t_{n+1},] \to [X_1, X_3];\ \rho \text{ is admissible}\big\},$$
$$\Gamma^+(X_1, X_2, X_3, \gamma_1, \dots, \gamma_k, \lambda) = \big\{\rho \in \Gamma(X_1, X_2, X_3, \gamma_1, \dots, \gamma_k, \lambda);\ \rho(t_{n+1}) = X_3\big\},$$
$$\Gamma^0(X_1, X_2, X_3, \gamma_1, \dots, \gamma_k, \lambda) = \big\{\rho \in \Gamma(X_1, X_2, X_3, \gamma_1, \dots, \gamma_k, \lambda);\ \rho(t_{n+1}) = X_2\big\},$$
$$\Gamma^-(X_1, X_2, X_3, \gamma_1, \dots, \gamma_k, \lambda) = \big\{\rho \in \Gamma(X_1, X_2, X_3, \gamma_1, \dots, \gamma_k, \lambda);\ \rho(t_{n+1}) = X_1\big\}.$$

For $\rho \in \Gamma(X_1, X_2, X_3, \gamma_1, \dots, \gamma_k, \lambda)$ we denote by δ_l, $l = 1, \dots, k$ the slopes of the line segments of ρ on the interval $[t_{n+\frac{l}{k}}, t_{n+\frac{l+1}{k}}]$, $l = 0, \dots, k-1$, and the associated β_l are defined by $\beta_l = \frac{\lambda\gamma_l}{\delta_l}$.

For each $i \in \mathbb{Z}$ let $\rho_i \in \Gamma(x_{i-1/2}, x_{i+1/2}, x_{i+3/2}, \gamma_1, \dots, \gamma_k, \lambda)$ satisfying

(i) $\exists j \in \{-, 0, +\}$ such that $\rho_i \in \Gamma^j(x_{i-1/2}, x_{i+1/2}, x_{i+3/2}, \gamma_1, \dots, \gamma_k, \lambda)$ $\forall i \in \mathbb{Z}$.
(ii) The slopes of the ρ_i's are the same for all i's and are denoted by $\delta_1, \dots, \delta_k$.

The ρ_i's will be the lateral boundaries of the control volumes used to define the finite volume scheme.

We assume that $\{\rho_i\}_{i\in\mathbb{Z}}$ satisfy the CFL condition

$$\gamma_l \sup_{u\in[-M,M]} \big|f'(u)\big| \leqslant \min\big(|\beta_l|, (1 - |\beta_l|)\big) \quad \text{for } 1 \leqslant l \leqslant k. \tag{5.1}$$

With the notations as in (4.2) we can now define the k-step scheme as follows. Given $\{u_i^n\}$ with $|u_i^n| \leqslant M$, define inductively for $1 \leqslant l \leqslant k-1$,

$$u_i^{n+\frac{l}{k}} = \begin{cases} H_{\delta_l}(u_i^{n+\frac{l-1}{k}}, u_{i+1}^{n+\frac{l-1}{k}}) & \text{if } \delta_l > 0, \\ H_{\delta_l}(u_{i-1}^{n+\frac{l-1}{k}}, u_i^{n+\frac{l-1}{k}}) & \text{if } \delta_l < 0, \end{cases} \qquad l = 1, \dots, k-1. \tag{5.2}$$

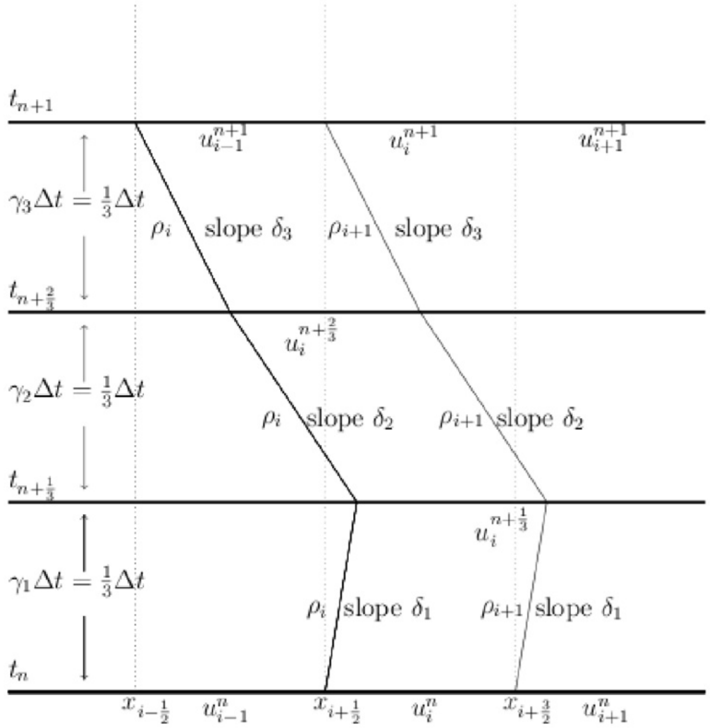

Fig. 5.1. Control volumes for a 3-step monotonization scheme, $\gamma_1 = \gamma_2 = \gamma_3 = \frac{1}{3}$, $\beta_1 = \frac{1}{3}$, $\beta_2 = -\frac{5}{6}$, $\beta_3 = -\frac{1}{2}$. $\rho_i \in \Gamma^-(x_{i-1/2}, x_{i+1/2}, x_{i+3/2}, \gamma_1, \gamma_2, \gamma_3, \lambda)$.

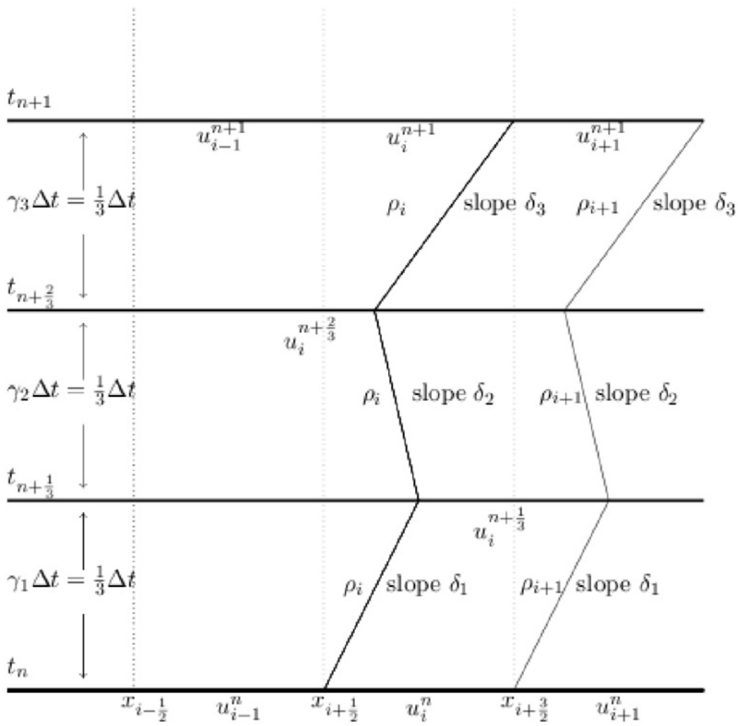

Fig. 5.2. Control volumes for a 3-step monotonization scheme, $\gamma_1 = \gamma_2 = \gamma_3 = \frac{1}{3}$, $\beta_1 = \frac{1}{2}$, $\beta_2 = -\frac{1}{3}$, $\beta_3 = \frac{5}{6}$. $\rho_i \in \Gamma^+(x_{i-1/2}, x_{i+1/2}, x_{i+3/2}, \gamma_1, \gamma_2, \gamma_3, \lambda)$.

Then

$$u_i^{n+1} = \begin{cases} H_{\delta_k}(u_{i-1}^{n+\frac{k-1}{k}}, u_i^{n+\frac{k-1}{k}}) & \text{if } \rho_i(t_{n+1}) = x_{i+3/2}, \\ H_{\delta_k}(u_i^{n+\frac{k-1}{k}}, u_{i+1}^{n+\frac{k-1}{k}}) & \text{if } \rho_i(t_{n+1}) = x_{i-1/2}, \\ H_{\delta_k}(u_{i-1}^{n+\frac{k-1}{k}}, u_i^{n+\frac{k-1}{k}}) & \text{if } \rho_i(t_{n+1}) = x_{i+1/2} \text{ and } \delta_k < 0, \\ H_{\delta_k}(u_i^{n+\frac{k-1}{k}}, u_{i+1}^{n+\frac{k-1}{k}}) & \text{if } \rho_i(t_{n+1}) = x_{i+1/2} \text{ and } \delta_k > 0. \end{cases} \tag{5.3}$$

As in Theorem 3.1, under the CFL condition (5.1), it follows easily that $H_{\delta_l}(a,b)$ is monotone in each of its variable and $H_{\delta_l}(a,a) = a$. Hence the scheme (5.2) and (5.3) converges to a unique entropy solution of problem (1.1), (1.2) if $\|u_0\| \leqslant M$ and $u_0 \in BV(\mathbb{R})$.

Example. 1. If $\delta_l \geqslant 0$, $l = 1, \dots, k$, then scheme (5.2), (5.3) can be written as follows.

Define for $l \geqslant 3$,

$$\overline{H}_2(X_1, X_2, X_3) = H_{\delta_2}\big(H_{\delta_1}(X_1, X_2), H_{\delta_1}(X_2, X_3)\big),$$
$$\overline{H}_l(X_1, X_2, \dots, X_{l+1}) = H_{\delta_l}\big(\overline{H}_{l-1}(X_1, \dots, X_l), \overline{H}_{l-1}(X_2, \dots, X_{l+1})\big),$$

and

$$F(X_1, \dots, X_k) = \gamma_1 f_{\delta_1}(X_1) + \gamma_2 f_{\delta_2}\big(H_1(X_1, X_2)\big) - \frac{X_1}{\lambda} + \sum_{l=2}^{k-1} \gamma_{l+1} f_{\delta_{l+1}}\big(\overline{H}_l(X_1, \dots, X_{l+1})\big). \tag{5.4}$$

Then

$$u_i^{n+1} = u_i^n - \lambda\big(F\big(u_i^n, u_{i+1}^n, \dots u_{i+k-1}^n\big) - F\big(u_{i-1}^n, \dots u_{i+k-2}^n\big)\big). \tag{5.5}$$

If $k = 2$, $\beta_l = \gamma_l = \frac{1}{2}$ then scheme (5.5) coincides with scheme (3.8), (3.9).

If $\beta_l = \gamma_l = \frac{1}{k}$ for $1 \leqslant l \leqslant k$, then the CFL condition (5.1) gives

$$\frac{\lambda}{k} \sup_{u \in [-M,M]} \big|f'(u)\big| \leqslant \min\left(\frac{1}{k}, \left(1 - \frac{1}{k}\right)\right) = \frac{1}{k}.$$

Hence the CFL condition reads now $\lambda \sup_{u \in [-M,M]} |f'(u)| \leqslant 1$.

If $k = 2$, $\delta_l > 0$, $l = 1, 2$, then scheme (5.5) can be written as

$$\begin{aligned} u_i^{n+1/2} &= H_{\delta_1}\big(u_i^n, u_{i+1}^n\big) = u_i^n - \gamma_1\lambda\big(f_{\delta_1}\big(u_{i+1}^n\big) - f_{\delta_1}\big(u_i^n\big)\big) \\ &= (1 - \beta_1)u_i^n + \beta_1 u_{i+1}^n - \gamma_1\lambda\big(f\big(u_{i+1}^n\big) - f\big(u_i^n\big)\big), \\ u_i^{n+1} &= (1 - \beta_2)u_{i-1}^{n+1/2} + \beta_2 u_i^{n+1/2} - \gamma_2\lambda\big(f\big(u_i^{n+1/2}\big) - f\big(u_{i-1}^{n+1/2}\big)\big). \end{aligned}$$

Furthermore if we let $\beta_1 = \beta_2 = \frac{1}{2}$, $\gamma_1 = \frac{3}{4}$, $\gamma_2 = \frac{1}{4}$, then the CFL condition (5.1) becomes

$$\lambda \sup_{u \in [-M,M]} \big|f'(u)\big| \leqslant 2/3.$$

2. We consider now the case when the slopes δ_l's change sign. Let $k = 3$, $\gamma_1 + \gamma_2 + \gamma_3 = 1$, $\beta_1 = |\beta_2| + |\beta_3|$ with $\beta_2 \leqslant 0$, $\beta_3 \leqslant\leqslant 0$, then

$$\begin{aligned} u_i^{n+1/3} &= u_i^n - \gamma_1\lambda\big(f_{\delta_1}\big(u_{i+1}^n\big) - f_{\delta_1}\big(u_i^n\big)\big), \\ u_i^{n+2/3} &= u_i^{n+1/3} - \gamma_2\lambda\big(f_{\delta_2}\big(u_i^{n+1/3}\big) - f_{\delta_2}\big(u_{i-1}^{n+1/3}\big)\big), \\ u_i^{n+1} &= u_i^{n+2/3} - \gamma_3\lambda\big(f_{\delta_3}\big(u_i^{n+2/3}\big) - f_{\delta_3}\big(u_{i-1}^{n+2/3}\big)\big). \end{aligned}$$

Let

$$F(X_1, X_2, X_3) = \gamma_1 f_{\delta_1}(X_3) + \gamma_2 f_{\delta_2}\big(H_{\delta_1}(X_2, X_3)\big) + \gamma_3 f_{\delta_3}\big(H_{\delta_2}\big(H_{\delta_1}(X_1, X_2), H_{\delta_1}(X_2, X_3)\big)\big),$$

then the scheme reads

$$u_i^{n+1} = u_i^n - \lambda\big(F\big(u_{i-1}^n, u_i^n, u_{i+1}^n\big) - F\big(u_{i-2}^n, u_{i-1}^n, u_i^n\big)\big).$$

6. Extension to systems

Consider a hyperbolic system of conservation laws

$$U_t + F(U)_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{6.1}$$

$$U(x,0) = U_0, \quad x \in \mathbb{R}, \tag{6.2}$$

where U is a n-vector and $F : \mathbb{R}^n \to \mathbb{R}^n$ a C^1-map.

For $\alpha \in \mathbb{R}$ let $X = x + \alpha t$, $\tau = t$, $V(X,\tau) = U(x,t)$. Then V satisfies

$$V_\tau + (F(V) - \alpha V)_X = 0, \quad X \in \mathbb{R},\ \tau > 0, \tag{6.3}$$

$$V(X,0) = U_0(X), \quad X \in \mathbb{R}. \tag{6.4}$$

If $\lambda(U)$ is an eigenvalue of $F'(U)$, then $(\lambda(U) - \alpha)$ is an eigenvalue of $F'(U) - \alpha I$. Hence if the eigenvalues of $F'(U)$ are bounded, then we can choose $|\alpha|$ large enough such that all the eigenvalues corresponding to (6.3) are positive. Therefore, if we have an L^∞-bound for a solution of (6.1), (6.2) then we can convert it to a solution of (6.3), (6.4) with all eigenvalues positive.

Furthermore if we define $F_\alpha(U) = F(U) - \frac{U}{\alpha}$ for $\alpha \neq 0$, then we can define the scheme (3.4) for the system (6.1), (6.2) provided that all the waves are trapped as before. In the same way k-step schemes can be defined for systems. Their advantage is that they are point evaluation schemes. One can expect a better accuracy by going to k-step schemes and choosing proper γ_l's and δ_l's.

This may also extend to the multidimensional case and with a diffusion term on the right-hand side.

7. Conclusion

Using the technique of monotonization we showed how to construct a family of multistep schemes with only point value evaluations of the flux function. This family includes the Force and proposes an alternative for the Musta schemes. For all these schemes we proved convergence of the approximate solution to the entropy solution of the continuous problem. We also gave hints on how to extend them to systems and high resolution schemes. In forthcoming papers we will extend these schemes to higher resolution schemes and to the discontinuous flux case. We will also give an example of application to a 2×2 system of conservation laws representing a problem of polymer flooding.

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